Math. Proc. Camb. Phil. Soc. (1988), **103**, 133
Printed in Great Britain

Packing regularity of sets in n-space

BY XAVIER SAINT RAYMOND
Département de mathématiques, Université Paris-Sud, 91405 Orsay, France

AND CLAUDE TRICOT
Département de mathématiques, Université du Québec à Montréal, Montréal, P.Q. H3C 3P8, Canada

(*Received* 17 *March* 1987; *revised* 28 *May* 1987)

1. *Introduction*

The notion of packing measure, introduced in [**12**], [**13**] and [**10**], has been used by comparison with Hausdorff measure to study the regularity and rectifiability of sets in the plane [**11**]. Since a few technical mistakes can be found in [**10**], lemma 5·11 and [**11**], lemma 3·2, we wish in this paper to give the corresponding exact proofs, together with a natural development of this theory in higher dimensions.

Let us indicate first the notation used throughout the paper: n is a fixed integer, and $\mathbf{R}^n$ the euclidean n-dimensional space. The euclidean distance between two points x and x' of $\mathbf{R}^n$ is denoted by $\operatorname{dist}(x, x')$; $B_r(x)$ is the closed ball $\{x' \in \mathbf{R}^n : \operatorname{dist}(x, x') \leqslant r\}$; for any $E \subset \mathbf{R}^n$, $\operatorname{diam} E = \sup\{\operatorname{dist}(x, x') : x, x' \in E\}$ is the diameter of E. $\mathbf{M}$ is the family of all positive, finite and borelian measures in $\mathbf{R}^n$, and Φ is the family of all continuous, increasing functions $\phi : \mathbf{R}^+ \to \mathbf{R}$ such that $\phi(0) = 0$ and $\phi^* = \limsup_{t \to 0+} \phi(2t)/\phi(t)$ is finite. Typical examples of such functions are $\phi(t) = t^\alpha, \alpha > 0$, where $\phi^* = 2^\alpha$.

If $\mathbf{B}$ is a family of bounded subsets of $\mathbf{R}^n$, we denote $\bigcup_{B \in \mathbf{B}} B$ by $\bigcup \mathbf{B}$, $\sup_{B \in \mathbf{B}} \operatorname{diam} B$ by $\|\mathbf{B}\|$, $\sum_{B \in \mathbf{B}} \phi(\operatorname{diam} B)$ by $\phi(\mathbf{B})$. If $\rho > 0$, $\mathbf{B}$ is a *ρ-covering* of E if $\|\mathbf{B}\| \leqslant \rho$ and $E \subset \bigcup \mathbf{B}$; $\mathbf{B}$ is a *ρ-packing* of E if $\|\mathbf{B}\| \leqslant \rho$, $\bar{B} \cap \bar{E} \neq \varnothing$ for every $B \in \mathbf{B}$, and the sets of $\mathbf{B}$ are disjoint.

We recall that the classical *Hausdorff ϕ-measure* of E is defined as

$$m^\phi(E) = \lim_{\rho \to 0} [\inf\{\phi(\mathbf{B}) : \mathbf{B} \ \rho\text{-covering of } E\}].$$

Together with this measure, we shall use the two symmetrical following:

$$C^\phi(E) = \lim_{\rho \to 0} [\inf\{\phi(\mathbf{B}) : \mathbf{B} \ \rho\text{-covering of } E \text{ by closed balls centred in } E\}]$$

$$P^\phi(E) = \lim_{\rho \to 0} [\sup\{\phi(\mathbf{B}) : \mathbf{B} \ \rho\text{-packing of } E \text{ by closed balls centred in } E\}].$$

These are not measures: C^ϕ is not monotone (see Section 4) and P^ϕ is not countably subadditive [**10**]. But they both give rise to measures, called respectively the *centred covering ϕ-measure* and the *centred packing ϕ-measure*, as follows:

$$c^\phi(E) = \sup\{C^\phi(F) : F \subset E\} \quad \text{and} \quad p^\phi(E) = \inf\{\sum_k P^\phi(E_k) : E \subset \bigcup_k E_k\}.$$

Here E is an arbitrary subset of $\mathbf{R}^n$. The construction of p^ϕ from a pre-measure is classical [**9**]. For this particular measure the main reference is [**10**]. The set function

https://doi.org/10.1017/S0305004100064690 Published online by Cambridge University Press

c^ϕ is new, and we will show that it is indeed a measure in Section 3. They constitute our basic tools for the study of rectifiability: the reader will verify that they have perfectly symmetrical behaviour, in particular with respect to the density inequalities below (Theorem 1·1).

Note that we specify 'closed balls' in the definitions of C^ϕ and P^ϕ only for further convenience: using open balls would not change the results, by continuity of ϕ. Similarly, the usual 'open' density ratios (as in [**1**]) are identical to our 'closed' ones which occur in Section 5.

If $\phi(t) = t^\alpha, \alpha > 0$, we will rather use the notations m^α, C^α, P^α, c^α and p^α. We recall that m^ϕ and p^ϕ are metric outer measures ([**10**], lemma 5·1), and so is c^ϕ (Section 3). If μ is one of them, and E is such that $\mu(E) < \infty$, then the restriction $\mu \mid E$ belongs to $\mathbf{M}$. The inequalities

$$m^\phi(E) \leqslant c^\phi(E) \leqslant p^\phi(E) \tag{1}$$

are always true (see Section 3), and the equality $m^\phi(E) = p^\phi(E)$ involves strong rectifiability conditions which have been investigated in [**11**] for the case $\phi(t) = t$, $n = 2$.

A property is μ-a.s. if it is true at all points but on a set of μ-measure zero. We will see that m^ϕ-a.s. is equivalent to c^ϕ-a.s. as a consequence of Lemma 3·3. In this paper the results are generally true p^ϕ-a.s., instead of m^ϕ-a.s. as in the classical geometric measure theory. This is more restrictive: there are sets such that for the same ϕ, $m^\phi(E) = 0$ and $p^\phi(E) = \infty$ ([**11**], section 5). Such sets are not considered as negligible in the present study.

