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On certain subshifts and their associated monoids

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Abstract. Within a subclass of monoids (with zero) a structural characterization is given of those that are associated to topologically transitive subshifts with Property (A).

1. *Introduction*

Let Σ be a finite alphabet, and let S_Σ be the shift on the shift space $\Sigma^{\mathbb{Z}}$,

$$S_\Sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}, \quad (x_i)_{i\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}.$$

In symbolic dynamics one studies subshifts, by which are meant the dynamical systems (C, S), where C is an S_Σ-invariant closed subset of $\Sigma^{\mathbb{Z}}$, and S is the restriction of S_Σ to C. An introduction to the theory of subshifts is in [**Kit98**, **LM95**] (see also [**Bea93**, **BP98**]). Here we continue the investigation of the semigroup invariants of subshifts as begun in [**Kri00**]. There a partially ordered set $\mathcal{P}_C$ was invariantly associated to a subshift. Also an invariant property (A) of subshifts was described. If a subshift has Property (A) then a semigroup (with zero) $\mathcal{S}_C$ is invariantly associated to C. The semigroup $\mathcal{S}_C$ is a monoid if and only if $\mathcal{P}_C$ is a singleton set and then $\mathcal{P}_C$ contains the unit of $\mathcal{S}_C$.

Let $\mathcal{S}$ be a semigroup (with zero). We denote by $\Gamma_-(a)$, $\Gamma(a)$, $\Gamma_+(a)$ the left context, context, and right context (in $\mathcal{S}$) of an element a of $\mathcal{S}$,

$$\begin{aligned}
\Gamma_-(a) &= \{g^- \in \mathcal{S} : \ g^- a \neq 0\},\\
\Gamma(a) &= \{(g^-, g^+) \in \mathcal{S}\times\mathcal{S} : \ g^- a g^+ \neq 0\},\\
\Gamma_+(a) &= \{g^+ \in \mathcal{S} : \ a g^+ \neq 0\}.
\end{aligned}$$

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Denoting by $[a]$ the set of elements in $\mathcal{S}$ that have the same context as $a \in \mathcal{S}$, the set $[\mathcal{S}] = \{[a] : a \in \mathcal{S}\}$ is a semigroup (with zero), where

$$[a][b] = [ab], \quad a, b \in \mathcal{S},$$

(see [**Pin97**, §2.2]). We define subsemigroups $\mathcal{S}^-$ and $\mathcal{S}^+$ of $\mathcal{S}$ by

$$\mathcal{S}^- = \bigcap_{b \in \mathcal{S}\setminus\{0\}} \{a \in \mathcal{S} : ba \neq 0\}, \quad \mathcal{S}^+ = \bigcap_{b \in \mathcal{S}\setminus\{0\}} \{a \in \mathcal{S} : ab \neq 0\}.$$

In §2 we show for a topologically transitive subshift C with Property (A), that the projection of $\mathcal{S}_C$ onto $[\mathcal{S}_C]$ is an isomorphism. We also show, for a topologically transitive subshift C with Property (A) such that

$$\mathcal{S}_C^- \cap \mathcal{S}_C^+ \neq \emptyset,$$

that $\mathcal{S}_C$ is a monoid (with zero), and that

$$\mathcal{S}_C^- \cap \mathcal{S}_C^+ = \{\mathbf{1}\},$$

or, equivalently, that

$$0 \in \Bigg(\bigcap_{a^+ \in \mathcal{S}_C^+ \setminus \{\mathbf{1}\}} \mathcal{S}_C^- a^+ \Bigg) \cap \Bigg(\bigcap_{a^- \in \mathcal{S}_C^- \setminus \{\mathbf{1}\}} a^- \mathcal{S}_C^+ \Bigg).$$

In the converse direction we consider in §3 finitely generated monoids $\mathcal{M}$ (with zero) such that

$$\mathcal{M} \setminus \{0\} = \mathcal{M}^+ \mathcal{M}^-, \tag{$\star$}$$

and such that

$$\{0, \mathbf{1}\} \subset \Bigg(\bigcap_{a^+ \in \mathcal{M}^+ \setminus \{\mathbf{1}\}} \mathcal{M}^- a^+ \Bigg) \cap \Bigg(\bigcap_{a^- \in \mathcal{M}^- \setminus \{\mathbf{1}\}} a^- \mathcal{M}^+ \Bigg). \tag{$\star\star$}$$

For a finite generating set Δ of a monoid $\mathcal{M}$ such that ($\star$) and ($\star\star$) hold, we show that the topologically transitive subshift $C(\Delta) \subset \Delta^{\mathbb{Z}}$ with admissible words $(\delta_i)_{1 \leq i \leq I} \in \Delta^{[1,I]}$, $I \in \mathbb{N}$, given by

$$\prod_{1 \leq i \leq I} \delta_i \neq 0,$$

has Property (A), and that

$$\mathcal{S}_{C(\Delta)} = [\mathcal{M}].$$

As a consequence, a finitely generated monoid (with zero) $\mathcal{M}$, such that ($\star$) and ($\star\star$) hold, is associated to a topologically transitive subshift with Property (A) if and only if the projection of $\mathcal{M}$ onto $[\mathcal{M}]$ is an isomorphism.

