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Compressive and rarefactive ion-acoustic solitons in a two component electron plasma*

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Abstract. In this paper an analysis is given for the conditions under which compressive and rarefactive ion-acoustic solitons may exist in a plasma containing two electron populations, one hot and the other cool. Compressive solitons exist in the collective Mach number range $1 < M < M_c$, which, for a given abundance ratio of the cool population, translates into an annular region in the ion-acoustic Mach numbers space. The upper limit M_c corresponds to a degenerate sonic point for the protons in which they tend to an infinite compression and are brought to rest at the wave centre. To form a charge neutral point in the rarefactive regime (a prerequisite to the existence of rarefactive solitons) it is necessary that the 'scale height' of the hot electrons be sufficiently large that their number density, which is nearly constant, approximately balances that of the protons, before the cool electrons experience their 'atmospheric lid'. Furthermore, a sufficient condition for the existence of rarefactive solitons is found by requiring a solution to the wave structure equation at the 'lid'. The case of very hot electrons (highly sub-ion-acoustic) is considered, and various parameter regimes in which compressive and rarefactive solitons exist are evaluated in this super-hot electron approximation, both for the polytropic case, $\gamma \neq 1$, and the isothermal case, $\gamma = 1$. This approximation is formally equivalent to the problem of a bi-ion plasma, including a super-massive negative species.

1. Introduction

Over the last few years there have been a number of studies into the properties of fully nonlinear stationary waves in plasmas (e.g. McKenzie et al. 2004a; Verheest et al. 2004). These studies, which use a fluid dynamic viewpoint that emphasizes the nature of the flow patterns of the species in the various types of waves, provide further insight into their properties, and highlight the critical role played by the sonic points of the species at which the flow becomes choked. (It is this trans-sonic

* It is a pleasure to dedicate this paper to Alan Cairns on the occasion of his 60th birthday.

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choked flow feature that limits the amplitude of solitons.) Using this approach, McKenzie and Doyle (2003) have provided a unified fluid dynamic viewpoint for the dust-acoustic, ion-acoustic and Langmuir-acoustic solitons, and more recently McKenzie et al. (2004b) have provided an analysis of the parameter regimes in which compressive and rarefactive ion-acoustic solitons exist in a bi-ion plasma. In this work we adopt a similar approach to the problem of compressive and rarefactive ion-acoustic solitons in a plasma with two electron populations, one hot and the other cool.

Buti (1980), Nishihara and Tajiri (1981) and Tajiri and Nishihara (1985) have previously studied ion-acoustic solitons in a two-electron-temperature plasma, including rarefactive solitons. These are characterized both by a dip in the density of all species and a negative electrostatic potential.

The numerical studies of Baboolal et al. (1990) and related work, reviewed by Hellberg et al. (1992), included a comprehensive parameter survey, setting out existence domains and cut-off criteria for both rarefactive and compressive ion-acoustic solitons in two-electron plasmas. These were presented as curves in the two-dimensional parameter space of normalized soliton potential and cool electron density fraction, Mach number being an additional parameter defining the curves.

Studying an isothermal plasma ($\gamma = 1$), Baboolal et al. (1990) showed that a cut-off for rarefactive solitons occurred when double layers were formed, with the double layer amplitude rising approximately linearly with the cool electron fraction, $f = n_{c0}/n_{p0}$, and a further cut-off was reported to occur for values of f and amplitude ϕ for which the cool electron density $n_c \to 0$. Importantly, it was found that the region in parameter space (ϕ, f) in which rarefactive solitons exist decreased with decreasing electron temperature ratio (T_h/T_c), and vanished for T_h/T_c somewhere between 5 and 15. This was consistent with the result of Bezzerides et al. (1978) for rarefaction shocks in such a plasma, which were shown to have a cut-off at $T_h/T_c \simeq 10$. However, the physical mechanism underlying the existence criteria for rarefactive solitons was not totally transparent.

Based on these results, we here consider the simplification that the hot electrons are 'super-hot', and show that in order to create a rarefactive charge neutral point (CNP) (out of equilibrium), which is a prerequisite for the formation of a rarefactive soliton, the 'scale height' of the hot electron population needs to be sufficiently large to balance the 'gentle decay' of the proton population before the cool electron population experiences its 'atmospheric lid'. The sufficient condition for existence of rarefactive solitons is found from the solution to the wave structure equation at the 'lid'.