After the technical lemmas given in Section 2, we prove in Section 3 that c^ϕ is a metric outer measure verifying the inequalities (1). An example illustrating these results is studied in Section 4. Section 5 proves our basic theorem involving density ratios, namely:

THEOREM 1·1. *Let E be a Borel set, $\mu \in \mathbf{M}$,*

$$\underline{d}^\phi_\mu(x) = \liminf_{r \to 0} \mu(B_r(x))/\phi(2r), \quad \bar{d}^\phi_\mu(x) = \limsup_{r \to 0} \mu(B_r(x))/\phi(2r).$$

Then

(i) *If $c^\phi(E) < \infty$,*

$$c^\phi(E) \inf_{x \in E} \bar{d}^\phi_\mu(x) \leqslant \mu(E) \leqslant c^\phi(E) \sup_{x \in E} \bar{d}^\phi_\mu(x). \tag{2}$$

(ii) *If $p^\phi(E) < \infty$,*

$$p^\phi(E) \inf_{x \in E} \underline{d}^\phi_\mu(x) \leqslant \mu(E) \leqslant p^\phi(E) \sup_{x \in E} \underline{d}^\phi_\mu(x). \tag{3}$$

Thus the measures c^ϕ and p^ϕ really play symmetric roles in this theory, and this is confirmed by further results. The remainder of this paper is devoted to corollaries of this theorem: persistence of the density in Section 6, elementary density bounds in Section 7, implications of the condition $0 < m^\phi(E) \leqslant p^\phi(E) < \infty$ in Section 8, and in Section 9 the implications of the more restrictive conditions $c^\phi(E) = p^\phi(E) < \infty$ or $m^\phi(E) = p^\phi(E) < \infty$. In particular we show that the following results, conjectured in [**11**], are true:

(C_1). *If $0 < m^\alpha(E) = p^\alpha(E) < \infty$, then α is an integer.*

(C_2). *If $k \in \mathbf{N}$, and $p^k(E) < \infty$ then the condition $m^k(E) = p^k(E)$ is true if and only if E is (p^k, k)-rectifiable.*

https://doi.org/10.1017/S0305004100064690 Published online by Cambridge University Press

We recall that E is (μ, k)-rectifiable if there exists a sequence of lipschitzian maps $f_i: \mathbf{R}^k \to \mathbf{R}^n$ and bounded subsets A_i of $\mathbf{R}^k$ such that

$$\mu(E \setminus \bigcup f_i(A_i)) = 0.$$

Finally, Section 10 is another interesting application of Theorem 1·1: we relax the condition on ball packings, and consider families of balls centred in E, which are not necessarily disjoint, but such that the intersection of any two of them contains no point of E (*pseudo-packings*). We prove that the measure constructed from these pseudo-packings is equal to our packing measure p^ϕ.

For old results or other developments, the interested reader is referred to [2], [3], [4]. We are deeply grateful to P. Mattila and S. J. Taylor for very helpful comments.

2. *Technical Lemmas*

We give here two generalizations of the Vitali covering theorem, whose proofs can be found in [4] (theorem 3·2·9 for our Lemma 2·1 and theorems 3·2·1 and 6·2·3 for our Lemma 2·2).

LEMMA 2·1. *Let $E \subset \mathbf{R}^n$ and, for all $x \in E$, let $\mathbf{F}_x$ be a sequence of closed balls $B_{r_k(x)}(x)$ such that $\lim r_k(x) = 0$. Then, from $\mathbf{F} = \bigcup_{x \in E} \mathbf{F}_x$ we can select a packing $\mathbf{B} = \{B_i\}$ such that for any j:*

$$E \setminus \left(\bigcup_{i=1}^{j} B_i \right) \subset \bigcup_{i=j+1}^{\infty} B_i^*$$

where B_i^* is the ball with the same centre as B_i and diameter 5 diam B_i.

LEMMA 2·2. *Let $E \subset \mathbf{R}^n$, $\mu \in \mathbf{M}$ and, for all $x \in E$, let $\mathbf{F}_x$ be a sequence of closed balls $B_{r_k(x)}(x)$ such that $\lim r_k(x) = 0$. Then, from $\mathbf{F} = \bigcup_{x \in E} \mathbf{F}_x$ we can select a packing $\mathbf{B}$ of E such that*

$$\mu(E \setminus (\bigcup \mathbf{B})) = 0.$$

3. *The covering measure c^ϕ*

The definition of c^ϕ is given in the introduction.

LEMMA 3·1. *For all $\phi \in \Phi$, c^ϕ is a metric outer measure.*

Proof. The definition implies immediately that c^ϕ is monotone. It is clear also that $c^\phi(\varnothing) = 0$ and that $c^\phi(E \cup F) = c^\phi(E) + c^\phi(F)$ whenever dist $(E, F) > 0$, since it is true for C^ϕ. All we have to prove is the countable subadditivity.

First, C^ϕ is subadditive: if $\mathbf{B}_k$ is a covering of E_k, then $\bigcup_k \mathbf{B}_k$ is a covering of $\bigcup_k E_k$ and $\phi(\bigcup_k \mathbf{B}_k) \leqslant \Sigma_k \phi(\mathbf{B}_k)$; therefore

$$C^\phi(\bigcup_k E_k) \leqslant \sum_k C^\phi(E_k).$$

Now for c^ϕ: if $F \subset \bigcup_k E_k$ is such that $C^\phi(F) \geqslant c^\phi(\bigcup_k E_k)$-$\epsilon$, then

$$\begin{aligned} c^\phi(\bigcup_k E_k) &\leqslant C^\phi(F) + \epsilon = C^\phi(\bigcup_k (F \cap E_k)) + \epsilon \quad \text{since } F \subset \bigcup_k E_k, \\ &\leqslant \sum_k C^\phi(F \cap E_k) + \epsilon \leqslant \sum_k c^\phi(E_k) + \epsilon \quad \text{since } F \cap E_k \subset E_k. \end{aligned}$$

Since ϵ is arbitrarily small, this ends the proof.

https://doi.org/10.1017/S0305004100064690 Published online by Cambridge University Press

Remark 3·2. Lemma 3·1 implies that c^ϕ is also borelian; thus, for any Borel set B one can find a sequence of closed sets $F_k \subset B$ such that $c^\phi(F_k) \to c^\phi(B)$.

The fundamental inequalities (1) are included in the following:

LEMMA 3·3. *For all* $\phi \in \Phi$ *and* $E \subset \mathbf{R}^n$,

$$(1/\phi^*)\, c^\phi(E) \leqslant m^\phi(E) \leqslant C^\phi(E) \leqslant c^\phi(E) \leqslant p^\phi(E) \leqslant P^\phi(E).$$

(*a*) *Proof of* $(1/\phi^*)\, c^\phi(E) \leqslant m^\phi(E)$.