In §4 we give examples. These we describe by confluent systems of string-rewriting rules.

2. *Topologically transitive subshifts with Property (A)*

We fix notation and terminology. Given a subshift $C \subset \Sigma^{\mathbb{Z}}$, we set, for $x \in C$,

$$x_{[i,k]} = (x_i)_{i \leq j \leq k},$$

and we set

$$C_{[i,k]} = \{x_{[i,k]} : x \in C\}, \quad i, k \in \mathbb{Z}, i \leq k.$$

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We use similar notation also for blocks,

$$b_{[i',k']} = (b_j)_{i' \le j \le k'}, \quad b \in C_{[i,k]},\ i \le i' \le k' \le k,\ i, k \in \mathbb{Z}.$$

The left context of a block $a \in C_{[i,k]}$, $i, k \in \mathbb{Z}$, $i \le k$, is denoted by $\Gamma^-(a)$:

$$\Gamma^-(a) = \bigcup_{J \in \mathbb{N}} \{b \in C_{[i-J,i)} : (b, a) \in C_{[i-J,k]}\}.$$

The notation $\Gamma^+(a)$ has the time symmetric meaning. The context of a block $a \in C_{[i,k]}$, i, $k \in \mathbb{Z}$, $i \le k$, is denoted by $\Gamma(a)$:

$$\Gamma(a) = \bigcup_{J,M \in \mathbb{N}} \{(b, c) \in C_{[i-J,i)} \times C_{(k,k+M]} :\ (b, a, c) \in C_{[i-J,k+M]}\}.$$

We also set

$$\omega^+(a) = \bigcap_{b \in \Gamma_-(a)} \{c \in \Gamma^+(a) :\ b \in \Gamma^-(a, c)\}, \quad a \in C_{[i,k]},\ i, k \in \mathbb{Z},\ i \le k.$$

The notation $\omega^-(a)$ has the time symmetric meaning.

We denote the (shortest) period of a periodic point p of C by $\Pi(p)$.

We recall from [**Kri00**] the construction of the partially ordered set $\mathcal{P}_C$. For a subshift $C \subset \Sigma^{\mathbb{Z}}$, set

$$X_n^-(C) = \bigcap_{i \in \mathbb{Z}} \{x \in C :\ x_i \in \omega^+(x_{[i-n,i)})\}, \quad n \in \mathbb{N},$$

with the time symmetric meaning for $X_n^+(C)$. We set

$$X_n(C) = X_n^-(C) \cap X_n^+(C), \quad n \in \mathbb{N}.$$

Also set

$$X^+(C) = \bigcup_{n \in \mathbb{N}} X_n^+(C).$$

The notation $X^-(C)$ has the time symmetric meaning. Also set

$$X(C) = \bigcup_{n \in \mathbb{N}} X_n(C).$$

In this paper we consider subshifts C such that $X(C)$ is dense in C. We denote by $P(X_n(C))$ the set of periodic points in $X_n(C)$, $n \in \mathbb{N}$, and also set

$$P(X(C)) = \bigcup_{n \in \mathbb{N}} P(X_n(C)).$$

Introduce a reflexive and transitive relation $\lesssim_C$ into the set $P(X(C))$. For $u, u' \in P(X(C))$, $u \gtrsim_C u'$ will mean that there exists a point in $X(C)$ that is negatively asymptotic to the orbit of u and positively asymptotic to the orbit of u'. Denote the resulting equivalence relation by $\sim_C$ and the set of $\sim_C$-equivalence classes by $\mathcal{P}_C$. The partial order that is induced by $\lesssim_C$ on $\mathcal{P}_C$ is also denoted by $\lesssim_C$.

We recall from [**Kri00**] the definition of Property (A). For $n \in \mathbb{N}$, a subshift $C \subset \Sigma^{\mathbb{Z}}$ has property (a, n, H), $H \in \mathbb{N}$, if for $h, \widetilde{h} \ge 3H$ and for $I_-, I_+, \widetilde{I}_-, \widetilde{I}_+ \in \mathbb{Z}$, such that

$$I_+ - I_-, \widetilde{I}_+ - \widetilde{I}_- \ge 3H,$$

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and for

$$a \in X_n(C)_{(I_-,I_+]}, \quad \widetilde{a} \in X_n(C)_{(\widetilde{I}_-,\widetilde{I}_+]},$$

such that

$$a_{(I_-,I_-+H]} = \tilde{a}_{(\widetilde{I}_-,\widetilde{I}_++H]}, \quad a_{(I_+-H,I_+]} = \tilde{a}_{(\widetilde{I}_+-H,\widetilde{I}_+]},$$

one has that a and $\tilde{a}$ have the same context. A subshift $C \subset \Sigma^{\mathbb{Z}}$ has Property (A) if there are H_n, $n \in \mathbb{N}$, such that X has the properties (a, n, H_n), $n \in \mathbb{N}$.

