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Applications of Intuitionistic Temporal Logic to Temporal Answer Set Programming[*]

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Abstract

The relationship between intuitionistic or intermediate logics and logic programming has been extensively studied, prominently featuring Pearce's equilibrium logic and Osorio's safe beliefs. Equilibrium logic admits a fixpoint characterization based on the logic of here-and-there, akin to theory completion in default and autoepistemic logics. Safe beliefs are similarly defined via a fixpoint operator, albeit under the semantics of intuitionistic or other intermediate logics. In this paper, we investigate the logical foundations of Temporal Answer Set Programming through the lens of Temporal Equilibrium Logic, a formalism combining equilibrium logic with linear-time temporal operators. We lift the seminal approaches of Pearce and Osorio to the temporal

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setting, establishing a formal correspondence between temporal intuitionistic logic and temporal logic programming. Our results deepen the theoretical underpinnings of Temporal Answer Set Programming and provide new avenues for research in temporal reasoning.

KEYWORDS: temporal answer set programming, temporal equilibrium logic, fixpoint characterization

1 Introduction

Temporal logic programming, introduced in the late 1980s (Abadi and Manna 1989), augments logic programming with temporal modal operators, primarily those from linear-time temporal logic (LTL; Pnueli 1977). Although this field experienced substantial research activity throughout the 1980s and 1990s, its momentum eventually waned. More recently, the advent of answer set programming (ASP; Lifschitz 2019), and particularly its demonstrated efficacy in modeling and resolving temporal scenarios, has sparked renewed interest in these foundational approaches to temporal logic programming.

Early approaches to time representation in ASP (Gelfond and Lifschitz 1993) relied on variables ranging over finite subsets of the natural numbers. Although straightforward, this methodology lacked the dedicated language constructs and specialized inference mechanisms characteristic of LTL. Consequently, it remains infeasible to represent or reason about properties of reactive systems over infinite traces, such as safety (e.g., "Is a specific state reachable?") or liveness (e.g., "Does a condition hold infinitely often?"). Furthermore, establishing the unsolvability of planning problems becomes substantially more difficult.

To overcome these limitations, several extensions of ASP with temporal operators have been investigated. For instance, Eiter and Šimkus (2009) extended logic programs with function symbols to model both past and future temporal references. Other modal-inspired approaches typically adopt a temporal or dynamic modal logic (Pnueli 1977; Harel *et al*. 2000) as a monotonic basis, subsequently introducing nonmonotonicity via an established ASP semantics (Lifschitz 2010). As an example, Giordano *et al*. (2013) generalized the traditional reduct-based semantics (Gelfond and Lifschitz 1988) to a logic programming fragment equipped with dynamic logic operators (Harel *et al*. 2000). Similarly, Aguado *et al*. (2013) integrated LTL with equilibrium logic (EL; Pearce 2006), the predominant logical characterization of stable models and answer sets. This latter framework was subsequently adapted to finite traces in (Aguado *et al*. 2023).

Equilibrium logic builds upon here-and-there logic (HT; Heyting 1930) by imposing a minimal model selection criterion to capture answer sets. In his seminal work, Pearce (2006) provided an alternative formulation of equilibrium logic based on fixpoints. Analogous to the treatment of default and autoepistemic logics (Marek and Truszczyński 1993), this characterization relies on theory extensions, or completions, rather than direct semantic minimization.

A related fixpoint characterization, termed safe beliefs, was introduced by Osorio *et al*. (2005). Instead of HT, their approach employs intuitionistic logic (INT; Mints 2000) as its monotonic foundation. Crucially, they demonstrated that INT can be substituted with any (intermediate) logic X satisfying $\mathrm{INT} \subseteq \mathrm{X} \subseteq \mathrm{HT}$ without altering the resulting safe

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beliefs. Consequently, safe beliefs provide a robust framework for investigating properties of (temporal) logic programs from a broader logical standpoint, facilitating novel and insightful program transformations.

In this paper, we extend both Pearce's and Osorio's fixpoint-based characterizations to the temporal domain. Regarding Pearce's approach, we demonstrate that theory completions coincide with temporal equilibrium models when HT is superseded by Temporal Here-and-There logic (THT; Balbiani and Diéguez 2016).

Extending Osorio's approach to the temporal setting presents several notable challenges.

First, Osorio's work heavily relies on fundamental properties of propositional intuitionistic and intermediate logics (Gabbay 1981). One pivotal property is that satisfiability (or consistency) in INT is preserved across all intermediate logics. Furthermore, if a formula is satisfiable, it is guaranteed to hold in a finite model. Unfortunately, these properties generally fail in the temporal case. To circumvent this, we identify an intuitionistic temporal logic that preserves these characteristics, serving as the "weakest" intuitionistic base logic in our framework.

Second, Osorio's approach depends on a syntactic consequence relation and Hilbert-style axiomatic systems for INT. However, to the best of our knowledge, no sound and complete axiomatic system currently exists for an intuitionistic version of LTL.[1] Alternative attempts introduce the co-implication connective (Rauszer 1974) into the language (Fernández-Duque *et al*. 2024; Aguilera *et al*., 2022, 2025). Additionally, Osorio's method employs syntactic transformations to eliminate propositional variables. These transformations cannot be directly lifted to the temporal setting, as the truth values of propositional variables vary dynamically over time.

In light of these challenges, we adopt a strictly semantic approach. We reformulate Osorio's fundamental results using a semantic entailment relation, which we subsequently generalize to the temporal case. A critical component of this strategy is the application of bisimulations for both intuitionistic (Patterson 1997) and intuitionistic temporal logics (Balbiani *et al*. 2020).

Beyond reformulating Osorio's results semantically, we formally define the notion of an X-temporal safe belief set, where X is any intermediate temporal logic. We establish two primary results: first, that THT-temporal safe belief sets exactly correspond to temporal equilibrium models; and second, that substituting THT with any weaker intermediate temporal logic yields the identical set of safe beliefs.

The remainder of this paper is organized as follows. Section 2 provides the background on propositional intuitionistic and equilibrium logics. Section 3 introduces Pearce's theory completions alongside our semantic reformulation of Osorio's safe beliefs. Section 4 reviews intuitionistic and intermediate temporal logics, detailing the specific technical results employed in our framework. Section 5 presents our primary contribution: lifting

[1] In Boudou *et al*. (2019), four axiomatic systems for intuitionistic temporal logics are studied. It is shown that each generates a different logic, inducing distinct confluence properties. However, their completeness remains unaddressed. A sound and complete axiomatic system for the $\Box$-free fragment of ITL^{e} is presented in Diéguez and Fernández-Duque (2018).

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Pearce's theory completions and Osorio's safe beliefs to the temporal domain. Finally, we conclude the paper with a brief discussion and directions for future research.

2 Intuitionistic and intermediate logics

Given a countable, possibly infinite set $\mathbb{P}$ of atoms, also called the *alphabet*, our basic language $\mathcal{L}_p$ consists of formulas generated by the following grammar:

$$\varphi ::= p \in \mathbb{P} \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi$$

The negation connective is defined in terms of implication as $\neg\varphi := \varphi \rightarrow \bot$. A *(propositional) theory* is a possibly infinite set of propositional formulas.

Formulas of $\mathcal{L}_p$ are interpreted over partially ordered sets. An *intuitionistic frame* is a tuple $\mathfrak{F} = (W, \preccurlyeq)$, where W is a non-empty set of (Kripke) worlds and $\preccurlyeq \subseteq W \times W$ is a partial order. Given a frame $\mathfrak{F} = (W, \preccurlyeq)$, a world $w \in W$ is $\preccurlyeq$-*maximal* if there is no $v \in W$ such that $w \neq v$ and $w \preccurlyeq v$. Because maximality is exclusively associated with the relation $\preccurlyeq$ throughout this work, we simply use the term *maximal*.

Given a frame $\mathfrak{F} = (W, \preccurlyeq)$, we say that a subset $U \subseteq W$ is an *upset* of $\mathfrak{F}$ if for every $w, v \in W$ we have that if both $w \in U$ and $w \preccurlyeq v$ then $v \in U$. Moreover, a frame $\mathfrak{F}' = (U, \preccurlyeq')$ is called a *generated subframe* of $\mathfrak{F}$ if $U \subseteq W$ is an upset of $\mathfrak{F}$ and $\preccurlyeq'$ is the restriction of $\preccurlyeq$ to U, that is, $\preccurlyeq' = \preccurlyeq \cap (U \times U)$. Finally, given $w \in W$, we define the subframe generated by w as the generated subframe $\mathfrak{F}' = (\{u \in W \mid w \preccurlyeq u\}, \preccurlyeq')$.

We say that an intuitionistic frame $\mathfrak{F} = (W, \preccurlyeq)$ is of *depth* n, $depth(\mathfrak{F}) = n$ in symbols, if there is a chain of n worlds in $\mathfrak{F}$ and no chain of more than n worlds. Whenever $\mathfrak{F}$ contains an n-world chain for every $n < \omega$, we say that $\mathfrak{F}$ is of *infinite depth* ∞. Given a frame $\mathfrak{F} = (W, \preccurlyeq)$ and $w \in W$, we denote the *depth* of the subframe generated by w as $depth((\mathfrak{F}, w))$.

An *intuitionistic model*, or simply *model*, is a tuple $\mathfrak{M} = \langle (W, \preccurlyeq), V \rangle$ consisting of a frame $(W, \preccurlyeq)$ equipped with a monotone valuation function $V :\rightarrow W2^{\mathbb{P}}$. That is, if $w \preccurlyeq v$, then $V(w) \subseteq V(v)$ for all $w, v \in W$. The satisfaction relation (denoted by $\models$) of a formula φ at $w \in W$ is defined inductively by:

1. $\mathfrak{M}, w \models p$ iff $p \in V(w)$
2. $\mathfrak{M}, w \not\models \bot$
3. $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
4. $\mathfrak{M}, w \models \varphi \vee \psi$ iff $\mathfrak{M}, w \models \varphi$ or $\mathfrak{M}, w \models \psi$
5. $\mathfrak{M}, w \models \varphi \rightarrow \psi$ iff for all $v \succcurlyeq w$, if $\mathfrak{M}, v \models \varphi$, then $\mathfrak{M}, v \models \psi$

A formula φ is satisfied in an intuitionistic model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, in symbols $\mathfrak{M} \models \varphi$, if $\mathfrak{M}, w \models \varphi$ for some $w \in \mathfrak{F}$. A formula φ is satisfied on an intuitionistic frame $\mathfrak{F}$, if there exists a model $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ such that $\mathfrak{M} \models \varphi$. A formula φ is valid on an intuitionistic frame $\mathfrak{F}$, in symbols $\mathfrak{F} \models \varphi$, if for all models $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$, we have $\mathfrak{M} \models \varphi$. In the case of a theory Γ, we say that $\mathfrak{M}, w \models \Gamma$ if $\mathfrak{M}, w \models \varphi$ for all $\varphi \in \Gamma$. Similarly, Γ is said to be *consistent*, if there is a model $\mathfrak{M}$ and a world w such that $\mathfrak{M}, w \models \Gamma$. Finally we define the INT as

$$\mathrm{INT} := \{\varphi \in \mathcal{L}_p \mid \mathfrak{F} \models \varphi\},$$

where $\mathfrak{F}$ is an intuitionistic frame.

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2.1 Intermediate logics

An *intermediate logic*[2] in the language $\mathcal{L}_p$ is any set of formulas X satisfying the following conditions:

1. $\mathrm{INT} \subseteq \mathrm{X} \subseteq \mathrm{CL}$, where CL stands for *classical logic*,
2. X is closed under *modus ponens*, that is $\varphi, \varphi \to \psi \in \mathrm{X}$ implies $\psi \in \mathrm{X}$,
3. X is closed under *uniform substitution*, that is $\varphi \in \mathrm{X}$ implies $\varphi\mathbf{s} \in \mathrm{X}$ for any $\varphi \in \mathcal{L}_p$ and substitution $\mathbf{s}$.[3]

A *proper intermediate logic* is an intermediate logic different from CL. Broadly speaking, intermediate logics are obtained by adding formulas (that are classically valid) to INT as axiom schemas (Gabbay 1981, Chapter 2). In this way, they impose restrictions on $\preccurlyeq$. Therefore, given an intermediate logic X, we define the *class* of X-frames as the set of all intuitionistic frames $(W, \preccurlyeq)$ where $\preccurlyeq$ satisfies the restriction induced by the schemas used to generate X. As in the intuitionistic case, an intermediate logic X can be defined as

$$\mathrm{X} := \{\varphi \in \mathcal{L}_p \mid \mathfrak{F} \models \varphi\},$$

where $\mathfrak{F}$ is an X-frame.

To give an example, the logic of the *weak exclude middle* (Jankov 1968; Gabbay 1981) (KC) is obtained by adding the axiom $\neg p \vee \neg\neg p$ to INT and it is characterized by intuitionistic frames $(W, \preccurlyeq)$ satisfying the following frame condition: there exists $u \in W$ such that $v \preccurlyeq u$ for all $v \in W$.[4] Another family of intermediate logics, denoted by $\mathrm{BD_n}$, are obtained by adding an instance of the axiom schema $\boldsymbol{bd_n}$ to INT. For a given $n \geq 1$, such a family of axioms is recursively defined as follows:

$$\begin{aligned} \boldsymbol{bd_1} &:= p_1 \vee \neg p_1 \\ \boldsymbol{bd_{n+1}} &:= p_{n+1} \vee (p_{n+1} \to \boldsymbol{bd_n}). \end{aligned}$$

The axiom $\boldsymbol{bd_n}$ induces the following property on intuitionistic frames.