Let $h > \phi^*$, and $\rho > 0$ be such that

$$t \leqslant \rho \Rightarrow \phi(2t)/\phi(t) \leqslant h.$$

Any bounded set B containing $x \in E$ is included in $B_r(x)$, with $r = \operatorname{diam} B$, so that

$$\phi(\operatorname{diam} B_r(x)) = \phi(2 \operatorname{diam} B) \leqslant h\phi(\operatorname{diam} B),$$

if $r \leqslant \rho$. From there we get $C^\phi(F) \leqslant h m^\phi(F)$ for all $F \subset E$, which gives the desired inequality.

(*b*) *Proof of* $c^\phi(E) \leqslant p^\phi(E)$.

Note that the following arguments replace those of lemma 5·11 of [**10**], where the use of open balls was erroneous when applying Vitali's theorem. It suffices to show that for every $E \subset \mathbf{R}^n$,

$$C^\phi(E) \leqslant P^\phi(E); \tag{4}$$

indeed, if (4) is proved, we will have for all $F \subset E$

$$C^\phi(F) \leqslant P^\phi(F) \leqslant P^\phi(E)$$

by monotonicity of P^ϕ. Therefore

$$c^\phi(E) = \sup\{C^\phi(F) : F \subset E\} \leqslant P^\phi(E).$$

Since c^ϕ is countably subadditive, this will give the desired result.

To prove (4), we may assume that $P^\phi(E) < \infty$. Let $\rho_0 > 0$ be such that

$$\phi(2t)/\phi(t) \leqslant 1 + \phi^*$$

for all $t \leqslant \rho_0$; fix $\epsilon > 0$, then fix $\rho_1 = \rho_1(\epsilon) > 0$ such that $C^\phi(E) \leqslant \phi(\mathbf{B}) + \epsilon$ for any ρ_1-covering $\mathbf{B}$ of E by centred closed balls, and finally $\rho_2 = \rho_2(\epsilon) > 0$ such that $\phi(\mathbf{B}) \leqslant P^\phi(E) + \epsilon$ for any ρ_2-packing $\mathbf{B}$ of E by centred closed balls. From the family

$$\{B_r(x) : x \in E \quad \text{and} \quad r \leqslant \min(\rho_0/8, \rho_1/10, \rho_2/2)\},$$

we can select, as in Lemma 2·1, a packing $\mathbf{B} = \{B_i\}$ of E such that

$$E \setminus \left(\bigcup_{i=1}^{j} B_i \right) \subset \bigcup_{i=j+1}^{\infty} B_i^*$$

for all j. Thus $\mathbf{B}$ is a ρ_2-packing of E, which implies that $\phi(\mathbf{B}) \leqslant P^\phi(E) + \epsilon$. The family $\mathbf{B}'_j = \mathbf{B} \cup \{B_i^* : i \geqslant j+1\}$ is a ρ_1-covering of E, such that

$$\begin{aligned} C^\phi(E) &\leqslant \phi(\mathbf{B}'_j) + \epsilon = \phi(\mathbf{B}) + \sum_{i=j+1}^{\infty} \phi(\operatorname{diam} B_i^*) + \epsilon \\ &\leqslant P^\phi(E) + \sum_{i=j+1}^{\infty} \phi(\operatorname{diam} B_i^*) + 2\epsilon. \end{aligned}$$

https://doi.org/10.1017/S0305004100064690 Published online by Cambridge University Press

Since ϕ is monotone, $\operatorname{diam} B_i \leqslant \rho_0/4$ implies

$$\phi(\operatorname{diam} B_i^*) \leqslant \phi(8 \operatorname{diam} B_i) \leqslant (1+\phi^*)^3 \operatorname{diam} B_i,$$

so that the series $\Sigma \phi(\operatorname{diam} B_i^*)$ converges: we get, by letting $j \to \infty$,

$$C^\phi(E) \leqslant P^\phi(E) + 2\epsilon,$$

hence (4).

(c) *Other inequalities of Lemma* 3·3.

Immediate from the definitions.

Remark 3·4. A direct consequence of Lemma 3·3 is that m^ϕ-a.s. is equivalent to c^ϕ-a.s., and that p^ϕ-a.s. implies m^ϕ-a.s.

4. *An example*

To give an illustration of the notions developed in Section 3, we continue the study of the self-similar set E of [11], section 5. This set is constructed in the plane by packing regular hexagons in an equilateral triangle T of side 1 (see Fig. 1). Let m, C,

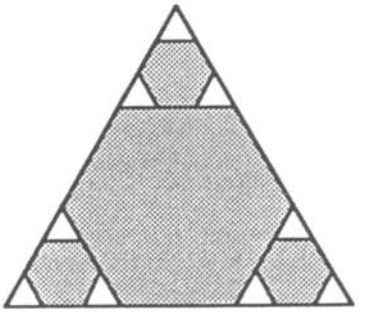


Fig. 1

c and p denote the (pre)-measures m^ϕ, C^ϕ, c^ϕ and p^ϕ with $\phi(t) = t$ (for which $\phi^* = 2$). It has been proved that $m(E) = 1$ and $p(E) = 4$, and that the natural measure expanded on E by giving the weight 3^{-k} to each fundamental triangle of side 3^{-k} coincides with the linear Hausdorff measure m. We will continue this list of properties by proving that

$$C(E) \geqslant 5/4. \tag{5}$$

This inequality has two interesting consequences. First, C is not monotone: if F denotes the set of all centres of the fundamental triangles, $E \cup F$ can be covered by 3^k centred balls of diameter $(2/\sqrt{3})\,3^{-k}$, so that $C(E \cup F) \leqslant 2/\sqrt{3} < 5/4$. Second, the measures m, c and p may take finite, distinct values on the same set: indeed Lemma 3·3 shows that $C(E) \leqslant c(E) \leqslant 2m(E)$, and then

$$m(E) < 5/4 \leqslant c(E) \leqslant 2 < p(E).$$

To get (5), it will be sufficient to prove the following inequality, for all $x \in E$:

$$\sup_{1/3 \leqslant r \leqslant 1} m(E \cap B_r(x))/2r \leqslant 4/5. \tag{6}$$

https://doi.org/10.1017/S0305004100064690 Published online by Cambridge University Press

Indeed, (6) and the self-similarity of E imply

$$\sup_{0<r\leqslant 1} m(E\cap B_r(x))/2r \leqslant 4/5$$

for all $x\in E$. Given any covering $\mathbf{B}$ by closed balls centred in E, it follows that

$$\sum_{B\in\mathbf{B}} \operatorname{diam} B \geqslant (5/4) \sum_{B\in\mathbf{B}} m(E\cap B) \geqslant 5/4,$$

hence (5).

