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On reversible maps and symmetric periodic points

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Abstract. In reversible dynamical systems, it is of great importance to understand symmetric features. The aim of this paper is to explore symmetric periodic points of reversible maps on planar domains invariant under a reflection. We extend Franks' theorem on a dichotomy of the number of periodic points of area-preserving maps on the annulus to symmetric periodic points of area-preserving reversible maps. Interestingly, even a non-symmetric periodic point guarantees infinitely many symmetric periodic points. We prove an analogous statement for symmetric odd-periodic points of area-preserving reversible maps isotopic to the identity, which can be applied to dynamical systems with double symmetries. Our approach is simple, elementary, and far from Franks' proof. We also show that a reversible map has a symmetric fixed point if and only if it is a twist map which generalizes a boundary twist condition on the closed annulus in the sense of Poincaré–Birkhoff. Applications to symmetric periodic orbits in reversible dynamical systems with two degrees of freedom are briefly discussed.

1. *Main results and applications*

The study of symmetric features of reversible dynamical systems is of great importance. This paper is majorly concerned with symmetric periodic points of reversible maps on planar domains, which can be applied to find symmetric periodic orbits of reversible dynamical systems with two degrees of freedom possessing global surfaces of section invariant under symmetries; see below. Among other studies of symmetric periodic points, we refer the reader to [**Bir15**, **DeV54**, **Dev76**], which are related to our approach.

Let I be the reflection on $\mathbb{R}^2$ given by

$$I : \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (-x, y).$$

A connected planar domain $\Omega \subset \mathbb{R}^2$ is said to be *invariant* if $I(\Omega) = \Omega$. Then I descends to a reflection I_Ω on an invariant domain Ω. A homeomorphism f on an invariant domain

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(Ω, I_Ω) obeying the condition

$$f \circ I_\Omega = I_\Omega \circ f^{-1}$$

is called a *reversible map*. A point $z \in \Omega$ is called a *symmetric periodic point* of f if

$$f^k(z) = z, \quad f^\ell(z) = I_\Omega(z) \quad \text{for some } k, \ell \in \mathbb{N}.$$

The minimal number $k \in \mathbb{N}$ satisfying the requirement is called a *period* of a symmetric fixed point z of f. In particular, if $k = \ell = 1$, z is called a *symmetric fixed point*. We point out that symmetric fixed points lie on the fixed locus Fix I_Ω of I_Ω. A point meeting the first condition is called a periodic point of f. Throughout this paper, we always assume that $\Omega \subset \mathbb{R}^2$ is an invariant connected domain with a reflection I_Ω and the following invariant domains are of interest to us:

$$A := \{z \in \mathbb{R}^2 \mid 1 \le |z| \le 2\}, \quad S := \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le 1\}, \quad D := \{z \in \mathbb{R}^2 \mid |z| \le 1\}$$

and

$$\mathring{A} := \{z \in \mathbb{R}^2 \mid 1 < |z| < 2\}, \quad \mathring{D} := \{z \in \mathbb{R}^2 \mid |z| < 1\}.$$

A beautiful theorem by Franks [**Fra92**, **Fra96**] states that every area-preserving homeomorphism on A or $\mathring{A}$ with a periodic point has infinitely many interior periodic points. Our first result is to extend this dichotomy to symmetric periodic points of reversible maps.

THEOREM 1.1. *Every area-preserving reversible map on A or $\mathring{A}$ is either periodic-point-free or has infinitely many interior symmetric periodic points.*

It is worth emphasizing that even a *non-symmetric* periodic point also guarantees infinitely many interior symmetric periodic points. Although the theorem is stated for A and $\mathring{A}$, it carries over to the half-closed annuli $\{z \in \mathbb{R}^2 \mid 1 \le |z| < 2\}$ and $\{z \in \mathbb{R}^2 \mid 1 < |z| \le 2\}$. We will show in Proposition 5.3 that every area-preserving reversible map on D or $\mathring{D}$ has at least one interior symmetric fixed point. The following corollary is a direct consequence of these results.

COROLLARY 1.2. *Every area-preserving reversible map on D or $\mathring{D}$ has either precisely one interior symmetric fixed point with no other periodic points or infinitely many interior symmetric periodic points.*

Next we show analogous results for symmetric periodic orbits with odd period. Symmetric odd-periodic points have an intimate relation with centrally symmetric periodic orbits, as described in the next subsection.

THEOREM 1.3. *Any area-preserving reversible map on A or $\mathring{A}$ isotopic to the identity with an odd-periodic point (not necessarily symmetric or interior) has infinitely many interior symmetric odd-periodic points. In consequence, every area-preserving, orientation-preserving reversible map on D or $\mathring{D}$ has either precisely one interior symmetric fixed point with no other odd-periodic points or infinitely many interior symmetric odd-periodic points.*

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In view of the Poincaré–Birkhoff theorem, we expect that there exists a symmetric fixed point on A under a boundary twist condition. Recall that a homeomorphism $f : A \to A$ isotopic to the identity is said to satisfy the *boundary twist condition* if f rotates two boundary circles of A in opposite angular directions. To be precise, there is a lift $F = (F_1, F_2) : S \to S$ of f such that

$$F_1(x, 0) < x < F_1(x, 1), \quad x \in \mathbb{R}.$$

THEOREM 1.4. *(Birkhoff) Let f be a reversible map on A isotopic to the identity with the boundary twist condition. Then there is an interior symmetric fixed point of f on each connected component of* Fix I_A.

We point out that here and in the following theorem a reversible map is not even required to have no wandering point, which is a weaker condition than being area preserving; see [**Fra88**]. In [**Bir15**], Birkhoff showed the existence of infinitely many symmetric periodic orbits and studied their rotation numbers. There was no precise statement like Theorem 1.4 but presumably he knew that this is true. We will outline two proofs of Theorem 1.4, which are on the lines of [**Bir13**, **Bir15**].

In fact, we find a necessary and sufficient condition on the existence of a symmetric fixed point of a reversible map isotopic to the identity on an arbitrary invariant connected domain Ω and, in particular, the boundary twist condition on A implies the sufficient condition. Note that the fixed locus of I_Ω is composed of disjoint intervals Y_i, i.e. Fix $I_\Omega = \bigsqcup_{1 \le i \le n} Y_i$.

THEOREM 1.5. *Let $f : \Omega \to \Omega$ be a reversible map isotopic to the identity. Then f has a symmetric fixed point on Y_i if and only if $f(Y_i) \cap Y_i \neq \emptyset$ for $i \in \{1, \ldots, n\}$. Similarly, the existence of an interior symmetric fixed point of f on Y_i is equivalent to $f(Y_i) \cap Y_i \cap \mathring{\Omega} \neq \emptyset$.*

In general, it is not easy to find an analogue of the boundary twist condition for an arbitrary possibly non-closed planar domain; see [**Fra88**] for the case of the open annulus. However, in view of this theorem, a reversible map $f : \Omega \to \Omega$ is reasonable to be called a *twist map* if Fix I_Ω is twisted; explicitly, if some component of Fix I_Ω has a self-intersection point by f.

Apart from these results, we will also show some classical fixed point theorems for symmetric fixed points of reversible maps and some of them will play key roles in the proof of our main results.

The Poincaré–Birkhoff theorem has been proved and generalized in many papers, e.g. [**Bir13**, **Bir15**, **Bir25**, **BN77**, **Neu77**, **Car82**, **Din83**, **Fra88**, **YZ95**, **Gui97**, **LCW09**, **HMS12**, **KS13**, **PR13**]. Franks' theorem also has been pursued further in e.g. [**FH03**, **LeC06**, **CKRTZ12**, **Ker12**]. There has been a remarkable new approach using pseudoholomorphic curves to understand smooth dynamics on planar domains by Bramham and Hofer [**BH12**] (see also the literature cited therein). We treat the Poincaré–Birkhoff theorem and Franks' theorem in their original forms for brevity but we expect that such generalized works might have reversible counterparts as well.