Theorem 1 (*Chagrov and Zakharyaschev 1997*).
An intuitionistic frame $\mathfrak{F} = (W, \preccurlyeq)$ *validates* $\boldsymbol{bd_n}$ *iff* $\mathit{depth}(\mathfrak{F}) \leq n$, *i.e, iff* $\mathfrak{F}$ *satisfies the following condition*

$$\forall w_0, \cdots, \forall w_n \left(\left(\bigwedge_{i=0}^{n-1} w_i \preccurlyeq w_{i+1} \right) \to \bigvee_{i \neq j} (w_i = w_j) \right). \tag{1}$$

The strongest proper intermediate logic is the logic of HT, which has been studied in the literature by different authors (Heyting 1930; Gödel 1932; Smetanich 1960). This logic is obtained by adding the axiom schema (Hosoi 1966)

$$p \vee (p \to q) \vee \neg q \tag{2}$$

[2] Chagrov and Zakharyaschev (1997) made a distinction between *super-intuitionistic* and intermediate logics but they also mention that in the propositional case, these two notions are practically identical.
[3] A *substitution* $\mathbf{s}$ is a mapping $\mathbf{s} : \mathbb{P} \to \mathcal{L}_p$ and $\varphi\mathbf{s}$ is defined by induction on the construction of φ: $\varphi\mathbf{s} = \mathbf{s}(p)$, $\bot\mathbf{s} = \bot$ and $(\varphi \odot \psi)\mathbf{s} = \varphi\mathbf{s} \odot \psi\mathbf{s}$ for $\odot \in \{\wedge, \vee, \to\}$.
[4] In the literature, this frame condition is usually called *topwidth 1* (Gabbay 1981).

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to INT and it is characterized by frames of the form $(\{0,1\}, \preccurlyeq)$ where $\preccurlyeq = \{(0,0),(1,1),(0,1)\}$. Broadly speaking, the world 0 (resp. 1) refers to the world "here" (resp. "there").

We write $\models_{\mathrm{X}}$ to specify the satisfaction relation in a concrete intermediate logic X. When the underlying logic is clear from the context, we omit X from the satisfaction relation. The notions of satisfiability, validity and consistency are defined in an analogous way as with INT. However, we prefix them with the logic when necessary. For instance, when Γ is consistent in the logic X, we say that Γ is X-consistent.

Consistency in any intermediate logic X is equivalent to consistency in classical logic. This is because, if a theory Γ is classically satisfiable, it is trivially satisfiable in a one-world model; conversely, if Γ is X-consistent, there exists a finite model satisfying all formulas in Γ at a world x. Hence, a maximal world of the submodel generated by x is a classical world that, in addition, satisfies Γ.

Lemma 1 (*van Dalen 1989; Osorio et al. 2005*).
Let X *and* Y *be two intermediate logics, and let* Γ *be a theory. Then* Γ *is* X*-consistent iff* Γ *is* Y*-consistent.*

Definition 1 (*Local semantic consequence*).
Let X *be any intermediate logic and let* Γ *and* ψ *be a propositional theory and a propositional formula, respectively. We define* ψ *as a* local semantic consequence *of* Γ, *written* $\Gamma \models_{\mathrm{X}} \psi$, *if, for each* X*-model* $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ *and for each world* $w \in \mathfrak{F}$,

$$(\mathfrak{M}, w \models \varphi \text{ for all } \varphi \in \Gamma) \text{ implies } \mathfrak{M}, w \models \psi.$$

When ψ *is replaced by a theory* Δ, *we say that* Δ *is local semantic consequence of* Γ, *written* $\Gamma \models_{\mathrm{X}} \Delta$, *if for each model* X*-model* $\mathfrak{M} = \langle \mathfrak{F}, V \rangle$ *and for each world* $w \in \mathfrak{F}$,

$$(\mathfrak{M}, w \models \varphi \text{ for all } \varphi \in \Gamma) \text{ implies } (\mathfrak{M}, w \models \psi, \text{ for all } \psi \in \Delta).$$

Proposition 1.
Let $\mathrm{X} \subseteq \mathrm{Y}$ *be two intermediate logics. For all theories* Γ *and all formulas* φ, $\Gamma \models_{\mathrm{X}} \varphi$ *implies* $\Gamma \models_{\mathrm{Y}} \varphi$.

We denote by $Cn_{\mathrm{X}}(\Gamma) := \{\varphi \in \mathcal{L}_p \mid \Gamma \models_{\mathrm{X}} \varphi\}$ the set of consequences obtained from Γ within the intermediate logic X.

2.2 Bisimulations for intuitionistic logic

In intuitionistic and modal logic, a world w in a model $\mathfrak{M}$ cannot see the entire Kripke model, it can only explore it *locally*, step by step, by means of the different accessibility relations ($\preccurlyeq$ in the case of INT). The concept of bisimulation states that, if two worlds w and w' are *bisimilar*, it does not matter how one can try to explore from w and w' using modal formulas, since they are behaviorally indistinguishable with respect to their logical properties. Bisimulation in intuitionistic propositional logic was studied by Patterson (1997). Formally, given two models $\mathfrak{M}_1 = \langle (W_1, \preccurlyeq_1), V_1 \rangle$ and $\mathfrak{M}_2 = \langle (W_2, \preccurlyeq_2), V_2 \rangle$, a bisimulation $\mathcal{Z}$ is a relation on $W_1 \times W_2$ that satisfies the following properties:

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C1 if $w_1 \mathcal{Z} w_2$ then $V_1(w_1) = V_2(w_2)$
C2 if $w_1 \mathcal{Z} w_2$ then for all $v_1 \in W_1$, if $w_1 \preccurlyeq_1 v_1$, then there exists $v_2 \in W_2$ such that $w_2 \preccurlyeq_2 v_2$ and $v_1 \mathcal{Z} v_2$
C3 if $w_1 \mathcal{Z} w_2$ then for all $v_2 \in W_2$, if $w_2 \preccurlyeq_2 v_2$, then there exists $v_1 \in W_1$ such that $w_1 \preccurlyeq_1 v_1$ and $v_1 \mathcal{Z} v_2$

Condition C1 ensures that two bisimilar worlds satisfy the same atoms. Condition C2 expresses that if we move forward from w_1 to a new world v_1, we can find a matching move from w_2 to a world v_2 that is (logically) indistinguishable from v_1. In other words, the second model can *imitate* the movements of the first. Condition C3 is the mirror image of C2: now the second model makes a move and the first one must be able to imitate it.

Given two models $\mathfrak{M}_1 = \langle (W_1, \preccurlyeq_1), V_1 \rangle$ and $\mathfrak{M}_2 = \langle (W_2, \preccurlyeq_2), V_2 \rangle$ with $w_1 \in W_1$ and $w_2 \in W_2$, we say that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are *bisimilar*, if there exists a bisimulation $\mathcal{Z}$ between W_1 and W_2 such that $w_1 \mathcal{Z} w_2$. The following lemma states that two bisimilar Kripke worlds satisfy the same formulas.

Lemma 2 (*Patterson 1997*).
Given two models $\mathfrak{M}_1 = \langle (W_1, \preccurlyeq_1), V_1 \rangle$ *and* $\mathfrak{M}_2 = \langle (W_2, \preccurlyeq_2), V_2 \rangle$ *and a bisimulation* $\mathcal{Z}$ *on* $W_1 \times W_2$, *then for all* $w_1 \in W_1$ *and for all* $w_2 \in W_2$, *if* $w_1 \mathcal{Z} w_2$ *then for all* $\varphi \in \mathcal{L}_p$, $\mathfrak{M}_1, w_1 \models \varphi$ *iff* $\mathfrak{M}_2, w_2 \models \varphi$.

The proof of the lemma is done by structural induction and it uses C1 to prove the case of the propositional variables while conditions C2 and C3 are used to prove the case of implication.

Osorio *et al*. (2005) show that the notion of X-safe beliefs is independent of the intermediate logic X; replacing X with any proper intermediate logic leaves the set of safe beliefs unchanged. Their approach, however, relies on a syntactic entailment relation and a Hilbert-style axiomatization of INT. Here, we establish the same result semantically via bisimulations. Our first lemma demonstrates that if a world w in an intuitionistic model $\mathfrak{M}$ satisfies all instances of the axiom $\neg p \vee \neg\neg p$, then all maximal worlds in the subframe generated by w can be merged into a single world.

Proposition 2.
Let $\mathfrak{M} = \langle (W, \preccurlyeq), V \rangle$ *be an intuitionistic model and let* $w \in W$ *be such that* $\mathfrak{M}, w \models \{\neg p \vee \neg\neg p \mid p \in \mathbb{P}\}$. *It follows that all maximal* $\preccurlyeq$*-worlds in the subframe generated by* x *satisfy the same propositional variables.*

Lemma 3.
Let $\mathfrak{M} = \langle (W, \preccurlyeq), V \rangle$ *an intuitionistic model and let* $w \in W$ *be such that* $\mathfrak{M}, w \models \{\neg p \vee \neg\neg p \mid p \in \mathbb{P}\}$. *There exists a model* $\mathfrak{M}' = \langle (W', \preccurlyeq'), V' \rangle$, $w' \in W'$ *and a bisimulation* $\mathcal{Z} W \times W'$ *such that* $w \mathcal{Z}' w'$ *and the subframe generated by* w' *has a unique maximal world with respect to* $\preccurlyeq'$.

Proof.
From $\mathfrak{M}, w \models \{\neg p \vee \neg\neg p \mid p \in \mathbb{P}\}$ and Proposition 2, all maximal worlds in the subframe generated by w satisfy the same propositional variables. Let us define now the model $\mathfrak{M}' := \langle (W', \preccurlyeq'), V' \rangle$ as follows:

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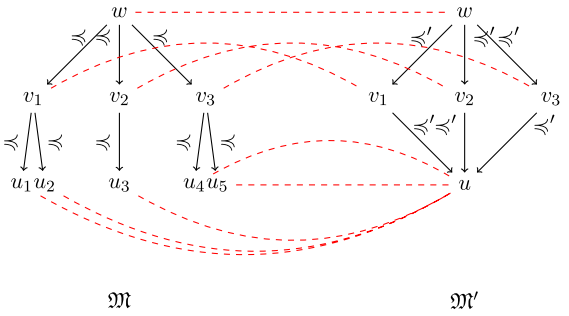


Fig 1. A bisimulation relation, represented in red dashed lines, among two intuitionistic models $\mathfrak{M}$ (on the left) and $\mathfrak{M}'$ (with a unique maximal world, on the right). We assume that, $\mathfrak{M}, w \models \{\neg p \vee \neg\neg p \mid p \in \mathbb{P}\}$ and we define V' as $V'(w) := V(w)$, $V'(v_i) := V(v_i)$ for all $i \in \{1, 2, 3\}$ and $V'(u)$ can be set (for instance) to $V(u_1)$. Reflexivity and transitivity of $\preccurlyeq$ and $\preccurlyeq'$ is not represented for the sake of readability.

- $W' = \{v \in W, u \mid w \preccurlyeq v$ and v is not $\preccurlyeq$-maximal $\}$, where $u \notin W$ is a fresh world.
- $v \preccurlyeq' v'$ if $v, v' \in W$ and $v \preccurlyeq v'$ or $v' = u$; $u \preccurlyeq' u$.
- $V'(u) := V(x)$, where $x \in W$, $w \preccurlyeq x$ and x is $\preccurlyeq$-maximal; $V'(v) := V(v)$, for all $v \in W'$ with $v \neq u$.

It can be checked that $\mathfrak{M}'$ is an intuitionistic model and, moreover, that there exists a relation $\mathcal{Z} W \times W'$, displayed in red dashed lines in Figure 1. In general, we map maximal worlds in W to $u \in W'$ while the remaining worlds $v \in W$ that belong to the subframe generated by w are mapped to themselves in W' (where they also belong by construction). The reader can easily check that $\mathcal{Z}$ is a bisimulation. □

The notion of bisimulation can also be used to contract intuitionistic models to HT models. In the following lemma, we identify the condition under which such contraction is possible. Before presenting our result, we introduce the following notation.

Definition 2.
Let $\mathfrak{M} = \langle (W, \preccurlyeq), V \rangle$ be an intuitionistic model and let $w \in W$. We define the sets

$$\preccurlyeq(w) := \{v \in W \mid w \preccurlyeq v\}$$
$$\prec(w) := \{v \in W \mid w \preccurlyeq v \text{ and } v \neq w\}.$$

Clearly, $w \in \preccurlyeq(w)$ while $w \notin \prec(w)$.

Lemma 4 (Contraction lemma).
Let $\mathfrak{M} = \langle (W, \preccurlyeq), V \rangle$ be an intuitionistic model, let $T \subseteq \mathbb{P}$ and let $w \in W$ satisfying the following conditions:

1. *$\preccurlyeq(w)$ has a unique maximal world, denoted by u, and*
2. *$V(v) = T$, for all $v \in \prec(w)$.*

Then, there exists a HT *model $\mathfrak{M}' = \langle (\{0, 1\}, \preccurlyeq'), V' \rangle$ and a bisimulation $\mathcal{Z} W \times \{0, 1\}$ such that $w \mathcal{Z}$.*

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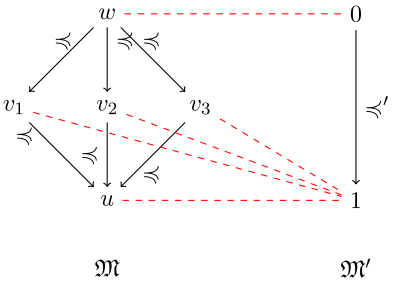


Fig 2. An intuitionistic model $\mathfrak{M}$, a HT model $\mathfrak{M}'$ and a bisimulations $\mathcal{Z}$ (in red dashed lines) among them. As preconditions, $V(v) = T$ for all $v \in W$ with $v \neq w$, $V'(0) = V(w)$ and $V'(1) = V(u)$. Reflexivity and transitivity of $\preccurlyeq$ and $\preccurlyeq'$ are not represented for the sake of readability.