We recall from [**Kri00**] the construction of the associated semigroup. For a subshift $C \subset \Sigma^{\mathbb{Z}}$ with Property (A), we denote by $Y(C)$ the set of points in C that are left asymptotic to a point in $P(X(C))$ and also right asymptotic to a point in $P(X(C))$. Let $y, \widetilde{y} \in Y(C)$, let y be left asymptotic to $p^{(-)} \in P(X(C))$ and right asymptotic to $p^{(+)} \in P(X(C))$, let $\widetilde{y}$ be left asymptotic to $\widetilde{p}^{(-)} \in P(X(C))$ and right asymptotic to $\widetilde{p}^{(+)} \in P(X(C))$, and let $p^{(-)} \sim_C \widetilde{p}^{(-)}$ and $p^{(+)} \sim_C \widetilde{p}^{(+)}$. The points y and $\tilde{y}$ are said to be $\approx_C$-equivalent if the following holds. Given that C has properties (a, n, H_n), $H_n \geq n$, $n \in \mathbb{N}$, one has for $n \in \mathbb{N}$ such that

$$p^{(-)}, \widetilde{p}^{(-)}, p^{(+)}, \widetilde{p}^{(+)} \in X_n(C),$$

and for I_-, I_+, $\widetilde{I}_-$, $\widetilde{I}_+ \in \mathbb{Z}$, $I_- < I_+$, $\widetilde{I}_- < \widetilde{I}_+$, such that

$$y_{(-\infty,I_-)} = p^{(-)}_{(-\infty,0)}, \quad y_{(I_+,\infty)} = p^{(+)}_{(0,\infty)},$$
$$\widetilde{y}_{(-\infty,\widetilde{I}_-)} = \widetilde{p}^{(-)}_{(-\infty,0)}, \quad \widetilde{y}_{(\widetilde{I}_+\infty)} = \widetilde{p}^{(+)}_{(0,\infty)},$$

that

$$a \in C_{[I_--3H_n,I_++3H_n]}$$

and

$$\widetilde{a} \in C_{[\tilde{I}_--3H_n,\widetilde{I}_++3H_n]},$$

such that

$$a_{[I_--H_n,I_++H_n]} = y_{[I_--H_n,I_++H_n]}, \quad \widetilde{a}_{[\widetilde{I}_--H_n,\widetilde{I}_++H_n]} = \widetilde{y}_{[\widetilde{I}_--H_n,\widetilde{I}_++H_n]},$$

and

$$a_{[I_--3H_n,I_-)} \in X_n(C)_{[I_--3H_n,I_-)}, \quad \widetilde{a}_{[\widetilde{I}_--3H_n,\widetilde{I}_-)} \in X_n(C)_{[\widetilde{I}_--3H_n,\widetilde{I}_-)},$$
$$a_{(I_+,I_++3H_n]} \in X_n(C)_{(I_+,I_++3H_n]}, \quad \widetilde{a}_{(\widetilde{I}_+,\widetilde{I}_++3H_n]} \in X_n(C)_{(\widetilde{I}_+,\widetilde{I}_++3H_n]},$$

and

$$a_{[I_--3H_n,I_--2H_n)} = \widetilde{a}_{[\widetilde{I}_--3H_n,\widetilde{I}_--2H_n)},$$
$$a_{(I_++2H_n,I_++3H_n]} = \widetilde{a}_{(\widetilde{I}_++2H_n,\widetilde{I}_++3H_n]},$$

have the same context. The set $[Y(C)]_{\approx}$ carries the structure of a semigroup. Let $u, v \in Y(C)$, let u be right asymptotic to $q \in P(X(C))$ and let v be left asymptotic to $r \in P(X(C))$. If here $q \gtrsim r$, then $[u]_{\approx}[v]_{\approx}$ is equal to $[y]_{\approx}$, where y is any point in Y, such that there is an $n \in \mathbb{N}$ such that $q, r \in X_n(C)$, together with $I, J \in \mathbb{Z}$ and $\widehat{I}, \widehat{J} \in \mathbb{Z}$, $\widehat{J} - \widehat{I} > 3H_n$, such that

$$u_{(I,\infty)} = q_{(I,\infty)}, \quad v_{(-\infty,J]} = r_{(-\infty,J]},$$
$$y_{(-\infty,\widehat{I}+H_n]} = u_{(-\infty,I+H_n]}, \quad y_{(\widehat{J}-H_n,\infty)} = v_{(J-H_n,\infty)},$$

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and

$$y_{(\widehat{I},\widehat{J}]} \in X_n(C)_{(\widehat{I},\widehat{J}]},$$

provided that such a point y exists. If such a point y does not exist, $[u]_{\approx}[v]_{\approx}$ is zero. Also, in the case that $q \not\gtrsim r$, $[u]_{\approx}[v]_{\approx}$ is zero. If $\mathcal{P}_C$ is a singleton set $\{p\}$, then $\mathcal{S}_C$ is a monoid (with zero) whose unit is given by $[p]_{\approx_C}$.