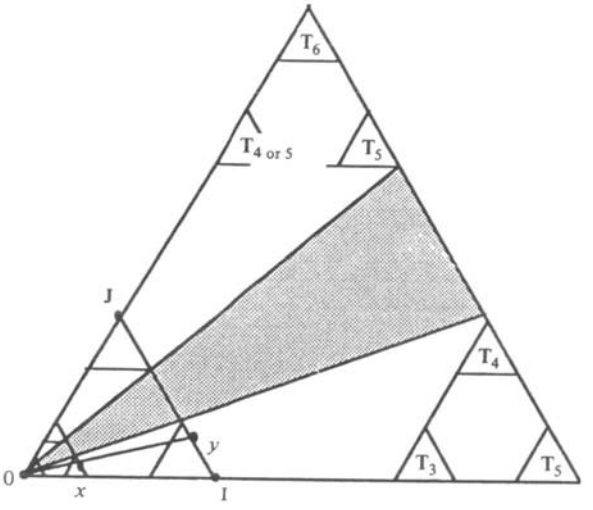


Fig. 2

Now we prove (6). Let $x\in E$, and O, I, J be the vertices of the fundamental triangle of side $1/3$ containing x, the point O being a vertex of T. The line passing through O and x cuts the segment IJ at a point y. Since $1/3 \leqslant r \leqslant 1$, we have

$$E\cap B_r(x) \subset E\cap B_r(y),$$

so that we need only to prove (6) for y. We observe that y cannot lie in the middle third of the segment IJ, and therefore assume that dist $(y, I) \leqslant 1/9$. Finally, let us call T_i all fundamental triangles of side $1/9$ which do not meet $B_{i/9}(y)$ but meet $B_{(i+1)/9}(y)$ (see Fig. 2). It follows that

$$\begin{aligned}
&r\in[3/9, 4/9] \Rightarrow m(E\cap B_r(y)) \leqslant 4/9 \quad \text{and} \quad m(E\cap B_r(y))/2r \leqslant 2/3,\\
&r\in[4/9, 5/9] \Rightarrow m(E\cap B_r(y)) \leqslant 6/9 \quad \text{and} \quad m(E\cap B_r(y))/2r \leqslant 3/4,\\
&r\in[5/9, 6/9] \Rightarrow m(E\cap B_r(y)) \leqslant 8/9 \quad \text{and} \quad m(E\cap B_r(y))/2r \leqslant 4/5,\\
&r\in[6/9, 9/9] \Rightarrow m(E\cap B_r(y)) \leqslant 1 \quad \text{and} \quad m(E\cap B_r(y))/2r \leqslant 3/4,
\end{aligned}$$

hence (6).

Remark. There is no doubt that the bound $5/4$ could be increased by refining these computations.

https://doi.org/10.1017/S0305004100064690 Published online by Cambridge University Press

5. *Fundamental density inequalities*

Here we give the proof of Theorem 1·1.

(*a*) *Proof of* $c^\phi(E)\inf_{x\in E}\bar{d}_\mu^\phi(x)\leqslant\mu(E)$.

According to Remark 3·2, there exists a sequence (E_k) of compact subsets of E such that $c^\phi(E_k)\to c^\phi(E)$. Since $\inf_{x\in E_k}\bar{d}_\mu^\phi(x)\geqslant\inf_{x\in E}\bar{d}_\mu^\phi(x)$ and $\mu(E_k)\leqslant\mu(E)$, it suffices to prove

$$hc^\phi(E)\leqslant\mu(E)$$

for every compact set E such that $\bar{d}_\mu^\phi(x)>h>0$ for all $x\in E$, that is

$$hC^\phi(F)\leqslant\mu(E) \tag{7}$$

for any $F\subset E$.

Let $\epsilon>0$, and $\rho>0$ be such that $C^\phi(F)\leqslant\phi(\mathbf{B})+\epsilon$ for all ρ-coverings of F by closed balls centred in F. From the family

$$\{B_r(x): x\in F, 2r\leqslant\rho, \mu(B_r(x))\geqslant h\phi(2r)\}$$

we can select (Lemma 2·1) a packing $\mathbf{B}=\{B_i\}$ of F such that

$$F\setminus\left(\bigcup_{i=1}^{j}B_i\right)\subset\bigcup_{i=j+1}^{\infty}B_i^*$$

for all j. Since $\phi(\mathbf{B})\leqslant\sum_{i=1}^\infty\mu(B_i)/h\leqslant\mu(E(\rho))/h<\infty$, where $E(\rho)$ denotes the closed ρ-neighbourhood of E, we get $\lim_j\sum_{i=j+1}^\infty\phi(\operatorname{diam}B_i)=0$. With an argument similar to the one used in part (*b*) of the proof of Lemma 3·3, it follows that

$$C^\phi(F)\leqslant\phi(\mathbf{B})+\epsilon.$$

Now we use the compactness of E, and the inequality $h\phi(\mathbf{B})\leqslant\mu(E(\rho))$, to get, by letting ρ tend to 0:

$$C^\phi(F)\leqslant\mu(E)/h+\epsilon,$$

hence (7).

(*b*) *Proof of* $\mu(E)\leqslant c^\phi(E)\sup_{x\in E}\bar{d}_\mu^\phi(x)$.

Suppose $\bar{d}_\mu^\phi(x)<h<\infty$ for all $x\in E$, and let

$$E_k=\{x\in E: r\leqslant 1/k\Rightarrow\mu(B_r(x))\leqslant h\phi(2r)\}.$$

Let $\epsilon>0$. For any k, there exists a $(2/k)$-covering $\mathbf{B}$ of E_k by closed balls centred in E_k such that $\phi(\mathbf{B})\leqslant C^\phi(E_k)+\epsilon$. Then

$$\mu(E_k)\leqslant\sum_{B\in\mathbf{B}}\mu(B)\leqslant h\phi(\mathbf{B})\leqslant h(C^\phi(E_k)+\epsilon)\leqslant h(c^\phi(E)+\epsilon).$$

Since ϵ is arbitrarily small and $E_k\uparrow E$, this gives the desired result.

(*c*) *Proof of* $p^\phi(E)\inf_{x\in E}\underline{d}_\mu^\phi(x)\leqslant\mu(E)$.