It is remarkable that in this highly subsonic (or 'super-hot') electron approximation the analysis is formally analogous to that of a bi-ion plasma in which one ion species is cold, negatively charged and 'super-massive'.

2. The wave structure equation (general case)

The structure of a one-dimensional, electrostatic, progressive wave of the form $f(x + Ut)$ in a multi-component plasma is governed by conservation of total momentum flux of the system and the various Bernoulli-like energy equations for each of the constituents. In the wave frame (where $\partial/\partial t = 0$) conservation of momentum flux, in which total (dynamic plus thermal) plasma pressure is balanced

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by the electric stresses, may be expressed (e.g. McKenzie and Doyle 2003) by

$$\frac{\epsilon_0 E^2}{2} \equiv \frac{\epsilon_0}{2}\left(\frac{d\phi}{dx}\right)^2 = U^2 \sum_j m_j n_{j0} P_j(u_j), \tag{2.1a}$$

in which E and ϕ are the electric field and potential, respectively, and U is the wave speed. The sum is taken over all species, j. The normalized individual species momentum flux, $P_j(u_j)$, is given by

$$P_j(u_j) = (u_j - 1) + \frac{1}{\gamma_j M_j^2}\left(\frac{1}{u_j^{\gamma_j}} - 1\right), \tag{2.1b}$$

where the normalized species flow speed $u_j \equiv u_{jx}/U$. Adiabatic flow has been assumed, i.e. $p_j \propto n_j^{\gamma_j} \propto u_j^{-\gamma_j}$, and the species Mach number M_j is defined as

$$M_j^2 = \frac{U^2 m_j}{\gamma_j k T_j}, \tag{2.1c}$$

where T_j is the species temperature. The first term on the right-hand side of (2.1b) is the normalized change in the dynamic pressure and the second term is the change in thermal pressure. The equations of motion for each species readily integrate to yield Bernoulli-like energy equations which, in normalized form, may be written as

$$\frac{m_j}{Z_j}\epsilon_j(u_j) = m_{\rm p}\epsilon_{\rm p}(u_{\rm p}) = -m_{\rm e}\epsilon_{\rm ec}(u_{\rm ec}) = -m_{\rm e}\epsilon_{\rm eh}(u_{\rm eh}) = -e\phi/U^2, \tag{2.2a}$$

in which, for species j, Z_j is the charge number and m_j the mass, the subscripts p, ec, eh relate to the protons, cool and hot electrons, respectively, and $\epsilon_j(u_j)$ is the normalized Bernoulli energy, given by

$$\epsilon_j(u_j) = \frac{1}{2}\left(u_j^2 - 1\right) + \frac{1}{(\gamma_j - 1)M_j^2}\left(\frac{1}{u_j^{\gamma_j - 1}} - 1\right). \tag{2.2b}$$

Here the first term on the right-hand side is the change in kinetic energy while the second term is the change in the enthalpy. Equations (2.2) lend themselves to the graphical interpretation shown in Fig. 1, which indicates whether a species is accelerated or decelerated according as to whether its flow speed is 'supersonic' ($M_j > 1$) or 'subsonic' ($M_j < 1$), the sign of its charge and the nature of the potential (hill or dip). Note that the energy function exhibits a minimum at the species sonic point, $u_j = 1/M_j^{2/(\gamma_j+1)}$. It is this choked flow feature which, in principle, restricts the maximum amplitude of the wave and thereby places constraints on the collective Mach number for the formation of ion-acoustic solitons.

3. Cold protons and two electron components

Let us now consider the case of a plasma consisting of a cold ($M_{\rm p} \to \infty$) proton population and two electron populations, one hot (h) and the other cool (c), but both sufficiently subsonic ($M_{\rm h,c} \ll 1$) to permit the neglect of their dynamic pressures relative to their thermal pressures. In this situation the Bernoulli momentum and

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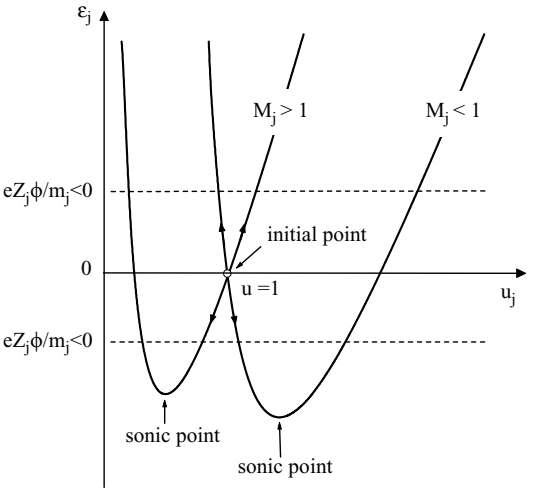