LEMMA 2.1. *Let C be a topologically transitive subshift such that*

$$X(C) \neq \emptyset.$$

Then $Y(C)$ is dense in C.

Proof. It is enough to consider the case that $X_1(C) \neq \emptyset$. Let u be a forward and backward transitive point of C, and let $q \in P(X_1(C))$. Let further $v \in C$, $K \in \mathbb{N}$. Then there are J_-, J_+, $I \in \mathbb{Z}$, such that

$$J_- \leq I - K, \quad J_+ \geq I + K,$$

and such that

$$u_{I+k} = v_k, \quad -K \leq k \leq K,$$

and

$$u_{J_-} = u_{J_+} = q_0.$$

One defines a point y in C by

$$\begin{aligned} y_i &= q_{i-J_-+I}, \quad i \leq J_- - I, \\ y_i &= u_{i+I}, \quad J_- - I \leq i \leq J_+ - I, \\ y_i &= q_{i-J_++I}, \quad J_+ - I \leq i, \end{aligned}$$

and then has

$$y_i = v_i, \quad -K \leq i \leq K,$$

and

$$y \in Y(C). \qquad \square$$

LEMMA 2.2. *Let $C \subset \Sigma^{\mathbb{Z}}$ be a subshift with properties (a, n, H_n), $n \in \mathbb{N}$. Let $n \in \mathbb{N}$, let $i_\circ$, $j_\circ \in \mathbb{Z}$, $j_\circ - i_\circ > 3H_n$, and let $r \in P(X_n(C))$ and u, $v \in Y(C)$ be such that*

$$u_{[i_\circ,\infty)} = r_{[i_\circ,\infty)}, \quad v_{(-\infty,j_\circ]} = r_{(-\infty,j_\circ)},$$

and

$$[u]_{\approx(C)}[v]_{\approx(C)} \neq 0. \tag{2.1}$$

Then

$$(u_{(-\infty,i_\circ)}, r_{[i_\circ,j_\circ)}, v_{[j_\circ,\infty)}) \in C. \tag{2.2}$$

Proof. By (2.1) there are I, $J \in \mathbb{Z}$, and $\widehat{I}$, $\widehat{J} \in \mathbb{Z}$, $\widehat{J} - \widehat{I} > 3H_n$, such that

$$u_{(I,\infty)} = r_{(I,\infty)}, \quad v_{(-\infty,J]} = r_{(-\infty,J]},$$

$$y_{(-\infty,\widehat{I}+H_n]} = u_{(-\infty,I+H_n]}, \quad y_{(\widehat{J}-H_n,\infty)} = v_{(J-H_n,\infty)},$$

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and

$$y_{(\widehat{I},\widehat{J}]} \in X_n(C)_{(\widehat{I},\widehat{J}]}.$$

Let $k \in \mathbb{Z}_+$, be such that $J + k\Pi(r) - I > 3H_n$. By property (a, n, H_n),

$$\Gamma(y_{[\widehat{I},\widehat{J})}) = \Gamma(r_{[I,J+k\Pi(r))}).$$

For the proof of (2.2) again apply property (a, n, H_n). □

THEOREM 2.3. *Let C be a topologically transitive subshift with Property (A). Then the projection of $\mathcal{S}_C$ onto $[\mathcal{S}_C]$ is an isomorphism.*

Proof. Let $y, y' \in Y(C)$ be such that

$$y \not\approx_C y'. \tag{2.3}$$

The task is to show that

$$\Gamma([y]_{\approx_C}) \neq \Gamma([y']_{\approx_C}). \tag{2.4}$$

Let y be left asymptotic to $q \in P(X(C))$ and right asymptotic to $r \in P(X(C))$, and let y' be left asymptotic to $q' \in P(X(C))$ and right asymptotic to $r' \in P(X(C))$. If $q \not\sim_C q'$, then $[q]_{\approx_C} \notin \Gamma^-([y]_{\approx_C})$, if $q' \not\sim_C q$, then $[q']_{\approx_C} \notin \Gamma^-([y]_{\approx_C})$, if $r \not\sim_C r'$, then $[q]_{\approx_C} \notin \Gamma^+([y]_{\approx_C})$, and if $q' \not\sim_C q$, then $[q']_{\approx_C} x \notin \Gamma^+([y]_{\approx_C})$. In the case that $q \sim_C q'$ and $r \sim_C r'$, replacing, if necessary, y' by another representative of $[y']_{\sim_C}$, we can continue with the assumption that y' is left asymptotic to the orbit of q and right asymptotic to the orbit of r. Let $n \in \mathbb{N}$ be such that