As in (*a*), it suffices to prove that

$$hp^\phi(E)\leqslant\mu(E)$$

https://doi.org/10.1017/S0305004100064690 Published online by Cambridge University Press

for any compact set E such that $\underline{d}_\mu^\phi(x) > h > 0$ for all $x \in E$. Let

$$E_k = \{x \in E : r \leqslant 1/k \Rightarrow \mu(B_r(x)) \geqslant h\phi(2r)\},$$

and $0 < \rho \leqslant 2/k$. Any ρ-packing $\mathbf{B}$ of E_k by centred closed balls satisfies

$$h\phi(\mathbf{B}) \leqslant \sum_{B \in \mathbf{B}} \mu(B) \leqslant \mu(E_k(\rho)).$$

Since E is closed, letting $\rho \to 0$ gives

$$hp^\phi(E_k) \leqslant hP^\phi(E_k) \leqslant \mu(\bar{E}_k) \leqslant \mu(E)$$

hence

$$hp^\phi(E) = \lim_k hp^\phi(E_k) \leqslant \mu(E).$$

(*d*) *Proof of* $\mu(E) \leqslant p^\phi(E) \sup_{x \in E} \underline{d}_\mu^\phi(x)$.

It is enough to show that

$$\mu(E) \leqslant hP^\phi(E) \tag{8}$$

for any E such that $\underline{d}_\mu^\phi(x) < h < \infty$ for all $x \in E$. Let $\epsilon > 0$, and $\rho > 0$ be such that $\phi(\mathbf{B}) \leqslant P^\phi(E) + \epsilon$ for any ρ-packing $\mathbf{B}$ of E by centred closed balls. By Lemma 2·2, we can select from the family

$$\{B_r(x) : x \in E, r \leqslant \rho/2, \mu(B_r(x)) \leqslant h\phi(2r)\}$$

a ρ-packing $\mathbf{B}$ of E such that $\mu(E) = \mu(E \cap (\bigcup \mathbf{B}))$. Then

$$\mu(E) \leqslant \sum_{B \in \mathbf{B}} \mu(B) \leqslant h\phi(\mathbf{B}) \leqslant h(P^\phi(E) + \epsilon)$$

which gives (8). This achieves the proof.

6. *Persistence of the density*

From now on, E will denote a Borel subset of $\mathbf{R}^n$, and ϕ a fixed element of Φ. Throughout this section, $\mu \in \mathbf{M}$, and ν is the restricted measure $\mu \mid E$.

COROLLARY 6·1. $c^\phi(E) < \infty \Rightarrow \underline{d}_\mu^\phi(x) = \underline{d}_\nu^\phi(x)$ c^ϕ*-a.s. on* E, *and* $\bar{d}_\mu^\phi(x) = \bar{d}_\nu^\phi(x)$ c^ϕ*-a.s. and* μ*-a.s. on* E.

Proof. Let λ be the measure defined by $\lambda(F) = \mu(F \backslash E)$ for all Borel F. Since $\mu = \nu + \lambda$ we have for all x

$$\underline{d}_\nu^\phi(x) \leqslant \underline{d}_\mu^\phi(x) \leqslant \underline{d}_\nu^\phi(x) + \bar{d}_\lambda^\phi(x)$$

and

$$\bar{d}_\nu^\phi(x) \leqslant \bar{d}_\mu^\phi(x) \leqslant \bar{d}_\nu^\phi(x) + \bar{d}_\lambda^\phi(x).$$

If $E_k = \{x \in E : \bar{d}_\lambda^\phi(x) \geqslant 1/k\}$, then (2) implies

$$c^\phi(E_k)/k \leqslant \lambda(E_k) = 0,$$

so that $\bar{d}_\lambda^\phi(x) = 0$ c^ϕ-a.s. on E. Hence $\underline{d}_\mu^\phi(x) = \underline{d}_\nu^\phi(x)$ and $\bar{d}_\mu^\phi(x) = \bar{d}_\nu^\phi(x)$ c^ϕ-a.s. on E.

Let $F = \{x \in E : \bar{d}_\nu^\phi(x) \neq \bar{d}_\mu^\phi(x)\}$. For every $x \in F$, $\bar{d}_\nu^\phi(x) < \bar{d}_\mu^\phi(x)$, therefore $\bar{d}_\nu^\phi(x)$ is finite. If $F_k = \{x \in F : \bar{d}_\nu^\phi(x) \leqslant k\}$, then (2) implies

$$\mu(F_k) = \nu(F_k) \leqslant kc^\phi(F_k) = 0.$$

Since $F = \bigcup F_k$, $\mu(F) = 0$. Therefore $\bar{d}_\mu^\phi(x) = \bar{d}_\nu^\phi(x)$ μ-a.s. on E.

https://doi.org/10.1017/S0305004100064690 Published online by Cambridge University Press

COROLLARY 6·2. $p^\phi(E) < \infty \Rightarrow \underline{d}^\phi_\mu(x) = \underline{d}^\phi_\nu(x)$ p^ϕ-*a.s. and* μ-*a.s. on* E.

Proof. Let $G = \{x \in E : \underline{d}^\phi_\nu(x) \neq \underline{d}^\phi_\mu(x)\}$: G is the countable union of all G_q with $q = (r, s) \in \mathbf{Q}^2$ such that $0 < r < s < \infty$ and

$$G_q = \{x \in E : \underline{d}^\phi_\nu(x) \leqslant r < s \leqslant \underline{d}^\phi_\mu(x)\}.$$

Then (3) implies

$$sp^\phi(G_q) \leqslant \mu(G_q) = \nu(G_q) \leqslant rp^\phi(G_q),$$

which is possible only if $p^\phi(G_q) = \mu(G_q) = 0$. This finishes the proof.

In order to get further results for Hausdorff and packing measures, we consider the particular measures m^ϕ, c^ϕ, p^ϕ, and the corresponding density ratios $d^\phi_m(x, E)$, $d^\phi_c(x, E)$, $d^\phi_p(x, E)$.

Corollaries 6·1 and 6·2 give directly, for any $F \subset E$:

COROLLARY 6·3. *If* $m^\phi(E) < \infty$, *then* $\underline{d}^\phi_m(x, F) = \underline{d}^\phi_m(x, E)$ m^ϕ-*a.s. on* F, *and the same holds for* $\bar{d}^\phi_m$, $\underline{d}^\phi_c$ *and* $\bar{d}^\phi_c$.