Figure 1. The behaviour of the energy function ϵ as a function of u_j for supersonic ($M_j > 1$) and subsonic ($M_j < 1$) initial conditions. If $M_j > 1$, positively (negatively) charged particles are decelerated (accelerated) in a potential hill ($\phi > 0$), which is depicted as intersections between horizontal straight lines and the energy function curve. If $M_j < 1$, positively (negatively) charged particles are accelerated (decelerated) in a potential hill ($\phi > 0$). Note that the energy function attains a minimum value where the flow speed becomes sonic.

energy functions for the three species become:

$$P_{\rm p}(u_{\rm p}) = u_{\rm p} - 1, \quad \epsilon_{\rm p}(u_{\rm p}) = \frac{1}{2}\left(u_{\rm p}^2 - 1\right) = -\frac{e\phi}{m_{\rm p}U^2} = -\phi_{\rm NORM} \tag{3.1a}$$

$$\frac{m_{\rm e}}{m_{\rm p}} P_{{\rm e}j}(u_{{\rm e}j}) = \frac{1}{\gamma_{{\rm e}j} M_{{\rm p}j}^2}\left(\frac{1}{u_{{\rm e}j}^{\gamma_{{\rm e}j}}} - 1\right) \quad (j = {\rm h}, {\rm c}), \tag{3.1b}$$

$$\frac{m_{\rm e}}{m_{\rm p}} \epsilon_{{\rm e}j}(u_{{\rm e}j}) = \frac{1}{(\gamma_{{\rm e}j} - 1) M_{{\rm p}j}^2}\left(\frac{1}{u_{{\rm e}j}^{\gamma_{{\rm e}j}-1}} - 1\right) = \frac{e\phi}{m_{\rm p}U^2} = \phi_{\rm NORM}, \tag{3.1c}$$

Here $M_{{\rm p}j}$, defined by

$$M_{{\rm p}j}^2 = \frac{m_{\rm p}U^2}{\gamma_{{\rm e}j} k T_{{\rm e}j}}, \tag{3.1d}$$

is the ion-acoustic Mach number appropriate to the hot ($j = {\rm h}$) or cool ($j = {\rm c}$) electron species with temperatures $T_{\rm h}$ or $T_{\rm c}$, respectively. Substituting (3.1a)–(3.1c) for the momentum functions into the conservation of total momentum (2.1a) gives the wave structure equation, which may be written as the following differential equation for ϕ, namely

$$\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 = R(\phi), \tag{3.2a}$$

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with the wave structure function

$$R(\phi) = (\sqrt{1-2\phi} - 1) + \frac{(1-f)}{\gamma_{\rm h} M_{\rm ph}^2}\left\{\left[1 + (\gamma_{\rm h} - 1)M_{\rm ph}^2\phi\right]^{\gamma_{\rm h}/(\gamma_{\rm h}-1)} - 1\right\}$$

$$+ \frac{f}{\gamma_{\rm c} M_{\rm pc}^2}\left\{\left[1 + (\gamma_{\rm c} - 1)M_{\rm pc}^2\phi\right]^{\gamma_{\rm c}/(\gamma_{\rm c}-1)} - 1\right\}, \tag{3.2b}$$

in which x is normalized to $U/\omega_{\rm p}$, where $\omega_{\rm p}$ is the proton plasma frequency, f is the fraction of cool electrons to protons, $n_{\rm ec0}/n_{\rm p0}$, so that $1-f$ is the fraction of hot electrons to protons, $n_{\rm eh0}/n_{\rm p0}$. Here the relation for $\omega_{\rm p}$ is given by

$$\omega_{\rm p}^2 = n_{\rm p0} e^2/\epsilon_0 m_{\rm e}. \tag{3.3}$$

The quantity $R(\phi)$ is the negative of what is often called the Sagdeev, or pseudo-potential, since (3.2a) is structurally similar to that of a particle moving in a 'potential well', of the form $V = -R(\phi)$. However, we prefer to call $R(\phi)$ the plasma momentum function, which, in fact, it is, with the first term on the right-hand side of (3.2b) representing the proton dynamic pressure, and the second and third terms representing, respectively, the hot and cool electron thermal pressures. All of these terms are expressed as functions of ϕ through the energy equations (3.1a)–(3.1d).