Proof.
Let us define $\mathfrak{M}' = \langle(\{0,1\}, \preccurlyeq'), V'\rangle$ as $V'(0) := V(w)$ and $V'(1) := V(u)$. Let us define the relation $\mathcal{Z}W \times \{0,1\}$ as $\mathcal{Z} = \{((w,0),(v,1)) \mid v \in \prec(w)\}$. It can be checked that $\mathcal{Z}$ is a bisimulation among $\mathfrak{M}$ and $\mathfrak{M}'$. Figure 2 shows both $\mathfrak{M}$ and $\mathfrak{M}'$ together with $\mathcal{Z}$ (in red dashed lines). □

2.3 Equilibrium logic

The logic of here-and-there (HT; Heyting 1930) is fundamental in logic programming, since it serves as the basis for equilibrium logic (EL; Pearce, 1997, 2006), the most prominent logical characterization of *stable models* and *answer sets* (Gelfond and Lifschitz 1988). EL extends HT via a model selection criterion that induces nonmonotonicity. For any two HT models $\mathfrak{M}_1 = (\mathfrak{F}, V_1)$ and $\mathfrak{M}_2 = (\mathfrak{F}, V_2)$, we define a partial order $\mathfrak{M}_1 \leq \mathfrak{M}_2$ holding when $V_1(1) = V_2(1)$ and $V_1(0) \subseteq V_2(0)$. Strict inequality $\mathfrak{M}_1 < \mathfrak{M}_2$ holds if $\mathfrak{M}_1 \leq \mathfrak{M}_2$ and $V_1 \neq V_2$. A model $\mathfrak{M} = (\mathfrak{F}, V)$ is called *total* if $V(0) = V(1)$.

Equilibrium models are then defined as follows.

Definition 3 (*Pearce 2006*).
A total HT *model* $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ *is an equilibrium model of a formula* φ *if*

1. $\mathfrak{M}, 0 \models_{\mathrm{HT}} \varphi$, *and*
2. $\mathfrak{M}$ *is* $\leq$*-minimal, that is there is no* $\mathfrak{M}' < \mathfrak{M}$ *such that* $\mathfrak{M}', 0 \models_{\mathrm{HT}} \varphi$.

3 Two fixpoint characterizations of propositional equilibrium logic

This section presents two characterizations of equilibrium models (and, consequently, answer sets) that we extend to the temporal setting in Section 5. The first characterization, originally defined by Pearce (2006), is based on the concept of theory completions, which has also been used in autoepistemic and default logic (Besnard 1989; Marek and Truszczyński 1993).

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3.1 Pearce's fixpoint characterization

Definition 4 (*Pearce 2006*).
Let Γ *be a theory. A set* E *of formulas extending* Γ *is said to be a* completion *of* Γ *iff*

$$E = Cn_{\mathrm{HT}}(\Gamma \cup \{\neg\psi \mid \psi \notin E\}).$$

Equilibrium models correspond precisely to completions in the propositional case. For any model $\mathfrak{M}$, we define

$$Th(\mathfrak{M}) := \{\psi \mid \mathfrak{M}, 0 \models_{\mathrm{HT}} \psi\}.$$

The relation between equilibrium models and theory completions is made precise next.

Proposition 3 (*Pearce 1999b*).
For any theory Γ, *there is a one-to-one correspondence between the equilibrium models of* Γ *and the completions of* Γ. *In particular,* $E = Th(\mathfrak{M})$ *for some equilibrium model* $\mathfrak{M}$ *of* Γ. *Similarly, any total* HT *model* $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ *is an equilibrium model of* Γ *iff* $V(0) = V(1) = E \cap \mathbb{P}$ *for some completion* E *of* Γ.

3.2 INT-safe beliefs

A slightly different fixpoint characterization of equilibrium logic can be given in terms of INT-*safe beliefs* (Osorio *et al.* 2005), which are typically defined in terms of entailment in intuitionistic and intermediate logics. In this section, we provide a semantic reformulation of INT-safe beliefs, offering a different perspective that is useful for our subsequent development of temporal safe beliefs.

Definition 5.
A set T *of atoms is said to be an* INT-safe belief set *of a theory* Γ *if*

- $\Gamma \cup \{\neg\neg p \mid p \in T\} \cup \{\neg p \mid p \notin T\}$ *is* INT-*consistent and*
- $\Gamma \cup \{\neg\neg p \mid p \in T\} \cup \{\neg p \mid p \notin T\} \models_{\mathrm{INT}} T$.

Analogous definitions can be reproduced for intermediate logics by replacing INT by any intermediate logic X. In fact, HT-safe beliefs correspond to equilibrium models, as stated in the following lemma.

Lemma 5 (*Osorio et al. 2005; Pearce 2006*).
There is a one-to-one correspondence between both the HT-*safe beliefs sets of a theory* Γ *and its equilibrium models. That is, for each* HT-*safe belief set* T, *we can construct an equilibrium model* $\mathfrak{M} = \langle \mathfrak{F}, V\rangle$ *where* $V(0) = V(1) = T$. *Conversely, given* $\mathfrak{M}$, *we can extract the* HT-*safe belief set* $T = V(0)$.

Osorio *et al.* (2005) show that safe beliefs are independent of the chosen intermediate logic, as stated in Lemmas 6 (below) and 7.

Lemma 6 (*Osorio et al. 2005*).
Let T *be a set of atoms and let* X *and* Y *be two proper intermediate logics such that* $\mathrm{X} \subseteq \mathrm{Y}$. *For any propositional theory* Γ, *if* T *is a* X-*safe belief set of* Γ, *then* T *is a* Y-*safe belief set of* Γ.

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In Osorio *et al.* (2005), Lemma 7 is proved by using arguments from proof theory. We provide a model-theoretic proof based on Lemma 4, which is easier to extend to temporal equilibrium logic.

Lemma 7.
Let T be a set of atoms and let X *and* Y *be two proper intermediate logics satisfying* $\mathrm{X} \subseteq \mathrm{Y}$. *For any propositional theory* Γ, *if T is a* Y*-safe belief set of* Γ *then T is a* X*-safe belief set of* Γ.

Proof.
Let us assume that T is a Y-safe belief of Γ. It holds that

(a) $\Gamma \cup \{\neg\neg p \mid p \in T\} \cup \{\neg p \mid p \notin T\}$ is Y-consistent and
(b) $\Gamma \cup \{\neg\neg p \mid p \in T\} \cup \{\neg p \mid p \notin T\} \models_{\mathrm{Y}} T$.

From

$$\Gamma \cup \{\neg\neg p \mid p \in T\} \cup \{\neg p \mid p \notin T\} \tag{3}$$

being Y-consistent and Lemma 1, it follows that (3) is both X-consistent and INT-consistent. From the second item, the fact that $\mathrm{Y} \subseteq \mathrm{HT}$ and Proposition 1 it follows

$$\Gamma \cup \{\neg\neg p \mid p \in T\} \cup \{\neg p \mid p \notin T\} \models_{\mathrm{HT}} T. \tag{4}$$

Let $\mathfrak{M} = \langle (W, \preccurlyeq), V \rangle$ be any intuitionistic model and $w \in W$ satisfying

(c) $\mathfrak{M}, w \models_{\mathrm{INT}} \Gamma$,
(d) $\mathfrak{M}, w \models_{\mathrm{INT}} \{\neg\neg p \mid p \in T\}$ and
(e) $\mathfrak{M}, w \models_{\mathrm{INT}} \{\neg p \mid p \notin T\}$.

Items (d) and (e) imply that $\mathfrak{M}, w \models_{\mathrm{INT}} \{\neg p \vee \neg\neg p \mid p \in \mathbb{P}\}$. By Lemma 3 there exists an intuitionistic $\mathfrak{M}' = \langle (W', \preccurlyeq'), V' \rangle$, $w' \in W'$ and a bisimulation $\mathcal{Z} W \times W'$ such that $w \mathcal{Z}'$ and the subframe generated by w' has an unique maximal world. By Lemma 2, $\mathfrak{M}', w' \models_{\mathrm{INT}} \Gamma$, $\mathfrak{M}', w' \models_{\mathrm{INT}} \{\neg\neg p \mid p \in T\}$ and $\mathfrak{M}', w' \models_{\mathrm{INT}} \{\neg p \mid p \notin T\}$.

Let us denote by $u' \in W'$ the (unique) maximal world in the subframe generated by w'. By the monotonicity property of INT, $\mathfrak{M}', u' \models_{\mathrm{INT}} \{\neg\neg p \mid p \in T\}$ and $\mathfrak{M}', u' \models_{\mathrm{INT}} \{\neg p \mid p \notin T\}$. Since u' is maximal $V'(u') = T$. In addition, we prove by induction on $depth(((W', \preccurlyeq'), v'))$, that for all $v' \in \preccurlyeq'(w')$ (which includes the case w' as well), $\mathfrak{M}', v' \models_{\mathrm{INT}} T$.

1. If $depth((W', v')) = 1$ then v' is maximal so $v' = u'$ (and we reason as in the base case).
2. For the inductive step, let us assume that $depth(((W', \preccurlyeq'), v')) = n + 1$ and the claim holds for every $x \in \prec'(v')$, that is, $depth(((W', \preccurlyeq'), x)) \leq n$.

(a) If $\prec'(v') = \emptyset$, v' is maximal so $V'(v') = V'(u') = T$.
(b) If $\prec'(v') \neq \emptyset$ then, by induction hypothesis, $\mathfrak{M}', x \models T$ for all $x \in \prec'(v')$. From $\mathfrak{M}', w' \models_{\mathrm{INT}} \{\neg p \mid p \notin T\}$ and $w' \preccurlyeq' v'$, $\mathfrak{M}', x \not\models p$ for all $x \in \prec'(v')$ and all $p \in \mathbb{P} \setminus T$. Therefore, $V'(x) = T$ for all for all $x \in \prec'(v')$. By Lemma 4, there exists a HT model $\mathfrak{M}'' = \langle (\{0, 1\}, \preccurlyeq''), V'' \rangle$ and a bisimulation $\mathcal{Z} \subseteq W' \times \{0, 1\}$ such that $v' \mathcal{Z} 0$. By Lemma 2, $\mathfrak{M}'', 0 \models_{\mathrm{HT}} \Gamma$, $\mathfrak{M}'', 0 \models_{\mathrm{HT}} \{\neg\neg p \mid p \in T\}$ and $\mathfrak{M}'', 0 \models_{\mathrm{HT}} \{\neg p \mid p \notin T\}$. Thank to (4) it follows $\mathfrak{M}'', 0 \models_{\mathrm{HT}} T$. From $v' \mathcal{Z} 0$ and Lemma 2, $\mathfrak{M}', v' \models_{\mathrm{INT}} T$.

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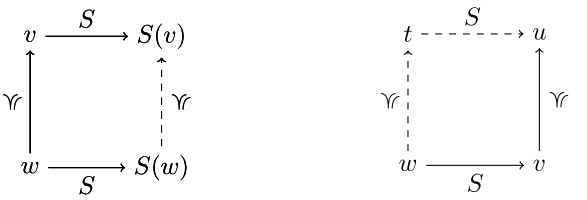


Fig 3. Diagrams associated to forward and backward confluence. The above diagrams can always be completed if S is forward or backward confluent (represented by means of dashed arrows).

Therefore, $\mathfrak{M}', w' \models_{\mathrm{INT}} T$. Since $w \mathcal{Z}' \mathfrak{M}, w \models_{\mathrm{INT}} T$. Since $\mathfrak{M}$ was chosen arbitrarily, it follows that $\Gamma \cup \{\neg\neg p \mid p \in T\} \cup \{\neg p \mid p \notin T\} \models_{\mathrm{INT}} T$. By Proposition 1, $\Gamma \cup \{\neg\neg p \mid p \in T\} \cup \{\neg p \mid p \notin T\} \models_{\mathrm{X}} T$. Therefore, T is a X-safe belief of Γ. □

Note that the combination of Lemmas 6 and 7 allows us to replace HT by any proper intermediate logic without altering the set of equilibrium models.

4 Temporal intuitionistic and intermediate logics

Our temporal language $\mathcal{L}_t$ consists of formulas generated by the following grammar:

$$\varphi ::= p \in \mathbb{P} \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \circ\varphi \mid \varphi \,\mathbb{U}\, \varphi \mid \varphi \,\mathbb{R}\, \varphi$$

This extends our basic language with temporal modal operators $\circ$, $\mathbb{U}$ and $\mathbb{R}$. The intended meaning of these operators is the following: $\circ\varphi$ means that φ is true at the next time point. $\varphi \,\mathbb{U}\, \psi$ means that φ is true until ψ is true. For $\varphi \,\mathbb{R}\, \psi$ the meaning is not as direct as for the previous operators. That is, $\varphi \,\mathbb{R}\, \psi$ means that ψ is true until both φ and ψ become true simultaneously or ψ is true forever. We also define several common derived operators like the Boolean connectives $\top := \neg\bot$, $\neg\varphi := \varphi \rightarrow \bot$, $\varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$, and the unary temporal operators $\Box\varphi := \bot \,\mathbb{R}\, \varphi$ (always afterwards) and $\Diamond\varphi := \top \,\mathbb{U}\, \varphi$ (eventually afterwards). A *(temporal) theory* is a possibly infinite set of temporal formulas.