COROLLARY 6·4. *If* $p^\phi(E) < \infty$, *then* $\underline{d}^\phi_m(x, F) = \underline{d}^\phi_m(x, E)$ p^ϕ-*a.s. on* F, *and the same holds for* $\underline{d}^\phi_c$, $\underline{d}^\phi_p$ *and* $\bar{d}^\phi_p$.

Let us recall that, according to Lemma 3·3, the condition $m^\phi(E) < \infty$ is equivalent to the condition $c^\phi(E) < \infty$, that m^ϕ-a.s. and c^ϕ-a.s. are equivalent, and both conclusions of Corollaries 6·3 and 6·4 hold when $p^\phi(E) < \infty$.

7. *Elementary density bounds*

We can deduce from Theorem 1·1 some natural bounds for the density ratios d^ϕ_m, d^ϕ_c and d^ϕ_p.

COROLLARY 7·1. *If* $c^\phi(E) < \infty$, *then* $\bar{d}^\phi_c(x, E) = 1$ c^ϕ-*a.s. on* E.

With Lemma 3·3, this implies the well-known result $1/\phi^* \leqslant \bar{d}^\phi_m(x, E) \leqslant 1$ m^ϕ-a.s. on E. The proof, which makes use of (2), is similar to the following:

COROLLARY 7·2. *If* $p^\phi(E) < \infty$, *then* $\underline{d}^\phi_p(x, E) = 1$ p^ϕ-*a.s. on* E.

Proof. Let $F_k = \{x \in E : \underline{d}^\phi_p(x, E) \geqslant 1 + 1/k\}$ and apply (3) to $\mu = p^\phi | E$ on F_k to get

$$(1 + 1/k)\, p^\phi(F_k) \leqslant p^\phi(F_k),$$

so that $p^\phi(F_k) = 0$. Hence $\underline{d}^\phi_p(x, E) \leqslant 1$ p^ϕ-a.s. on E. To get $\underline{d}^\phi_p(x, E) \geqslant 1$, take $G_k = \{x \in E : \underline{d}^\phi_p(x, E) \leqslant 1 - 1/k\}$ and apply (3) again.

COROLLARY 7·3. *If* $p^\phi(E) < \infty$, *then* $\underline{d}^\phi_c(x, E) \leqslant 1 \leqslant \bar{d}^\phi_p(x, E)$ p^ϕ-*a.s. on* E, *and* $0 < \underline{d}^\phi_c(x, E) \leqslant \bar{d}^\phi_p(x, E) < \infty$ c^ϕ-*a.s. on* E.

The inequality $1 \leqslant \bar{d}^\phi_p(x, E)$ is a consequence of Corollary 7·2 and has been included here only for the symmetry of the formulas.

Proof. Take $F_k = \{x \in E : \underline{d}^\phi_c(x, E) \geqslant 1 + 1/k\}$ and use (3) with $\mu = c^\phi | E$ to get

$$(1 + 1/k)\, p^\phi(F_k) \leqslant c^\phi(F_k),$$

https://doi.org/10.1017/S0305004100064690 Published online by Cambridge University Press

hence $p^\phi(F_k) = 0$, and $\underline{d}_c^\phi(x, E) \leqslant 1$ p^ϕ-a.s. on E. Finally, let $G_k = \{x \in E : \underline{d}_c^\phi(x, E) \leqslant 1/k\}$: thus $\{x \in E : \underline{d}_c^\phi(x, E) = 0\}$ is contained in G_k for any k, and (3) gives

$$c^\phi(G_k) \leqslant p^\phi(G_k)/k \leqslant p^\phi(E)/k$$

so that $\underline{d}_c^\phi(x, E) > 0$ c^ϕ-a.s. on E. Similarly, one can get $\bar{d}_p^\phi(x, E) < \infty$ c^ϕ-a.s. on E using $H_k = \{x \in E : \bar{d}_p^\phi(x, E) \geqslant k\}$ and (2).

8. *A weak notion of regularity*

In [11], section 5, a set E was described such that $m^1(E) = 0$ and $p^1(E) = \infty$. No doubt we could find such an example for any $\phi \in \Phi$. Sets E such that $m^\phi(E) > 0$ and $p^\phi(E) < \infty$ for some ϕ may be considered as more regular. It is interesting to give other characterizations of such sets:

COROLLARY 8·1. *Suppose* $p^\phi(E) < \infty$. *The following three statements are equivalent*:

(i) $m^\phi(E) = 0$.

(ii) $\underline{d}_m^\phi(x, E) = 0$ p^ϕ-*a.s. on* E.

(iii) $\bar{d}_p^\phi(x, E) = \infty$ p^ϕ-*a.s. on* E.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is obtained with the same arguments as for the following equivalence (i) $\Leftrightarrow$ (iii).

For (i) $\Rightarrow$ (iii), we set $E_k = \{x \in E : \bar{d}_p^\phi(x, E) \leqslant k\}$. Then (2) gives

$$p^\phi(E_k) \leqslant k c^\phi(E_k) \leqslant k \phi^* m^\phi(E_k) = 0,$$

so that $\bar{d}_p^\phi(x, E) = \infty$ p^ϕ-a.s. on E.

Conversely, we set $F_k = \{x \in E : \bar{d}_p^\phi(x, E) \geqslant k\}$ and obtain from (2) that for any k,

$$m^\phi(E) \leqslant c^\phi(F_k) + p^\phi(E \backslash F_k) \leqslant p^\phi(E)/k.$$

The lemma is proved.

Very similar arguments can be used for

COROLLARY 8·2. *Suppose* $p^\phi(E) < \infty$. *The following three statements are equivalent*:

(*i*) $\forall F \subset E$, $m^\phi(F) = 0 \Leftrightarrow p^\phi(F) = 0$.

(ii) $\underline{d}_m^\phi(x, E) > 0$ p^ϕ-*a.s. on* E.

(iii) $\bar{d}_p^\phi(x, E) < \infty$ p^ϕ-*a.s. on* E.

9. *Strong notions of regularity*

The regularity of a set is generally defined by a convergence to 1 of density ratios. As in [11], it may also be characterized by the equality between two measures. Rectifiability is related to the Lipschitz properties of the surfaces, generalizing the notion of Jordan arc of finite length. Let us investigate these different notions by using m^ϕ, c^ϕ and p^ϕ.

