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Two-dimensional links and diffeomorphisms on 3-manifolds

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Abstract. For diffeomorphisms on 3-manifolds, two-dimensional links composed of tori (and possibly one Klein bottle) embedded in $S^2 \times S^1$ appear in a natural way to describe the topological position of two-dimensional unstable manifolds of saddles in the basin of a periodic sink. For such a link $\mathcal{E}$ we build a diffeomorphism $f_{\mathcal{E}}$ which is used as a canonical model for 'pieces of the dynamics', namely the basin of attractors consisting of a sink and a finite set of two-dimensional unstable manifolds of saddles.

0. *Introduction*

The classification of Morse–Smale diffeomorphisms on 3-manifolds up to topological conjugacy needs to take into account the topological configuration of the invariant manifolds of the saddles. Thus knots appear in a natural way as topological invariants, even in the simplest case: when the non-wandering set is reduced to a unique repelling fixed point, a unique saddle periodic orbit with a two-dimensional stable manifold and two attracting periodic orbits, [**BoGr**] shows that a knot (in $S^2 \times S^1$) is canonically associated to each topological conjugacy class and that this knot is a complete invariant of the class (see [**BoGr**]).

Here we deal with diffeomorphisms having more than one saddle. We show that two-dimensional links in $S^2 \times S^1$ made of tori and possibly one Klein bottle naturally arise as invariants of topological conjugacy classes of diffeomorphisms.

Let us be more precise. Let f be an orientation-preserving diffeomorphism of some compact 3-manifold M. Let ω be some attracting hyperbolic fixed point of f and $\Sigma = \{\sigma_k\}$ an f-invariant finite set of periodic hyperbolic saddle points with two-dimensional unstable

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manifolds such that $W^u(\sigma_k) \setminus \{\sigma_k\}$ is contained in the basin of the attractor ω. The quotient space $N = (W^s(\omega) \setminus \{\omega\})/f$ is diffeomorphic to $S^2 \times S^1$ (see for instance [**BoGr**]). By hypothesis $W^u(\sigma_k) \setminus \{\sigma_k\}$ is included in $W^s(\omega) \setminus \{\omega\}$, and its projection E_k in N is a torus or a Klein bottle smoothly embedded in N according to whether f preserves the orientation on $W^u(\sigma_k)$ or not. We show that the image of the fundamental group of E_k in $\mathbb{Z} = \pi_1(S^2 \times S^1)$ is not trivial. Since all the unstable manifolds $W^u(\sigma_k)$ are disjoint, the collection $\{E_k\}$ forms a two-dimensional link $\mathcal{E}(\omega, \Sigma)$ in N (see the precise definition in §1.3). The identification diffeomorphism between N and $S^2 \times S^1$ is not unique: given two of them, they differ by an element in the group $\mathrm{Diff}^*_+(S^2 \times S^1)$ of orientation-preserving diffeomorphisms of $S^2 \times S^1$ whose action on the fundamental group is trivial. As a consequence, we say that two two-dimensional links are equivalent if there exists an element of $\mathrm{Diff}^*_+(S^2 \times S^1)$ which sends one link onto the other. This equivalence relation differs from the isotopy equivalence by the action of $\pi_0(\mathrm{Diff}^*_+(S^2 \times S^1))$ which is $\mathbb{Z}/2\mathbb{Z}$ (see [**BoGr**]). We will show (see Lemma 1.24) that the equivalence class of the link $\mathcal{E}(\omega, \Sigma)$ is an invariant under topological conjugacy.

Intuitively, this invariant describes the topological position of the invariant manifolds of the saddles into the fundamental domain at the sink. In the case of flows, the existence in the neighborhood of a sink, of 2-spheres transverse to the flow indicates that the corresponding two-dimensional link is always trivial (the unique invariant is the number of components of the link). As a consequence, the problem of classifying the flows is of global nature. We refer the reader to [**Fl**] for the classification of gradient-like flows on 3-manifolds.

We will see that $\mathcal{E}(\omega, \Sigma)$ contains at most one Klein bottle. Moreover, Proposition 1.4 shows that all embedded Klein bottles in $S^2 \times S^1$ are equivalent (up to $\mathrm{Diff}^*_+(S^2 \times S^1)$). As a consequence we obtain the following.

COROLLARY 0.1. *Let f be an orientation-preserving diffeomorphism on a 3-manifold M and let ω be a hyperbolic attracting periodic orbit of f of period p. Then there is at most one periodic orbit of saddle σ having a two-dimensional unstable manifold such that $W^u(\sigma) \setminus \sigma$ is contained in the basin of ω, and such that f^p reverses the orientation of $W^u(\sigma)$. Moreover, the period of σ is equal to p.*

A two-dimensional link in $S^2 \times S^1$ can be really sophisticated (see Figure 1), so that it is natural to ask which ones appear as links associated to a diffeomorphism at a sink. In fact, we show that *all* two-dimensional links are realizable in this way by very simple Morse–Smale diffeomorphisms of the 3-sphere. More precisely, we get the following.

THEOREM 1. *Given any two-dimensional link $\mathcal{E} \subset S^2 \times S^1$ whose connected components are tori and possibly one Klein bottle with non-trivial fundamental group in $\mathbb{Z} = \pi_1(S^2 \times S^1)$, there is a Morse–Smale diffeomorphism $f_{\mathcal{E}} : S^3 \to S^3$ such that:*

(1) *the non-wandering set $\Omega(f)$ consists of one fixed sink ω, a set Σ of saddles σ_i with two-dimensional unstable manifolds, and some periodic sources;*

(2) *there are no heteroclinic intersections, so that for any i one has $W^u(\sigma_i) \setminus \{\sigma_i\} \subset W^s(\omega)$;*

(3) *$\mathcal{E}(\omega, \Sigma)$ is equivalent to $\mathcal{E}$.*

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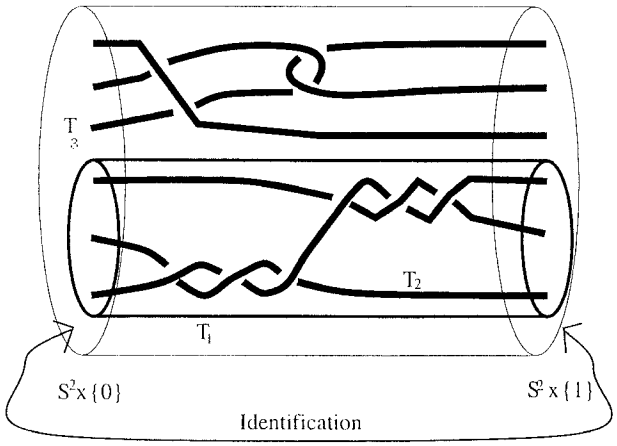


FIGURE 1. An example of a two-dimensional link in $S^2 \times S^1$ with three components.

A Morse–Smale diffeomorphism on a compact manifold M satisfying properties (1) and (2) of the theorem is called a $\mathcal{G}$-diffeomorphism (see Definition 2.1).

The $\mathcal{G}$-diffeomorphisms $f_{\mathcal{E}}$ that we build for Theorem 1 can be used as 'canonical models' for 'pieces of the dynamics' of any diffeomorphism.

More precisely, let us consider a diffeomorphism $f : M \to M$ on some compact 3-manifold M and let ω be some hyperbolic fixed sink of f. Let $\Sigma = \{\sigma_1, \ldots, \sigma_k\}$ be an f-invariant set of periodic hyperbolic saddles such that the unstable manifold $W^u(\sigma_i)$ is two-dimensional and $W^u(\sigma_i) \setminus \{\sigma_i\}$ is contained in the basin of ω, $W^s(\omega)$. Notice that $\Omega = \omega \cup \bigcup_{i=1}^{k} W^u(\sigma_i)$ is a topological (non-transitive) attractor for f, and let $W^s(\Omega)$ denote its basin.

We first notice that the link $\mathcal{E}(\omega, \Sigma)$ is invariant by topological conjugacy on the basin of Ω (see Lemma 1.24). Moreover it is a complete invariant, as stated in the following theorem.

THEOREM 2. *With the notation above, the restriction of f to the basin $W^s(\Omega)$ is conjugate to the restriction of the model $f_{\mathcal{E}}$ to $S^3 \setminus \{\alpha_1, \ldots, \alpha_l\}$ where the α_i's are all sources of $f_{\mathcal{E}}$, and where $\mathcal{E}$ is any link equivalent to $\mathcal{E}(\omega, \Sigma)$.*

As a consequence, we first obtain that the model map $f_{\mathcal{E}}$ of an equivalence class of link $\mathcal{E}$ is unique up to topological conjugacy. Then we obtain a classification of the 3-manifolds admitting Morse–Smale diffeomorphisms without any heteroclinic points or curves.

COROLLARY 0.2. *A compact orientable 3-manifold M admits orientation-preserving Morse–Smale diffeomorphisms without any heteroclinic point or curve if and only if M is diffeomorphic to the sphere S^3 or to the connected sum of finitely many copies of $S^2 \times S^1$:*

$$M = S^3 \quad or \quad M = \sharp_{i=1}^{k}(S^2 \times S^1 \times \{i\}).$$

This classification has also been obtained in [**BoGrMePé**] by very different methods. For gradient-like flows, the fact that the local invariant at the sink is always trivial (see the

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discussion above) allows us to get the previous result by using arguments in the spirit of Morse Theory: see the work [**dR**].

1. *Topological context*

Let S^n denote the unit sphere in $\mathbb{R}^{n+1}$ and let D^2 denote the closed unit disk in $\mathbb{R}^2$. We define the standard torus $T^2 = S^1 \times S^1$. Finally, let $\mathbf{K}$ stand for the standard Klein bottle. In general, we call a sphere, torus, solid torus or Klein bottle any smooth embedding of S^n, T^2, $D^2 \times S^1$ or $\mathbf{K}$, respectively.

Recall that the Klein bottle is the square $[0, 1] \times [0, 1]$ with pairwise opposite sides identified through the relations

$$(x, 0) \equiv (x, 1) \quad \text{and} \quad (0, y) \equiv (1, 1 - y).$$

Let a be the homotopy class on $\mathbf{K}$ of the closed curve $t \mapsto (t, \frac{1}{2})$, $t \in S^1$, and let b be the homotopy class of the closed curve $t \mapsto (0, \frac{1}{2} + t)$. Then $\pi_1(\mathbf{K})$ is the group $\langle a, b | aba^{-1} = b^{-1} \rangle$. One verifies easily that each element z of this group can be written in a unique way as $z = a^n b^m$, $m, n \in \mathbb{Z}$ and that $\langle b \rangle \simeq \mathbb{Z}$ is the kernel of the natural morphism $\pi_1(\mathbf{K}) \to H_1(\mathbf{K}, \mathbb{R})$.

In this paper we will consider special classes of torus embeddings in $S^2 \times S^1$, namely those whose fundamental group embeds non-trivially. More precisely, we have the following definition.

Definition 1.1. Given an embedding T of the torus T^2 into $S^2 \times S^1$, we say that T *has type* $k \in \mathbb{N}$ if the linear map $T_* : \pi_1(T^2) \longrightarrow \mathbb{Z} = \pi_1(S^2 \times S^1)$ has image $k\mathbb{Z}$.