Soliton pulse-like solutions of (3.2b) necessarily require $R(\phi)$ to have a double, positive, zero at the initial point ($x = -\infty$) where $\phi = 0$ and the normalized speeds, u_j, are unity. This requirement is equivalent to the linear stationary waves being of an evanescent character. Thus, it may be stated that solitons propagate in regimes in which linear waves cannot propagate. The centre of the pulse, $\phi = \phi_{\rm m}$, is given by a simple zero of $R(\phi)$, if it exists, at which point $d\phi/dx = 0$. This, however, implies that $R(\phi)$ must exhibit a maximum somewhere in the transition, where $\partial R/\partial\phi = 0$, namely where

$$\frac{1}{\sqrt{1-2\phi}} = (1-f)\left[1 + (\gamma_{\rm h} - 1)M_{\rm ph}^2\phi\right]^{1/(\gamma_{\rm h}-1)} + f\left[1 + (\gamma_{\rm c} - 1)M_{\rm pc}^2\phi\right]^{1/(\gamma_{\rm c}-1)}. \tag{3.4}$$

This equation defines the CNP (charge neutral point) ($\phi_{\rm CNP}$) in the transition, with the left-hand side representing the proton number density, and the right-hand side representing the sum of the hot and cool electron number densities. Expansion of $R(\phi)$ near $\phi = 0$ immediately yields the necessary condition for a double positive zero, as

$$M^2 \equiv M_{\rm ph}^2(1-f) + M_{\rm pc}^2 f > 1, \tag{3.5}$$

which also serves to define the collective Mach number M of the wave which, therefore, must be supersonic in this, the 'collective' sense. That is, for a soliton to exist, the speed of the structure must exceed the ion-acoustic speed for a plasma with effective temperature $T_{\rm eff}$, given by

$$\frac{1}{T_{\rm eff}} = \frac{f}{T_{\rm c}} + \frac{1-f}{T_{\rm h}}. \tag{3.6}$$

From (3.5) it follows that the collective sonic curve, $M = 1$, is an ellipse in $(M_{\rm ph}, M_{\rm pc})$ space for given f, as shown in Fig. 2.

The physical significance of (3.4) for $\phi_{\rm CNP}$ is that it gives the potential at the CNP in the transition at which the electric stresses must maximize before reaching the centre of the wave. An examination of the CNP equation (3.4) in the compressive regime $\phi > 0$, in which the protons are decelerated, shows that there is always a

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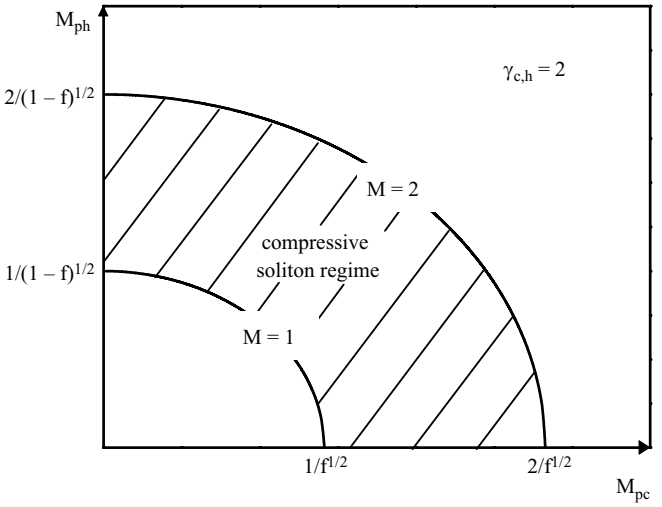


Figure 2. The compressive soliton regime annular region between the two ellipses $M = 1$ and 2 in $(M_{\rm ph}, M_{\rm pc})$ space for the special case $\gamma_{\rm c,h} = 2$.