PROPOSITION 9·1. *Let* E *and* ϕ *be such that* $p^\phi(E) < \infty$. *The following three statements are equivalent*:

(i) $c^\phi(E) = p^\phi(E)$.

(ii) $\underline{d}_c^\phi(x, E) = \bar{d}_c^\phi(x, E) = 1$ p^ϕ-*a.s. on* E.

(iii) $\underline{d}_p^\phi(x, E) = \bar{d}_p^\phi(x, E) = 1$ p^ϕ-*a.s. on* E.

The proof is similar to that of the following:

https://doi.org/10.1017/S0305004100064690 Published online by Cambridge University Press

PROPOSITION 9·2. *Let E and ϕ be such that $p^\phi(E) < \infty$. The following two statements are equivalent*:

(i) $m^\phi(E) = p^\phi(E)$.

(ii) $\underline{d}^\phi_m(x, E) = \bar{d}^\phi_m(x, E) = 1 \quad p^\phi$-*a.s. on* E.

Proof. (i) $\Rightarrow$ (ii): Notice first that (i) is equivalent to:

$$\forall F \subset E, \quad m^\phi(F) = p^\phi(F),$$

so that m^ϕ-a.s. and p^ϕ-a.s. are equivalent on E. Then, Corollary 7·1 implies that $\underline{d}^\phi_m(x, E) \leqslant \bar{d}^\phi_m(x, E) \leqslant 1$ p^ϕ-a.s. on E. Let

$$E_k = \{x \in E : \underline{d}^\phi_m(x, E) \leqslant 1 - 1/k\}.$$

Using (3) with $\mu = m^\phi \mid E$ gives

$$m^\phi(E_k) \leqslant (1 - 1/k)\, p^\phi(E_k).$$

Since $m^\phi(E_k) = p^\phi(E_k)$, this is possible only if $p^\phi(E_k) = 0$. Hence $\underline{d}^\phi_m(x, E) \geqslant 1$ p^ϕ-a.s. on E.

(ii) $\Rightarrow$ (i): Let $F = \{x \in E : \underline{d}^\phi_m(x, E) = 1\}$. We have $p^\phi(F) = p^\phi(E)$. Use (3) with $\mu = m^\phi \mid E$ to get

$$p^\phi(F) \leqslant m^\phi(F).$$

Hence $m^\phi(E) \geqslant m^\phi(F) \geqslant p^\phi(E)$, which ends our proof.

Together with (1), the following result solves the conjecture (C_1) of the introduction:

COROLLARY 9·3. *If $0 < c^\alpha(E) = p^\alpha(E) < \infty$, then $\alpha \in \mathbf{N}$.*

Proof. Indeed, Proposition 9·1 gives $\underline{d}^\alpha_c(x, E) = \bar{d}^\alpha_c(x, E) = 1$ c^α-a.s. on E. If $c^\alpha(E) > 0$, this is possible only if α is an integer ([5], theorem 1).

Now we make use of a recent result of D. Preiss [8] which completes the proof of a well-known conjecture ([3], 3·3·22; [6], theorem 2·1):

If μ is a finite Borel measure, and k an integer, then $\mathbf{R}^n$ is (μ, k)-rectifiable if and only if $\underline{d}^k_\mu(x) = \bar{d}^k_\mu(x)$ μ-a.s. on $\mathbf{R}^n$.

For the (p^k, k)-rectifiability, our results allow us to replace this condition on the density ratios by a simple condition on the measures c^k and p^k:

COROLLARY 9·4. *If $p^k(E) < \infty$, E is (p^k, k)-rectifiable if and only if $c^k(E) = p^k(E)$.*

Proof. Apply Preiss's result with $\mu = p^k \mid E$, Corollary 7·2 and Proposition 9·1.

Finally, Conjecture (C_2) follows from Corollary 9·4 and the next:

COROLLARY 9·5. *The two following conditions are equivalent*:

(i) $c^\alpha(E) = p^\alpha(E)$.

(ii) $m^\alpha(E) = p^\alpha(E)$.

Proof. Let us assume (i). Since $m^\alpha \sim c^\alpha$ (Lemma 3·3), it suffices to consider the case $0 < c^\alpha(E) = p^\alpha(E) < \infty$, and therefore the case $\alpha = k \in \mathbf{N}$ (Corollary 9·3). Using Corollary 9·4, we know that there exists a sequence (E_i) of lipschitzian manifolds such that $p^k(E \setminus \bigcup E_i) = 0$. Therefore $m^k(E \setminus \bigcup E_i) = 0$: E is (m^k, k)-rectifiable. From [6],

https://doi.org/10.1017/S0305004100064690 Published online by Cambridge University Press

theorem 2·1, we have $\underline{d}_m^k(x,E) = \bar{d}_m^k(x,E) = 1$ m^k-a.s. on E, or equivalently c^k-a.s. on E. According to (i) and (1), c^k-a.s. on E is equivalent to p^k-a.s. on E (same argument as in the proof of Proposition 9·2). Then (ii) follows from Proposition 9·2. The implication (ii) $\Rightarrow$ (i) is obvious, from (1).

10. *Pseudo-packings*

We call a *pseudo-packing* of E any family $\mathbf{B}$ of bounded subsets of $\mathbf{R}^n$ such that, if $B, B' \in \mathbf{B}$, $\bar{B} \cap \bar{E} \neq \varnothing$ and $B \cap B' \cap E = \varnothing$. Thus every packing is a pseudo-packing. Let us note for further use that any pseudo-packing $\mathbf{B}$ of E by balls centred in E has the following property: if $B_r(x), B_s(y) \in \mathbf{B}$, then $y \notin B_r(x)$.

This notion of pseudo-packing gives rise to a new pre-measure for any E:

$$R^\phi(E) = \lim_{\rho \to 0} [\sup \{\phi(\mathbf{B}) : \mathbf{B} \ \rho\text{-pseudo-packing of } E \text{ by closed balls centred in } E\}],$$

and therefore to a measure

$$r^\phi(E) = \inf\{\sum_k R^\phi(E_k) : E \subset \bigcup_k E_k\}.$$

It is clear that $p^\phi(E) \leqslant r^\phi(E)$ for any E. With help of Theorem 1·1 we can prove that the converse inequality is true:

PROPOSITION 10·1. *For any E and ϕ, $p^\phi(E) = r^\phi(E)$.*

Proof. (*a*) We notice first that

$$r^\phi(E) \leqslant (1+\phi^*)\, p^\phi(E): \tag{9}$$

indeed, every pseudo-packing $\mathbf{B}$ of E with sufficiently small norm can be changed, by taking the balls of same centre and half the diameter, into a packing $\mathbf{B}'$ such that

$$\phi(\mathbf{B}) \leqslant (1+\phi^*)\, \phi(\mathbf{B}');$$

hence (9).