By abuse of language, T might denote the image $T(T^2)$ and might itself be called a torus.

1.1. *Bounding tori.* In this section, we will make constant use of the following proposition.

PROPOSITION 1.2. *Let $T : T^2 \to S^2 \times S^1$ (respectively $T : T^2 \to D^2 \times S^1$) be a smooth embedding. If $T_*(\pi_1(T^2))$ is non-trivial in $\pi_1(S^2 \times S^1)$ (respectively in $\pi_1(D^2 \times S^1)$) then T bounds a solid torus in $S^2 \times S^1$ (respectively $D^2 \times S^1$).*

The proof of Proposition 1.2 has been written in [**BoGr**] in the case of an embedding in $S^2 \times S^1$, and the proof in the case of an embedding in $D^2 \times S^1$ follows exactly the same argument.

Remark 1.3. Recall (see [**Ro**]) that any embedded torus in S^3 bounds a solid torus. As the following example shows, this is not true in $S^2 \times S^1$. Take a fiber $F = S^2 \times \{t_0\}$ and construct a torus T by removing two disks and adding a 2-handle. Note that, by consideration of orientability, T has to have type 0. Now, T cannot be the boundary of a 3-chain, since it is homologous to F, which is not homologous to zero in $S^2 \times S^1$. One can also easily build a torus embedded in $S^2 \times S^1$ homotopic to a point, which does not bound a solid torus. Notice that in both examples the tori have type 0. The connection of these facts with Proposition 1.2 can be understood by describing the relation between the type and the second homology group of a smoothly embedded torus $T : T^2 \to S^2 \times S^1$. We have:

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- if T has non-trivial type then the image of $H_2(T^2, \mathbb{Z})$ in $H_2(S^2 \times S^1, \mathbb{Z})$ is trivial (this follows (for instance) directly from Proposition 1.2);
- if T has trivial type then either T is homotopic to a point or the image of $H_2(T^2, \mathbb{Z})$ in $H_2(S^2 \times S^1, \mathbb{Z})$ is not trivial.

Proof of the second statement. As the fundamental group of T vanishes in $\pi_1(S^2 \times S^1)$, there is a lift $\tilde{T}$ of T into the universal cover $S^2 \times \mathbb{R}$ of $S^2 \times S^1$, and $\tilde{T}$ is an embedded torus. If it separates the two ends of $S^2 \times \mathbb{R}$ then the intersection number of T with a circle $\{p\} \times S^1$ is equal to the intersection number of $\tilde{T}$ with a straight line $\{p\} \times \mathbb{R}$ which is 1, so that T is not zero in $H_2(S^2 \times S^1, \mathbb{Z})$. In the other case, there is some embedded line L disjoint from $\tilde{T}$ and joining the two ends of $S^2 \times \mathbb{R}$. Moreover, one can choose L coinciding with the straight line $\{p\} \times \mathbb{R}$ out of some compact set. If we identify $S^2 \times \mathbb{R}$ with $S^3 \setminus \{N, S\}$ then the closure of L in S^3 is a smooth segment $\bar{L} = L \cup \{N, S\}$. Then $\tilde{T}$ is included in $S^3 \setminus \bar{L}$ which is diffeomorphic to $\mathbb{R}^3$. So $\tilde{T}$ is homotopic to zero in $S^2 \times \mathbb{R}$ and T is homotopic to zero in $S^2 \times S^1$. □

1.2. *Embedded Klein bottles in $S^2 \times S^1$.* While it is impossible to embed the Klein bottle in $\mathbb{R}^3$, the following is an explicit example of an embedding of $\mathbf{K}$ into $S^2 \times S^1$.

Consider two antipodal points on the 2-sphere S^2, the north pole N and the south pole S, and call $\mathcal{R}_{(NS)}(t)$ the rotation with axis (NS) and angle πt. Choose a great circle C_0 in S^2 passing through N and S. In $S^2 \times S^1$, consider

$$\bigcup_{0 \le t \le 1} \mathcal{R}_{(NS)}(t)(C_0) \times \{2\pi t\}.$$

This is a closed surface and in fact it is a Klein bottle. In the following, this model will be denoted by $\mathcal{K}$.

PROPOSITION 1.4. *For any Klein bottle K smoothly embedded in $S^2 \times S^1$ there is an orientation preserving diffeomorphism $\phi \in \mathrm{Diff}_+(S^2 \times S^1)$ such that $\phi \circ K = \mathcal{K}$. Moreover, one can choose ϕ such that $\phi_* : \pi_1(S^2 \times S^1) \to \pi_1(S^2 \times S^1)$ is the identity.*

Proof. The proof of this proposition is the aim of this section.

Let V be some tubular neighborhood of K. As K is a non-orientable submanifold of the orientable manifold $S^2 \times S^1$ it is not transversally orientable, so that V is the non-orientable fibration by segments $[-1, 1]$ over K, such that the action of $\pi_1(K)$ on the orientations of the fibers coincides with the action of $\pi_1(K)$ on the local orientations of the tangent bundle of K.

As V retracts by deformation on K, its fundamental group is the same as $\mathbf{K}$ so we will always denote its generators by a and b.

In particular, the boundary ∂V is some embedded torus T in $S^2 \times S^1$. One can choose the generators c and d of $\pi_1(T)$ such that the natural morphism (induced by the inclusion $T \subset V$) on the fundamental groups is defined by $c \mapsto a^2$ and $d \mapsto b$. Moreover, this morphism is injective. Thus we will identify $c = a^2$, $d = b$ and $\pi_1(T) = a^2\mathbb{Z} \oplus b\mathbb{Z}$.

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LEMMA 1.5. *The image in $\pi_1(S^2 \times S^1)$ of the fundamental group of the torus $T = \partial V$ is not trivial. More precisely,*

$$T_*(a^2) \neq 0 \in \pi_1(S^2 \times S^1) \quad \textit{and} \quad T_*(b) = 0 \in \pi_1(S^2 \times S^1).$$

Proof. Consider the pre-image of K in the universal cover $\Pi : S^2 \times \mathbb{R} \to S^2 \times S^1$: as there is no embedded Klein bottle in $S^2 \times \mathbb{R} \subset S^3$, a connected component $\tilde{K}$ of $\Pi^{-1}(K)$ is a non-trivial cover of K, so that there is some automorphism ψ of the universal cover $\Pi : S^2 \times \mathbb{R} \to S^2 \times S^1$ leaving $\tilde{K}$ invariant. As ψ has infinite order, $\tilde{K}$ is not compact. Consider now the lift $\tilde{V}$ of V which is a tubular neighborhood of $\tilde{K}$. Each connected component of its boundary is a connected component of $\Pi^{-1}(T)$ and is not compact. As a consequence, the morphism $T_* : \pi_1(T) \to \pi_1(S^2 \times S^1) = \mathrm{Aut}(\Pi)$ is not trivial (where $\mathrm{Aut}(\Pi)$ denotes the group of the automorphisms of the universal cover $\Pi : S^2 \times \mathbb{R} \to S^2 \times S^1$).

Furthermore, as the group of automorphisms of Π is isomorphic to $\mathbb{Z}$, two elements of $\pi_1(K)$ in the same homology class in $H_1(K, \mathbb{R})$ have the same image by T_*. So the image of b is zero. □

COROLLARY 1.6. *The complement of the interior of a tubular neighborhood of K is diffeomorphic to $D^2 \times S^1$.*

Proof. From Proposition 1.2 we get that T splits $S^2 \times S^1$ into two components, one of them being diffeomorphic to $D^2 \times S^1$ and one of them being the tubular neighborhood V of K. Clearly V is not diffeomorphic to $D^2 \times S^1$. □

Remark 1.7. The tubular neighborhood V is diffeomorphic to the quotient of $T^2 \times [0, 1]$ by the equivalence relation $((x, y, 0) \equiv (x + 1/2, -y, 0))$ (where x and y belong to $S^1 = \mathbb{R}/\mathbb{Z}$).

LEMMA 1.8. *There is an orientation-reversing diffeomorphism φ of $T_1 = \partial(D^2 \times S^1)$ conjugate to the map $(x, y) \mapsto (x + 1/2, -y)$, and there is a homeomorphism $h : S^2 \times S^1 \to D^2 \times S^1/\varphi$ such that $h(K) = T_1/\varphi$, where $D^2 \times S^1/\varphi$ denotes the quotient of $D^2 \times S^1$ by the equivalence relation whose classes are $\{x, \phi(x)\}$ for $x \in T_1 = \partial(D^2 \times S^1)$ which is reduced to $\{x\}$ if $x \notin \partial(D^2 \times S^1)$.*

Proof. Let $\phi_0 : D^2 \times S^1 \to S^2 \times S^1$ be an orientation-preserving diffeomorphism onto the complement of the interior of the tubular neighborhood V. By definition, a tubular neighborhood is a fibration over K whose fibers are segments. Let us denote by ϕ_1 the diffeomorphism of ∂V which sends the extremity of a fiber onto the other extremity of the same fiber. By the Remark 1.7, ϕ_1 is conjugated to $(x, y) \mapsto (x + 1/2, -y)$. The diffeomorphism φ is defined on $\partial D^2 \times S^1$ as $\phi_0^{-1} \circ \phi_1 \circ \phi_0$ and so is conjugated to $(x, y) \mapsto (x + 1/2, -y)$. Finally, the announced homeomorphism h^{-1} is obtained by composing ϕ_0 by a retraction of each fiber of V on the corresponding point of K. □

COROLLARY 1.9. *If K_1 and K_2 are two embeddings of $\mathbf{K}$ in $S^2 \times S^1$ then $K_1 \cap K_2 \neq \emptyset$.*

Proof. If $K_1 \cap K_2 = \emptyset$ then K_2 is contained in the complement of a tubular neighborhood of K_1, which is a solid torus. This is a contradiction, since no Klein bottle can be embedded in a solid torus. □

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COROLLARY 1.10. *The image of the generator $a \in \pi_1(K)$ in $\pi_1(S^2 \times S^1)$ is ± 1.*

Proof. Let h_1 denote the automorphism of the universal cover $\Pi : S^2 \times \mathbb{R} \to S^2 \times S^1$ corresponding to the homotopy class $1 \in \mathbb{Z} = \pi_1(S^2 \times S^1)$.

Let $k = |T_*(a)| \in \mathbb{N}$. The pre-image $\Pi^{-1}(K) \subset S^2 \times \mathbb{R}$ consists of k connected components $\tilde{K}^1, \ldots, \tilde{K}^k$, each of them being invariant by h_1^k; more precisely let k denote the smallest period of each component $\tilde{K}^i$. We deduce that K, which is by construction the quotient $(\bigcup_1^k \tilde{K}^i)/h_1$, is also the quotient $\tilde{K}^i/h_1^k$.

Now we consider the quotient $S^2 \times \mathbb{R}/h_1^k$. It is diffeomorphic to $S^2 \times S^1$ and the projection of $\Pi^{-1}(K)$ in this quotient consists of k disjoint copies of the Klein bottle, smoothly embedded in $S^2 \times \mathbb{R}/h_1^k$. By Corollary 1.9, we get that $k = 1$. □

We can now finish the proof of Proposition 1.4: $S^2 \times S^1$ is obtained by gluing V with $D^2 \times S^1$ by some diffeomorphism ϕ of the boundary $T = \partial V$. Recall that $\pi_1(T)$ is $a^2\mathbb{Z} \oplus b\mathbb{Z}$ (where a and b are generators of $\pi_1(K)$ which we have chosen).