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1.1. *Applications to symmetric periodic orbits.* Let M be a three-dimensional smooth manifold with a smooth vector field X. An embedded compact Riemann surface $\Sigma \hookrightarrow M$ with boundary is called a *global surface of section* for X if (Σ, X) meets the following requirements.

(1) The boundary of Σ consists of periodic orbits, called the *spanning orbits*.
(2) The vector field X is transversal to the interior $\mathring{\Sigma}$ of Σ.
(3) Every orbit of X, except the spanning orbits, passes through $\mathring{\Sigma}$ in forward and backward time.

When we study dynamical systems with two degrees of freedom, the existence of global surfaces of section reduces the complexity by one dimension. A notion of global surfaces of section was introduced by Poincaré [**Poi99**] and further developed by Birkhoff [**Bir17**]. In the spirit of Poincaré's ground-breaking idea, our results help us to find symmetric periodic orbits of reversible dynamical systems with global surfaces of section invariant under symmetries. If M carries a smooth involution ρ with $\rho_* X = -X$, a global surface of section Σ invariant under ρ, i.e. $\rho(\Sigma) = \Sigma$, is called an *invariant global surface of section*. Note that the unique spanning orbit $\gamma : \mathbb{R}/T\mathbb{Z} \to M$ of an invariant global disk-like surface of section is automatically *symmetric*, i.e. $\rho(\gamma(\mathbb{R}/T\mathbb{Z})) = \gamma(\mathbb{R}/T\mathbb{Z})$, or equivalently if $\rho(\gamma(t)) = \gamma(-t)$, $t \in \mathbb{R}/T\mathbb{Z}$, up to reparametrization. The condition $\rho^* X = -X$ yields that the induced Poincaré return map $f : \mathring{\Sigma} \to \mathring{\Sigma}$ is reversible with respect to the involution $\rho|_\Sigma$. A crucial observation is that there is the following one-to-one correspondence:

$$\left\{\begin{matrix}\text{geometrically distinct sym-}\\ \text{metric periodic orbits of } X\end{matrix}\right\} \setminus \left\{\begin{matrix}\text{spanning}\\ \text{orbits of } \Sigma\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}f\text{-orbits of symmetric}\\ \text{periodic points of } f\end{matrix}\right\}.$$

Accordingly, reversible dynamical systems with invariant global surfaces of section, for instance the planar restricted three-body problem or the Störmer problem, are of interest to us. As observed in [**Bir15**, **Con63**, **McG69**], there are invariant global surfaces of section in the restricted three-body problem sometimes and the Poincaré return maps are area preserving. Although they were not interested in whether their global surfaces of section are invariant, one can easily check that these are indeed invariant. Another example is the Störmer problem; see [**Stö07**, **Bra70**]. The global surface of section in the Störmer problem constructed in [**DeV54**] is also invariant.

More generally, Frauenfelder and the author [**FK16**] proved that if a hypersurface in $\mathbb{R}^4$ bounds a strictly convex domain and is invariant under the involution $\rho := \mathrm{diag}(-1, -1, 1, 1)$, there exists an invariant global disk-like surface of section for the standard Reeb vector field and the Poincaré return map is area preserving, which generalizes pioneering work of Hofer *et al* [**HWZ98**, **HWZ03**]. This shows that a global disk-like surface of section in [**AFFHvK12**] is also invariant. As a consequence of Corollary 1.2, there are either two or infinitely many symmetric periodic orbits on such a hypersurface and the latter happens under the presence of a non-symmetric periodic orbit.

Suppose that a hypersurface $M \subset \mathbb{R}^4$ is centrally symmetric, i.e. $-\mathrm{Id}_{\mathbb{R}^4}(M) = M$. The standard Reeb vector field X on M automatically satisfies $(-\mathrm{Id}_{\mathbb{R}^4})^* X = X$. A periodic orbit which is symmetric with respect to $-\mathrm{Id}_{\mathbb{R}^4}$ is called *centrally symmetric*. A centrally symmetric hypersurface M which is symmetric with respect to ρ as well is also interesting as the regularized energy hypersurface of the planar restricted three-body problem is so;

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see [**AFFHvK12**]. If a periodic orbit is symmetric with respect to ρ as well as $-\mathrm{Id}_{\mathbb{R}^4}$, it is called *doubly symmetric*. The geometric motion of doubly symmetric periodic orbits and that of ρ-symmetric centrally non-symmetric periodic orbits are considerably different; see [**Kan12**]. In this situation, if Σ is a ρ-invariant global surface of section, so is $-\mathrm{Id}_{\mathbb{R}^4}(\Sigma)$. In consequence, the Poincaré (half-return) map $f : \mathring{\Sigma} \to -\mathrm{Id}_{\mathbb{R}^4}(\mathring{\Sigma}) = \mathring{\Sigma}$ (the identification by $-\mathrm{Id}_{\mathbb{R}^4}$) encodes the qualitative properties of the doubly symmetric aspect of X, and in particular we have the following correspondence:

$$\left\{\begin{array}{l}\text{geometrically distinct doubly}\\ \text{symmetric periodic orbits of } X\end{array}\right\} \setminus \left\{\begin{array}{l}\text{spanning}\\ \text{orbits of } \Sigma\end{array}\right\} \longleftrightarrow \left\{\begin{array}{l}f\text{-orbits of symmetric}\\ \text{odd-periodic points of } f\end{array}\right\}.$$

In view of this correspondence, Theorem 1.3 helps us to detect doubly symmetric periodic orbits. We expect that a hypersurface in $\mathbb{R}^4$ bounding a strictly convex domain and symmetric with respect to both ρ and $-\mathrm{Id}_{\mathbb{R}^4}$ has a ρ-invariant global disk-like surface of section with the doubly symmetric spanning orbit.

We can apply this to symmetric closed geodesics on Riemannian 2-spheres. A combination of Franks' work [**Fra92**] and Bangert's work [**Ban93**] (see also [**Hin93**]) proves the existence of infinitely many closed geodesics on every Riemannian 2-sphere. In particular, Franks' work applies to a Riemannian 2-sphere with Birkhoff's global annulus-like surface of section. Let $\gamma : S^1 \to S^2$ be a simple closed geodesic with respect to a Riemannian metric g. It separates S^2 into two disks and we denote one of them by D_1. Then

$$\Sigma_\gamma := \{(x, v) \in S_g S^2 \mid x \in \gamma(S^1),\ 0 \le \angle(\dot\gamma_x, v) \le \pi\}$$

is an embedded surface in the unit tangent bundle $S_g S^2$ diffeomorphic to the closed annulus with boundary $\dot\gamma(S^1) \sqcup \dot{\bar\gamma}(S^1)$, where $\bar\gamma(t) := \gamma(-t)$. Birkhoff showed in [**Bir27**] that Σ_γ is a global surface of section and the Poincaré return map on it extends to the boundary of Σ_γ if γ meets certain properties, which are true for (S^2, g) with positive curvature. Moreover, the extended Poincaré return map on Σ_γ is area preserving and isotopic to the identity.

There is a refinement of this in the symmetric case as follows. Suppose that there exists an involutive orientation-reversing isometry R on (S^2, g), i.e. $R^2 = \mathrm{Id}_{S^2}$, $\mathrm{Fix}\, R \cong S^1$, and $R^* g = g$. We call a closed geodesic γ *symmetric* if $R(\gamma(t)) = \gamma(-t)$. Assume that there is a simple symmetric closed geodesic γ which defines a global surface of section Σ_γ. Then Σ_γ is invariant under the involution R_*. Hence, the following corollary is an immediate consequence of Theorems 1.1 and 1.4; see the proof of [**Fra92**, Theorem 4.1] for details.

COROLLARY 1.6. *There are infinitely many symmetric closed geodesics on a symmetric (S^2, g, R) equipped with an invariant Birkhoff's global annulus-like surface of section.*

1.2. *Questions.* We expect that an analogue of Theorem 1.3 in the absence of reversibility is also true. To be precise, we expect that every area-preserving homeomorphism f on A or $\mathring{A}$ isotopic to the identity has infinitely many odd-periodic points provided there is a single odd-periodic point. More generally, it is conceivable that the following claim is true. Assume that an area-preserving homeomorphism f on A or $\mathring{A}$ isotopic to the identity has a k-periodic point for some $k \in \mathbb{N}$. For a given $n \in \mathbb{N}$, if n and

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k are relatively prime, f has infinitely many periodic orbits with period relatively prime to n. This is motivated by the case S^3 with the standard $\mathbb{Z}/n$-action. Note that this covers the centrally symmetric case when $n = 2$.