Formulas of $\mathcal{L}_t$ are interpreted over intuitionistic temporal frames. An *intuitionistic temporal frame* is a tuple $\mathfrak{D} = (W, \preccurlyeq, S)$, where W is a non-empty set of (Kripke) worlds, $\preccurlyeq$ is a partial order, and S is a function from W to W satisfying the *forward confluence* condition: If $w \preccurlyeq v$ then $S(w) \preccurlyeq S(v)$ for all $w, v \in W$. Conversely, the *backward confluence* condition stipulates that if $S(w) = v$ and $v \preccurlyeq u$, then there exists $t \in W$ such that $w \preccurlyeq t$ and $S(t) = u$ for all $w, v, u \in W$. If $\mathfrak{D}$ satisfies both confluence conditions, we call $\mathfrak{D}$ a *persistent intuitionistic temporal frame*. Figure 3a (resp. Figure 3b) shows a graphical version of the forward (resp. backward) confluence condition.

An *intuitionistic temporal model*, or simply *model*, is a tuple $\mathfrak{M} = \langle \mathfrak{D}, V \rangle$ consisting of an intuitionistic temporal frame $\mathfrak{D} = (W, \preccurlyeq, S)$ equipped with a valuation function $V :\rightarrow W2^{\mathbb{P}}$ that is *monotone* in the sense that if $w \preccurlyeq v$ then $V(w) \subseteq V(v)$ for all $w, v \in W$. In the standard way, we define $S^0(w) := w$ and $S^{k+1}(w) := S\left(S^k(w)\right)$ for $k \geq 0$. Regarding the satisfaction relation, the propositional connectives are satisfied as in INT (see Section 2). The satisfaction of the temporal connectives is presented below.

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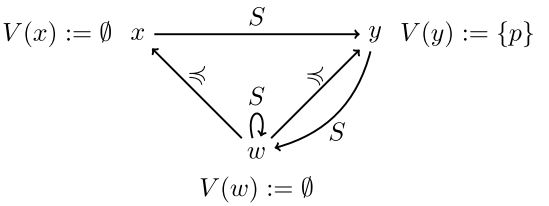


Fig 4. Example of an ITL^{e} model $\mathfrak{M} = ((W, \preccurlyeq, S), V)$, where reflexivity and transitivity for $\preccurlyeq$ are not represented.

6. $\mathfrak{M}, w \models \circ\varphi$ iff $\mathfrak{M}, S(w) \models \varphi$
7. $\mathfrak{M}, w \models \varphi \,\mathbb{U}\, \psi$ iff there exists $k \geq 0$ such that $\mathfrak{M}, S^k(w) \models \psi$ and $\mathfrak{M}, S^i(w) \models \varphi$ for all $i \in [0..k)$
8. $\mathfrak{M}, w \models \varphi \,\mathbb{R}\, \psi$ iff for all $k \geq 0$, either $\mathfrak{M}, S^k(w) \models \psi$ or $\mathfrak{M}, S^i(w) \models \varphi$ for some $i \in [0..k)$.

Figure 4 illustrates the satisfaction relation '$\models$' (Balbiani *et al.* 2020). Note that $\mathfrak{M}, x \models \circ p$ but $\mathfrak{M}, x \not\models p$, while $\mathfrak{M}, y \models p$ but $\mathfrak{M}, y \not\models \circ p$. From this, it follows that $\mathfrak{M}, w \not\models (\circ p \to p) \vee (p \to \circ p)$. We refer to the intuitionistic temporal logic interpreted over the class of intuitionistic temporal frames as *intuitionistic temporal logic* (ITL^{e}). Formally, defined as

$$\text{ITL}^{e} := \{\varphi \in \mathcal{L}_t \mid \mathfrak{D} \models \varphi\},$$

where $\mathfrak{D}$ is an intuitionistic temporal frame. If in addition $\mathfrak{D}$ is persistent, we denote by

$$\text{ITL}^{p} := \{\varphi \in \mathcal{L}_t \mid \mathfrak{D} \models \varphi\}$$

the *persistent intuitionistic temporal logic* (ITL^{p}), i.e., the intuitionistic temporal logic interpreted over the class of the intuitionistic persistent frames.

The following proposition shows that ITL$^{e} \neq$ ITL^{p}.

Proposition 4 (*Balbiani et al. 2020*).
The formulas $(\circ p \to \circ q) \to \circ\,(p \to q)$ *and* $(\Diamond p \to \Box q) \to \Box\,(p \to q)$ *are valid over the class of persistent intuitionistic temporal frames.*

The formulas presented in the proposition above are valid in ITL^{p} but not in ITL^{e}. This leads to the following result.

Corollary 1 (*Balbiani et al. 2020*).
ITL$^{e} \neq$ ITL^{p}.

We remark that intuitionistic temporal frames impose minimal conditions on S and $\preccurlyeq$ in order to preserve the monotonicity of truth of formulas, in the sense that if $\mathfrak{M}, w \models_{\text{ITL}^{e}} \varphi$ and $w \preccurlyeq v$ then $\mathfrak{M}, v \models_{\text{ITL}^{e}} \varphi$. In the propositional case, the monotonicity property is guaranteed by the use of a monotone valuation. In the temporal case, we additionally require the forward confluence property, which relates $\preccurlyeq$ and S. Forward confluence ensures the satisfaction of temporal formulas is also monotone with respect to $\preccurlyeq$. The backward confluence property, while not required for monotonicity, allows us to show that maximal points are preserved under the temporal successor relation: if w is a $\preccurlyeq$-maximal point, then $S(w)$ is also maximal (see Proposition 7).

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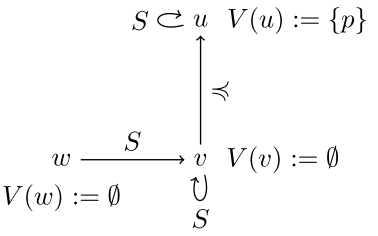


Fig 5. Example of an ITL$^{\text{e}}$ model satisfying $\Gamma := \{\neg\circ p, \neg\circ\neg p\}$. Reflexivity and transitivity of $\preccurlyeq$ are omitted for the sake of clarity.

Proposition 5 (*monotonicity; Balbiani et al. 2020*).
Let $\mathfrak{M} = \langle (W, \preccurlyeq, S), V\rangle$ *be an intuitionistic temporal model. For any* $w, v \in W$, *if* $w \preccurlyeq v$ *then for any temporal formula* φ, $\mathfrak{M}, w \models \varphi$ *implies* $\mathfrak{M}, v \models \varphi$.

Contrary to the non-temporal case (see Section 2), consistency in ITL$^{\text{e}}$ cannot be reduced to consistency in plain LTL. We provide here the counterexample proposed by Balbiani *et al.* (2020): consider $\Gamma := \{\neg\circ p, \neg\circ\neg p\}$. In LTL, this theory is equivalent to $\{\circ\neg p, \circ p\}$ and it is inconsistent. However, the ITL$^{\text{e}}$ model $\mathfrak{M}$ shown in Figure 5 satisfies Γ at the world w (in symbols, $\mathfrak{M}, w \models \Gamma$). Note that, in this case, S is forward, but not backward, confluent. Hence, the decidability of the satisfiability problem in ITL$^{\text{e}}$ is not a corollary of the LTL case. When adding the backward confluence property, Γ becomes inconsistent. However, adding such condition (i.e. replacing ITL$^{\text{e}}$ by ITL$^{\text{p}}$) does not allow us to reduce ITL$^{\text{p}}$-consistency to LTL-consistency either. To show this, we consider the theory $\Gamma := \{\Box\neg\neg p, \neg\Box p\}$. In LTL, Γ is equivalent to $\{\Box p, \neg\Box p\}$ and it is clearly inconsistent. Now, consider the model $\mathfrak{M} = \langle (\mathbb{N} \times \mathbb{N}, \preccurlyeq, S), V\rangle$ shown in Figure 6. In this case, the valuation V is defined as $V((i, j)) := \{p\}$ if $i < j$ and $\emptyset$ otherwise. In fact, $\mathfrak{M}$ is a ITL$^{\text{p}}$ model since it possesses both forward and backward confluence properties. Moreover,

1. For every $i \geq 0$ there exist $j > 0$ such that $\mathfrak{M}, (i, j) \models p$, so $\mathfrak{M}, (i, 0) \models \neg\neg p$, for every $i \geq 0$. Consequently, $\mathfrak{M}, (0, 0) \models \Box\neg\neg p$.
2. For every $j \geq 0$ there exist $i \geq 0$ such that $\mathfrak{M}, (i, j) \not\models p$. Therefore, $\mathfrak{M}, (0, j) \not\models \Box p$, for every $j \geq 0$. Consequently, $\mathfrak{M}, (0, 0) \models \neg\Box p$.

As a consequence, $\mathbf{M}, (0, 0) \models \Gamma$. We remark that the depth of $\preccurlyeq$ is not finite in $\mathfrak{M}$. Such a counterexample leads to the following result.

Proposition 6.
ITL$^{\text{p}}$*-consistency cannot be reduced to* LTL*-consistency.*

4.1 Intermediate temporal logics

As we have shown in the previous section, intermediate temporal logics defined extending ITL$^{\text{p}}$ with new axioms may not preserve consistency. There is one extra condition that we need to impose on the intuitionistic temporal frames: *the intuitionistic depth* ($\preccurlyeq$-depth, for short) *must be finite*. Finite depth can be achieved by forcing ITL$^{\text{p}}$ to validate the

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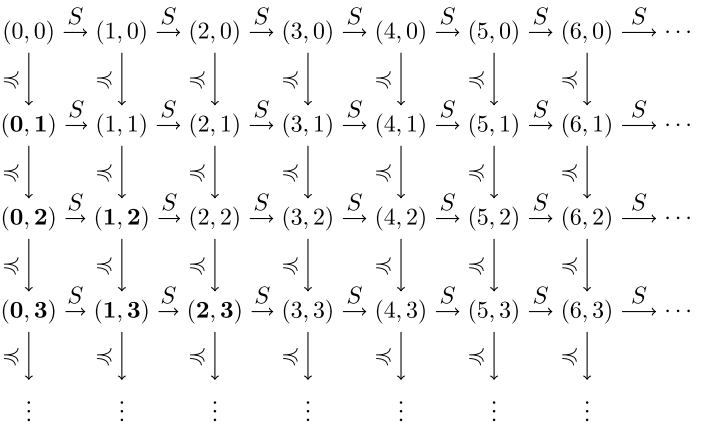


Fig 6. A ITL$^{\mathrm{P}}$ model $\mathfrak{M}$ satisfying $\Gamma := \{\{\Box\neg\neg p, \neg\Box p\}\}$ at $(0,0)$. The proposition p is true in the worlds displayed boldface while false in those that are not. Reflexivity and transitivity of $\preccurlyeq$ are not represented for the sake of readability.

schema $\boldsymbol{bd_n}$, for some $n \geq 1$, as shown in Lemma 1. We define the family ITL$^{\mathrm{BD_n}}$ of intuitionistic temporal logics as

$$\mathrm{ITL}^{\mathrm{BD_n}} := \{\varphi \in \mathcal{L}_t \mid \mathfrak{D} \models \varphi\},$$

where $\mathfrak{D} = (W, \preccurlyeq, S)$ is a ITL$^{\mathrm{P}}$ frame with $depth((W, \preccurlyeq)) \leq n$.

Observation 1.
ITL$^{\mathrm{BD_1}}$ *corresponds to* LTL, *that is,* ITL$^{\mathrm{BD_1}}$ *frames* $\mathfrak{D} = (W, \preccurlyeq, S)$ *such that* $depth((W, \preccurlyeq)) = 1$.[5]

Proposition 7.
For all ITL$^{\mathrm{BD_n}}$ *frame* $\mathfrak{D} = (W, \preccurlyeq, S)$ *and for all* $w \in W$, *if* w *is maximal w.r.t.* $\preccurlyeq$ *then* $S(w)$ *is maximal.*

Note that the proposition above can be proved only if the backward-confluence property holds. In addition, using the result above, we can prove that ITL$^{\mathrm{BD_n}}$-consistency can be reduced to LTL-consistency.

Lemma 8.
Any temporal theory Γ *is* ITL$^{\mathrm{BD_n}}$*-consistent iff* Γ *is* LTL*-consistent.*

To the best of our knowledge, the family of intermediate temporal logics has not been defined in the literature. We define those logics as extensions of ITL$^{\mathrm{BD_n}}$.[6]

Definition 6 (*Intermediate temporal logic*).
An intermediate temporal logic *in the language* $\mathcal{L}_t$ *is any set of formulas* X *satisfying the following conditions:*

1. ITL$^{\mathrm{BD_n}} \subseteq \mathrm{X} \subseteq$ LTL
2. X *is closed under* modus ponens, *that is* $\varphi, \varphi \to \psi \in \mathrm{X}$ *implies* $\psi \in \mathrm{X}$

[5] Note that the axiom schema $\boldsymbol{bd_1}$ corresponds to the excluded middle axiom, which is added to INT to get classical logic and to ITL$^{\mathrm{e}}$ to obtain LTL.
[6] In other words, we want ITL$^{\mathrm{BD_n}}$ to play the same role as INT does in the propositional case.