CNP in the regime $0 < \phi < \frac{1}{2}$, subject to the necessary condition (3.5) being satisfied. This occurs because although the proton number density initially increases more slowly than the sum of the electron densities, the protons undergo an infinite compression as the limit $\phi \to \frac{1}{2}$ is approached, thereby ensuring the formation of a compressive CNP.

A second requirement for the formation of a compressive soliton is that $R(\phi)$ does, actually, possess a simple zero in the compressive range which is limited to $\phi \leqslant \frac{1}{2}$, where the upper limit $\frac{1}{2}$ corresponds to the proton fluid being brought to a complete standstill (an infinite compression, $u_{\rm p} = 0$) at the wave centre. Thus we also require that $R(\frac{1}{2}) \leqslant 0$, which yields

$$\frac{(1-f)}{\gamma_{\rm h} M_{\rm ph}^2}\left\{\left[1+\frac{(\gamma_{\rm h}-1)}{2}M_{\rm ph}^2\right]^{\gamma_{\rm h}/(\gamma_{\rm h}-1)}-1\right\} + \frac{f}{\gamma_{\rm c} M_{\rm pc}^2}\left\{\left[1+\frac{(\gamma_{\rm c}-1)}{2}M_{\rm pc}^2\right]^{\gamma_{\rm c}/(\gamma_{\rm c}-1)}-1\right\} \leqslant 1. \quad (3.7)$$

For given values of $\gamma_{\rm h}, \gamma_{\rm c}$ and f this equation defines an 'elliptical' curve in the Mach numbers $(M_{\rm ph}, M_{\rm pc})$ parameter space in the same way as the necessary condition (3.5). Compressive solitons exist in the annulus of this parameter space lying between these two curves (see, for example, Fig. 2 for the special case of $\gamma_{\rm c,h} = 2$).

In the rarefactive regime, $\phi < 0$, in which the protons are accelerated, it is not immediately obvious under which conditions it is possible to form a rarefactive root of the CNP equation (3.4). However, note that for a general value of $\gamma_{\rm c,h} \neq 1$, both electron populations exhibit a classical 'atmospheric lid' at which their densities drop to zero at $\phi = -1/(\gamma_j - 1)M_{{\rm p}j}^2$. This is a characteristic feature of all polytropic density fall-offs under a conservative force, such as gravity or, as in this case, an electrostatic field. For $\gamma_{\rm c,h} = 1$, an exact 'lid' is not achieved, but as observed by

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Baboolal et al. (1990), the electron number density $n_{ec} \to 0$ yields an apparent cut-off, which they took to occur at $n_{ec} \leqslant 10^{-4}$.

Therefore, this property indicates that favourable conditions for the formation of a rarefactive CNP will occur if the 'scale height' of the hot electron component is sufficiently large (requiring $M_{pj}^2 \ll 1$, i.e. highly 'subsonic') to enable the hot electron population to balance the more gentle decline of the proton population, which goes as $(-\phi)^{1/2}$ for large negative ϕ, before the cool electron population experiences its lid.

This is consistent with earlier observations (Baboolal et al. 1990; Nishihara and Tajiri 1981; Bezzerides et al. 1978) that required $T_h \gg T_c$ for the existence of rarefactive solitons.

In order to illustrate this requirement we consider the extreme case of one population of the electrons being 'very hot', that is to say $M_{ph}^2 \to 0$. In this case, the plasma momentum function $R(\phi)$ becomes

$$R(\phi) = \sqrt{1-2\phi} - 1 + (1-f)\phi + \frac{f}{\gamma_c M_{pc}^2}\left\{\left[1 + (\gamma_c - 1)M_{pc}^2\phi\right]^{\gamma_c/(\gamma_c-1)} - 1\right\}, \quad (3.8)$$

and the CNP equation (3.4) may be written

$$\frac{1}{f}\left[\frac{1}{\sqrt{1-2\phi}} - (1-f)\right] = \left[1 + (\gamma_c - 1)M_{pc}^2\phi\right]^{1/(\gamma_c-1)}. \quad (3.9)$$

This latter equation immediately shows that a necessary condition for the formation of a rarefactive CNP(charge neutral point) is

$$\frac{1}{(\gamma_c - 1)M_{pc}^2} > \frac{1}{2}\left[\frac{1}{(1-f)^2} - 1\right], \quad (3.10a)$$

or, equivalently,

$$M_{pc}^2 < \frac{2}{(\gamma_c - 1)}\frac{(1-f)^2}{f(2-f)}. \quad (3.10b)$$

We emphasize that relations (3.10) provide a necessary condition.