$$q, r \in P(X_n(C)),$$

and let $H_n \in \mathbb{N}$ be such that the subshift C has property (a, n, H_n). Interchanging, if necessary, y and y', one has as a consequence of property (a, n, H_n) and of (2.3) that there are $K \in \mathbb{N}$, $K \geq 3H_n\Pi(q), 3H_n\Pi(r)$, and

$$(x^-, x^+) \in C_{(-\infty,-K)} \times C_{(K,-\infty)},$$

such that

$$y_{(-\infty,-K+3H_n\Pi(q))} = q_{(-\infty,0)}, \quad y_{(K-3H_n\Pi(r),\infty)} = r_{[0,\infty)},$$

and also

$$y'_{(-\infty,-K+3H_n\Pi(q))} = q_{(-\infty,0)}, \quad y'_{(K-3H_n\Pi(r),\infty)} = r_{[0,\infty)},$$

and such that one has, setting

$$z = (x^-, y_{[-K,K]}, x^+), \quad z' = (x^-, y'_{[-K,K]}, x^+),$$

that $z \in C$, but

$$z' \notin C. \tag{2.5}$$

By Lemma 2.1 there are points $z^{(m)} \in Y(C)$, $m \in \mathbb{N}$, such that

$$z^{(m)}_{[-K-m,K+m]} = z_{[-K-m,K+m]}, \quad m \in \mathbb{N}. \tag{2.6}$$

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We construct points $u^{(m)}$, $v^{(m)} \in Y(C)$, $m \in \mathbb{N}$, by setting

$$u^{(m)} = (z^{(m)}_{(-\infty,-K)}, q_{[-K,\infty)}), \quad v^{(m)} = (r_{(-\infty,K]}, z^{(m)}_{(K,\infty)}).$$

The construction is such that

$$[u^{(m)}]_{\approx_C}[y]_{\approx_C}[v^{(m)}]_{\approx_C} = [z^{(m)}]_{\approx_C}, \quad m \in \mathbb{N}. \tag{2.7}$$

By (2.5) and (2.6) there is an $m_\circ \in \mathbb{N}$ such that

$$(z^{(m)}_{(-\infty,-K)}, y'_{[-K,K]}, z^{(m)}_{(K,\infty)}) \notin C, \quad m \geq m_\circ. \tag{2.8}$$

Therefore

$$[u^{(m)}]_{\approx_C}[y]_{\approx_C}[v^{(m)}]_{\approx_C} = 0, \quad m \geq m_\circ, \tag{2.9}$$

since, as a consequence of Lemma 2.2,

$$[u^{(m)}]_{\approx_C}[y]_{\approx_C}[v^{(m)}]_{\approx_C} \neq 0,$$

would contradict (2.8). By (2.7) and (2.9), (2.4) is shown. □

LEMMA 2.4. *Let C be a topologically transitive subshift with Property (A), and let $y \in Y(C)$ be such that*

$$[y]_{\approx_C} \in \mathcal{M}^-_C. \tag{2.10}$$

Then

$$y \in X^-(C).$$

Proof. Let $y \in C$ be left asymptotic to $q \in P(X(C))$ and right asymptotic to $r \in P(X(C))$, and let $n \in \mathbb{N}$ be such that

$$q, r \in P(X_n(C)).$$

Let C have property (a, n, H_n). Let $K \in \mathbb{N}$ be such that

$$K > 3H_n\Pi(q),$$

and

$$y_{(-\infty,-K)} = q_{(-\infty,0)}, \quad y_{[K,\infty)} = r_{[0,\infty)}.$$

We claim that

$$y \in X^-_{3K+n}(C).$$

To justify this claim we let $z \in C$ be such that

$$z_{[-2K,\infty)} = q_{[-K,\infty)},$$

and we prove that

$$(z_{(-\infty,-K)}, y_{[-K,\infty)}) \in C. \tag{2.11}$$

By Lemma 2.1 there are points $z^{(m)} \in Y(C)$, $m \in \mathbb{N}$, such that

$$z^{(m)}_{[-K-m,\infty)} = z_{[-K-m,\infty)}, \quad m \in \mathbb{N}. \tag{2.12}$$

By (2.10),

$$z^{(m)}[y]_{\approx_C} \neq 0,$$

and by Lemma 2.2 then

$$(z^{(m)}_{(-\infty,-K)}, y_{[-K,\infty)}) \in C,$$

and (2.11) follows now by (2.12). □

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THEOREM 2.5. *Let C be a topologically transitive subshift with Property (A) such that*

$$\mathcal{M}_C^- \cap \mathcal{M}_C^+ \neq \emptyset. \tag{2.13}$$

Then $\mathcal{M}_C$ is a monoid (with zero), and

$$\mathcal{M}_C^- \cap \mathcal{M}_C^+ = \{\mathbf{1}\}.$$

Proof. By (2.13) there is a $y \in Y(C)$ such that

$$[y]_{\approx} \in \mathcal{M}_C^- \cap \mathcal{M}_C^+. \tag{2.14}$$

By Lemma 2.4 we have $y \in X(C)$. Let y be left asymptotic to $q \in P(X(C))$ and right asymptotic to $r \in P(X(C))$. Equation (2.14) implies for all periodic point $p \in X(C)$ that $p \gtrsim_C q$ and $r \gtrsim_C p$. In particular, $q \sim_C r$ and it follows that all periodic points $p \in X(C)$ are $\sim_C$-equivalent and $\approx_C$-equivalent to y. □