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3. X *is closed under* necessitation, *that is* $\psi \in \mathrm{X}$ *implies* $\circ\psi \in \mathrm{X}$ *and* $\Box\psi \in \mathrm{X}$
4. X *is closed under* uniform substitution, *that is* $\varphi \in \mathrm{X}$ *implies* $\varphi\mathbf{s} \in \mathrm{X}$ *for any* $\varphi \in \mathcal{L}_t$ *and a substitution* $\mathbf{s}$

As in the propositional case, an intermediate temporal logic is said to be *proper* if it is different from LTL. Assuming that any intermediate temporal logic extends $\mathrm{ITL}^{\mathrm{BD_n}}$, the following result directly follows from Lemma 8.

Corollary 2.
Let X *be any intermediate temporal logic. Any temporal theory* Γ *is* X*-consistent iff* Γ *is* LTL*-consistent.*

The following proposition shows that the depth of a generated (intuitionistic) subframe does not increase between two worlds x and $S(x)$.

Proposition 8.
For any intuitionistic temporal frame $\mathfrak{D} = (W, \preccurlyeq, S)$ *and for an* $w \in W$ *and for any* $n \geq 1$, *if* $depth(((W, \preccurlyeq), w)) \leq n$ *then* $depth(((W, \preccurlyeq), S(w))) \leq n$.

Similar to the propositional case, intermediate temporal logics defined as extensions of $\mathrm{ITL}^{\mathrm{BD_n}}$ satisfy the following proposition.

Proposition 9.
Let X *and* Y *be two intermediate temporal logics satisfying* $\mathrm{X} \subseteq \mathrm{Y}$. *A theory* Γ *is* X*-consistent iff* Γ *is* Y*-consistent.*

Furthermore, semantic entailment is preserved when strengthening the logic.

Proposition 10.
Let X *and* Y *be two intermediate temporal logics satisfying* $\mathrm{X} \subseteq \mathrm{Y}$. *For any theories* Γ *and* Δ,

$$\Gamma \models_{\mathrm{X}} \Delta \text{ implies } \Gamma \models_{\mathrm{Y}} \Delta.$$

The strongest proper intermediate temporal logic is the logic of here-and-there (THT; Balbiani and Diéguez 2016). In our setting, this logic is obtained by adding Axiom (2) to $\mathrm{ITL}^{\mathrm{BD_2}}$.[7] THT frames are of the form $(W, \preccurlyeq, S)$ where

$$\begin{aligned} W &= \mathbb{N} \times \{0, 1\}, \\ \preccurlyeq &= \{((i,h),(i,t)) \mid ((i,h),(i,t)) \in W \times W \text{ such that } h \leq t\} \text{ and} \\ S &= \{((i,k),(i+1,k)) \mid ((i,k),(i+1,k)) \in W \times W\}. \end{aligned}$$

In the definition above, pairs of the form $(i, 0)$ (resp. $(i, 1)$) represent the "here" (resp. "there") world at each time point i.

4.2 Bisimulations for intuitionistic temporal logics

The notion of intuitionistic temporal bisimulation was introduced in Balbiani *et al.* (2020) and we use it in what follows to contract intuitionistic temporal models into THT. Given

[7] It can also be obtained by adding Axiom (2) to $\mathrm{ITL}^{\mathrm{P}}$. However, in this paper, we take $\mathrm{ITL}^{\mathrm{BD_n}}$ as the root of our constructions.

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two intuitionistic temporal models $\mathfrak{M}_1 = \langle (W_1, \preccurlyeq_1, S_1), V_1 \rangle$ and $\mathfrak{M}_2 = \langle (W_2, \preccurlyeq_2, S_2), V_2 \rangle$, a relation $\mathcal{Z} W_1 \times W_2$ is an intuitionistic temporal bisimulation, if it satisfies Conditions C1-C3 together with the following ones:

C5 If $w_1 \mathcal{Z} w_2$ then $S(w_1) \mathcal{Z} (w_2)$;
C6 If $w_1 \mathcal{Z} w_2$ then for all $k_1 \geq 0$ there exists $k_2 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that

(a) $v_2 \preccurlyeq S^{k_2}(w_2)$, $S^{k_1}(w_1) \preccurlyeq v_1$ and $v_1 \mathcal{Z} w_2$ and
(b) for all $j_2 \in [0..k_2)$ there exists $j_1 \in [0..k_1)$ and $(u_1, u_2) \in W_1 \times W_2$ such that $S^{j_1}(w_1) \preccurlyeq u_1$, $u_2 \preccurlyeq S^{j_2}(w_2)$ and $u_1 \mathcal{Z} v_2$.

C7 If $w_1 \mathcal{Z} w_2$ then for all $k_2 \geq 0$ there exists $k_1 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that

(a) $v_1 \preccurlyeq S^{k_1}(w_1)$, $S^{k_2}(w_2) \preccurlyeq v_2$ and $v_1 \mathcal{Z} w_2$ and
(b) for all $j_1 \in [0..k_1)$ there exists $j_2 \in [0..k_2)$ and $(u_1, u_2) \in W_1 \times W_2$ such that $S^{j_2}(w_2) \preccurlyeq u_2$, $u_1 \preccurlyeq S^{j_1}(w_1)$ and $u_1 \mathcal{Z} v_2$.

C8 If $w_1 \mathcal{Z} w_2$ then for all $k_2 \geq 0$ there exists $k_1 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that

(a) $v_2 \preccurlyeq S^{k_2}(w_2)$, $S^{k_1}(w_1) \preccurlyeq v_1$ and $v_1 \mathcal{Z} w_2$ and
(b) for all $j_1 \in [0..k_1)$ there exists $j_2 \in [0..k_2)$ and $(u_1, u_2) \in W_1 \times W_2$ such that $S^{j_1}(w_1) \preccurlyeq u_1$, $u_2 \preccurlyeq S^{j_2}(w_2)$ and $u_1 \mathcal{Z} v_2$.

C9 If $w_1 \mathcal{Z} w_2$ then for all $k_1 \geq 0$ there exists $k_2 \geq 0$ and $(v_1, v_2) \in W_1 \times W_2$ such that

(a) $v_1 \preccurlyeq S^{k_1}(w_1)$, $S^{k_2}(w_2) \preccurlyeq v_2$ and $v_1 \mathcal{Z} w_2$ and
(b) for all $j_2 \in [0..k_2)$ there exists $j_1 \in [0..k_1)$ and $(u_1, u_2) \in W_1 \times W_2$ such that $S^{j_2}(w_2) \preccurlyeq u_2$, $u_1 \preccurlyeq S^{j_1}(w_1)$, and $u_1 \mathcal{Z} v_2$.

Conditions C5–C9 play the same role as conditions C2–C3 in the case of INT but they affect the temporal modalities. More precisely, Conditions C6 and C7 (resp. conditions C8 and C9) simulate the behavior of the until (resp. release) operator. Note that, due confluence of both $\preccurlyeq$ and S, the forth and back conditions for the binary temporal modalities involve both relations. Condition C5, which is used to simulate the next modality, is not divided into two conditions because the next operator is interpreted in terms of a function.

Given two intuitionistic temporal models $\mathfrak{M}_1 = \langle (W_1, \preccurlyeq_1, S_1), V_1 \rangle$ and $\mathfrak{M}_2 = \langle (W_2, \preccurlyeq_2, S_2), V_2 \rangle$ with $w_1 \in W_1$ and $w_2 \in W_2$, we say that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are bisimilar, if there exists an intuitionistic temporal bisimulation $\mathcal{Z}$ between W_1 and W_2 such that $w_1 \mathcal{Z} w_2$. The following lemma states that two bisimilar Kripke worlds satisfy the same temporal formulas.

Lemma 9 (*Balbiani et al. 2020*).
Given two models $\mathfrak{M}_1 = \langle (W_1, \preccurlyeq_1, S_1), V_1 \rangle$ *and* $\mathfrak{M}_2 = \langle (W_2, \preccurlyeq_2, S_2), V_2 \rangle$ *and a bisimulation* $\mathcal{Z}$ *on* $W_1 \times W_2$, *we have for all* $w_1 \in W_1$ *and for all* $w_2 \in W_2$, *if* $w_1 \mathcal{Z} w_2$ *then for all* $\varphi \in \mathcal{L}_t$, $\mathfrak{M}_1, w_1 \models \varphi$ *iff* $\mathfrak{M}_2, w_2 \models \varphi$.

In the proof of the previous lemma, Condition C1 is used to establish the case of propositional variables. Conditions C3 and C2 are used to handle the case of implication. Conditions C7 and C6 are employed for the $\mathbb{U}$ operator, while Conditions C9 and C8 are used for the $\mathbb{R}$ operator. Finally, Condition C5 is used in the proof of the case of the $\circ$ connective.

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In Balbiani *et al.* (2020), where $\mathrm{ITL}^{\mathrm{e}}$ and $\mathrm{ITL}^{\mathrm{p}}$ are studied in detail, the authors proved that both satisfiability (resp. validity) on arbitrary $\mathrm{ITL}^{\mathrm{e}}$ models is equivalent to satisfiability (resp. validity) on the so-called *expanding models*, defined in the following theorem.

Theorem 2 (*Boudou et al. 2017; Balbiani et al. 2020*).
Every $\mathrm{ITL}^{\mathrm{e}}$ *model* $\mathfrak{M}$ *can be unfolded into an expanding model* $\mathfrak{M}' = \langle (W', \preccurlyeq', S'), V' \rangle$, *which satisfies the following properties:*

1. *For all* $i \geq 0$ *and* $w' \in W'$, *the intuitionistic subframe generated by each* $S'^{i}(w')$ *is a tree,*
2. *the sequence of trees induced by* $S'^{i}(w')$ *is a sequence of disjoint trees, and*
3. *for all* $v' \in W'$ *and for all* $i \geq 0$, *if* $S'^{i}(w') \preccurlyeq v'$ *then* $S'^{i+1}(w') \preccurlyeq S'(v')$, *that is,* $S'(v')$ *falls in the tree generated by* $S'^{i+1}(w')$.

We refer the reader to Balbiani *et al.* (2020) for more details about such unfolding. Since $\mathrm{ITL}^{\mathrm{BD_n}}$ is contained in $\mathrm{ITL}^{\mathrm{e}}$ then the construction above can be also applied to models of finite $\preccurlyeq$-depth. From now on, when displaying the $\mathrm{ITL}^{\mathrm{BD_n}}$ models, we consider expanding models.

Bisimulations for intuitionistic temporal logics allow us to extend Lemmas 3 and 4 to the temporal case.

Lemma 10.
Let $\mathfrak{M} = \langle (W, \preccurlyeq, S), V \rangle$ *be an* $\mathrm{ITL}^{\mathrm{BD_n}}$ *model and let* $w \in W$ *be such that* $\mathfrak{M}, w \models \{\Box(\neg p \vee \neg\neg p) \mid p \in \mathbb{P}\}$. *Then, there exists an* $\mathrm{ITL}^{\mathrm{BD_n}}$ *model* $\mathfrak{M}' = \langle (W', \preccurlyeq', S'), V' \rangle$, $w' \in W'$ *and a bisimulation* $\mathcal{Z} W \times W'$ *such that, for all* $i \geq 0$ *both* $S^i(w) \mathcal{Z}'^{i}(w')$ *and each (intuitionistic) subframe generated by each* $S'^{i}(w')$ *has an unique maximal world,* u_i, *w.r.t.* $\preccurlyeq'$.

Proof.
Since $\mathfrak{M}, w \models \{\Box(\neg p \vee \neg\neg p) \mid p \in \mathbb{P}\}$ implies $\mathfrak{M}, S^i(w) \models \{\neg p \vee \neg\neg p \mid p \in \mathbb{P}\}$, for all $i \geq 0$. Since $\mathfrak{M}$ is an $\mathrm{ITL}^{\mathrm{BD_n}}$ model, the intuitionistic depth of $(W, \preccurlyeq)$ is finite and, because of Proposition 2, all maximal worlds in $\preccurlyeq(S^i(w))$ satisfy the same propositional variables.

Let us define the $\mathrm{ITL}^{\mathrm{BD_n}}$ model $\mathfrak{M}' := \langle (W', \preccurlyeq', S'), V' \rangle$, where its corresponding frame is defined as:

$$
\begin{aligned}
W' &:= \{v, u_i \mid i \geq 0,\ v \text{ is } \preccurlyeq\text{-maximal in} \preccurlyeq(S^i(w)) \text{ and } u_i \notin W \text{ is a fresh world}\};\\
\preccurlyeq' &:= \{(u_i, u_i), (v, u_i) \mid i \geq 0,\ v, u_i \in W' \text{ and } S^i(w) \preccurlyeq v\}\\
&\quad \cup \{(v, v') \mid v, v' \in W' \text{ and } v \preccurlyeq v'\};\\
S' &:= \{(u_i, u_{i+1}), (v, u_{i+1}) \mid i \geq 0,\ S^i(w) \preccurlyeq v \text{ and } S(v) = x, \text{ with } x \preccurlyeq\text{-maximal}\}\\
&\quad \cup \{(v, v') \mid (v, v') \in W' \times W' \text{ and } vSv'\}.
\end{aligned}
$$

We verify that it $(W', \preccurlyeq', S')$ is an $\mathrm{ITL}^{\mathrm{BD_n}}$ frame. Since $\preccurlyeq$ is a partial order relation, by construction, $\preccurlyeq'$ is. Since S is a function, it can be verified that S' is a function as well. Also by construction, $(W', \preccurlyeq')$ is of finite depth. We readily check that S' and $\preccurlyeq'$ are forward and backward confluent. For the forward confluence let us consider $v, v' \in W$ satisfying $v \preccurlyeq' v'$. In order to prove that $S'(v) \preccurlyeq' S'(v')$ we need to consider several cases.