CLAIM 1.1. *The image by ϕ_* of $\partial D^2 \times \{0\}$ is homotopic to b or b^{-1}.*

Proof. Notice that $\partial D^2 \times \{0\}$ is homotopic to zero in $D^2 \times S^1$ and *a fortiori* in $S^2 \times S^1$. So its image by ϕ_* belongs to the kernel of the inclusion of $\pi_1(T)$ in $\pi_1(S^2 \times S^1)$. By Lemma 1.5, then $\phi_*(\partial D^2 \times \{0\})$ belongs to $b\mathbb{Z}$. The image by ϕ of a simple closed curve is a simple closed curve, so $\phi_*(\partial D^2 \times \{0\}) = b^{\pm 1}$. □

Notice that $\phi_*^{-1}(a^2)$ is a longitude of $D^2 \times S^1$. As there is a diffeomorphism of $D^2 \times S^1$ sending an arbitrary longitude on $\{x\} \times S^1$, $x \in \partial D^2$ (up to a change of the orientation of the longitude), we get that $(S^2 \times S^1, K)$ is diffeomorphic to the manifold obtained by gluing V with $D^2 \times S^1$ by some diffeomorphism ψ sending b on $\partial D^2 \times \{0\}$ and a^2 on the longitude $\{x\} \times S^1$, that is $(S^2 \times S^1, K)$ is diffeomorphic to $(S^2 \times S^1, \mathcal{K})$. Notice that since there exists an orientation-reversing diffeomorphism of $S^2 \times S^1$ preserving $\mathcal{K}$, we can now choose the diffeomorphism ψ which preserves the orientation. Finally, there is an orientation-preserving diffeomorphism of $S^2 \times S^1$ leaving $\mathcal{K}$ invariant and whose action on $\pi_1(S^2 \times S^1) \simeq \mathbb{Z}$ sends a positive generator to a negative generator; so up to composition of ψ by such a diffeomorphism, one can assume that the action of ψ on $\pi_1(S^2 \times S^1)$ is trivial. □

1.3. *Two-dimensional link associated to a sink.* A collection $\mathcal{E} = \{E_1, \ldots, E_k\}$ in $S^2 \times S^1$ in which at most one element E_i is a Klein bottle and all other elements are tori is called a *two-dimensional link* if no two components of the link intersect and each one has type different from 0 (that is, the image of its fundamental group in $\pi_1(S^2 \times S^1)$ is not trivial). Recall that a torus is the image of an embedding of the standard torus T^2.

Remark 1.11. We are aware that the expression 'two-dimensional link' usually refers to a collection of 2-spheres embedded in a 4-manifold. Nevertheless, since in this paper no 4-manifolds are involved, we think that the reader will not be confused by our definition, which is very natural in our context.

We will denote by $\tilde{\mathcal{E}}$ the equivalence class of $\mathcal{E}$ under ambient isotopy equivalence.

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LEMMA 1.12. *Let $\mathcal{E} = \{E_1, \dots, E_k\}$ be a two-dimensional link.*

(i) *If the link contains a Klein bottle, each torus E_i bounds a unique solid torus Γ_i, and any two of them are either disjoint or included one in the other, that is*

$$\Gamma_i \subset \Gamma_j \quad or \quad \Gamma_j \subset \Gamma_i \quad or \quad \Gamma_i \cap \Gamma_j = \emptyset.$$

(ii) *If none of the components of $\mathcal{E}$ is a Klein bottle, there is a family $\Gamma_i \subset S^2 \times S^1$ of smoothly embedded solid tori such that E_i is the boundary of Γ_i and such that any two of them are either disjoint or included one in the other.*

Proof. (i) Assume that E_k is a Klein bottle. If E_i, $i \neq k$, bounds two solid tori then E_k is contained in one of them. As a Klein bottle cannot be embedded in a solid torus, that gives a contradiction. So each E_i, $i \neq k$, bounds a unique solid torus Γ_i which is the component of $S^2 \times S^1 \setminus E_i$ which does not contain E_k.

Assume that $\Gamma_i \cap \Gamma_j \neq \emptyset$. As the boundaries E_i and E_j cannot intersect, one deduces that (up to interverting i and j) $E_i \subset \Gamma_j$. Then one of the components of $S^2 \times S^1 \setminus E_i$ is included in Γ_j so it is disjoint from E_k; therefore its closure is $\Gamma_i \subset \Gamma_j$.

(ii) Each torus E_i bounds one or two solid tori. Let $\mathcal{C}$ be the set of all these solid tori, and consider $C_0 \in \mathcal{C}$ as a maximal element for the partial order given by inclusion. Then for each E_i we define the solid torus Γ_i as follows. If E_i is contained in C_0 then it bounds a solid torus $\Gamma_i \subset C_0$ (see Proposition 1.2). If E_i is not contained in C_0 and so is disjoint from C_0 then the connected component of $S^2 \times S^1 \setminus E_i$ containing C_0 cannot be a solid torus, because C_0 is maximal for the inclusion. Then E_i bounds a unique solid torus Γ_i and Γ_i is disjoint from C_0.

Assume now that for some i and j, $\Gamma_i \cap \Gamma_j \neq \emptyset$. As in the previous case, up to exchanging i and j, one gets that $E_i \subset \Gamma_j$. If Γ_i or Γ_j coincides with C_0 then we conclude that it contains the other, since C_0 is maximal. In the other case, Γ_i and Γ_j are the closures of the connected components of $S^2 \times S^1 \setminus E_i$ and $S^2 \times S^1 \setminus E_j$, respectively, which do not contain ∂C_0, and so $\Gamma_i \subset \Gamma_j$. □

Remark 1.13. In the case of Lemma 1.12(ii), the family of solid tori Γ_i is not necessarily unique.

Definition 1.14. We say that a family $\{\Gamma_i\}$ of solid tori is adapted to the link $\mathcal{E} = \{E_i\}$ or $\mathcal{E} = \{K\} \cup \{E_i\}$, if $E_i = \partial\Gamma_i$ and $\Gamma_i \subset \Gamma_j$ or $\Gamma_j \subset \Gamma_i$ or $\Gamma_i \cap \Gamma_j = \emptyset$ for all i and j.

We say that the family $\{\Gamma_i\}$ is *well-ordered* if the indexation is increasing for the partial order given by inclusion, that is

$$i < j \Longrightarrow \Gamma_i \subset \Gamma_j \quad \text{or} \quad \Gamma_i \cap \Gamma_j = \emptyset.$$

Definition 1.15. We say that the link $\mathcal{E} = \{E_1, \dots, E_k\}$ is *well ordered* if it satisfies the following conditions:

- if it contains a Klein bottle then it is the component E_k and there exists a well-ordered family of solid tori adapted to $\{E_1, \dots, E_{k-1}\}$;
- if it does not contain a Klein bottle then there exists a well-ordered family of solid tori adapted to $\{E_1, \dots, E_k\}$.

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Let f be an orientation-preserving diffeomorphism on a compact 3-manifold M having an attracting hyperbolic fixed point ω. We call *the space of orbits* of f at ω the quotient space

$$N(f,\omega) = (W^s(\omega) \setminus \{\omega\})/f.$$

Each point in $N(f,\omega)$ is an orbit $\{x, f(x), \dots, f^n(x), \dots\}$.

LEMMA 1.16. *The space of orbits $N(f,\omega)$ is diffeomorphic to $S^2 \times S^1$. More precisely, let $\tau : (p,t) \mapsto (p,t+1)$ be the translation on $S^2 \times \mathbb{R}$. There exists an orientation-preserving diffeomorphism $\Psi(f,\omega) : W^s(\omega) \setminus \{\omega\} \longrightarrow S^2 \times \mathbb{R}$ such that*

$$\Psi(f,\omega) \circ f = \tau \circ \Psi(f,\omega).$$

Thus $\Psi(f,\omega)$ induces a diffeomorphism $\psi(f,\omega)$ on the quotient spaces

$$\psi(f,\omega) : N(f,\omega) \longrightarrow S^2 \times S^1.$$

(See [**BoGr**] for a proof.)

Remark 1.17. An arc in $W^s(\omega) \setminus \{\omega\}$ joining a point x to $f(x)$ projects into $N(f,\omega)$ onto a closed path whose homotopy class is 1. More generally, if an arc joins two points of the same orbit, then it projects onto a closed arc and its homotopy class is given by the number of iterations between two points of the same orbit.

Definition 1.18. A *positive generator* of $\pi_1(N(f,\omega))$ is the homotopy class of a path corresponding to a segment in $W^s(\omega) \setminus \{\omega\}$ joining some point x to $f(x)$.

Remark 1.19. If $\psi(f,\omega) : N(f,\omega) \to S^2 \times S^1$ and $\psi'(f,\omega) : N(f,\omega) \to S^2 \times S^1$ are two diffeomorphisms given by Lemma 1.16, then $\phi = \psi'(f,\omega) \circ \psi^{-1}(f,\omega)$ lies in the group $\mathrm{Diff}^*_+(S^2 \times S^1)$ of orientation-preserving diffeomorphisms on $S^2 \times S^1$ whose action on the fundamental group of $S^2 \times S^1$ is trivial. It was shown in [**BoGr**] that this group contains exactly two connected components, that is $\pi_0(\mathrm{Diff}^*_+(S^2 \times S^1)) = \mathbb{Z}/2\mathbb{Z}$.

LEMMA 1.20. *Choose some diffeomorphism $\psi : N(f,\omega) \to S^2 \times S^1$. Suppose that f has an invariant set of saddle periodic points $\Sigma = \{\sigma_1, \dots, \sigma_k\}$, each with a two-dimensional unstable manifold such that $W^u(\sigma_i) \setminus \{\sigma_i\}$ is contained in the basin of ω. Then, the image by ψ of their projections into $N(f,\omega)$ gives a two-dimensional link $\mathcal{E}_f(\omega, \Sigma, \psi) \subset S^2 \times S^1$.*

Proof. Suppose that σ_i is a saddle fixed point. Then the quotient space $(W^u(\sigma_i) \setminus \{\sigma_i\})/f$ is a torus or a Klein bottle according to whether the restriction of f to $W^u(\sigma_i)$ preserves the orientation on $W^u(\sigma_i)$ or not. Moreover, the image of the fundamental group of the quotient space $(W^u(\sigma_i) \setminus \{\sigma_i\})/f$ in $\pi_1(S^2 \times S^1)$ is onto.

Suppose now that σ_i is a periodic point of period $l > 1$. Then the projection into $N(f,\omega)$ of $W^u(\sigma_i) \setminus \{\sigma_i\}$ is the quotient $(W^u(\sigma_i) \setminus \{\sigma_i\})/f^l$, which could be *a priori* a torus or a Klein bottle, as in the previous case. However, the image of its fundamental group in $\mathbb{Z} = \pi_1(S^2 \times S^1)$ is $l\mathbb{Z}$. Then, by Corollary 1.10, $(W^u(\sigma_i) \setminus \{\sigma_i\})/f^l$ cannot be a Klein bottle and so it is a torus.