3. *Subshifts constructed from a class of monoids*

PROPOSITION 3.1. *Let $\mathcal{M}$ be a monoid such that*

$$\mathcal{M} \setminus \{0\} = \mathcal{M}^+\mathcal{M}^-, \tag{$\star$}$$

and such that

$$\{0, \mathbf{1}\} \subset \Big(\bigcap_{a^+ \in \mathcal{M}^+ \setminus \{\mathbf{1}\}} \mathcal{M}^- a^+ \Big) \cap \Big(\bigcap_{a^- \in \mathcal{M}^- \setminus \{\mathbf{1}\}} a^- \mathcal{M}^+ \Big). \tag{$\star\star$}$$

Then for $f \in \mathcal{M} \setminus \{0\}$, the elements $a^+ \in \mathcal{M}^+$ and $a^- \in \mathcal{M}^-$, such that $f = a^+a^-$, are uniquely determined by f.

Proof. For $f \in \mathcal{M} \setminus \{0\}$, let $a^+, b^+ \in \mathcal{M}^+$ and $a^-, b^- \in \mathcal{M}^-$ be given such that

$$f = a^+a^- = b^+b^-. \tag{3.1}$$

By ($\star\star$) there are $c^+, d^+ \in \mathcal{M}^+$, such that

$$a^-c^+ = b^-d^+ = \mathbf{1}. \tag{3.2}$$

Then

$$b^-c^+ \in \mathcal{M}^+. \tag{3.3}$$

Otherwise, by ($\star$) there would be a $g^+ \in \mathcal{M}^+$ and a $g^- \in \mathcal{M}^- \setminus \{\mathbf{1}\}$, such that

$$b^-c^+ = g^+g^-, \tag{3.4}$$

and by ($\star\star$) there would then be an $h^+ \in \mathcal{M}^+ \setminus \{\mathbf{1}\}$ such that

$$g^-h^+ = 0. \tag{3.5}$$

One now derives a contradiction from (3.1): on the one hand one has by (3.2) that

$$a^+a^-c^+h^+ = a^+h^+ \in \mathcal{M}^+,$$

and on the other hand one has by (3.4) and (3.5) that

$$b^+b^-c^+h^+ = b^+g^+g^-h^+ = 0.$$

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From (3.1) and (3.2),

$$a^+ = b^+ b^- c^+, \tag{3.6}$$

and by the same argument,

$$b^+ = a^+ a^- d^+. \tag{3.7}$$

From (3.6) and (3.7),

$$b^+ = b^+ b^- c^+ a^- d^+,$$

which implies as a consequence of $(\star\star)$ that

$$b^- c^+ a^- d^+ = \mathbf{1},$$

and in the same way it follows from (3.3) that

$$a^- d^+ = \mathbf{1}.$$

From (3.7) one then has that $b^+ = a^+$ and from $(\star\star)$ also that $b^- = a^-$. □

THEOREM 3.2. *Let $\mathcal{M}$ be a monoid such that $(\star)$ and $(\star\star)$ hold, and let Δ be a finite generating set of $\mathcal{M}$. Then the subshift $C(\Delta)$ has Property (A), and*

$$\mathcal{S}_{C(\Delta)} = [\mathcal{M}].$$

Proof. For $f \in \mathcal{M} \setminus \{0\}$, choose $I_f \in \mathbb{N}$ and $\delta_i^{(f)} \in \Delta$, $1 \le i \le I_f$, such that

$$\prod_{1 \le i \le I_f} \delta_i^{(f)} = f,$$

and set

$$w^{(f)} = (\delta_i^{(f)})_{1 \le i \le I_f}.$$

Let H, $I \in \mathbb{N}$, $I \ge 2H$, and let $u \in C(\Delta)_{[1,I]}$ be such that

$$u_{(H,I]} \in \omega^+(u_{(1,H]}), \quad u_{[1,I-H]} \in \omega^-(u_{(I-H,I]}). \tag{3.8}$$