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1. If $v = u_i$, for some $i \geq 0$, by construction, $v' = u_i$ too. Therefore, $u_{i+1} = S'(v) \preccurlyeq' S'(v') = u_{i+1}$.
2. If $v \neq u_i$ but $v' = u_i$ then $S'(v') = u_{i+1}$ and $v \in W$. If $S(v)$ is maximal then $u_{i+1} = S'(v) \preccurlyeq S'(v') = u_{i+1}$ by definition. If not, take $S'(v) := S(v) \in W'$. By construction, $S'(v) \preccurlyeq u_{i+1}$.
3. If $v \neq u_i$ and $v' \neq u_i$ it follows that $v \preccurlyeq v'$. Since $\preccurlyeq$ and S are forward confluent then $S(v) \preccurlyeq S(v')$. If either $S(v)$ or $S(v')$ are maximal w.r.t. $\preccurlyeq$, we follow a similar reasoning as in the previous two items in order to check $S'(v) \preccurlyeq' S'(v')$. If neither $S(v)$ nor $S(v')$ are maximal then $S'(v) = S(v)$ and $S'(v') = S(v')$ so $S'(v) \preccurlyeq' S'(v')$ by construction.

In any case we conclude that, if $v \preccurlyeq' v'$ then $S'(v) \preccurlyeq S'(v')$ as requested.

For the backward confluence, let us take three arbitrary worlds $v, y, z \in W'$ satisfying $S'(v) = y \preccurlyeq' z$. We show that there exists $t \in W'$ such $v \preccurlyeq' t$ and $S'(t) = z$. As for the forward case, we proceed by cases.

1. If $y = u_{i+1}$ then $z = u_{i+1}$ by construction. Since $S'(v) = u_{i+1}$ then take $t := u_i$. By definition, $v \preccurlyeq' u_i$ and $S'(u_i) = u_{i+1} = z$.
2. If $y \neq u_{i+1}$ and $z = u_{i+1}$ then take $t := u_i$. $v \preccurlyeq' t$ by construction and $S'(u_i) = u_{i+1} = z$.
3. If $y \neq u_{i+1}$ and $z \neq u_{i+1}$ then $y, z \in W$, $y \preccurlyeq z$ and z is not maximal w.r.t. $\preccurlyeq$. Since $y \neq u_{i+1}$ then $S'(v) = y \neq u_{i+1}$. Therefore, $v \in W$. By construction, $S(v) = y$. Because of the backward confluence property, there exist $x \in W$ such that $v \preccurlyeq x$ and $S(x) = z$. The world x cannot be maximal, otherwise z would be maximal in view of Proposition 7 and it would not belong to W'. By construction, $x \in W'$ and $S'(x) = z$. Set $t := x$ so we would get $v \preccurlyeq' x$ and $S'(x) = z$.

As a consequence, $(W', \preccurlyeq', S)$ is an $\mathrm{ITL}^{\mathrm{BD_n}}$ frame.

Let us define V' as $V'(v) := V(v)$ if $v \neq u_i$ and $V'(u_i) := V(x)$ with x being any maximal world in $\preccurlyeq(S^i(w))$,[8] for all $i \geq 0$. We define the relation $\mathcal{Z} W \times W'$ as follows:

$$\begin{aligned}\mathcal{Z} := &\{(v, v) \mid v \in W \text{ and } v \in W'\}\\ &\cup \{(v, u_i) \mid v \in W, S^i(w) \preccurlyeq v \text{ and } v \text{ is maximal w.r.t. } \preccurlyeq\}.\end{aligned}$$

It can be checked that $\mathcal{Z}$ is a bisimulation. Figure 7 provides an example of an $\mathrm{ITL}^{\mathrm{BD_n}}$ model $\mathfrak{M}$ and its bisimilar model $\mathfrak{M}'$ owning an unique $\preccurlyeq'$-maximal world u_i per time instant. □

Lemma 11 (*Contraction lemma for* $\mathrm{ITL}^{\mathrm{BD_n}}$).
Let $\mathfrak{M} = \langle (W, \preccurlyeq, S), V\rangle$ *be an* $\mathrm{ITL}^{\mathrm{BD_n}}$ *model and let* $w \in W$. *If for all* $i \geq 0$

1. *there exists an unique maximal world, denoted by* u_i, *in* $\preccurlyeq(S^i(w))$ *and*
2. $V(v) = T_i \subseteq \mathbb{P}$ *for all* $S^i(w) \prec v$

then there exists a THT *model* $\mathfrak{M}' = \langle (\mathbb{N} \times \{0, 1\}, \preccurlyeq', S'), V'\rangle$ *and a bisimulation* $\mathcal{Z} W \times (\mathbb{N} \times \{0, 1\})$ *such that, for all* $i \geq 0$, $S^i(w) \mathcal{Z}' {}^i((0, 0))$ *and* $S^i(u_0) \mathcal{Z}' {}^i((0, 1))$.

[8] Note that, since $\mathfrak{M}$ is a $\mathrm{ITL}^{\mathrm{BD_n}}$ model, such x always exists. Moreover, since by assumption $\mathfrak{M}, S^i(w) \models \{\neg p \vee \neg\neg p \mid p \in \mathbb{P}\}$, all $\preccurlyeq$-maximal worlds in $\preccurlyeq(S^i(w))$ satisfy the same propositions.

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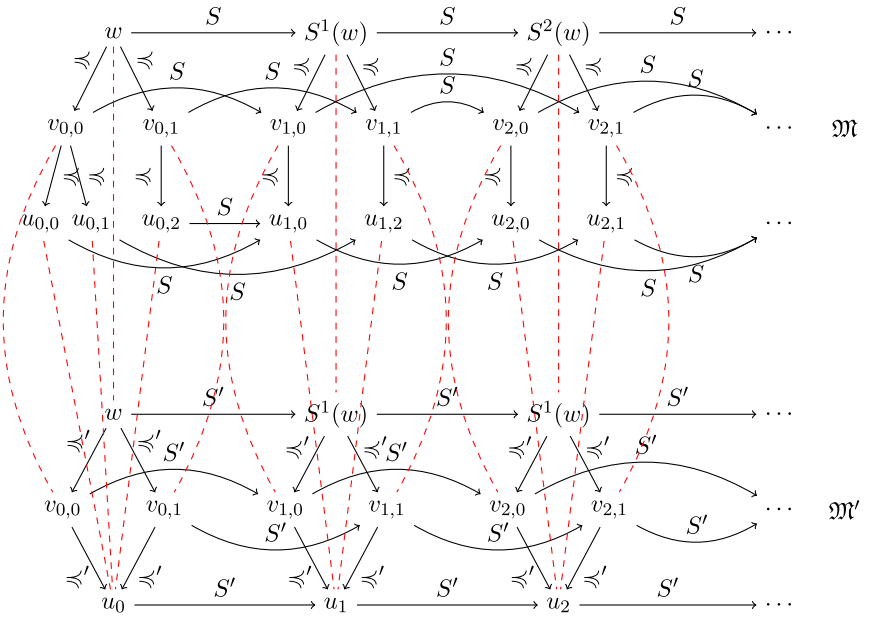


Fig 7. Two $\mathrm{ITL}^{\mathrm{BD_n}}$ models $\mathfrak{M} = \langle (W, \preccurlyeq, S), V\rangle$ and $\mathfrak{M}' = \langle (W', \preccurlyeq', S'), V'\rangle$. Under the assumption that $\mathfrak{M}, w \models \{\Box(\neg p \vee \neg\neg p) \mid p \in \mathbb{P}\}$, for all $i \geq 0$, all maximal worlds in $\preccurlyeq(S^i(w))$ satisfy the same atoms. By setting $V'(v) := V(v)$ for every world $v \in W \cap W'$ and, for all $i \geq 0$, $V'(u_i) := V(x)$, with x a maximal world in $\preccurlyeq(S^i(w))$, it can be verified that the relation $\mathcal{Z}$ displayed in terms of red dashed lines is a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$. The reflexivity and transitivity of $\preccurlyeq$ and $\preccurlyeq'$ is not represented for the sake of readability.

Proof.
Let us define the THT model $\mathfrak{M}' = \langle (\mathbb{N} \times \{0,1\}, \preccurlyeq', S'), V'\rangle$ where V' is defined as $V'((i,0)) := V(S^i(w))$ and $V'((i,1)) := V(u_i)$, for all $i \geq 0$.[9] Let us define now the relation $\mathcal{Z} W \times (\mathbb{N} \times \{0,1\})$ as

$$\mathcal{Z} = \{(S^i(w), (i,0)), (v, (i,1)) \mid i \geq 0 \text{ and } v \in \prec(S^i(w))\}. \tag{5}$$

It can be checked that $\mathcal{Z}$ is an intuitionistic temporal bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$. The condition for the propositional variables is satisfied because of Condition 2. The other conditions can be easily checked. Figure 8 shows an example of how a bisimulation between an $\mathrm{ITL}^{\mathrm{BD_n}}$ model $\mathfrak{M}$ and a THT model $\mathfrak{M}'$, which can be constructed whenever $\mathfrak{M}$ satisfies the preconditions 1 and 2 stated in this lemma. □

4.3 Temporal equilibrium logic

Given two THT models $\mathfrak{M}' = \langle (W, \preccurlyeq, S), V'\rangle$ and $\mathfrak{M} = \langle (W, \preccurlyeq, S), V\rangle$, we define $\mathfrak{M}' \leq \mathfrak{M}$ if $V'((i,1)) = V((i,1))$ and $V'((i,0)) \subseteq V((i,0))$ for all $i \geq 0$, and $\mathfrak{M}' = \mathfrak{M}$ if $V'((i,x)) = V((i,x))$ for all $i \geq 0$ and $x \in \{0,1\}$. Strict inequality $\mathfrak{M}' < \mathfrak{M}$ is defined as

[9] All u_is exist because of Condition 1.

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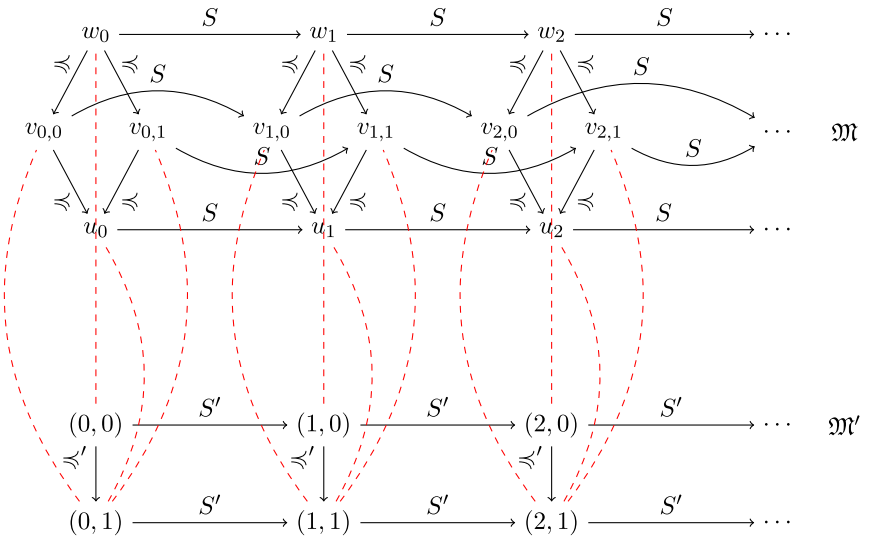


Fig 8. Example of model contraction. $\mathfrak{M} = \langle (W, \preccurlyeq, S), V\rangle$ is an $\text{ITL}^{\text{BD}_\text{n}}$ model and $\mathfrak{M}' = \langle (\mathbb{N} \times \{0,1\}, \preccurlyeq', S'), V'\rangle$ is a THT model. Under the assumption that every world $v \in \prec(S^i(w))$ satisfies exactly the same set of propositional variables, we can set $V'((i,0)) := V(w_i)$ and $V'((i,1)) := V(u_i)$, for all $i \geq 0$. The relation $\mathcal{Z}$, displayed in red dashed lines, is a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}'$. The reflexivity and transitivity of $\preccurlyeq'$ and $\preccurlyeq$ is not represented for the sake of readability.

$\mathfrak{M}' \leq \mathfrak{M}$ and $\mathfrak{M}' \neq \mathfrak{M}$. Finally, we also say that $\mathfrak{M}$ is *total* if $V((i,0)) = V((i,1))$, for all $i \geq 0$.

The following result is a corollary of Proposition 5.