To finish the proof just note that the projections of any two different two-dimensional unstable manifolds cannot intersect. □

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Definition 1.21. We say that two isotopy classes of links $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}'$ are equivalent if for $\mathcal{E} \in \tilde{\mathcal{E}}$ and $\mathcal{E}' \in \tilde{\mathcal{E}}'$ there exists a diffeomorphism $\phi \in \mathrm{Diff}^*_+(S^2 \times S^1)$ such that $\phi(\mathcal{E}) = \mathcal{E}'$.

It is clear that this definition does not depend on the choice of the links in each isotopy class.

Remark 1.22. From Remark 1.19, the group $\pi_0(\mathrm{Diff}^*_+(S^2 \times S^1)) = \mathbb{Z}/2\mathbb{Z}$ acts in a natural way on the set of isotopy classes of links and two isotopy classes are equivalent if and only if they belong to the same orbit of this action.

Remark 1.23. One can give a natural but very particular example of a two-dimensional link in $S^2 \times S^1$ by considering the boundary of a tubular neighborhood of a geometrical braid of S^2 (see [**Bi**] for a definition and a survey of the elementary properties of the subject of braid groups of a sphere). Those two-dimensional links have no Klein bottle component and all the components bound disjoint solid tori. It is clear from the definitions that if two braids are equivalent then they are also equivalent, when understood as two-dimensional links, in the sense of Definition 1.21.

Nevertheless, sphere braids are a very particular class of two-dimensional links. For instance, any non-trivial knot in $S^2 \times S^1$ which is homotopic to the trivial knot $\{x\} \times S^1$ cannot represent a braid.

Let us call $\tilde{\mathcal{E}}_f(\omega, \Sigma, \psi)$ the isotopy class of $\mathcal{E}_f(\omega, \Sigma, \psi)$. We denote by $\mathcal{E}_f(\omega, \Sigma)$ the equivalence class of $\tilde{\mathcal{E}}_f(\omega, \Sigma, \psi)$ (this class does not depend on the choice of the diffeomorphism ψ), and by abuse of language we will call it the link associated to f at ω, Σ.

The link $\mathcal{E}_f(\omega, \Sigma)$ is an invariant under topological conjugacy in the neighborhood of the sink ω.

LEMMA 1.24. *Let f and f' be two diffeomorphisms of 3-manifolds, let ω and ω' be hyperbolic fixed sinks of f and f', respectively, and let Σ and Σ' be invariant sets of hyperbolic saddle periodic points of f and f' such that for each periodic point σ in Σ or $\sigma' \in \Sigma'$ the unstable manifold $W^u(\sigma)$ (respectively $W^u(\sigma')$) is two-dimensional and the punctured unstable manifold $W^u(\sigma) \setminus \{\sigma\}$ (respectively $W^u(\sigma') \setminus \{\sigma'\}$) is contained in the basin of ω (respectively ω').*

Assume that there is an orientation-preserving homeomorphism $h : W^s(\omega) \to W^s(\omega')$, conjugating f and f' and mapping the punctured unstable manifolds of the saddles in Σ bijectively onto the punctured unstable manifolds of the saddles in Σ'. Then $\mathcal{E}_f(\omega, \Sigma) = \mathcal{E}_{f'}(\omega', \Sigma')$.

Proof. Choose two diffeomorphisms $\psi : (W^s(\omega) \setminus \{\omega\})/f \to S^2 \times S^1$ and $\psi' : (W^s(\omega') \setminus \{\omega'\})/f' \to S^2 \times S^1$.

Since the homeomorphism h conjugates f to f', it projects onto a homeomorphism $\tilde{h} : (W^s(\omega) \setminus \{\omega\})/f \to (W^s(\omega') \setminus \{\omega'\})/f'$. Let $\bar{h} : S^2 \times S^1 \to S^2 \times S^1$ be defined by $\bar{h} = \psi' \circ \tilde{h} \circ \psi^{-1}$. The action $\bar{h}_*$ induced by $\bar{h}$ on the fundamental group of $S^2 \times S^1$ sends a positive generator onto a positive generator. This homeomorphism $\bar{h}$ induces a one-to-one correspondence between the components of the link $\mathcal{E}_f(\omega, \Sigma, \psi)$ and the components of the link $\mathcal{E}_{f'}(\omega', \Sigma', \psi')$, that is $\bar{h}(\mathcal{E}_f(\omega, \Sigma, \psi)) = \mathcal{E}_{f'}(\omega', \Sigma', \psi')$. To see that the links

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$\mathcal{E}_f(\omega, \Sigma, \psi)$ and $\mathcal{E}_{f'}(\omega', \Sigma', \psi')$ are equivalent, it is enough to see that $\bar{h}$ is isotopic to a diffeomorphism, which will necessarily lie into $\mathrm{Diff}^*_+(S^2 \times S^1)$. This follows from the following theorem.

THEOREM. *Let M and N be two differentiable compact 3-manifolds and let $h : M \to N$ be a homeomorphism. Then h is isotopic to a diffeomorphism.*

Proof. Let $\mathcal{C}$ be a differentiable structure on M and $\mathcal{C}'$ a differentiable structure on N and let $h : (M, \mathcal{C}) \to (N, \mathcal{C}')$ be a homeomorphism. The differentiable structure $\mathcal{C}$ on M induces through h a differentiable structure $\mathcal{C}_M$ on N and the map $\hat{h} : (M, \mathcal{C}) \to (N, \mathcal{C}_M)$ defined by $\hat{h}(x) = h(x)$ is a diffeomorphism. There exists only one differentiable structure on the 3-manifold N, that is there exists a diffeomorphism $g : (N, \mathcal{C}') \to (N, \mathcal{C}_M)$ which is isotopic (through homeomorphisms) to Identity (see for instance [**Mu**]). Then h is isotopic to the diffeomorphism $g^{-1} \circ \hat{h}$. □

Then the link $\mathcal{E}_f(\omega, \Sigma, \psi)$ is isotopic to the link $\mathcal{E}_{f'}(\omega', \Sigma', \psi')$ and so $\mathcal{E}_f(\omega, \Sigma) = \mathcal{E}_{f'}(\omega', \Sigma')$. □

Before proving our main results, let us present a direct corollary of the proof of Proposition 1.4.

1.4. *Unicity of the saddle inducing a Klein bottle.* In this section we prove Corollary 0.1 which we now recall.

COROLLARY 0.1. *Let f be an orientation-preserving diffeomorphism on a 3-manifold M and let ω be a hyperbolic attracting periodic orbit of f of period p. Then there is at most one periodic orbit of saddle σ, having a two-dimensional unstable manifold such that $W^u(\sigma) \setminus \sigma$ is contained in the basin of ω, and such that f^p reverses the orientation of $W^u(\sigma)$. Moreover, the period of σ is equal to p.*

Proof. Up to changing f to f^p we assume that ω is a fixed sink. For each two-dimensional unstable punctured manifold $W^u(\sigma) \setminus \sigma$ contained in the basin of ω, its projection into $N(f, \omega)$ is a Klein bottle if f reverses the orientation on $W^u(\sigma)$. Corollary 1.9 ensures that there is at most one Klein bottle in a two-dimensional link, so that there is at most one such orbit σ. Let us denote the projection of $W^u(\sigma) \setminus \sigma$ in $N(f, \omega)$ by K_σ.

Moreover, Corollary 1.10 asserts that the image of the fundamental group of K_σ is $\pi_1(N(f, \omega))$, that is K_σ contains a generator of $\pi_1(N(f, \omega))$. This generator lifts into $W^u(\sigma)$ to a segment joining a point x to $f(x)$, so σ is a fixed point. □

Remark 1.25. When a link $\mathcal{E}$ contains a Klein bottle, the tori contained in $\mathcal{E}$ necessarily have even type. We deduce the following property on the periods of the saddles.

Let ω be a fixed sink and Σ an invariant set of saddles σ such that $W^u(\sigma) \setminus \{\sigma\} \subset W^s(\omega)$. If there is $\sigma_0 \in \Sigma$ such that f reverses the orientation on $W^u(\sigma_0)$, then the period of each $\sigma \in \Sigma \setminus \{\sigma_0\}$ is even. (If ω is periodic of period p then the period of such a σ is a multiple of $2p$.)

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2. *Fundamental construction: proof of Theorem 1*

In this section, given a two-dimensional link $\mathcal{E}$ in $S^2 \times S^1$, we construct a canonical example of a diffeomorphism $f_{\mathcal{E}}$ on S^3 with a unique sink, such that the link associated to f at the sink is equivalent to $\mathcal{E}$. More precisely, we have the following definition.

Definition 2.1. Let $f : M \to M$ be a diffeomorphism of a compact 3-manifold M. We say that f is a *$\mathcal{G}$-diffeomorphism* if it is a Morse–Smale diffeomorphism without any heteroclinic intersections, such that the non-wandering set $\Omega(f)$ contains exactly one attracting orbit which is a fixed point and each saddle periodic orbit has a two-dimensional unstable manifold.

If f is a $\mathcal{G}$-diffeomorphism then $\mathcal{E}(f)$ stands for $\mathcal{E}_f(\omega, \Sigma)$, where Σ is the set of all saddles of f, and ω is the sink of f.

Definition 2.2. Given any two-dimensional link $\mathcal{E}$ of $S^2 \times S^1$ a *model map* for $\mathcal{E}$ is a $\mathcal{G}$-diffeomorphism f such that $\mathcal{E}(f)$ is the equivalence class of $\mathcal{E}$.

Now, Theorem 1 can be restated as follows:

any two-dimensional link $\mathcal{E}$ admits a model $f_{\mathcal{E}} : S^3 \to S^3$.

The proof will run by induction on the number of components of the link $\mathcal{E}$.

2.1. *$\mathcal{G}$-diffeomorphism associated to a torus or a Klein bottle in $S^2 \times S^1$.*

PROPOSITION 2.3. *Consider E as a torus or a Klein bottle in $S^2 \times S^1$ with type $l \neq 0$ (in the case of a Klein bottle, $l = 1$). Then, there exists a $\mathcal{G}$-diffeomorphism f_E defined on S^3 with one l-periodic saddle orbit S, whose unstable manifold projects into the space of orbits at the sink A onto a diffeomorphic image of E. In other words, $\mathcal{E}(f_E)$ is equivalent to E.*

Proof.

First case: E is the Klein bottle. By Proposition 1.4, it is enough to exhibit one example of a $\mathcal{G}$-diffeomorphism f_E which has one saddle fixed point with a two-dimensional unstable manifold on which f_E reverses the orientation.