We apply Proposition 3.1. We let

$$a^-(-), c^-, a^-(+) \in \mathcal{M}^-, \quad a^+(-), c^+, a^+(+) \in \mathcal{M}^+$$

be given by

$$\prod_{1 \le i \le H} u_i = a^+(-)a^-(-), \qquad \prod_{H < i \le I-H} u_i = c^+ c^-, \qquad \prod_{I-H < i \le I} u_i = a^+(+)a^-(+),$$

and then let $b^- \in \mathcal{M}^-$ and $b^+ \in \mathcal{M}^+$ given by

$$a^-(-)c^+ c^- a^+(+) = b^+ b^-.$$

Then $b^+ = \mathbf{1}$, for otherwise there would exist by $(\star\star)$ a $g^- \in \mathcal{M}^-$ and an $h^- \in \mathcal{M}^-$ such that $g^- a^+(-) = 1$ and $h^- b^+ = 0$, and then

$$(w^{(h^-)}, w^{(g^-)}) \in \Gamma^-(u_{[1,H]}),$$

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but

$$(w^{(h^-)}, w^{(g^-)}) \notin \Gamma^-(u),$$

contradicting (3.8). By the symmetric argument one has that $b^- = \mathbf{1}$. It follows for $v(-) \in C(\Delta)_{(-I_-,1]}$, $v(+) \in C(\Delta)_{(I,I+I_+]}$, I_-, $I_+ \in \mathbb{N}$, that $(v(-), v(+)) \in \Gamma(u)$ if and only if

$$\Big(\prod_{-I_- < i \le 1} v_i(-)\Big) a^+(-)a^-(+) \Big(\prod_{I < i \le I+I_+} v_i(+)\Big) \neq 0,$$

and this shows that $C(\Delta)$ has property (a, H, H). It also follows that a periodic point p is in $X(C(\Delta))$ if and only if p has a period Π such that with $a^- \in \mathcal{M}^-$ and $a^+ \in \mathcal{M}^+$, given by

$$\prod_{1 \le i \le \Pi} p_i = a^+ a^-,$$

one has $a^- a^+ = \mathbf{1}$.

We show that all periodic points in $X(C(\Delta))$ are $\sim_C$-equivalent. For this let q and r be periodic points in $X(C(\Delta))$, and let Π_q be a period of q, and $\Pi_r q$ a period of r, such that with $a^-(q)$, $a^-(r) \in \mathcal{M}^-$ and $a^+(q)$, $a^+(r) \in \mathcal{M}^+$, given by

$$\prod_{1 \le i \le \Pi_q} q_i = a^+(q)a^-(q), \qquad \prod_{1 \le i \le \Pi_r} r_i = a^+(r)a^-(r),$$

one has

$$a^-(q)a^+(q) = \mathbf{1}, \quad a^-(r)a^+(r) = \mathbf{1}.$$

By $(\star\star)$ there are $b^- \in \mathcal{M}^-$ and $b^+ \in \mathcal{M}^+$ such that

$$a^-(q)b^+ = \mathbf{1}, \quad b^- a^+(r) = \mathbf{1}.$$

The point $x \in C(\Delta)$, which is given by setting

$$x_{[1, I_{b^+b^-}]} = w^{(b^+b^-)}$$

and

$$x_i = \begin{cases} q_i & \text{if } i < 0, \\ r_{i - I_{b^-b^+}} & \text{if } i > I_{b^-b^+}, \end{cases}$$

is in $X(C)$.

For $f \in \mathcal{M}$, let $x^{(f)}$ be the point in $Y(C(\Delta))$ that is given by setting

$$x^{(f)}_{(-(k+1)I_{\mathbf{1}}, kI_{\mathbf{1}}]} = w^{(\mathbf{1})}, \quad k \in \mathbb{N},$$
$$x^{(f)}_{[1, I_f]} = w^{(f)},$$
$$x^{(f)}_{(I_f + kI_{\mathbf{1}}, I_f + (k+1)I_{\mathbf{1}})} = w^{(\mathbf{1})}, \quad k \in \mathbb{Z}_+.$$

The $\approx_C$-class of $x^{(f)}$ does not depend on the choice of $w^{(f)}$, and a homomorphism φ of $\mathcal{S}_{C(\Gamma)}$ onto $[\mathcal{M}]$ is obtained by setting

$$\varphi(f) = [x^{(f)}]_{\approx_C}, \quad f \in \mathcal{M}.$$

The homomorphism φ is injective since, for f, $f' \in \mathcal{M}$,

$$\Gamma(f) = \Gamma(f'),$$

if and only if

$$\Gamma(x^{(f)}_{[-kI_{\mathbf{1}}, I_f + kI_{\mathbf{1}}]}) = \Gamma(x^{(f')}_{[-kI_{\mathbf{1}}, I_{f'} + kI_{\mathbf{1}}]}), \quad k \in \mathbb{Z}_+. \qquad \square$$

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We note at this point that the subshifts $C(\Delta)$ are coded systems in the sense of [**BH86**]. Also compare the definition of the subshifts $C(\Delta)$ with how sofic systems were originally defined [**Wei73**].

PROPOSITION 3.3. *Let $\mathcal{M}$ be a monoid such that ($\star$) and ($\star\star$) hold. Let the mappings*

$$a^+ \to \Gamma^-(a^+)\ (a^+ \in \mathcal{M}^+), \quad a^- \to \Gamma^+(a^-)\ (a^- \in \mathcal{M}^-),$$

be injective. Then the projection of $\mathcal{M}$ onto $[\mathcal{M}]$ is an isomorphism.