Corollary 3 (*Satisfaction of negation*).
For any THT *model* $\mathfrak{M} = \langle (W, \preccurlyeq, S), V\rangle$, *for any* $i \geq 0$ *and for all* $\varphi \in \mathcal{L}_t$, $\mathfrak{M}, (i,0) \models \neg\varphi$ *iff* $\mathfrak{M}, (i,1) \not\models \varphi$

Definition 7.
We say that a total THT *model* $\mathfrak{M} = \langle (W, \preccurlyeq, S), V\rangle$ *is an equilibrium logic of a temporal formula* φ *if*

1. $\mathfrak{M}, (0,0) \models \varphi$ *and*
2. *there is no* THT *model* $\mathfrak{M}'$ *such that* $\mathfrak{M}' < \mathfrak{M}$ *and* $\mathfrak{M}', (0,0) \models \varphi$.

Temporal Equilibrium Logic (TEL for short) is the nonmonotonic logic induced by the temporal equilibrium models.

5 Two fixpoint characterizations of temporal equilibrium logic

In this section, we extend the fixpoint characterizations presented in Section 3 to the temporal case. In order to extend Pearce's characterization to the TEL case, we need to reformulate some of his definitions. In this section, given a THT model $\mathfrak{M}$, we *redefine*

$$Th(\mathfrak{M}) := \{\varphi \mid \mathfrak{M}, (0,0) \models_{\text{THT}} \varphi\}.$$

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Proposition 11.
Let $\mathfrak{M} = \langle (W, \preccurlyeq, S), V\rangle$ *be a temporal equilibrium model of* Γ. *For every* THT *model* $\mathfrak{M}' = \langle (W, \preccurlyeq, S), V'\rangle$, *if* $\mathfrak{M}', (0,0) \models \Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathfrak{M})\}$, *then* $V((i,1)) = V'((i,1)) = V'((i,0))$ *for all* $i \geq 0$.

Proof.
Assume towards a contradiction that $\mathfrak{M}', (0,0) \models \Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathfrak{M})\}$ but there exists $i \geq 0$ such that not $V((i,1)) = V'((i,1)) = V'((i,0))$. We first consider the case where $V((i,1)) \neq V'((i,1))$. There are two cases:

- If $V((i,1)) \not\subseteq V'((i,1))$, there exists some $p \in V((i,1))$ such that $p \notin V'((i,1))$. Since $p \in V((i,1))$, then $\mathfrak{M}, (0,1) \models \circ^i p$. Since $\mathfrak{M}$ is a total model, it follows that $\mathfrak{M}, (0,0) \models \circ^i p$ and $\mathfrak{M}, (0,0) \not\models \neg\circ^i p$. Therefore, $\neg\circ^i p \notin Th(\mathfrak{M})$. Since $\mathfrak{M}', (0,0) \models \Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathfrak{M})\}$, then $\mathfrak{M}', (0,0) \models \neg\neg\circ^i p$. By Proposition 5, $\mathfrak{M}', (0,1) \models \neg\neg\circ^i p$. Since the world $(0,1)$ is a classical world, $\mathfrak{M}', (0,1) \models \circ^i p$ so $p \in V'((i,1))$: a contradiction.
- If $V((i,1)) \not\supseteq V'((i,1))$, there exists some $p \in V'((i,1))$ such that $p \notin V((i,1))$. Since $p \notin V((i,1))$, then $\mathfrak{M}, (0,1) \not\models \circ^i p$. By Proposition 5, $\mathfrak{M}, (0,0) \not\models \circ^i p$. Therefore, $\circ^i p \notin Th(\mathfrak{M})$. Since $\mathfrak{M}', (0,0) \models \Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathfrak{M})\}$ then $\mathfrak{M}', (0,0) \models \neg\circ^i p$. By the satisfaction relation it follows that $\mathfrak{M}', (0,1) \not\models \circ^i p$, so $p \notin V'((i,1))$: a contradiction.

Therefore, we can assume that $V((i,1)) = V'((i,1))$, for all $i \geq 0$. For the case, $V'((i,0)) \neq V'((i,1))$, we can conclude that $V'((i,0)) \subset V'((i,1))$. Therefore, $\mathfrak{M}' < \mathfrak{M}$. Since $\mathfrak{M}$ is a temporal equilibrium model of Γ then, $\mathfrak{M}', (0,0) \not\models \Gamma$, so $\mathfrak{M}', (0,0) \not\models \Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathfrak{M})\}$: a contradiction. □

In the temporal case, we can obtain the same result by replacing HT for THT as underlying logic, as stated in the following proposition.

Lemma 12.
For any theory Γ *and any total* THT *model* $\mathfrak{M}$, *the following items are equivalent:*

1. $\mathfrak{M}$ *is a temporal equilibrium model of* Γ
2. $\Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathfrak{M})\} \models_{\text{THT}} \varphi$ *iff* $\varphi \in Th(\mathfrak{M})$ *for all* $\varphi \in \mathcal{L}_t$.

Proof.
To prove that Item 1 implies Item 2 we assume that Item 1 holds but 2 does not. Then, $\mathfrak{M}$ is a temporal equilibrium model of Γ but there exists a formula $\varphi \in \mathcal{L}_t$ for which one of the following two cases hold:

- $\Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathfrak{M})\} \models_{\text{THT}} \varphi$ but $\varphi \notin Th(\mathfrak{M})$: in this case, since $\mathfrak{M}$ is a temporal equilibrium model of Γ then $\mathfrak{M}$ is total and, in addition, $\mathfrak{M}, (0,0) \models \Gamma$. We can easily check that $\mathfrak{M}, (0,0) \models \{\neg\varphi \mid \varphi \notin Th(\mathfrak{M})\}$. Therefore, $\mathfrak{M}, (0,0) \models \varphi$ which contradicts $\varphi \notin Th(\mathfrak{M})$.
- $\varphi \in Th(\mathfrak{M})$ but $\Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathfrak{M})\} \not\models_{\text{THT}} \varphi$: in this case, there exists $\mathfrak{M}' = \langle (W, \preccurlyeq, S), V'\rangle$ such that $\mathfrak{M}', (0,0) \models \Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathfrak{M})\}$ but $\mathfrak{M}', (0,0) \not\models \varphi$. From $\mathfrak{M}', (0,0) \models \Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathfrak{M})\}$ and Proposition 11 it follows $V' = V$. Therefore, $\mathfrak{M}, (0,0) \not\models \varphi$, which means that $\varphi \notin Th(\mathfrak{M})$: a contradiction.

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For the converse direction, let us assume towards a contradiction that $\mathfrak{M}$ is not an equilibrium model of Γ. We assume without loss of generality that $\mathfrak{M}$ is total but one of the following conditions fails.

- $\mathfrak{M}, (0,0) \not\models \Gamma$. Assume that $\Gamma \neq \emptyset$ so there exists $\varphi \in \Gamma$ such that $\mathfrak{M}, (0,0) \not\models \varphi$. This means that $\varphi \notin Th(\mathfrak{M})$. Since item 2 holds, $\Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathfrak{M})\} \not\models_{\mathrm{THT}} \varphi$. It follows that there exists $\mathfrak{M}' = \langle (W, \preccurlyeq, S), V' \rangle$ such that $\mathfrak{M}', (0,0) \models \Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathfrak{M})\}$ but $\mathfrak{M}', (0,0) \not\models \varphi$. Since $\mathfrak{M}', (0,0) \models \Gamma$ and $\varphi \in \Gamma$ then $\mathfrak{M}', (0,0) \models \varphi$. Since, $\varphi \notin Th(\mathfrak{M})$ then $\mathfrak{M}', (0,0) \models \neg\varphi$. From the two previous points we conclude that $\mathfrak{M}', (0,0) \models \bot$: a contradiction.
- $\mathfrak{M}, (0,0) \models \Gamma$ but there exists $\mathfrak{M}' = \langle (W, \preccurlyeq, S), V' \rangle$ such that $\mathfrak{M}' < \mathfrak{M}$ and $\mathfrak{M}', (0,0) \models \Gamma$. From $\mathfrak{M}' < \mathfrak{M}$ follows that there exists $i \geq 0$ and $\mathsf{o}^i p \in \mathcal{L}_t$ such that $\mathfrak{M}', (0,0) \not\models \mathsf{o}^i p$, but $\mathfrak{M}, (0,0) \models \mathsf{o}^i p$. Since $\mathsf{o}^i p \in Th(\mathfrak{M})$ then $\Gamma \cup \{\neg\varphi \mid \varphi \notin Th(\mathfrak{M})\} \models_{\mathrm{THT}} \mathsf{o}^i p$. It can be checked that $\mathfrak{M}', (0,0) \models \{\neg\varphi \mid \varphi \notin Th(\mathfrak{M})\}$. Therefore, $\mathfrak{M}', (0,0) \models \mathsf{o}^i p$, a contradiction.

□

5.1 Temporal safe beliefs

For extending Definition 5 to the temporal case, we need some extra definitions. Since in the temporal case the truth of an atom may vary along time, we define the so-called set of *temporal atoms* associated with a signature $\mathbb{P}$ (in symbols, $\mathbb{P}^{\mathsf{o}}$) as

$$\mathbb{P}^{\mathsf{o}} := \{\mathsf{o}^i p \mid p \in \mathbb{P} \text{ and } i \geq 0\}.$$

Clearly, for any $p \in \mathbb{P}$, $\mathsf{o}^0 p := p$, so $\mathbb{P} \subseteq \mathbb{P}^{\mathsf{o}}$.

Definition 8 ($\mathrm{ITL}^{\mathrm{BD_n}}$*-temporal safe belief*).
Let Γ be a temporal theory. The set $T \subseteq \mathbb{P}^{\mathsf{o}}$ is said to be a $\mathrm{ITL}^{\mathrm{BD_n}}$-temporal safe belief set *with respect to Γ if*

1. $\Gamma \cup \{\mathsf{o}^i \neg\neg p \mid \mathsf{o}^i p \in T\} \cup \{\mathsf{o}^i \neg p \mid \mathsf{o}^i p \notin T\}$ *is* $\mathrm{ITL}^{\mathrm{BD_n}}$*-consistent and*
2. $\Gamma \cup \{\mathsf{o}^i \neg\neg p \mid \mathsf{o}^i p \in T\} \cup \{\mathsf{o}^i \neg p \mid \mathsf{o}^i p \notin T\} \models_{\mathrm{ITL}^{\mathrm{BD_n}}} T$.

In the definition above, $\mathrm{ITL}^{\mathrm{BD_n}}$ can be exchanged by any other proper intermediate temporal logic X. In the particular case of THT, we can prove a correspondence between THT-temporal safe beliefs and temporal equilibrium models.

Definition 9.
Given a total THT *model $\mathfrak{M} = \langle (W, \preccurlyeq, S), V \rangle$ we define*

$$T := \{\mathsf{o}^i p \mid p \in V((i,0)) \text{ and } i \geq 0\}.$$

Clearly, $T \subseteq \mathbb{P}^{\mathsf{o}}$. Conversely, given T we retrieve $\mathfrak{M} = \langle (W, \preccurlyeq, S), V \rangle$ by setting

$$V((i,0)), V((i,1)) := \{p \mid \mathsf{o}^i p \in T\}, \text{ for all } i \geq 0.$$

Proposition 12.
For any temporal theory Γ, any total THT *model $\mathfrak{M} = \langle (W, \preccurlyeq, S), V \rangle$ and set $T \subseteq \mathbb{P}^{\mathsf{o}}$ related as described in Definition 9, the following items are equivalent:*

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1. $\mathfrak{M}$ *is a temporal equilibrium model of* Γ.
2. T *is a* THT*-temporal safe belief of* Γ.

Proof.
To prove that (1) implies (2), let us assume that T is not a THT-temporal safe belief of Γ. Let us assume that

$$\Gamma \cup \{\neg\neg \circ^i p \mid \circ^i p \in T\} \cup \{\neg \circ^i p \mid \circ^i p \notin T\}$$

is consistent but

$$\Gamma \cup \{\neg\neg \circ^i p \mid \circ^i p \in T\} \cup \{\neg \circ^i p \mid \circ^i p \notin T\} \not\models_{\mathrm{THT}} T.$$

This means that there exists a THT model $\mathfrak{M}' = \langle (W, \preccurlyeq, S), V' \rangle$ such that $\mathfrak{M}', (0,0) \models \Gamma$, $\mathfrak{M}', (0,0) \models \{\neg\neg \circ^i p \mid \circ^i p \in T\}$, $\mathfrak{M}', (0,0) \models \{\neg \circ^i p \mid \circ^i p \notin T\}$ but $\mathfrak{M}', (0,0) \not\models T$. From $\mathfrak{M}', (0,0) \models \{\neg\neg \circ^i p \mid \circ^i p \in T\}$ and $\mathfrak{M}', (0,0) \models \{\neg \circ^i p \mid \circ^i p \notin T\}$ we can conclude that $V((i,1)) = V'((i,1))$, for all $i \geq 0$.

From $\mathfrak{M}', (0,0) \not\models T$ it follows that $\mathfrak{M}', (0,0) \not\models \circ^i p$ for some $\circ^i p \in T$, with $i \geq 0$. This means that $\mathfrak{M}', (i,0) \not\models p$. From $\mathfrak{M}', (0,0) \models \{\neg\neg \circ^i p \mid \circ^i p \in T\}$ we conclude that $\mathfrak{M}', (0,0) \models \neg\neg \circ^i p$, so $\mathfrak{M}', (i,1) \models p$. Therefore, $\mathfrak{M}' < \mathfrak{M}$. Since $\mathfrak{M}', (0,0) \models \Gamma$, $\mathfrak{M}$ is not an equilibrium model of Γ: a contradiction.