We consider first the vector field Z on $\mathbb{R}^3$ given by

$$Z(x, y, z) = \frac{x(x-1)(1+x)}{1+x^2}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}.$$

The zeros of Z are $\sigma = (0, 0, 0)$ (which is a saddle with two-dimensional unstable manifold), $\alpha_1 = (1, 0, 0)$ and $\alpha_2 = (-1, 0, 0)$ (which are sources). The flow of Z admits a smooth compactification on $S^3 = \mathbb{R}^3 \cup \{\infty\}$ by adding a sink at infinity. The vector field Z is invariant under the rotation Θ with angle π around the vertical axis: $\Theta(x, y, z) = (-x, -y, z)$. Then, we set $f_E = \Theta \circ Z_1$ where Z_1 is the time 1-mapping of the flow of the vector field Z. Notice that:

- f_E is an orientation-preserving $\mathcal{G}$-diffeomorphism and $f_E^2 = Z_2$ where Z_2 is the time 2-mapping of the flow of Z;
- $\{\alpha_1, \alpha_2\}$ is a period 2 source;

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- the point $A = \infty$ is the unique sink of f_E;
- σ is the unique saddle point of f_E and its invariant manifolds coincide with its Z-invariant manifolds; the diffeomorphism f_E exchanges the two connected components, called *separatrices* by analogy with the two-dimensional vocabulary, of the one-dimensional stable manifold of σ; thus the restriction of f_E to the two-dimensional unstable manifold $W^u(\sigma)$ is orientation reversing; as a consequence the quotient $(W^u(\sigma) \setminus \{\sigma\})/f_{\mathcal{E}}$ is a Klein bottle.

We have shown that the link associated to f_E is a Klein bottle.

Second case: E is a torus. In this case, our construction is directly adapted from [**BoGr**]. Nonetheless, we recall it now because we will use it in the general case.

Let f_0 denote the time one mapping of the radial flow on S^3:

$$X(x, y, z) = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z},$$

where (x, y, z) are standard coordinates in $\mathbb{R}^3 \subset S^3$. Then f_0 has exactly one attracting fixed point $A = (0, 0, 0)$ and one repelling fixed point $R = \infty$. The basin of A is $B_A = S^3 \setminus \{R\}$. The diffeomorphism

$$\Phi : S^3 \setminus \{A, R\} \longrightarrow S^2 \times \mathbb{R}$$

$$u = (x, y, z) \mapsto \left(\frac{u}{\|u\|}, \log \|u\|\right)$$

conjugates f_0 to the translation $\tau : (p, t) \mapsto (p, t + 1)$.

Consider a mapping $\Gamma : D^2 \times S^1 \longrightarrow S^2 \times S^1$ whose image has boundary E: such a solid torus exists from Proposition 1.2. Since the torus E has type l, Γ lifts into $S^3 \setminus \{A, R\}$ to l disjoint infinite full cylinders $\tilde{\Gamma}_0, \ldots, \tilde{\Gamma}_{l-1} : D^2 \times \mathbb{R} \to S^3 \setminus \{A, R\}$, such that for any i the map $\tilde{\Gamma}_{i+1}^{-1} \circ f_0 \circ \tilde{\Gamma}_i : D^2 \times \mathbb{R} \to D^2 \times \mathbb{R}$ coincides with τ.

The construction of f_E will result from a modification of f_0 inside the cylinders $\tilde{\Gamma}_i$. For this aim, we now define a vector field Y on $D^2 \times \mathbb{R}$ using the classical Cherry construction (see Figure 2): Y coincides with the vertical ascendant vector field $\partial/\partial t$ out of some compact ball contained in the interior of $D^2 \times \mathbb{R}$; it has exactly two singularities which are hyperbolic: these are a saddle σ with two-dimensional unstable manifold and a source ω_2. Moreover, any orbit not in $\{\omega_2\} \cup W^s(\sigma)$ has its ω-limit set at $D^2 \times \{+\infty\}$ and each orbit not in $W^u(\omega_2) \cup W^u(\sigma)$ has its α-limit set in $D^2 \times \{-\infty\}$. Finally, there exists $K > 0$ such that the stable manifold $W^s(\sigma)$ of σ contains the half straight line $\{O\}\times]-K, -\infty[$ and the unstable manifold $W^u(\sigma)$ contains the half cylinder $C\times]K, +\infty[$, where $C \subset D^2$ is some circle.

The technical definition and construction of Y can be found for example in [**BoGr**].

Let us call g the time one mapping of the flow of Y. Notice that g and the translation τ on $S^2 \times \mathbb{R}$ coincide out of a compact set contained in the interior of $D^2 \times \mathbb{R}$.

We now define the map f_E on S^3 by setting:

- f_E equal to f_0 outside the cylinders $\bigcup_{i=0}^{l-1} \tilde{\Gamma}_i(D^2 \times \mathbb{R})$;
- inside each $\tilde{\Gamma}_i(D^2 \times \mathbb{R})$, $i \in \mathbb{Z}/l\mathbb{Z}$, $f_E = \tilde{\Gamma}_{i+1} \circ g \circ \tilde{\Gamma}_i^{-1}$.

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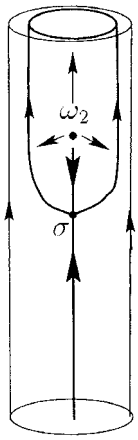


FIGURE 2. Phase portrait of the vector field Y.

The map f_E is a diffeomorphism because $\tilde{\Gamma}_{i+1} \circ g \circ \tilde{\Gamma}_i^{-1}$ and f_0 coincide out of l compact balls, each of them included in the interior of one of the $\tilde{\Gamma}_i$. It is now an exercise to show that f_E is a $\mathcal{G}$-diffeomorphism: it only remains to verify that the torus $\Gamma(f_E) \subset S^2 \times S^1$ induced by the unstable manifold $W^u(\sigma)$ is isotopic to E. This is almost immediate as f_E coincides with f_0 in a neighborhood of the sink A, and in this neighborhood $W^u(\sigma)$ is composed of one cylinder which is trivial in each (knotted) tube $\tilde{\Gamma}_i$. In the quotient space one gets a torus in the solid torus Γ which is parallel to the boundary E. □

2.2. *From a neighborhood of the repeller R of f_E to a neighborhood of the attractor A.* Let $E \subset S^2 \times S^1$ be a torus whose fundamental group is non-trivial in $\pi_1(S^2 \times S^1)$, and consider the model $f_E : S^3 \to S^3$ built above. We can identify $N(f_E, A)$ with $S^2 \times S^1$. Denote by Γ the solid torus bounded by E, which we used for building f_E. We have seen that the projection in $N(f_E, A)$ of $W^u(\sigma) \setminus \{\sigma\}$ is a torus E' contained in the interior of Γ and parallel to E. Let us denote by Γ' the solid torus bounded by E' and contained in Γ, and let $\Delta = N(f_E, A) \setminus \Gamma'$.

Let us consider the quotient space $N(f_E, R) = (W^u(R) \setminus \{R\})/f_E \simeq S^2 \times S^1$. Recall that a positive generator of $\pi_1(N(f_E, R))$ corresponds to a segment joining some point x to $f_E(x)$. The projection of $W^s(\sigma) \setminus \{\sigma\}$ into the quotient space $N(f_E, R)$ gives an embedded circle $\gamma(f_E)$. Notice that there is a canonical diffeomorphism $I : N(f_E, R) \setminus \gamma(f_E) \to \Delta$: one point $x \in N(f_E, R)$ corresponds to an orbit of f_E and $I(x)$ is its projection into $N(f_E, A)$; one can easily check that this projection belongs to Δ.

LEMMA 2.4. *With the notation above, for any neighborhood U of $\gamma(f_E)$ there is a diffeomorphism $\Psi : N(f_E, R) \to N(f_E, A)$ coinciding with I on $N(f_E, R) \setminus U$ and such that Γ' is a tubular neighborhood of $\Psi(\gamma(f_E))$.*

Proof. For U small enough, the lift of U into S^3 is contained in $\tilde{\Gamma}$ which is a union of the solid cylinders $\tilde{\Gamma}_i$. The lemma is therefore a direct consequence of a similar statement for the restriction of f_E^l to the solid cylinder $\tilde{\Gamma}_0$.

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Let Y be the cherry flow on $D^2 \times \mathbb{R}$, and $g = Y_1$. For N large g coincides with the translation τ out of $D^2 \times [-N, N]$. Then the quotient spaces $N(g, -) = (D^2 \times]-\infty, -N])/g$ and $N(g, +) = (D^2 \times [N, \infty[)/g$ are identified with $D^2 \times S^1$. Recall that one connected component of the stable one-dimensional manifold of the saddle σ of Y coincides with $\{0\} \times]-\infty, -N]$ and its unstable manifold coincides with $C \times [N, \infty[$ so that the corresponding projections into $N(g, -)$ and $N(g, +)$ are $\gamma_- = \{0\} \times S^1$ and $T_+ = C \times S^1$. Let us denote by $\Gamma_+ \subset N(g, +)$ the solid torus bounded by T_+. As above, there is a canonical diffeomorphism $I_0 : N(g, -) \setminus \gamma_- \to N(g, +) \setminus \Gamma_+$.

CLAIM 2.1. *With the notation above, for any neighborhood V of γ_- there is a diffeomorphism $\Psi_0 : N(g, -) \to N(g, +)$ coinciding with I_0 on $N(g, -) \setminus V$ such that Γ_+ is a tubular neighborhood of $\Psi_0(\gamma_-)$.*

The proof of this claim follows from the definition of the cherry flow Y. Now the proof of Lemma 2.4 is complete. □

Let $E \in S^2 \times S^1$ be a torus, f_E the model map associated to E and $\gamma(f_E)$ the knot in $(W^u(R) \setminus \{R\})/f_E \simeq S^2 \times S^1$. Suppose now that $E_1, \ldots, E_k$ is a collection of pairwise disjoint tori (and possibly one Klein bottle) which is included in the space of orbits of f_E at the repelling point R, $(W^u(R) \setminus \{R\})/f_E$, in such a way that the fundamental group of each of them is not trivial in $\pi_1(S^2 \times S^1)$ and they are all disjoint from $\gamma(f_E)$. Denote by E_0 the boundary of some small tubular neighborhood of $\gamma(f_E)$ which does not intersect any of the E_i's, $i \geq 1$, so that $E_0, E_1, \ldots, E_k$ form a two-dimensional link.

For $i = 1, \ldots, k$, let $\tilde{E}_i \subset W^u(R) \setminus \{R\}$ be the lift of E_i to S^3 by the natural projection. Each $\tilde{E}_i$ is a collection of disjoint cylinders, invariant by f_E and contained in $W^s(A) \setminus \{A\}$. Thus one can now consider the projection F_i of each $\tilde{E}_i$ into the space of orbits of f_E at A, $(W^s(A) \setminus \{A\})/f_E$, to obtain the following.

COROLLARY 2.5. *The collection $\{E, F_1, \ldots, F_k\}$ is a two-dimensional link which is equivalent to the link $\{E_0, E_1, \ldots, E_k\}$.*

Proof. Notice that $F_i = I(E_i)$ (with the notation of Lemma 2.4). Choosing the neighborhood U in Lemma 2.4 small enough to be disjoint from the torus E^i's, the diffeomorphism Ψ realizes the announced equivalence between the links $\{E, F_1, \ldots, F_k\}$ and $\{E_0, E_1, \ldots, E_k\}$. □

2.3. *$\mathcal{G}$-diffeomorphism associated to a two-dimensional link in $S^2 \times S^1$: proof of Theorem 1.* Let us recall the following statement.