It is also tempting to remove the assumption on the existence of Birkhoff's global surface of section in Corollary 1.6 by showing a reversible counterpart to a result in [**Ban93**].

2. *Reversible maps and involutions*

A remarkable property of reversible maps is that $f \circ I_\Omega$ is an involution if f is a reversible map on an invariant domain Ω. Symmetric fixed points of a reversible map f can be interpreted as intersection points of two fixed loci Fix f and Fix I_Ω. Therefore, to study a symmetric fixed point of f, it is crucial to observe the fixed locus $\mathrm{Fix}(f \circ I_\Omega)$ of the involution $f \circ I_\Omega$, since

$$\mathrm{Fix}\ f \cap \mathrm{Fix}\ I_\Omega = \mathrm{Fix}(f \circ I_\Omega) \cap \mathrm{Fix}\ I_\Omega.$$

Remark 2.1. We define an operation $\star$ on the space $\mathfrak{Inv}$ of involutions on Ω by

$$J \star K = J \circ K \circ J, \quad J, K \in \mathfrak{Inv},$$

so that $(\mathfrak{Inv}, \star)$ becomes a magma, i.e. $J \star K \in \mathfrak{Inv}$. We also consider the space $\mathfrak{Rev}$ of reversible maps on Ω and endow $\mathfrak{Rev}$ with a magma operation by

$$f \diamond g = f \circ g^{-1} \circ f, \quad f, g \in \mathfrak{Rev}.$$

Then we have a bijective magma homomorphism between them:

$$(\mathfrak{Rev}, \diamond) \to (\mathfrak{Inv}, \star), \quad f \mapsto f \circ I_\Omega.$$

In particular, any reversible map f is a composition of two involutions $f \circ I_\Omega$ and I_Ω, as observed by Birkhoff [**Bir15**].

Recall that a map $f : \Omega \to \Omega$ is said to be *conjugate* to a map $g : \Omega \to \Omega$ if there exists a homeomorphism $h : \Omega \to \Omega$ with $h \circ f = g \circ h$. If this is the case,

$$h(\mathrm{Fix}\ f) = \mathrm{Fix}\ g.$$

According to Brouwer [**Brou19**] and Kérékjartó [**Kér19**], any involution on $\mathbb{R}^2$ is conjugate to Id or $-$Id or I. We refer to [**CK94**] for a modern exposition for this result. Consequently, the fixed locus of an orientation-reversing involution on $\mathbb{R}^2$ is an embedded line and the fixed locus of an involution on a planar domain is a topological submanifold. The following lemma immediately follows from this observation.

LEMMA 2.2. *If J is an orientation-reversing involution on Ω,* Fix J *is a topological submanifold. If $\Omega = \mathbb{R}^2$,* Fix $J \cong \mathbb{R}$. *If $\Omega = D$,* Fix $J \cong [0, 1]$. *If $\Omega = S$ and J is isotopic to I_S,* Fix $J \cong [0, 1]$. *If $\Omega = A$ and J is isotopic to I_A,* Fix $J \cong [0, 1] \sqcup [0, 1]$.

Remark 2.3. In general, it is not easy to answer whether an involution has a fixed point. There is a Lefschetz fixed point theorem for involutions due to [**KK68**]. This states that if an involution J on a locally compact space X is fixed point free,

$$\Lambda_J := \sum_{k \geq 0} (-1)^k \mathrm{Tr}(J_* | H_k(X; \mathbb{R})) = 0.$$

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Even if Fix J is non-empty, it is not necessarily a topological submanifold, whereas the fixed locus of a smooth involution is always a smooth submanifold. There is the so-called *dog-bone space* B due to Bing which is not a topological manifold but $B \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^4$. Using this, one can construct a continuous involution J on $\mathbb{R}^4$ with Fix $J = B$. We refer the reader to [**Bin59**, §9] for details.

Remark 2.4. The condition $J \simeq I_A$ when $\Omega = A$ in the preceding lemma is indispensable. Note that an antisymplectic involution J on A defined by

$$(r, \theta) \mapsto (3 - r, \pi + \theta)$$

is fixed point free, where $(r, \theta) \in \mathbb{R}^2$ is the polar coordinate. In particular, the Lefschetz number Λ_J in Remark 2.3 vanishes.

The following two easy lemmas will play key roles. Note that any iteration f^i, $i \in \mathbb{Z}$, of a reversible map f is reversible again.

LEMMA 2.5. *If $f : \Omega \to \Omega$ is reversible,*

$$f^j(\mathrm{Fix}(f^i \circ I_\Omega)) = \mathrm{Fix}(f^{2j+i} \circ I_\Omega) \quad \textit{for all } i,\ j \in \mathbb{Z}.$$

Proof. For $z \in \mathrm{Fix}(f^i \circ I_\Omega)$, $f^j(z) \in \mathrm{Fix}(f^{2j+i} \circ I_\Omega)$ since

$$f^{2j+i} \circ I_\Omega(f^j(z)) = f^{2j+i} \circ f^{-j}(I_\Omega(z)) = f^j(f^i \circ I_\Omega(z)) = f^j(z).$$

To prove the converse, we pick $z \in \mathrm{Fix}(f^{2j+i} \circ I_\Omega)$. Then

$$f^i \circ I_\Omega(f^{-j}(z)) = I_\Omega \circ f^{-i-j}(f^{2j+i} \circ I_\Omega(z)) = I_\Omega \circ f^j \circ I_\Omega(z) = f^{-j}(z).$$

This implies that $f^{-j}(z) \in \mathrm{Fix}(f^i \circ I_\Omega)$ and thus $z \in f^j(\mathrm{Fix}\ f^i \circ I_\Omega)$. □

LEMMA 2.6. *Let $f : \Omega \to \Omega$ be a reversible map. If $z \in \mathrm{Fix}(f^k \circ I_\Omega) \cap \mathrm{Fix}(f^\ell \circ I_\Omega)$ for some $k > \ell \in \mathbb{Z}$, then z is a symmetric $(k - \ell)$-periodic point of f.*

Proof. Since $f^k \circ I_\Omega(z) = f^\ell \circ I_\Omega(z)$,

$$z = I_\Omega \circ f^{\ell-k} \circ I_\Omega(z) = f^{k-\ell}(z).$$

Moreover, by the assumption, $f^k \circ I_\Omega(z) = z$ and hence

$$f^{m(k-\ell)-k}(z) = f^{-k}(z) = I_\Omega(z)$$

for any $m \in \mathbb{Z}$. □

3. *On the fixed point index for reversible maps*

For any two smooth paths $\delta, \gamma : [0, 1] \to \Omega$ with $\delta(t) \neq \gamma(t)$ for every $t \in [0, 1]$, we define a path $\gamma - \delta : [0, 1] \to \Omega$ avoiding the origin by $(\gamma - \delta)(t) := \gamma(t) - \delta(t)$. The variation of angle of such two paths δ, γ is defined by

$$i(\delta, \gamma) := \frac{1}{2\pi} \int_{\gamma - \delta} d\theta,$$

where θ is the angular coordinate on $\mathbb{R}^2$. We list some useful properties of $i(\delta, \gamma)$.