Conversely, let us assume towards a contradiction that T is a THT-temporal safe belief of Γ but $\mathfrak{M} = \langle (W, \preccurlyeq, S), V \rangle$ is not a temporal equilibrium model of Γ. Assume, without loss a contradiction that $\mathfrak{M}$ is total. Since T is a THT-temporal safe belief of Γ then $\Gamma \cup \{\neg\neg \circ^i p \mid \circ^i p \in T\} \cup \{\neg \circ^i p \mid \circ^i p \notin T\}$ is consistent. Let $\mathfrak{M}' = \langle (W, \preccurlyeq, S), V' \rangle$ be such that $\mathfrak{M}', (0,0) \models \Gamma \cup \{\neg\neg \circ^i p \mid \circ^i p \in T\} \cup \{\neg \circ^i p \mid \circ^i p \notin T\}$. Since $\mathfrak{M}', (0,0) \models \{\neg\neg \circ^i p \mid \circ^i p \in T\} \cup \{\neg \circ^i p \mid \circ^i p \notin T\}$ then $V'(i,1) = V(i,1)$, for all $i \geq 0$. Since $\mathfrak{M}', (0,0) \models \Gamma$ then $\mathfrak{M}', (0,1) \models \Gamma$. Since $\mathfrak{M}$ is total and $V'(i,1) = V(i,1)$, for all $i \geq 0$ then $\mathfrak{M}, (0,0) \models \Gamma$. Since $\mathfrak{M}$ is not an equilibrium model of Γ, there exists $\mathfrak{M}'' = \langle (W, \preccurlyeq, S), V'' \rangle$ such that $\mathfrak{M}'' < \mathfrak{M}$ and $\mathfrak{M}'', (0,0) \models \Gamma$. However, this contradicts Condition 2 of Definition 8. □

To conclude this section, we show that temporal safe belief sets are preserved when changing the underlying logic. In this case, we extend the results shown in Section 3.

Lemma 13.
Let us consider $T \subseteq \mathbb{P}^{\circ}$ *and let* X *and* Y *be two proper intermediate temporal logics satisfying* $\mathrm{X} \subseteq \mathrm{Y}$. *For any temporal theory* Γ, *if* T *is a* X*-temporal safe belief of* Γ, *then* T *is a* Y*-temporal safe belief of* Γ.

Proof.
If T is a X-temporal safe belief of Γ, it follows that

1. $\Gamma \cup \{\circ^i \neg\neg p \mid \circ^i p \in T\} \cup \{\circ^i \neg p \mid \circ^i p \notin T\}$ is X-consistent and
2. $\Gamma \cup \{\circ^i \neg\neg p \mid \circ^i p \in T\} \cup \{\circ^i \neg p \mid \circ^i p \notin T\} \models_{\mathrm{X}} T$

From the first item and Proposition 9 it follows that

$$\Gamma \cup \{\circ^i \neg\neg p \mid \circ^i p \in T\} \cup \{\circ^i \neg p \mid \circ^i p \notin T\}$$

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is Y-consistent. From the second item and Proposition 10 we conclude that

$$\Gamma \cup \{\circ^i \neg\neg p \mid \circ^i p \in T\} \cup \{\circ^i \neg p \mid \circ^i p \notin T\} \models_{\mathrm{Y}} T.$$

As a consequence, T is a Y-safe belief of Γ. □

We prove the converse of Lemma 13 below.

Lemma 14.
Let us consider $T \subseteq \mathbb{P}^{\circ}$ and let X *and* Y *be two proper intermediate temporal logics satisfying* $\mathrm{X} \subseteq \mathrm{Y}$. *For any temporal theory* Γ, *if T is a* Y*-temporal safe belief set of* Γ, *then T is a* X*-temporal safe belief set of* Γ.

Proof.
Let us assume that T is a Y-temporal safe belief of Γ. It holds that

(a) $\Gamma \cup \{\circ^i \neg\neg p \mid \circ^i p \in T\} \cup \{\circ^i \neg p \mid \circ^i p \notin T\}$ is Y-consistent and
(b) $\Gamma \cup \{\circ^i \neg\neg p \mid \circ^i p \in T\} \cup \{\circ^i \neg p \mid \circ^i p \notin T\} \models_{\mathrm{Y}} T$.

From item (a) and Proposition 9 it follows that

$$\Gamma \cup \{\circ^i \neg\neg p \mid p \in T\} \cup \{\circ^i \neg p \mid p \notin T\}$$

is both X-consistent and $\mathrm{ITL}^{\mathrm{BD_n}}$-consistent. From $\mathrm{Y} \subseteq \mathrm{THT}$, the second item and Proposition 10

$$\Gamma \cup \{\circ^i \neg\neg p \mid \circ^i p \in T\} \cup \{\circ^i \neg p \mid \circ^i p \notin T\} \models_{\mathrm{THT}} T. \tag{6}$$

Let $\mathfrak{M} = \langle (W, \preccurlyeq, S), V \rangle$ be an $\mathrm{ITL}^{\mathrm{BD_n}}$ model and $w \in W$ satisfying

(c) $\mathfrak{M}, w \models_{\mathrm{ITL}^{\mathrm{BD_n}}} \Gamma$,
(d) $\mathfrak{M}, w \models_{\mathrm{ITL}^{\mathrm{BD_n}}} \{\circ^i \neg\neg p \mid \circ^i p \in T\}$ and
(e) $\mathfrak{M}, w \models_{\mathrm{ITL}^{\mathrm{BD_n}}} \{\circ^i \neg p \mid \circ^i p \notin T\}$.

From items (d) and (e) it follows that $\mathfrak{M}, w \models \{\Box(\neg\neg p \vee \neg p) \mid p \in \mathbb{P}\}$.

In view of Lemma 10, there exists an $\mathrm{ITL}^{\mathrm{BD_n}}$ model $\mathfrak{M}' = \langle (W', \preccurlyeq', S'), V' \rangle$, $w' \in W'$ and a bisimulation $\mathcal{Z} W \times W'$ such that, $w \mathcal{Z} w'$ and for each $i \geq 0$, the intuitionistic subframe generated by $S'^i(w')$ contains an unique maximal point that we name u'_i. By Lemma 9, $\mathfrak{M}', w' \models_{\mathrm{ITL}^{\mathrm{BD_n}}} \Gamma$, $\mathfrak{M}', w' \models_{\mathrm{ITL}^{\mathrm{BD_n}}} \{\circ^i \neg\neg p \mid \circ^i p \in T\}$ and $\mathfrak{M}', w' \models_{\mathrm{ITL}^{\mathrm{BD_n}}} \{\circ^i \neg p \mid \circ^i p \notin T\}$. We prove the following claim:

$$\text{For all } v' \in W', \text{ if } w' \preccurlyeq v' \text{ then } \mathfrak{M}', v' \models T. \tag{7}$$

The proof is done by induction in $depth(((W', \preccurlyeq'), v'))$.

1. If $depth(((W', \preccurlyeq'), v')) = 1$ then v' is maximal, so $v' = u'_0$. By monotonicity,

$$\mathfrak{M}', u'_0 \models \{\circ^i \neg\neg p \mid \circ^i p \in T \text{ and } i \geq 0\} \cup \{\circ^i \neg p \mid \circ^i p \notin T \text{ and } i \geq 0\}.$$

Because of Proposition 7, $S^i(u'_0)$ is $\preccurlyeq$-maximal, for all $i \geq 0$. Therefore,

$$\mathfrak{M}', S'^i(u'_0) \models \{p \mid \circ^i p \in T\} \cup \{\neg p \mid \circ^i p \notin T\},$$

for all $i \geq 0$. Hence, $V'(S'^i(u'_0)) = \{p \mid \circ^i p \in T\}$, for all $i \geq 0$ so $\mathfrak{M}', u'_0 \models T$.

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2. For the inductive step, let us assume that $depth(((W, \preccurlyeq'), v')) = n + 1$ and the claim holds for every $x \in W'$ satisfying $v' \preccurlyeq x$ and $depth(((W, \preccurlyeq'), x)) \leq n$ (so $v' \prec x$). By Proposition 8, $depth(((W, \preccurlyeq), S'^{i}(v'))) \leq n + 1$, for all $i \geq 0$.
   By induction hypothesis, for all $x \in \prec'(v')$, $\mathfrak{M}', x \models T$. Moreover, by monotonicity, for all $x \in \preccurlyeq'(v')$ (including v' itself), $\mathfrak{M}', x \models \{\circ^i \neg p \mid \circ^i p \notin T\}$.
   By the semantics, for all $i \geq 0$ and for all $y \preccurlyeq' (S'^{i}(v'))$, $\mathfrak{M}', y \models \{\neg p \mid \circ^i p \notin T\}$.
   By the semantics, for all $i \geq 0$ and for all $y \prec' (S'^{i}(v'))$, $\mathfrak{M}', y \models \{p \mid \circ^i p \in T\}$.
   From the two previous results, it follows that for all $i \geq 0$ and for all $y \in \prec'(S'^{i}(v'))$, $V'(y) = \{p \mid \circ^i p \in T\}$. Therefore, every point $y \in \prec'(S'^{i}(v'))$ satisfies exactly the set $\{p \mid \circ^i p \in T\}$. By Lemma 11 there exists a THT model $\mathfrak{M}'' = \langle (\mathbb{N} \times \{0, 1\}, \preccurlyeq'', S''), V'' \rangle$ and a bisimulation $\mathcal{Z} \subseteq W' \times (\mathbb{N} \times \{0, 1\})$ such that $v' \mathcal{Z} (0, 0)$. Because of items (c)–(e), the monotonicity property and Lemma 9,

$$\mathfrak{M}'', (0, 0) \models_{\mathrm{THT}} \Gamma \cup \{\circ^i \neg\neg p \mid \circ^i p \in T\} \cup \{\circ^i \neg p \mid \circ^i p \notin T\}.$$

   Since $\mathfrak{M}''$ is a THT model, $\mathfrak{M}'', (0, 0) \models_{\mathrm{THT}} T$ because of (6). From $v' \mathcal{Z} (0, 0)$ and Lemma 9 it follows $\mathfrak{M}', v' \models_{\mathrm{ITL}^{\mathrm{BD_n}}} T$.

As a consequence, $\mathfrak{M}', w' \models_{\mathrm{ITL}^{\mathrm{BD_n}}} T$. From $w \mathcal{Z} w'$ and Lemma 9, $\mathfrak{M}, w \models_{\mathrm{ITL}^{\mathrm{BD_n}}} T$. Since $\mathfrak{M}$ was chosen arbitrary it follows that

$$\Gamma \cup \{\circ^i \neg\neg p \mid \circ^i p \in T\} \cup \{\circ^i \neg p \mid \circ^i p \notin T\} \models_{\mathrm{ITL}^{\mathrm{BD_n}}} T,$$

so T is a $\mathrm{ITL}^{\mathrm{BD_n}}$-safe belief of Γ. By Lemma 13 and the fact that $\mathrm{ITL}^{\mathrm{BD_n}} \subseteq \mathrm{X}$,

$$\Gamma \cup \{\circ^i \neg\neg p \mid \circ^i p \in T\} \cup \{\circ^i \neg p \mid \circ^i p \notin T\} \models_{\mathrm{X}} T.$$

□

Lemmas 13 and 14 state that THT can be replaced by any proper intermediate temporal logic extending $\mathrm{ITL}^{\mathrm{BD_n}}$ without affecting the resulting safe beliefs.

Theorem 3.
For any intermediate temporal logic X *satisfying* $\mathrm{ITL}^{\mathrm{BD_n}} \subseteq \mathrm{X} \subseteq \mathrm{THT}$ *and for any theory* Γ, *the set of* X*-temporal safe beliefs of* Γ *coincide.*

6 Conclusions

In this paper, we revisited two well-known fixpoint characterizations of propositional equilibrium logic and answer sets. The first characterization, originally defined by Pearce (1999b, 2006) is based on the concept of theory completions, which has also been used in autoepistemic and default logic (Besnard 1989; Marek and Truszczyński 1993). We extended this characterization to the case of TEL.

The second characterization, introduced by Osorio *et al.* (2005) and known as safe belief sets, relates the equilibrium logic of arbitrary theories to syntactic entailment in INT.[10] The authors proved that INT can be replaced by any proper intermediate logic without changing the set of safe beliefs. Their results rely on syntactic transformations

[10] Similar results were also obtained by David Pearce for the case of disjunctive logic programs (Pearce 1999a).

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that cannot be easily reproduced in the temporal case. Therefore, as a first contribution, we reformulated Osorio et al.'s approach in terms of semantic consequence in INT.

We have identified a family of intuitionistic temporal logics, $\mathrm{ITL}^{\mathrm{BD_n}}$, for which we have defined a temporal extension of safe beliefs. We first show that any proper intermediate temporal logic extending $\mathrm{ITL}^{\mathrm{BD_n}}$ can be used instead, without affecting the resulting set of temporal safe beliefs. Moreover, we show that in the case of THT, temporal safe beliefs correspond to temporal equilibrium models.

We believe our results have fostered connections between temporal answer set programming and constructive modal logic, while also enhancing the visibility of THT within the field of constructive temporal logics. In future work, we plan to investigate intermediate logics not covered here, such as $\mathrm{ITL}^{\mathrm{e}}$, $\mathrm{ITL}^{\mathrm{p}}$, and real-valued Gödel temporal logics (Aguilera *et al*. 2025). Since consistency is not always preserved across these logics, it remains unclear whether the set of safe beliefs is preserved when using one of them as the monotonic basis for temporal equilibrium logic.

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Competing interests

The authors declare none.

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