THEOREM. *Let $\mathcal{E} = \{E_1, \ldots, E_k\}$ be a two-dimensional link in $S^2 \times S^1$. There exists a $\mathcal{G}$-diffeomorphism $f_{\mathcal{E}}$ on S^3 with k saddle orbits, such that the two-dimensional link $\mathcal{E}(f_{\mathcal{E}}, A)$ associated to $f_{\mathcal{E}}$ at the attracting fixed point A is equivalent to $\mathcal{E}$.*

Proof. The proof works by induction on the number k of components of the link. The case $k = 1$ has already been set up (see Proposition 2.3).

Suppose now that the theorem is true for a link made of k components and let $\mathcal{E} = \{E_0, E_1, \ldots, E_k\}$ be a two-dimensional link with $k + 1$ components. Up to reindexation,

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we suppose that this family is well ordered (see Definition 1.15) and we let $\{\Gamma_0, \dots, \Gamma_k\}$ be a family of solid tori, adapted to $\mathcal{E}$ and well-ordered. Thus, Γ_0 is included or disjoint from all the other solid tori.

Let us denote $\mathcal{F} = \{E_1, \dots, E_k\}$. By our induction hypothesis, we know that there exists a model map $f_{\mathcal{F}}$ for the link $\mathcal{F}$ and a model map f_{E_0} for E_0. The idea of the construction is to make a 'connected sum' of the diffeomorphisms f_{E_0} and $f_{\mathcal{F}}$, by removing a neighborhood of the attractor of $f_{\mathcal{F}}$ and a neighborhood of the repeller of f_{E_0} and identifying the corresponding fundamental domains. Here follows the precise construction.

Let us denote by A the unique fixed point attractor of the $\mathcal{G}$-diffeomorphism $f_{\mathcal{F}}$ and R the repelling fixed point of f_{E_0} corresponding in the construction of the model to the repeller of f_0. Recall that E_0 is a torus (not a Klein bottle). We called γ_{E_0} the (one-dimensional) knot induced by the projection into $(W^u(R) \setminus \{R\})/f_{E_0}$ of the stable manifold of the saddle orbit of f_{E_0}. Moreover, E_0 is equivalent to the boundary $\hat{E}_0$ of a tubular neighborhood of γ_{E_0}.

Recall also that we have chosen homeomorphisms, up to $\mathrm{Diff}^*_+(S^2 \times S^1)$, identifying $N(f_{E_0}, R) = (W^u(R) \setminus \{R\})/f_{E_0}$ and $N(f_{\mathcal{F}}, A) = (W^s(A) \setminus \{A\})/f_{\mathcal{F}}$ with $S^2 \times S^1$.

Then there is a diffeomorphism $h : N(f_{E_0}, R) \to N(f_{\mathcal{F}}, A)$ preserving the orientation and sending positive generators of $\pi_1(N(f_{E_0}, R))$ onto positive generators of $\pi_1(N(f_{\mathcal{F}}, A))$ (see Definition 1.18), and such that the two-dimensional link $\{\hat{E}_0, h^{-1}(E_1), \dots, h^{-1}(E_k)\}$ is equivalent to the link $\mathcal{E}$.

Consider some lift $\tilde{h} : W^u(R) \setminus \{R\} \to W^s(A) \setminus \{A\}$: it is a diffeomorphism conjugating the corresponding restrictions of f_{E_0} and $f_{\mathcal{F}}$. By using the homeomorphism $\tilde{h}$, we now glue $S^3 \setminus \{R\}$ with $S^3 \setminus \{A\}$. More precisely, we consider the quotient space $M_h = S^3 \setminus \{R\} \coprod S^3 \setminus \{A\}/\tilde{h}$ and we claim the following.

CLAIM 2.2. *M_h is a connected sum of two spheres S^3 and thus is diffeomorphic to the sphere S^3.*

Proof. We choose a fundamental domain Δ_R of the restriction f_{E_0} to $W^u(R) \setminus \{R\}$, such that Δ_R is diffeomorphic to $S^2 \times [0, 1]$ and such that the connected component of $S^3 \setminus \Delta_R$ containing R is a ball B_R. Then $\Delta_A = \tilde{h}(\Delta_R)$ is a fundamental domain of $f_{\mathcal{F}}$ on $W^u(A) \setminus \{A\}$ and the connected component of $S^3 \setminus \Delta_A$ containing A is a ball B_A. Then M_h is obtained by gluing $S^3 \setminus B_R$ and $S^3 \setminus B_A$ along the fundamental domains Δ_R and Δ_A *via* the diffeomorphism $\tilde{h}$. □

We call U and V the projections of $S^3 \setminus R$ and $S^3 \setminus A$, respectively, into M_h. Since f_{E_0} and $f_{\mathcal{F}}$ are conjugated by $\tilde{h}$, we can define a diffeomorphism f defined on M_h which coincides with f_{E_0} on U and with $f_{\mathcal{F}}$ on V.

CLAIM 2.3. *The diffeomorphism f is a $\mathcal{G}$-diffeomorphism which is a model map for the link $\mathcal{E}$.*

Proof. By construction, f has exactly one attractor, which is denoted $\omega \in U$ and its non-wandering set consists of a finite number of periodic orbits coming from f_{E_0} in U and $f_{\mathcal{F}}$ in V. Recall that there is only one saddle periodic orbit σ contained in U. Since U is invariant by f, the invariant manifolds of σ are contained in U. Given another

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saddle of f (necessarily in V) the intersection of its unstable manifold with U is its image by $\tilde{h}^{-1}$. As $h^{-1}(\mathcal{F})$ is disjoint from γ_{E_0}, we deduce that $W^s(\sigma)$ is disjoint from the unstable manifolds of the other saddles. Thus, f has no heteroclinic intersection and is a $\mathcal{G}$-diffeomorphism. To achieve the proof, we need to show that the two-dimensional link $\mathcal{E}(f,\omega)$ is the link $\mathcal{E}$. But this is a direct consequence of the choice of h using Lemma 2.4. □

□

3. *Proof of Theorem 2*

Let f be an orientation-preserving diffeomorphism on a 3-manifold M, ω an attracting fixed point of f and Σ an f-invariant set of saddle periodic points σ having a two-dimensional unstable manifold such that $W^u(\sigma) \setminus \{\sigma\}$ is contained in the basin of ω. We denote $W^u(\Sigma) = \bigcup_{\sigma\in\Sigma} W^u(\sigma)$. We choose a diffeomorphism $\psi : N(f,\omega) \longrightarrow S^2 \times S^1$, and we denote the bi-dimensional link induced by the unstable manifolds of the saddles in Σ by $\mathcal{E} = \mathcal{E}_f(\omega, \Sigma, \psi)$. In this section we denote the projection of the unstable manifold $W^u(\Sigma) \setminus \Sigma$ by $\mathbf{E} \subset N(f,\omega)$, so that $\mathcal{E}$ is the image by ψ of $\mathbf{E}$. Let us now consider the attractor $\Omega = \{\omega\} \cup \Sigma \cup \bigcup_{\sigma\in\Sigma} W^u(\sigma)$: its basin is $W^s(\Omega) = W^s(\omega) \cup \bigcup_{\sigma\in\Sigma} W^s(\sigma)$.

Then $f_{\mathcal{E}}$ stands for the model built in §2 and associated to a link equivalent to $\mathcal{E}$. We denote the set of periodic saddles of $f_{\mathcal{E}}$ by $\Sigma_{\mathcal{E}}$, the set of repelling periodic points of $f_{\mathcal{E}}$ by $\mathcal{R}_{\mathcal{E}}$ and the unique sink of $f_{\mathcal{E}}$ by A. We denote the projection of the unstable manifold $W^u(\Sigma_{\mathcal{E}}) \setminus \Sigma_{\mathcal{E}}$ by $\mathbf{E}_0 \subset N(f_{\mathcal{E}}, A)$.

To prove Theorem 2 we have to prove the following statement.

There exists a homeomorphism $\Theta' : W^s(\Omega) \longrightarrow S^3 \setminus \mathcal{R}_{\mathcal{E}}$ *such that* $\Theta' \circ f = f_{\mathcal{E}} \circ \Theta'$.

Proof. As the links associated to f, Σ and $f_{\mathcal{E}}, \Sigma_{\mathcal{E}}$ are equivalent by construction, there is an orientation-preserving diffeomorphism $\theta : N(f,\omega) \to N(f_{\mathcal{E}}, A)$ mapping $\mathbf{E}$ onto $\mathbf{E}_0$, and whose action on the fundamental groups of $N(f,\omega)$ and $N(f_{\mathcal{E}}, A)$ maps positive generators onto positive generators.

Consider a lift $\Theta : W^s(f,\omega) \setminus \{\omega\} \to W^s(f_{\mathcal{E}}, A) \setminus \{A\}$ of θ. Then Θ can be extended by continuity into an homeomorphism $\Theta : W^s(f,\omega) \to W^s(f_{\mathcal{E}}, A)$ just by letting $\Theta(\omega) = A$. Notice that Θ conjugates f and $f_{\mathcal{E}}$ on the basins of ω and A, respectively, and that $\Theta(W^u(\Sigma) \setminus \Sigma) = W^u(\Sigma_{\mathcal{E}}) \setminus \Sigma_{\mathcal{E}}$. The proof of Theorem 2 consists of showing that one can choose θ such that the lift Θ extends by continuity on $W^s(\Omega) = W^s(\omega) \cup W^s(\Sigma)$.

From the classical theory on hyperbolic systems (see for instance [**PaMe**]), we know that there exists an f-invariant neighborhood U of Σ endowed with an invariant two-dimensional foliation $\tilde{\mathcal{F}}^u$ admitting the unstable manifold of each saddle in Σ as a leaf. Moreover, each leaf of $\tilde{\mathcal{F}}^u$ cuts in exactly one point $W^s(\Sigma)$. Up to changing U into a smaller f-invariant neighborhood of Σ, we can assume that $U \subset W^s(\Omega)$, since $W^s(\Omega)$ is an invariant neighborhood of Σ.

As $\tilde{\mathcal{F}}^u$ is invariant by f, its restriction to $U \setminus \bigcup_{\sigma\in\Sigma} W^s(\sigma)$ (which is contained in $W^s(\omega) \setminus \{\omega\}$) induces by projection on $N(f,\omega)$ a foliation $\mathcal{F}^u$ defined in a neighborhood of $\mathbf{E}$.

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In the same way, a foliation $\tilde{\mathcal{F}}^u_{\mathcal{E}}$ defined on an invariant neighborhood of the set of the saddles of $f_{\mathcal{E}}$ induces on $N(f_{\mathcal{E}}, A)$ a foliation $\mathcal{F}^u_{\mathcal{E}}$ defined in a neighborhood of $\mathbf{E}_0$.

The main step of the proof is the following.