Proof. Let $a, \widetilde{a} \in \mathcal{M}$, $a \neq \widetilde{a}$, and $b^+, \widetilde{b}^+ \in \mathcal{M}^+$, $b^-, \widetilde{b}^- \in \mathcal{M}^-$ be given by

$$a = b^+b^-, \widetilde{a} = \widetilde{b}^+\widetilde{b}^-,$$

and consider the case that

$$b^+ \neq \widetilde{b}^+.$$

Let $c^- \in \mathcal{M}^-$ be such that

$$c^- \in \Gamma^-(b^+), \quad c^- \notin \Gamma^-(\widetilde{b}^+).$$

Then

$$(c^-, \mathbf{1}) \in \Gamma(a), \quad (c^-, \mathbf{1}) \notin \Gamma(\widetilde{a}). \qquad \square$$

4. *A class of examples*

Consider the case of a monoid $\mathcal{M}$ that satisfies ($\star$) and ($\star\star$) where the submonoids $\mathcal{M}^-$ and $\mathcal{M}^+$ are finitely and freely generated. With $N_-, N_+ \in \mathbb{N}$ and a generating set $\Delta_- = \{\delta^-(n_-) : 0 \leq n_- \leq N_-\}$ of $\mathcal{M}^-$ and a generating set $\Delta_+ = \{\delta^+(n_+) : 0 \leq n_+ \leq N_+\}$ of $\mathcal{M}^+$ there are then relations

$$\begin{aligned}&\delta^-(n_-)\delta^+(n_+)\\ &\quad = \Bigg(\prod_{I_+(n_-,n_+)\geq i_{+,n_-,n_+}>0} \delta^+(i_{+,n_-,n_+})\Bigg)\Bigg(\prod_{0<i_{+,n_-,n_+}\leq I_+(n_-,n_+)} \delta^-(i_{+,n_-,n_+})\Bigg),\\ &\qquad 0 \leq n_- \leq N_-,\ 0 \leq n_+ \leq N_+. \end{aligned} \tag{4.1}$$

This suggests considering the subclass of the class of monoids (with zero) that satisfy ($\star$) and ($\star\star$) which contains the monoids (with zero) that are generated by $\Delta_- \cup \Delta_+$ with the relations that are expressed by (4.1), in which case (4.1) is necessarily a confluent string-rewriting system. Conversely, given a confluent string-rewriting system (4.1) there are procedures to decide if it defines a monoid $\mathcal{M}$ (with zero) that satisfies ($\star\star$), and if the projection of $\mathcal{M}$ onto $[\mathcal{M}]$ is an isomorphism [**BO93**, §4.3].

For the examples, which we will write as string-rewriting systems, it can be verified by inspection that they describe monoids (with zero) $\mathcal{M}$ such that ($\star\star$) holds, and by means of Proposition 3.3 it can be verified that the projection of $\mathcal{M}$ onto $[\mathcal{M}]$ is an isomorphism.

A prototype example is the Dyck inverse monoid (the polycyclic monoid [**NP70**]), which is given by

$$\begin{aligned}&\Delta_- = \{\lambda, \lambda'\}, \quad \Delta_+ = \{\rho, \rho'\},\\ &\lambda\rho, \lambda'\rho' \to \mathbf{1},\\ &\lambda\rho', \lambda'\rho \to 0.\end{aligned}$$

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Examples where $\lambda\rho \in \{0, \mathbf{1}\}$, $\lambda \in \Delta_-$, $\rho \in \Delta_+$ (compare [**Kem96**]) are

$$\Delta_- = \{\lambda, \lambda', \lambda''\}, \quad \Delta_+ = \{\rho, \rho', \rho''\},$$
$$\lambda\rho', \lambda\rho, \lambda'\rho', \lambda''\rho'' \to \mathbf{1},$$
$$\lambda'\rho, \lambda\rho'', \lambda'\rho'', \lambda''\rho, \lambda''\rho' \to 0,$$

and

$$\Delta_- = \{\lambda, \lambda', \lambda''\}, \quad \Delta_+ = \{\rho, \rho', \rho''\},$$
$$\lambda\rho, \lambda'\rho, \lambda'\rho', \lambda''\rho'' \to \mathbf{1},$$
$$\lambda\rho', \lambda\rho'', \lambda'\rho'', \lambda''\rho, \lambda''\rho' \to 0.$$

Another example is

$$\Delta_- = \{\lambda, \lambda', \lambda''\}, \quad \Delta_+ = \{\rho, \rho', \rho''\},$$
$$\lambda\rho, \lambda'\rho', \lambda''\rho'' \to \mathbf{1},$$
$$\lambda\rho'', \lambda'\rho, \lambda''\rho' \to 0,$$
$$\lambda\rho' \to \lambda,$$
$$\lambda'\rho'' \to \lambda',$$
$$\lambda''\rho' \to \lambda''.$$

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