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(1) For any $\delta, \gamma : [0, 1] \to \Omega$ with $\delta(t) \neq \gamma(t)$,

$$i(\delta, \gamma) = i(\gamma, \delta).$$

(2) If $\delta_1, \delta_2, \gamma_1, \gamma_2 : [0, 1] \to \Omega$ with $\delta_j(t) \neq \gamma_j(t)$, $j = 1, 2$, satisfy $\delta_1(1) = \delta_2(0)$ and $\gamma_1(1) = \gamma_2(0)$,

$$i(\delta_1 * \delta_2, \gamma_1 * \gamma_2) = i(\delta_1, \gamma_1) + i(\delta_2, \gamma_2),$$

where $*$ stands for the catenation operation defined by

$$\gamma_1 * \gamma_2(t) := \begin{cases} \gamma_1(2t), & t \in [0, \frac{1}{2}], \\ \gamma_2(2t - 1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

(3) If $\delta, \gamma : [0, 1] \to \Omega$ with $\delta(t) \neq \gamma(t)$ loops, i.e. $\delta(0) = \delta(1)$, $\gamma(0) = \gamma(1)$,

$$i(\delta, \gamma) \in \mathbb{Z}.$$

(4) For any $\delta, \gamma : [0, 1] \to \Omega$ with $\delta(t) \neq \gamma(t)$,

$$i(\bar{\delta}, \bar{\gamma}) = -i(\delta, \gamma),$$

where $\bar{\delta}(t) := \delta(1 - t)$, $\bar{\gamma}(t) := \gamma(1 - t)$.

(5) Recall that we have assumed that Ω is invariant. For any $\delta, \gamma : [0, 1] \to \Omega$ with $\delta(t) \neq \gamma(t)$,

$$i(I_\Omega \circ \delta, I_\Omega \circ \gamma) = -i(\delta, \gamma).$$

(6) For continuous families of paths $\delta_s, \gamma_s : [0, 1] \to \Omega$, $s \in \mathbb{R}$, with $\delta_s(t) \neq \gamma_s(t)$ for any s, $i(\delta_s, \gamma_s)$ varies continuously with the parameter s.

PROPOSITION 3.1. *Let $f : \Omega \to \Omega$ be a reversible map isotopic to the identity. Then, for any loop $\gamma : S^1 \to \Omega \setminus \operatorname{Fix} f$,*

$$i(\gamma, f \circ \gamma) = i(I_\Omega \circ \bar{\gamma}, f \circ I_\Omega \circ \bar{\gamma}).$$

Let $h : \Omega \to \Omega$ be a reversible map isotopic to I_Ω. Then, for any loop $\gamma : S^1 \to \Omega \setminus \operatorname{Fix} h$,

$$i(\gamma, h \circ \gamma) = -i(I_\Omega \circ \bar{\gamma}, h \circ I_\Omega \circ \bar{\gamma}).$$

Proof. Observe that by reversibility of f,

$$i(I_\Omega \circ \bar{\gamma}, f \circ I_\Omega \circ \bar{\gamma}) = i(I_\Omega \circ \bar{\gamma}, I_\Omega \circ f^{-1} \circ \bar{\gamma}) = i(\gamma, f^{-1} \circ \gamma).$$

Let $f_s : \Omega \to \Omega$, $s \in [0, 1]$, be an isotopy between $f_0 = \mathrm{Id}_\Omega$ and $f_1 = f$. Since $i(f_s \circ \gamma, f_s \circ f^{-1} \circ \gamma)$ varies continuously in $s \in [0, 1]$ and it takes an integer value for every $s \in [0, 1]$, $i(f_s \circ \gamma, f_s \circ f^{-1} \circ \gamma)$ is constant. In particular,

$$i(\gamma, f^{-1} \circ \gamma) = i(f \circ \gamma, \gamma),$$

and the first assertion is proved. For the second identity, let $h_s : \Omega \to \Omega$, $s \in [0, 1]$, be an isotopy between $h_0 = I_\Omega$ and $h_1 = h$. In a similar vein, we compute

$$i(I_\Omega \circ \bar{\gamma}, h \circ I_\Omega \circ \bar{\gamma}) = i(\gamma, h^{-1} \circ \gamma) = -i(I_\Omega \circ \gamma, I_\Omega \circ h^{-1} \circ \gamma) = -i(h \circ \gamma, \gamma). \quad \square$$

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Recall that the *index* of an isolated interior fixed point $z \in \operatorname{Fix} f$ is defined by

$$i(f, z) := i(\gamma, f \circ \gamma)$$

for any simple sufficiently small loop $\gamma : S^1 \to \Omega \setminus \operatorname{Fix} f$ winding z counterclockwise. We will also use the fact that for any simply connected domain D_δ enclosed by a simple loop $\delta : S^1 \to \Omega \setminus \operatorname{Fix} f$ with finitely many fixed points of f,

$$i(\delta, f \circ \delta) = \sum_{x \in \operatorname{Fix} f \cap D_\delta} i(f, x).$$

COROLLARY 3.2. *Let $f : \Omega \to \Omega$ be a reversible map. If z is fixed by f, so is $I_\Omega(z)$. Moreover, if f is isotopic to the identity and $z \in \operatorname{Fix} f \cap \mathring{\Omega}$, $i(f, z) = i(f, I_\Omega(z))$. If f and I_Ω are isotopic, $i(f, z) = -i(f, I_\Omega(z))$*

Proof. If $f(z) = z$, $f \circ I_\Omega(z) = I_\Omega \circ f^{-1}(z) = I_\Omega(z)$. Assume that f is isotopic to the identity. The case that f is isotopic to I_Ω follows in a similar way. Suppose that $\gamma : S^1 \to \Omega \setminus \operatorname{Fix} f$ is a sufficiently small loop such that $i(f, z) = i(\gamma, f \circ \gamma)$ for $z \in \operatorname{Fix} f$. Then $I_\Omega \circ \bar{\gamma}$ is a sufficiently small loop surrounding $I_\Omega(x)$ counterclockwise, and hence using Proposition 3.1 we have

$$i(f, I_\Omega(z)) = i(I_\Omega \circ \bar{\gamma}, f \circ I_\Omega \circ \bar{\gamma}) = i(\gamma, f \circ \gamma) = i(f, z). \qquad \square$$

4. *Classical fixed point theorems for reversible maps*

The following statement can be thought of as Brouwer's fixed point theorem for symmetric fixed points of reversible maps. Note that any continuous map from D to itself is isotopic to Id_D (to I_D) provided that it is orientation preserving (reversing).

PROPOSITION 4.1. *Let $f : D \to D$ be a reversible map with finitely many fixed points. Then f has at least one symmetric fixed point.*

Proof. Suppose that there is no symmetric fixed point of f. Then, due to Corollary 3.2,

$$\sum_{z \in \operatorname{Fix} f} i(f, z) \in 2\mathbb{Z}.$$

This contradicts the Lefschetz–Hopf theorem, which implies that $\sum_{z \in \operatorname{Fix} f} i(f, z) = 1$. $\square$

Remark 4.2. There is a reversible map $f : D \to D$ with infinitely many fixed points but no symmetric fixed point. For simplicity, we consider the domain $B = \{(x, y) \in \mathbb{R}^2 \mid -2 \le x \le 2,\ 0 \le y \le 1\}$ instead of D. Let $f : B \to B$ be a reversible map given by

$$f(x, y) := \begin{cases} (4 + 3x, y), & x \in [-2, -1], \\ (\frac{4}{3} + \frac{1}{3}x, y), & x \in [-1, 2]. \end{cases}$$

Then $\operatorname{Fix} f = \{(-2, y) \cup (2, y) \mid 0 \le y \le 1\}$ but f has no symmetric periodic point.

The following is a version of Brouwer's lemma [**Brou12**] for reversible maps.