LEMMA 3.1. *Let $\theta : N(f, \omega) \to N(f_{\mathcal{E}}, A)$ be an orientation-preserving diffeomorphism sending positive generators of $\pi_1(N(f, \omega))$ onto positive generators of $\pi_1(N(f_{\mathcal{E}}, A))$, such that $\theta(\mathbf{E}) = \mathbf{E}_0$.*

There is an homeomorphism $\theta' : N(f, \omega) \to N(f_{\mathcal{E}}, A)$ coinciding with θ out of a tubular neighborhood of $\mathcal{E}_f(\omega, \Sigma)$, such that $\theta'(\mathbf{E}) = \mathbf{E}_0$ and the image foliation $\theta'_*(\mathcal{F}^u)$ coincides with $\mathcal{F}^u_{\mathcal{E}}$ in a neighborhood of $\mathcal{E}(f_E, A)$.

Before proving this lemma let us finish the proof of Theorem 2. Let $\Theta' : W^s(\omega) \to W^s(A)$ be a lift of θ' which coincides with Θ outside a small neighborhood of the unstable manifolds of the saddles $\sigma \in \Sigma$. Notice that, in an invariant neighborhood V of the unstable manifolds of the saddles of Σ, the homeomorphism Θ' sends the unstable leaves of the foliation $\tilde{\mathcal{F}}^u$ onto leaves of the foliation $\tilde{F}^u_{\mathcal{E}}$. Moreover, since the neighborhood V is invariant by f, its union with the stable manifolds of the saddles in Σ is a neighborhood of those stable manifolds. As a consequence, we can extend the homeomorphism Θ' to the stable manifolds of the saddles in the following way.

For each $\sigma \in \Sigma$, and for each point x in $W^s(\sigma)$, we let $F \in \tilde{\mathcal{F}}^u$ be the unstable leaf containing x. Then there exists a leaf $F' \in \tilde{\mathcal{F}}^u_{\mathcal{E}}$ such that

$$\Theta'(F \setminus \{x\}) \subset F'.$$

Let y be the unique intersection point of F' with the stable manifolds of the saddles of $f_{\mathcal{E}}$. We set

$$\Theta'(x) = y.$$

Θ' is clearly a well-defined bijection on the basin of Ω and it is a conjugacy between f and $f_{\mathcal{E}}$. The continuity derives from a standard argument and we refer the reader to the proof of the Stability Lemma in [**Pa**]. □

The end of this section is devoted to the proof of Lemma 3.1.

Proof of Lemma 3.1. For the proof we will use the following theorem, which results from the classical theory of foliations (see for instance [**CaLi**, Capitulo IV Teorema 2]).

THEOREM. *Let $C_1 \subset M_1$ and $C_2 \subset M_2$ be two diffeomorphic compact submanifolds of some k-manifolds M_1 and M_2 ($k > 1$). Assume that there exist local foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, defined in a neighborhood of C_1 and C_2 respectively, such that C_1 and C_2 are compact leaves of the corresponding foliations.*

Finally, let $H_0 : C_1 \to C_2$ be some diffeomorphism, and choose $x_1 \in C_1$ and $x_2 = h(x_1) \in C_2$. We consider local sections T_1 at x_1 transverse to $\mathcal{F}_1$ and T_2 at x_2 transverse to $\mathcal{F}_2$.

The following propositions are equivalent.

- *There exists a homeomorphism H defined on a neighborhood of C_1 onto a neighborhood of C_2, coinciding with H_0 on C_1 such that $H(\mathcal{F}_1) = \mathcal{F}_2$ in a neighborhood of C_1 and C_2 respectively;*

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- *There exists a homeomorphism $h : T_1 \longrightarrow T_2$ defined in a neighborhood of x_1, such that $h(x_1) = x_2$ and such that, for all $\gamma \in \pi_1(C_1, x_1)$, h conjugates the germs of holonomy mappings of γ on T_1 and $(H_0)_*(\gamma)$ on T_2.*

Remark 3.2. When the second item holds, we can choose H coinciding with h on T_1, in a neighborhood of x_1 in T_1.

LEMMA 3.3. *Let C be a compact leaf of the foliation $\mathcal{F}^u$ (or $\mathcal{F}^u_{\mathcal{E}}$). Its fundamental group admits a basis $\lambda, \gamma \in \pi_1(C)$ such that λ is homotopic to zero in $\pi_1(N(f, \omega))$ (respectively $\pi_1(N(f_{\mathcal{E}}, A))$) and γ is a positive generator of the image of $\pi_1(C)$ in $\pi_1(N(f, \omega))$ (respectively $\pi_1(N(f_{\mathcal{E}}, A))$). Then hol_λ is the identity map and hol_γ is a strict expansion.*

Proof. Let $\tilde{\lambda}$ be a lift of λ to the basin of ω. As λ is homotopically trivial in $N(f, \omega)$, $\tilde{\lambda}$ is a simple closed curve in the unstable manifold of the corresponding saddle σ. Since this unstable manifold is a simply connected leaf of $\tilde{\mathcal{F}}^u$, the holonomy of $\tilde{\lambda}$ and thus that of λ is trivial.

On the other hand, let $\tilde{\gamma}$ be a lift of γ. It is an arc q_0q_1 with $f(q_0) = q_1$. Let Σ_0 be a transverse section to $\tilde{\mathcal{F}}^u$ at q_0 and $\Sigma_1 = f(\Sigma_0)$. The map hol_γ lifts to the map

$$\widetilde{\mathrm{hol}}_{\tilde{\gamma}} : \begin{array}{l} \Sigma_0 \longrightarrow \Sigma_1 \\ \quad x \longmapsto y = F^u(x) \cap \tilde{\Sigma}_1. \end{array}$$

Since the restriction of f to $W^s(\sigma)$ is a contraction, the projection of $\widetilde{\mathrm{hol}}_{\tilde{\gamma}}(x)$ onto $W^s(\sigma)$, which is the same as the projection of x, is farther from σ than the projection of $f(x)$. Thus, when $\tilde{\Sigma}_0$ and $\tilde{\Sigma}_1$ are identified through f, the map $\mathrm{hol}_{\tilde{\gamma}}$ is an expansion.

The proof is the same for $\mathcal{F}^u_{\mathcal{E}}$. □

COROLLARY 3.4. *There is an orientation-preserving homeomorphism H from a neighborhood U_1 of C_1 on a neighborhood U_2 of C_2 coinciding with Θ on C_1 and such that $H(\mathcal{F}^u) = \mathcal{F}^u_{\mathcal{E}}$.*

Proof. From the classical theorem of foliations recalled just above and Lemma 3.3, it is enough to note that the restriction $\theta : \mathbf{E} \longrightarrow \mathbf{E}_0$ satisfies the following:

- θ sends the closed curves homotopic to zero in $N(f, \omega)$ onto null homotopic closed paths in $N(f_{\mathcal{E}}, A)$;
- given a component $E \in \mathbf{E}$, the positive generators of the image of $\pi_1(E)$ in $\pi_1(N(f, \omega))$ are sent on positive generators of the image of $\pi_1(\theta(E))$ in $\pi_1(N(f_{\mathcal{E}}, A))$. □

To achieve the proof of Lemma 3.1, it remains to show that there exists a homeomorphism θ' coinciding with θ outside a tubular neighborhood of $\mathbf{E}$ and with H inside a smaller tubular neighborhood of $\mathbf{E}$. This is a consequence of the fact that the restrictions to $\mathbf{E}$ of the maps θ and H are isotopic and also of the following topological lemma.

LEMMA 3.5. *Let $\theta : M_1 \longrightarrow M_2$ be a diffeomorphism between three-dimensional orientable manifolds and T_1 a torus embedded in M_1 and $T_2 = \theta(T_1)$. For $i = 1, 2$, consider tubular neighborhoods of T_i, V_i and V_i' such that $V_i' \subset \mathrm{int}(V_i)$ and $\theta(V_1) = V_2$.*

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Let $H : V_1' \longrightarrow V_2'$ be a homeomorphism such that $H(T_1) = T_2$ and H and θ are isotopic on T_1.

Then, there exists a homeomorphism θ' coinciding with θ outside V_1 and with H in a neighborhood of T_1.

Lemma 3.5 is a direct consequence of a general topological fact which is proved in Appendix A, see Proposition A.1.

The proof of Lemma 3.1 and Theorem 2 is complete. □

4. *Proof of Corollary 0.2*

Let M be an orientable compact 3-manifold and $f : M \to M$ an orientation-preserving Morse–Smale diffeomorphism without any heteroclinic intersection. Up to changing f by one of its positive iterates we can assume without loss of generality that every periodic orbit of f is a fixed point: from now on we always assume that every periodic orbit is a fixed point.

To prove Corollary 0.2, that is *M is either the sphere S^3 or the connected sum of copies of $S^2 \times S^1$*, we will use the following characterization.

THEOREM. *Let M be an orientable compact 3-manifold containing a family $S_1, \ldots, S_l$ of disjoint smoothly embedded 2-spheres such that each connected component of $M \setminus \bigcup_i S_i$ is diffeomorphic to the 3-sphere minus a finite set of points. Then M is either the sphere S^3 or the connected sum of copies of $S^2 \times S^1$.*

This result might be clear for topologists. We can show it by induction, using the fact that, given a connected oriented manifold N, the process of removing two disjoint 3-balls and gluing their boundaries to obtain an orientable manifold is equivalent to making the connected sum of N with $S^2 \times S^1$.

Let $\mathcal{A} = \{\alpha_1, \ldots, \alpha_{k_1}\}$, $\mathcal{L} = \{\lambda_1, \ldots, \lambda_{k_2}\}$, $\mathcal{S} = \{\sigma_1, \ldots, \sigma_{k_3}\}$ and $\mathcal{O} = \{\omega_1, \ldots, \omega_{k_4}\}$ be the sets of fixed points of f which are, respectively, the sources, the saddles with two-dimensional stable manifold, the saddles with two-dimensional unstable manifold and the sinks.

As there are no heteroclinic orbits, for each saddle point $\lambda \in \mathcal{L}$ there is a source $\alpha \in \mathcal{A}$ such that $W^s(\lambda) \setminus \lambda \subset W^u(\alpha)$ and for each saddle point $\sigma \in \mathcal{S}$ there is a sink $\omega \in \mathcal{O}$ such that $W^u(\sigma) \setminus \sigma \subset W^s(\omega)$.

For every source $\alpha \in \mathcal{A}$ we denote

$$\mathcal{L}_\alpha = \{\lambda \in \mathcal{L}, W^s(\lambda) \setminus \lambda \subset W^u(\alpha)\}.$$

In the same way, for every sink $\omega \in \mathcal{O}$ we denote

$$\mathcal{S}_\omega = \{\sigma \in \mathcal{S}, W^u(\sigma) \setminus \sigma \subset W^s(\omega)\}.$$

The families $\{\mathcal{L}_\alpha\}_{\alpha \in \mathcal{A}}$ and $\{\mathcal{S}_\omega\}_{\omega \in \mathcal{O}}$ are partitions of, respectively, $\mathcal{L}$ and $\mathcal{S}$.

If $\alpha_i \in \mathcal{A}$ is a source, we will denote

$$A_i = \alpha_i \cup \bigcup_{\lambda \in \mathcal{L}_{\alpha_i}} W^s(\lambda).$$

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Notice that A_i is a repeller of f and $W^u(A_i) = W^u(\alpha_i) \cup \bigcup_{\lambda \in \mathcal{L}_{\alpha_i}} W^u(\lambda)$.