(*b*) From (9) we deduce that

$$p^\phi(E) = 0 \Leftrightarrow r^\phi(E) = 0 \quad \text{and} \quad p^\phi(E) = \infty \Leftrightarrow r^\phi(E) = \infty.$$

We can therefore assume that $p^\phi(E) < \infty$, and obtain from Corollary 7·2 that $\underline{d}_p^\phi(x,E) = 1$ p^ϕ-a.s. on E. Let $h < 1$,

$$E_k = \{x \in E : r \leqslant 1/k \Rightarrow p^\phi(E \cap B_r(x)) \geqslant h\phi(2r)\}$$

and $F_k = E \backslash E_k$: we have

$$\lim_k p^\phi(E_k) = p^\phi(E), \quad \lim_k p^\phi(F_k) = 0 = \lim_k r^\phi(F_k), \quad \lim_k r^\phi(E_k) = r^\phi(E).$$

For any $2/k$-pseudo-packing $\mathbf{B}$ of E_k by centred closed balls,

$$\begin{aligned} h\phi(\mathbf{B}) &\leqslant \sum_{B \in \mathbf{B}} p^\phi(E \cap B) \\ &= \sum_{B \in \mathbf{B}} p^\phi(E_k \cap B) + \sum_{B \in \mathbf{B}} p^\phi(F_k \cap B). \end{aligned} \tag{10}$$

As $\mathbf{B}$ is a pseudo-packing of E_k, the $(E_k \cap B)$'s are disjoint, therefore the first sum in the right member of (10) is less than $p^\phi(E_k)$. But the $(F_k \cap B)$'s may overlap, and we need to make use of the following:

https://doi.org/10.1017/S0305004100064690 Published online by Cambridge University Press

LEMMA 10·2. *Let* $\mathbf{B} = (B_i)$ *be a family of closed balls of* $\mathbf{R}^n$ *such that for all* i, *the centre of* B_i *is not in* $\bigcup_{j \neq i} B_j$. *Then the intersection of any* 3^n *different balls of* $\mathbf{B}$ *is empty.*

We know that any pseudo-packing satisfies the assumption of Lemma 10·2: this implies

$$\sum_{B \in \mathbf{B}} p^\phi(F_k \cap B) \leqslant 3^n p^\phi(F_k),$$

so that

$$hr^\phi(E_k) \leqslant hR^\phi(E_k) \leqslant p^\phi(E_k) + 3^n p^\phi(F_k).$$

As k tends to ∞, we get

$$hr^\phi(E) = \lim_k hr^\phi(E_k) \leqslant p^\phi(E),$$

as was to be shown.

Proof of Lemma 10·2. *Let* $B_i = B_{r_i}(x_i)$. We choose a point 0 in $\mathbf{R}^n \backslash \{x_i\}$ as an origin, to get a vector space. Let $|x| = \operatorname{dist}(0, x)$. We set $\mathbf{B}_0 = \{B \in \mathbf{B} : 0 \in B\}$. All we have to prove is

$$\operatorname{Card} \mathbf{B}_0 < 3^n.$$

For all i, j such that $B_i, B_j \in \mathbf{B}_0$, let $y_i = x_i/|x_i|$ and $z_{i,j} = |x_i|\, y_j = (|x_i|/|x_j|)\, x_j$. Let us assume, for example, that $|x_j| \leqslant |x_i|$, so that the point x_j lies in the segment $[0, z_{i,j}]$. We know that $0 \in B_i$ and $x_j \notin B_i$. By convexity of B_i we deduce that

$$\operatorname{dist}(x_i, z_{i,j}) > r_i \geqslant |x_i|.$$

Therefore

$$\operatorname{dist}(y_i, y_j) = \operatorname{dist}(x_i, z_{i,j})/|x_i| > 1.$$

This shows that, for all i such that $B_i \in \mathbf{B}_0$, the balls $B_{1/2}(y_i)$ are disjoint. Since they are also disjoint from $B_{1/2}(0)$, and included in $B_{3/2}(0)$, it follows that

$$(\operatorname{Card} \mathbf{B}_0 + 1)\, 2^{-n} \leqslant (3/2)^n,$$

hence the result.

REFERENCES

[1] A. S. BESICOVITCH. On the fundamental properties of linearly measurable plane sets of points. *Math. Ann.* **98** (1928), 422–464.

[2] K. J. FALCONER. *The Geometry of Fractal Sets* (Cambridge University Press, 1985).

[3] H. FEDERER. *Geometric measure theory*. Grundlehren der Math. Wiss. **153** (Springer-Verlag 1969).

[4] M. DE GUZMAN. *Real Variable methods in Fourier Analysis*, Mathematics studies 46 (North-Holland, 1981).

[5] J. M. MARSTRAND. The (ϕ, s)-regular subsets of n-space. *Trans. Amer. Math. Soc.* **113** (1964), 369–392.

[6] P. MATTILA. Hausdorff m-regular and rectifiable sets in n-space. *Trans. Amer. Math. Soc.* **205** (1975), 263–274.

[7] E. F. MOORE. Density ratios and $(\phi, 1)$-rectifiability in n-space. *Trans. Amer. Math. Soc.* **55** (1944), 236–305.

[8] D. PREISS. Geometry of measures in $\mathbf{R}^n$: distribution, rectifiability and densities, preprint.

[9] C. A. ROGERS. *Hausdorff measures* (Cambridge University Press, 1970).

[10] S. J. TAYLOR and C. TRICOT. Packing measure and its evaluation for a brownian path. *Trans. Amer. Math. Soc.* **288** (1985), 679–699.

[11] S. J. TAYLOR and C. TRICOT. The packing measure of rectifiable subsets of the plane. *Math. Proc. Cambridge Philos. Soc.* **99** (1986), 285–296.

[12] C. TRICOT. *Sur la classification des ensembles boréliens de mesure de Lebesgue nulle*. Thèse de doctorat, Genève 1979.

[13] C. TRICOT. Two definitions of fractional dimension. *Math. Proc. Cambridge Philos. Soc.* **91** (1982), 57–74.

https://doi.org/10.1017/S0305004100064690 Published online by Cambridge University Press