PROPOSITION 4.3. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an orientation-preserving reversible map with a symmetric periodic point. Then f has a symmetric fixed point.*

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Proof. Let $(x, y) \in \mathbb{R}^2$ be a symmetric periodic point of f, i.e. $f^\ell(x, y) = I(x, y) = (-x, y)$ and $f^k(x, y) = (x, y)$ for some $\ell, k \in \mathbb{N}$. Since $f \circ I$ is an orientation-reversing involution, $\mathrm{Fix}(f \circ I)$ is an embedded line in the plane due to Lemma 2.2. We write $\alpha = \mathrm{Fix}(f \circ I)$. Assume by contradiction that $\alpha \cap \mathrm{Fix}\, I = \emptyset$; see Lemma 2.6. Note that $I(\alpha)$ is also an embedded line which does not intersect both $\mathrm{Fix}\, I$ and α. We assume that $I(\alpha) \subset \{(x, y) \in \mathbb{R}^2 \mid x < 0\}$. The case that $I(\alpha) \subset \{(x, y) \in \mathbb{R}^2 \mid x > 0\}$ also follows in a similar way. The line α (respectively $I(\alpha)$) divides the plane into two open regions α_+ and α_- (respectively $I(\alpha)_+$ and $I(\alpha)_-$). Let α_- and $I(\alpha)_+$ be regions containing the imaginary axis $\mathrm{Fix}\, I$. Since f is orientation preserving and $f(I(\alpha)) = \alpha$, f maps $I(\alpha)_\pm$ to $\alpha_\pm$, respectively. Now we show that (x, y) cannot be a symmetric periodic point. If $(x, y) \in I(\alpha)_-$, $f^n(x, y) \in \alpha_-$ for all $n \in \mathbb{N}$ but $f^\ell(x, y) = (-x, y) \in \alpha_+$ for some $\ell \in \mathbb{N}$. Thus, this does not happen and other cases $(x, y) \in I(\alpha) \cup I(\alpha)_+$ can be ruled out in a similar way. This contradiction proves the proposition. □

The above proposition does not remain true for general invariant domains. However, we are still able to find a symmetric fixed point in the presence of a certain symmetric 2-periodic point. Note that if $(0, y) \in \mathrm{Fix}\, I_\Omega$ is a symmetric 2-periodic point of a reversible map $f : \Omega \to \Omega$, $f(0, y) \in \mathrm{Fix}\, I_\Omega$ as well. Borrowing an idea of Brown [**Brow90**], if symmetric 2-periodic points $(0, y)$, $f(0, y) \in \mathrm{Fix}\, I_\Omega$ lie on the same connected component of $\mathrm{Fix}\, I_\Omega$, then we can find a symmetric fixed point of a reversible map f isotopic to the identity. This will be used later to prove Theorem 1.5.

PROPOSITION 4.4. *Suppose that a reversible map $f : \Omega \to \Omega$ isotopic to the identity has a symmetric 2-periodic point $(0, y) \in \mathrm{Fix}\, I_\Omega$. If $(0, y)$ and $f(0, y)$ are on the same connected component of $\mathrm{Fix}\, I_\Omega$, there exists a symmetric fixed point of f lying on $[(0, y), f(0, y)] \subset \mathrm{Fix}\, I_\Omega$.*

Proof. We may assume that f has no fixed point in a small open neighborhood of the interval $[(0, y), f(0, y)]$ since otherwise f has a fixed point on $[(0, y), f(0, y)]$, which is in turn a symmetric fixed point. We choose an arc $\gamma : [0, 1] \to \Omega$ such that

$$\gamma(0) = (0, y), \quad \gamma(1) = f(0, y), \quad \gamma(t) \cap (\mathrm{Fix}\, f \cup \mathrm{Fix}\, I_\Omega) = \emptyset, \quad t \in (0, 1)$$

and that the invariant domain D_γ enclosed by the loop $\gamma * (I_\Omega \circ \bar{\gamma})$ is simply connected and contains no fixed points of f. Since $f \circ \gamma(1) - \gamma(1) = (0, y) - f(0, y)$ and $f \circ \gamma(0) - \gamma(0) = f(0, y) - (0, y)$ point in opposite directions,

$$i(\gamma, f \circ \gamma) \in \tfrac{1}{2} + \mathbb{Z}.$$

Now we claim that

$$i(I_\Omega \circ \bar{\gamma}, f \circ I_\Omega \circ \bar{\gamma}) = i(\gamma, f \circ \gamma),$$

where $\bar{\gamma}(t) = \gamma(1 - t)$. Indeed,

$$\begin{aligned} i(I_\Omega \circ \bar{\gamma}, f \circ I_\Omega \circ \bar{\gamma}) &= -i(I_\Omega \circ \gamma, f \circ I_\Omega \circ \gamma) = -i(I_\Omega \circ \gamma, I_\Omega \circ f^{-1} \circ \gamma) \\ &= i(\gamma, f^{-1} \circ \gamma). \end{aligned}$$

Let $f_s : \Omega \to \Omega$, $s \in [0, 1]$, be an isotopy between $f_0 = \mathrm{Id}$ and $f_1 = f$. Since $f_s \circ f^{-1} \circ \gamma(1) - f_s \circ \gamma(1) = f_s(0, y) - f_s \circ f(0, y)$ and $f_s \circ f^{-1} \circ \gamma(0) - f_s \circ \gamma(0) =$

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$f_s \circ f(0, y) - f_s(0, y)$ point in opposite directions,

$$i(f_s \circ \gamma,\ f_s \circ f^{-1} \circ \gamma) \in \tfrac{1}{2} + \mathbb{Z}$$

for every $s \in [0, 1]$ and in particular

$$i(\gamma,\ f^{-1} \circ \gamma) = i(f \circ \gamma, \gamma).$$

This proves the claim. By the claim, we have

$$i(\gamma * (I_\Omega \circ \bar{\gamma}),\ f(\gamma * (I_\Omega \circ \bar{\gamma}))) = i(\gamma,\ f \circ \gamma) + i(I_\Omega \circ \bar{\gamma},\ f \circ I_\Omega \circ \bar{\gamma}) = 2i(\gamma,\ f \circ \gamma).$$

Using $i(\gamma,\ f \circ \gamma) \in \frac{1}{2} + \mathbb{Z}$, we deduce that

$$i(\gamma * (I_\Omega \circ \bar{\gamma}),\ f(\gamma * (I_\Omega \circ \bar{\gamma}))) \in 1 + 2\mathbb{Z},$$

and therefore there is a fixed point of f in D_γ. This contradiction shows that there exists at least one symmetric fixed point of f inside D_γ, and hence on $\mathrm{Fix}\, I_\Omega \cap D_\gamma = [(0, y),\ f(0, y)]$. □

5. *Proofs of the main results*

5.1. *Proofs of Theorem 1.1 and Corollary 1.2.*

PROPOSITION 5.1. *If an area-preserving reversible map f on A or $\mathring{A}$ has a periodic point, there is an interior symmetric periodic point of f.*

Proof. We only give a proof for A, which applies to $\mathring{A}$ as well. Assume by contradiction that there is no interior symmetric periodic point of f. We denote the connected components of $\mathrm{Fix}\, I_A$ by $Y_\pm := \{0\} \times \pm[1, 2]$. By Lemma 2.5, $f^m(Y_+) \subset \mathrm{Fix}(f^{2m} \circ I_A)$ for every $m \in \mathbb{N}$, and thus by Lemma 2.6 we have

$$f^m(Y_+) \cap Y_\pm \cap \mathring{A} = \emptyset, \quad m \in \mathbb{N}. \tag{5.1}$$

Two segments Y_+ and $f(Y_+)$ divide A into two closed regions A_- and A_+ such that $A_- \cup A_+ = A$ and $A_- \cap A_+ = Y_+ \cup f(Y_+)$. Since $f(Y_+) \cap Y_- \cap \mathring{A} = \emptyset$, A_- and A_+ have different areas, say $\mathrm{area}(A_-) < \mathrm{area}(A_+)$. We assume that $A_- \subset \{(x, y) \in \mathbb{R}^2 \mid x \leq 0\}$. The other case is proved in the same manner. Note that

$$\mathrm{area}(A_-) = \mathrm{area}(f^m(A_-)) < \frac{\mathrm{area}(A)}{2}, \quad m \in \mathbb{N} \cup \{0\}. \tag{5.2}$$

We claim that

$$f^m(A_-) \cap f^{m+1}(A_-) = f^m(Y_+), \quad m \in \mathbb{N} \cup \{0\}. \tag{5.3}$$

It suffices to prove the claim for $m = 0$. Indeed, if this is not true, $A_- \cap f(A_-) \supsetneq f(Y_+)$, and thus $f^2(Y_+) \cap Y_+ \cap \mathring{A} \neq \emptyset$ since $\mathrm{area}(A_-) + \mathrm{area}(f(A_-)) < \mathrm{area}(A)$. This contradicts (5.1) and hence (5.3) follows.