In the same way, if $\omega_i \in \mathcal{O}$ is an sink, we will denote

$$O_i = \omega_i \cup \bigcup_{\sigma \in \mathcal{S}_{\omega_i}} W^u(\sigma).$$

Notice that O_i is an attractor of f and $W^s(O_i) = W^s(\omega_i) \cup \bigcup_{\sigma \in \mathcal{S}_{\omega_i}} W^s(\sigma)$.

Finally, we set $A = \bigcup_{i \in \{1,\dots,k_1\}} A_i$ and $O = \bigcup_{i \in \{1,\dots,k_4\}} O_i$. We have

$$W^u(A) \cap W^s(O) = W^u(A) \setminus A = W^s(O) \setminus O.$$

LEMMA 4.1. *The space of orbits $(W^s(O) \setminus O)/f$ is diffeomorphic to the disjoint union of a finite family of copies of $S^2 \times S^1$.*

Proof. First notice that $W^s(O) \setminus O$ is the disjoint union of the spaces $W^s(O_i) \setminus O_i$, which are invariant by f: thus it suffices to show the lemma for $W^s(O_i) \setminus O_i$. So, we consider a sink ω_i.

Theorem 2 asserts that there is a two-dimensional link $\mathcal{E}$ such that $f|_{W^s(O_i)}$ is conjugated by some homeomorphism Θ to the restriction of $f_{\mathcal{E}}$ to $S^3 \setminus \mathcal{R}(f_{\mathcal{E}})$ where $\mathcal{R}(f_{\mathcal{E}})$ is the finite set of repelling points of $f_{\mathcal{E}}$. One deduces that $\Theta(W^s(O_i) \setminus O_i)$ coincides with $\bigcup_{\alpha \in \mathcal{R}(f_E)} W^u(\alpha) \setminus \alpha$, and Θ induces a homeomorphism $\theta : (W^s(O_i) \setminus O_i)/f \to \big(\bigcup_{\alpha \in \mathcal{R}(f_E)} W^u(\alpha) \setminus \alpha\big)/f_{\mathcal{E}}$. As we have shown (and already often used), for each $\alpha \in \mathcal{R}(f_E)$ the space of orbits $(W^u(\alpha) \setminus \alpha)/f_{\mathcal{E}}$ is a copy of $S^2 \times S^1$. □

COROLLARY 4.2. *Each connected component of $W^u(A) \cap W^s(O)$ is diffeomorphic to $S^2 \times \mathbb{R}$.*

Proof. Just notice that each connected component C of $W^u(A) \cap W^s(O)$ is periodic under f and that the natural projection $C \to C/f^p$ (where p in the period of C) is an infinite cyclic cover of one component of $(W^u(A) \cap W^s(O))/f = (W^s(O) \setminus O)/f$; Lemma 4.1 asserts that such a component is diffeomorphic to $S^2 \times S^1$. □

Remark 4.3. In the proof above, each component C contains at least one connected component of the one-dimensional invariant manifold of a saddle point (except in the trivial case), so that the period of C is at most two.

If C is a connected component of $W^u(A) \cap W^s(O)$, and so is diffeomorphic to $S^2 \times \mathbb{R}$, an *essential sphere* in C will be a smoothly embedded sphere in C which is isotopic to $S^2 \times \{0\} \subset S^2 \times \mathbb{R}$.

Let k be the number of connected components of $W^u(A) \cap W^s(O)$ and let $S_1, \dots, S_k$ be a family of smoothly embedded spheres in M such that each connected component C_i of $W^u(A) \cap W^s(O)$ contains exactly one S_i as an essential sphere. We finish the proof of Corollary 0.2 by the following lemma.

LEMMA 4.4. *Each component Δ of $M \setminus \bigcup_i S_i$ is contained in some $W^s(O_j)$ or in some $W^u(A_i)$. As a consequence Δ is diffeomorphic to a sphere minus a finite set of points.*

Proof. The component Δ contains in its closure one of the $S_i \subset C_i$. As S_i is an essential sphere in C_i and is the unique component of the boundary of Δ in C_i, one gets that one of

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the components of $C_i \setminus S_i$ is contained in Δ. As a consequence, there is some attractor O_j or some repeller A_j which intersects the closure of Δ. As the boundary $\partial\Delta$ is contained in the union of the spheres S_l and so is disjoint from the connected sets A_j or O_j, it follows that A_j or O_j is contained in Δ. We will assume for instance that $O_j \subset \Delta$.

To prove that $\Delta \subset W^s(O_i)$, it is now enough to see that the connected component of O_j in $W^s(O_j) \setminus \bigcup_i S_i$ is relatively compact in $W^s(O_j)$. To do this, using Theorem 2, we can consider that $W^s(O_j)$ is the sphere S^3 minus the set of repelling orbits of the model map $f_{\mathcal{E}}$ where $\mathcal{E}$ is the link associated to ω_j. In this model, each sphere S_i bounds a ball around one of the sources (and contained in the basin of this source), and for each source its basin is one component of $W^s(O_j) \setminus O_j$ and so contains one of the S_i. Then, the closure of Δ in the model does not contain any of the sources of $f_{\mathcal{E}}$ and is a compact set contained in $W^s(O_j)$. □

□

5. *Final comments*

Throughout this paper, we have concentrated on a two-dimensional situation, that is we have exhibited a complete topological invariant for the restriction of diffeomorphisms to the basin of an attractor made of an hyperbolic attracting fixed point together with some two-dimensional unstable manifolds of saddle periodic orbits converging to the fixed point attractor (and without heteroclinic intersection). What happens if we consider more generally an attractor made of an attracting fixed point and *one- and two*-dimensional unstable manifolds of saddles converging to the fixed point? In fact, our methods can be adapted directly to show that the link obtained by projection of the saddles' unstable manifolds, which is now made of tori, Klein bottles and circles, is a topological invariant. In this case, the situation is much more complicated: one-dimensional unstable manifolds project into two circles, each one corresponding to a separatrix. A neighborhood of the corresponding saddle is cut by the two-dimensional stable manifold. In order to get a complete invariant, we have to introduce gluing rules.

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A. *Appendix. Extension of local homeomorphisms in a neighborhood of codimension-one submanifolds*

PROPOSITION A.1. *Let N be any compact manifold. Fix $t_0 \in]0, 1[$ and let $h : N \times [-t_0, t_0] \to N \times [-1, 1]$ be some homeomorphism from $N \times [-t_0, t_0]$ onto its image. We assume that the restriction of h to $N \times \{0\}$ is a homeomorphism of $N \times \{0\}$ which is isotopic to the identity. Finally, we assume that h preserves the transverse orientation of $N \times \{0\}$, that is if $t \in [0, t_0]$ then $h(x, t)$ belongs to $N \times [0, 1]$.*

Then there exists a homeomorphism $H : N \times [-1, 1] \to N \times [-1, 1]$ which coincides with the identity map out of $N \times [-1/2, 1/2]$ and with h in a small neighborhood of $N \times \{0\}$.

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This proposition is a corollary of a more general and deeper result (see [**Da**, p. 107]), but it admits a short direct proof which we now give.

Proof. Let us first make some simplifications on the homeomorphism h.

First, the homeomorphism h can be readily taken to be the identity on $N \times \{0\}$ as the following arguments show. As the restriction of h to $N \times \{0\}$ is isotopic to the identity, there is a homeomorphism h_0 of $N \times [-1, 1]$ which coincides with the identity out of $N \times [-1/2, 1/2]$ and with h on $N \times \{0\}$: we can choose for instance h_0 which lets each $N \times \{t\}$ be invariant and such that the homeomorphism of N induced by restriction on $N \times \{t\}$ realizes (for $t \in [0, 1/2]$ and for $t \in [-1/2, 0]$) an isotopy of h to the identity map on $N \times \{0\}$.

Up to replacing h by $h_0^{-1} \circ h$, one can assume that h coincides with the identity map on $N \times \{0\}$.

The second simplification is the following. Let us denote by $h_+ : N \times [-1, t_0] \to N \times [-1, 1]$ the map coinciding with h on $N \times [0, t_0]$ and with the identity map on $N \times [-1, 0]$. As h coincides with the identity map on $N \times \{0\}$, we get that h_+ is a homeomorphism onto its image. One defines in the same way $h_- : N \times [-t_0, 1] \to N \times [-1, 1]$ as the homeomorphism coinciding with h for negative t and with the identity map for positive t. One verifies directly that $h = h_+ \circ h_-$.

Now, if there exist homeomorphisms H_+ and H_- of $N \times [-1, 1]$ coinciding with h_+ and h_- in a neighborhood of $N \times \{0\}$ and with the identity out of $N \times [-1/2, 1/2]$, then the composition map $H = H_+ \circ H_-$ will be the announced homeomorphism.

From now on, we assume that $h = h_+$, that is h coincides with the identity on $N \times [-1, 0]$, so that the domain of h is $N \times [-1, t_0]$. Now follows the practical construction of the announced homeomorphism H.

We fix $t_1 \in]0, t_0/2[$ such that $N \times [-1, t_1]$ is contained in $h(N \times [-1, t_0])$. We fix $t_2 \in]0, t_1[$ such that $N \times [1, t_2] \subset h^{-1}(N \times [-1, t_1])$.

Let φ denote a homeomorphism of $N \times [-1, 1]$ which coincides with the identity map out of $N \times [-3t_2/4, 3t_2/4]$ and with the map $(x, t) \mapsto (x, t - (t_2/4))$ on $N \times [-t_2/4, t_2/4]$. We assume furthermore that for each (x, t) its image $\varphi(x, t)$ belongs to $N \times [-1, t]$.

Let us consider the map $H = h \circ \varphi^{-1} \circ h^{-1} \circ \varphi$.

If $t \in [0, t_2/4]$ then $h^{-1} \circ \varphi(x, t) = h^{-1}(x, t - (t_2/4)) = (x, t - (t_2/4))$ because $t - (t_2/4) \leq 0$. As $t - (t_2/4) \in [-t_2/4, 0]$ one gets than $\varphi^{-1}(x, t - (t_2/4)) = (x, t)$. Thus

$$h \circ \varphi^{-1} \circ h^{-1} \varphi(x, t) = h(x, t) \quad \text{for } t \in \left[0, \frac{t_2}{4}\right].$$

For $t \geq t_1$ one has $\varphi(x, t) = (x, t)$. Assume that $(x, t) \in h(N \times [-1, t_0])$ (for example if $t = t_1$). Then $h^{-1} \circ \varphi(x, t) = h^{-1}(x, t) \in h^{-1}(N \times [t_1, 1[) \subset N \times [t_2, 1]$. By hypothesis, φ is the identity map on $N \times [t_2, 1]$, so that

$$h \circ \varphi^{-1} \circ h^{-1} \circ \varphi(x, t) = h \circ h^{-1}(x, t) = (x, t) \quad \text{for } t \geq t_0/2.$$

Notice that H coincides with the identity map on $t \leq 0$.

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So H is well-defined on $N \times [-t_1, t_1]$, and coincides with h on $N \times [-1, t_2/4]$ and with the identity map on $N \times \{-t_1, t_1\}$. Thus, H can be extended continuously by the identity map on $N \times [t_1, 1]$. Then H verifies all the announced properties. □

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