Now we are ready to prove the assertion. Suppose that $z_- \in Y_+ \cap f(Y_+)$. By (5.1), $z_- \in \partial A$. Observe that $f^\ell(A_-) \cap \{(x, y) \in A \mid x > 0\} \neq \emptyset$ for some $\ell \in \mathbb{N}$ by (5.2) and (5.3). If $\ell \in \mathbb{N}$ is the minimal number with this property, there exists a point

$$z_+ \in f^{\ell+1}(Y_+) \cap \{(x, y) \in A \mid x > 0\}.$$

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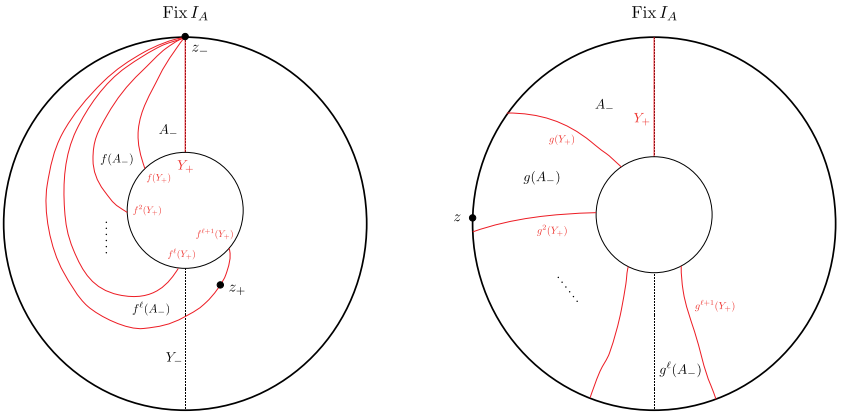


FIGURE 5.1. Iterated maps and fixed loci on A.

Since z_- is fixed by f, $f^{\ell+1}(Y_+)$ connects z_- and z_+, and therefore $f^{\ell+1}(Y_+) \cap Y_- \cap \mathring{A} \neq \emptyset$, which ensures the existence of an interior symmetric periodic point of f by Lemma 2.6. See the first picture of Figure 5.1.

Suppose that $Y_+ \cap f(Y_+) = \emptyset$. Let $z \in A$ be a periodic point of f, i.e. $f^k(z) = z$ for some $k \in \mathbb{N}$. Abbreviate $g := f^k$. Then

$$z \notin g^m(Y_+), \quad m \in \mathbb{N} \cup \{0\},$$

since otherwise $z = g(z) \in Y_+ \cap g(Y_+)$. We may assume that $z \in \{(x, y) \in A \mid x \leq 0\}$ as $I_A(z)$ is also a periodic point of f. As before, Y_+ and $g(Y_+)$ divide A into two closed regions and we call the smaller region A_- again. With g and this A_-, (5.1), (5.2), and (5.3) still hold. By (5.2) and (5.3),

$$\{(x, y) \in A \mid x \leq 0\} \subset \bigcup_{0 \leq m \leq \ell} g^m(A_-).$$

Hence, the periodic point $z \in g^m(A_-)$ for some $0 \leq m \leq \ell$. This contradicts that $z \in \operatorname{Fix} g$ since $g^m(A_-) \cap g^{m+1}(A_-) = g^{m+1}(Y_+)$ by (5.3) and $z \notin g^{m+1}(Y_+)$, $m \in \mathbb{N} \cup \{0\}$. See the second picture of Figure 5.1. □

THEOREM 5.2. *If an area-preserving reversible map f on A or $\mathring{A}$ has a periodic point, it possesses infinitely many interior symmetric periodic points.*

Proof. See Figure 5.2. We will prove the theorem only for A and the same proof goes through for $\mathring{A}$ as well. Due to Proposition 5.1, there is an interior symmetric periodic point of f. Suppose that there are only finitely many interior symmetric periodic points of f. By deleting interior symmetric periodic points from A, we obtain a punctured annulus, which we denote by M. Suppose that M has $n \geq 1$ punctures. Since f is a homeomorphism, f permutes n punctures and thus the iterated map $g := f^{2n!}$ fixes punctures pointwise. Moreover, g is isotopic to the identity and in particular g preserves the inner/outer boundary circles of M.

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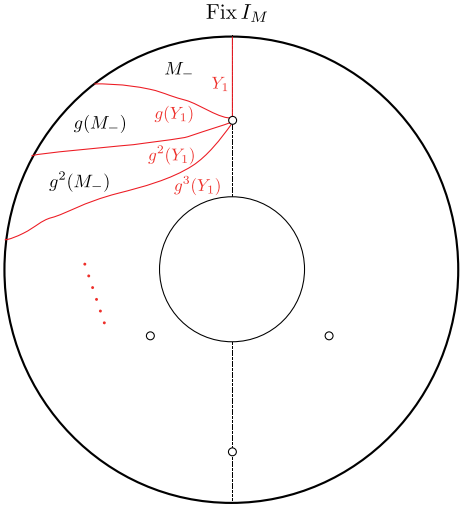


FIGURE 5.2. Iterated maps and fixed loci on a punctured annulus.

The fixed locus Fix I_M consists of several connected components. We denote by Y_1 the component at the very top, i.e. Y_1 connects the outer boundary of A and the very top puncture on Fix I_M or the inner boundary circle of A. We observe that

$$g^m(Y_1) \cap \text{Fix } I_M \cap \mathring{M} = \emptyset, \quad m \in \mathbb{N} \tag{5.4}$$

as in (5.1) due to Lemmas 2.5 and 2.6. Note that Y_1 and $g(Y_1)$ separate M into two closed regions M_- and M_+. Due to (5.4) with $m = 1$, M_- and M_+ have different areas, say $\text{area}(M_-) < \text{area}(M_+)$. Observe as before, see (5.3), that

$$g^m(M_-) \cap g^{m+1}(M_-) = g^m(Y_1), \quad m \in \mathbb{N} \cup \{0\}. \tag{5.5}$$

Since g fixes punctures pointwise, there is no puncture inside every $g^m(M_-)$, $m \in \mathbb{N} \cup \{0\}$, by (5.5). There is no loss of generality in assuming that $M_- \subset \{(x, y) \in M \mid x \le 0\}$. Due to (5.4) and the fact that there is at least one puncture on $\{(x, y) \in A \mid x \le 0\}$, every $g^m(Y_1) \subset \{(x, y) \in M \mid x \le 0\}$ for all $m \in \mathbb{N} \cup \{0\}$, which contradicts that g preserves the area of M_-. This completes the proof. □

One can use a covering picture to prove the above theorem; see Theorem 5.5.

PROPOSITION 5.3. *Every area-preserving reversible map f on D or $\mathring{D}$ has an interior symmetric fixed point.*

Proof. The proof is stated for D but the same argument works for $\mathring{D}$. As observed, it suffices to show that

$$\text{Fix}(f \circ I_D) \cap \text{Fix } I_D \cap \mathring{D} \neq \emptyset.$$

Suppose that f is orientation preserving. Then, since $f \circ I_D$ is an orientation-reversing, area-preserving involution, $\text{Fix}(f \circ I_D)$ is an embedded line in D (see Lemma 2.2), which divides D into two regions with the same area. Thus, it has to cross the interior part of Fix I_D.

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On the other hand, if f is orientation reversing, we consider f^2 instead. Then, by the above argument, f^2 has an interior symmetric fixed point, which in turn implies that f has a symmetric 2-periodic point on $\mathrm{Fix}\, I_D \cap \mathring{D}$. Applying Proposition 4.4, we can find an interior symmetric fixed point of f. □

5.2. *Proof of Theorem 1.3.*

PROPOSITION 5.4. *If an area-preserving reversible map f on A or $\mathring{A}$ isotopic to the identity has an odd-periodic point, there is an interior symmetric odd-periodic point of f.*

Proof. This follows from almost the same argument as in the proof of Proposition 5.1. Instead of repeating the argument, we briefly explain how the proof carries over to this proposition in the case of A. Let $z \in A$ be an ℓ-periodic point of f for some $\ell \in 2\mathbb{N} - 1$. We abbreviate $g := f^\ell$, so that $z \in \mathrm{Fix}\, g$. The first key point is that f is additionally assumed to be isotopic to the identity. This ensures that $\mathrm{Fix}(g \circ I_A)$ separates A into two regions, which are topologically disks, by Lemma 2.2. We choose a connected component Z of $\mathrm{Fix}(g \circ I_A)$. The second key fact is that points in

$$g^m(Z) \cap \mathrm{Fix}\, I_A \cap \mathring{A}, \quad m \in \mathbb{N} \cup \{0\}$$

are interior symmetric periodic points of f with odd period (divisor of $2\ell m + 1$) by Lemmas 2.5 and 2.6. Assume for contradiction that $g^m(Z) \cap \mathrm{Fix}\, I_A \cap \mathring{A} = \emptyset$ for every $m \in \mathbb{N} \cup \{0\}$. Recall that $Y_\pm$ are two connected components of $\mathrm{Fix}\, I_A$. The annulus A is divided into two regions with different areas by Z and Y_+. We call A_- the region with smaller area. The set $I_A(A_-)$ is enclosed by $I_A(Z)$ and Y_+ and $\mathrm{area}(I_A(A_-)) = \mathrm{area}(A_-)$. Since $g(I_A(Z)) = Z$, $g(I_A(A_-)) \cap A_- = Z$. More generally, we have that

$$g^m(A_-) \cap g^m(I_A(A_-)) = g^m(Y_+), \quad g^m(A_-) \cap g^{m+1}(I_A(A_-)) = g^m(Z), \quad m \in \mathbb{N} \cup \{0\}. \tag{5.6}$$

Now we are ready to prove the proposition. Suppose that $Z \cap Y_+ \cap \partial A \neq \emptyset$. Then, due to (5.6) and the area-preserving property of g, there exists $m \in \mathbb{N} \cup \{0\}$ such that $g^m(Z) \cap Y_- \cap \mathring{A} \neq \emptyset$, which contradicts the assumption; cf. the first picture in Figure 5.1. For a similar reason, $Z \cap Y_- \cap \partial A \neq \emptyset$ also leads to a contradiction. Thus, we suppose that $Z \cap \mathrm{Fix}\, I_A = \emptyset$. There is no loss of generality in assuming that $\mathrm{area}(A_-) \leq \mathrm{area}(A)/4$ by choosing the other connected component Z of $\mathrm{Fix}(g \circ I_A)$ if necessary. Together with (5.6) and the area-preserving property of g again, this yields that

$$A_- \cap g(A_-) = \emptyset, \quad I_A(A_-) \cap g(I_A(A_-)) = \emptyset. \tag{5.7}$$

Then $z \in \mathrm{Fix}\, g$ is contained in some $g^m(A_-)$ or $g^m(I_A(A_-))$, $m \in \mathbb{N} \cup \{0\}$. However, $g(z) = z$ contradicts (5.7); cf. the second picture in Figure 5.1. This shows that $g^m(Z) \cap \mathrm{Fix}\, I_A \cap \mathring{A} \neq \emptyset$ and hence f has an interior odd-symmetric point. □

It is conceivable that the following theorem also immediately follows from the proof of Theorem 5.2 with minor modifications. However, we provide a slightly different picture.

THEOREM 5.5. *Let f be an area-preserving reversible map on A or $\mathring{A}$ isotopic to the identity. If f has an odd-periodic point, there are infinitely many interior symmetric odd-periodic points.*

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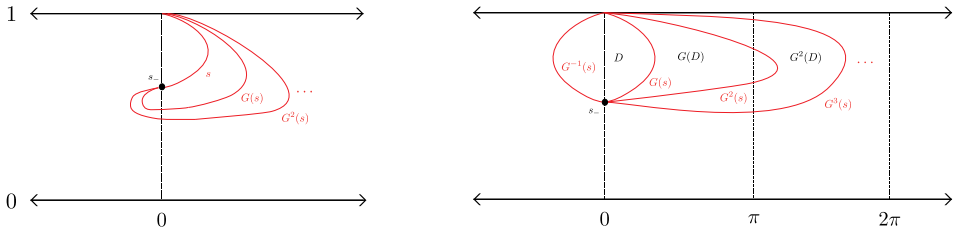


FIGURE 5.3. A piece of $\mathrm{Fix}(G^{2\ell+1} \circ I_S)$ for $\ell \in \mathbb{Z}$.

Proof. We prove the theorem for A and the case $\mathring{A}$ can be proved in a similar vein. Due to the previous proposition, we assume that there is an interior symmetric k-periodic point z of f for $k \in 2\mathbb{N} - 1$. Since k is odd, one of $f^\ell(z)$, $1 \le \ell \le k$ lies on $\mathrm{Fix}\, I_A$; we may assume that $z \in \mathrm{Fix}\, I_A \cap \mathring{A}$. Denote $g = f^k$. We consider the covering $S \to A$ and lift I_A to I_S. Thus, we can choose a lift $G : S \to S$ of $g : A \to A$ such that $G \circ I_S = I_S \circ G^{-1}$ and

$$\mathrm{Fix}(G \circ I_S) \cap \mathrm{Fix}\, I_S \neq \emptyset.$$

Suppose that the cardinality of the above set is finite, since otherwise there are infinitely many symmetric k-periodic points. Since f and hence G is isotopic to the identity, $\mathrm{Fix}(G \circ I_S)$ is a line connecting the upper boundary and the lower boundary of S; see Lemma 2.2. We choose the closed sub-segment s of $\mathrm{Fix}(G \circ I_S)$ with boundary $s_\pm$ satisfying

$$s \cap \mathrm{Fix}\, I_S \cap \mathring{S} = \{s_-\}, \quad s \cap (\mathbb{R} \times \{1\}) = \{s_+\}.$$

Note that $I_S(s) = G^{-1}(s)$, $s \cap I_S(s) \cap \mathring{S} = \{s_-\}$, and therefore

$$G^m(s) \cap G^{m+1}(s) \cap \mathring{S} = \{s_-\}, \quad m \in \mathbb{N} \cup \{0\}. \tag{5.8}$$

Without loss of generality, we assume that $s \subset [0, \infty) \times [0, 1]$. Since $G(I_S(s_+)) = s_+$ and G is isotopic to the identity,

$$G^\ell(s_+) \in [s_+, \infty) \times \{1\}, \quad \ell \in \mathbb{N}. \tag{5.9}$$

Suppose that there is $q_0 \in \mathbb{N}$ such that $G^{q_0}(s) \cap \mathrm{Fix}\, I_S \cap \mathring{S} \neq \{s_-\}$. Then there exists

$$(0, y_q) \in G^q(s) \cap \mathrm{Fix}\, I_S \cap \mathring{S}, \quad q \ge q_0$$

with $y_q > y_{q+1}$ different from s_- due to (5.8), (5.9), and $G^q(s_-) = s_-$. See the first picture of Figure 5.3. Due to Lemmas 2.5 and 2.6, this shows the existence of infinitely many interior symmetric odd-periodic points of f.

Suppose that $G^\ell(s) \cap \mathrm{Fix}\, I_S \cap \mathring{S} = \{s_-\}$ for every $\ell \in \mathbb{N}$. Let D be the domain enclosed by $s \cup I_S(s)$ and $\mathbb{R} \times \{1\}$. Observe that

$$G^\ell(D) \cap G^{\ell+1}(D) = G^{\ell+1}(s), \quad \ell \in \mathbb{N} \cup \{0\}$$

by (5.8); see the second picture of Figure 5.3. Since $\bigcup_{\ell \in \mathbb{N}} G^\ell(D) \subset [0, \infty) \times [0, 1]$ is connected and has infinite area, there exists $n(\ell) \in \mathbb{N}$, $\ell \in \mathbb{N}$, such that

$$\lim_{\ell \to \infty} n(\ell) = \infty, \quad G^\ell(s) \cap (\{n\pi\} \times [0, 1]) \cap \mathring{S} \neq \emptyset, \quad 0 \le \text{for all } n \le n(\ell).$$

Since the $\{n\pi\} \times [0, 1]$ are the lift of $\mathrm{Fix}\, I_A$, this shows that there are infinitely many interior symmetric odd-periodic points of f. □

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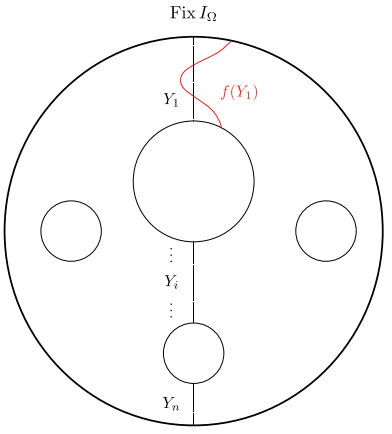


FIGURE 5.4. The self-intersections of Y_1 for a twist map f.

5.3. *Proofs of Theorems 1.4 and 1.5.* As we emphasized, we do *not* require here f to be area preserving. Recall that we denote Fix $I_\Omega = Y_1 \sqcup \cdots \sqcup Y_n$, where Y_i are disjoint intervals for an invariant connected possibly non-closed domain $\Omega \subset \mathbb{R}^2$.

THEOREM 5.6. *Let f be a reversible map on Ω isotopic to the identity. Then f has a (an interior) symmetric fixed point on Y_i if and only if $f(Y_i) \cap Y_i \neq \emptyset$ ($f(Y_i) \cap Y_i \cap \mathring{\Omega} \neq \emptyset$) for $i \in \{1, \ldots, n\}$.*

Proof. See Figure 5.4. If f has a symmetric fixed point $z \in Y_i$, $z = f(z) \in f(Y_i) \cap Y_i$ for $1 \leq i \leq n$. For the converse, suppose that $z \in f(Y_i) \cap Y_i$ for $1 \leq i \leq n$. Then $z = f(z_0)$ for some $z_0 \in Y_i$. We assume that $z \neq z_0$, since otherwise we are done. Since both z_0, $z \in Y_i \subset \text{Fix } I_\Omega$, we compute

$$f(z) = f \circ I_\Omega(z) = I_\Omega \circ f^{-1}(z) = I_\Omega(z_0) = z_0.$$

This implies that z is a symmetric 2-periodic point on Y_i with $f(z) \in Y_i$, and hence Proposition 4.4 guarantees the existence of a symmetric fixed point of f in between z and $f(z)$ and hence on Y_i. The assertion concerning an interior symmetric fixed point follows in the same way. □

Although Theorem 5.6 directly implies the theorem below, we outline two more elementary proofs of this. The first proof makes use of the fixed locus of the involution $f \circ I_A$ while the second proof uses the fixed point index.

THEOREM 5.7. *Let f be a reversible map on A isotopic to the identity with the boundary twist condition. Then there is a symmetric fixed point of f on each connected component of* Fix I_A.

Sketch of Proof 1. See Figure 5.5. We lift $f : A \to A$ to a reversible map

$$F = (F_1, F_2) : S = \mathbb{R} \times [0, 1] \to S$$

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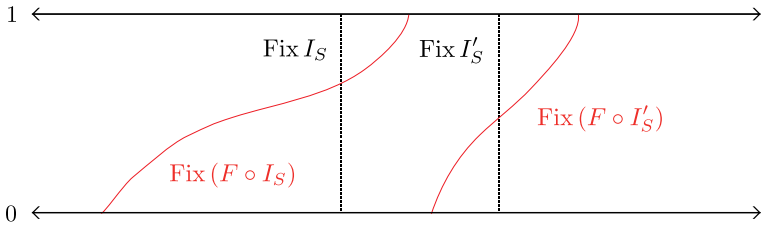


FIGURE 5.5. Fix$(F \circ I_S)$ under the boundary twist condition.

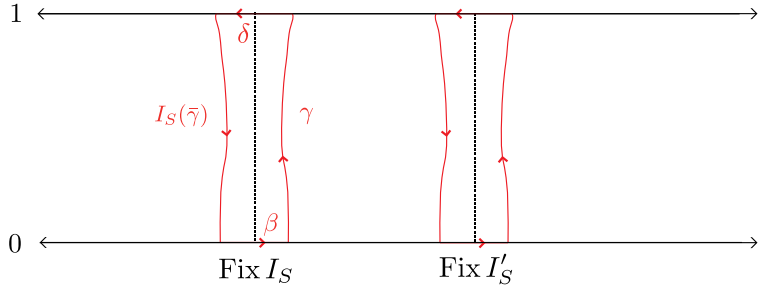


FIGURE 5.6. A loop enclosing Fix I_S.

such that $F_1(0, 1) \in (0, 2\pi]$. Then, by the boundary twist condition,

$$\mathrm{Fix}(F \circ I_S) \cap (\mathbb{R} \times \{1\}) \in (0, 2\pi], \quad \mathrm{Fix}(F \circ I_S) \cap (\mathbb{R} \times \{0\}) < 0.$$

Therefore, we have

$$\mathrm{Fix}(F \circ I_S) \cap \mathrm{Fix}\, I_S \cap \mathring{S} \neq \emptyset,$$

i.e. there is an interior symmetric fixed point of f on one connected component of Fix I_A. To find another interior symmetric fixed point of f on the other connected component of Fix I_A, consider the reflection $I'_S : (x, y) \mapsto (2\pi - x, y)$, which is another lift of I_A to S. Then F is also reversible with respect to I'_S and we have

$$\mathrm{Fix}(F \circ I'_S) \cap \mathrm{Fix}\, I'_S \cap \mathring{S} \neq \emptyset.$$

by the boundary twist condition again. This gives another symmetric fixed point of f. □

Sketch of Proof 2. See Figure 5.6. We lift $f : A \to A$ to $F : S \to S$, which satisfies the boundary twist condition as above. Suppose that F does not have any fixed points on Fix I_S. Then there is no fixed point of F inside $(-\epsilon, \epsilon) \times [0, 1]$ for small $\epsilon > 0$. Now we choose any path $\gamma : [0, 1] \to (0, \epsilon) \times [0, 1]$ from $\mathbb{R} \times \{0\}$ to $\mathbb{R} \times \{1\}$. Then, due to the boundary twist condition,

$$i(\gamma, F \circ \gamma) \in \tfrac{1}{2} + \mathbb{Z}. \tag{5.10}$$

Moreover, as in the proof of Proposition 4.4, we have

$$i(I_S \circ \bar{\gamma}, F \circ I_S \circ \bar{\gamma}) = i(\gamma, F \circ \gamma). \tag{5.11}$$

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We choose auxiliary paths β and δ in the boundary of S as in Figure 5.6 so that $\alpha := \gamma * \delta * I(\bar{\gamma}) * \beta$ forms a loop enclosing Fix I_S. Then, using (5.10) and (5.11), we compute

$$i(\alpha, F \circ \alpha) = i(I \circ \bar{\gamma}, F \circ I \circ \bar{\gamma}) + i(\gamma, F \circ \gamma) \in 1 + 2\mathbb{Z}.$$

Therefore, there is a fixed point of F inside the domain enclosed by the loop α. This contradiction shows the existence of a symmetric fixed point of f on one connected component of Fix I_A. In a similar way, we are able to find a fixed point of F on Fix I'_S, which gives another symmetric fixed point of f on the other connected component of Fix I_A. □

A standard argument shows the following; see for instance [**MS98**, Corollary 8.6].

COROLLARY 5.8. *Let $f : A \to A$ be a reversible map isotopic to the identity with the boundary twist condition. Then, for each $p/q \in \mathbb{Q}$ satisfying*

$$\min_{x\in\mathbb{R}}(F_1(x, 0) - x) < \frac{2\pi p}{q} < \max_{x\in\mathbb{R}}(F_1(x, 1) - x),$$

there is a symmetric q-periodic point of f with rotation number p/q on each connected component of Fix I_A.

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