Returning to (3.9), we note that the right-hand side represents the cool electron density, n_{ec}. At the lid, $\phi = \phi_L = -1/(\gamma_c - 1)M_{pc}^2$, the cool electron density goes to zero. Thus a rarefactive soliton will only exist if $R(\phi) = 0$ for negative potentials satisfying $|\phi| < |\phi_L|$, and thus the sufficient condition may be written as $R(\phi_L) \leqslant 0$. Substitution of ϕ_L in (3.8) reduces the cool electron term to $-f/\gamma_c M_{pc}^2$. Hence, $R(\phi_l) \leqslant 0$ yields

$$M_{pc}^2 \leqslant M_r^2 \equiv \frac{(\gamma_c - f)^2}{2\gamma_c(\gamma_c - 1)f}. \quad (3.11)$$

However, we also require the collective supersonic condition, relation (3.5), which, in this 'super-hot' electron population approximation, becomes

$$M_{pc}^2 > \frac{1}{f}. \quad (3.12)$$

The simultaneous satisfaction of the inequalities (3.11 and 3.12) requires $\gamma_c < 2$, and that the fraction of the cool electron population is less than a critical abundance f_c given by

$$f_c = \gamma_c - \sqrt{2\gamma_c(\gamma_c - 1)}. \quad (3.13)$$

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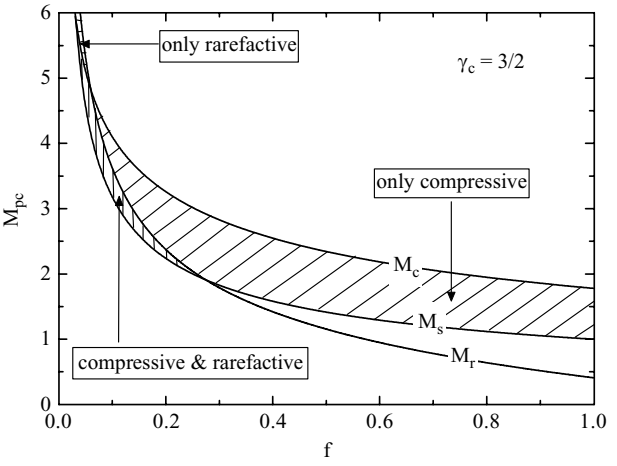


Figure 3. Regions in the parameter space (M_{pc}, f) in which the compressive and rarefactive solitons may exist for a plasma with $\gamma_c = 1.5$.

For the purpose of illustration we consider $\gamma_c = \frac{3}{2}$, in which case the various critical Mach numbers become

$$M_{pc}^2 > M_s^2 = \frac{1}{f}, \tag{3.14a}$$

$$M_{pc}^2 \leqslant M_c^2 = 6(\sqrt{1 + 4/3f} - 1), \tag{3.14b}$$

$$M_{pc}^2 \leqslant M_r^2 = \frac{3}{2f}\left(1 - \frac{2f}{3}\right)^2 \tag{3.14c}$$

and the critical abundance $f_c \simeq 0.28$. These expressions define the various regions of the parameter space (M_{pc}, f) in which the two types of solitons (compressive and rarefactive) may exist, as shown in Fig. 3.

Remarkably, it is found that in this highly subsonic (or 'super-hot') electron approximation the analysis is formally analogous to that of a bi-ion plasma in which one ion species is cool, negatively charged and super-massive (McKenzie et al. 2004b). Mathematically, this is the case because in both situations an important component in the wave structure equation is linearized. In the present problem, it is the fractional power-law term, describing the behaviour of the 'super-hot' electrons, while in the case of the bi-ion plasma, a square-root term describing the 'super-massive' ion component is linearized. This formal similarity will be pursued in another paper.

In the bi-ion case the massive negative ions, although undergoing almost imperceptible motions, nevertheless contribute finite dynamic pressure to the total plasma momentum function. Similarly, in this case the very hot electrons, although undergoing infinitesimal compression and expansion, do contribute finite thermal pressure that must be included in the total momentum function $R(\phi)$. In both cases, the number density of the special species (super-hot electrons or super-massive negative ions) remains approximately constant, thereby permitting the formation of a rarefactive CNP in certain parameter regimes, and hence solitons, as shown in Fig. 3.

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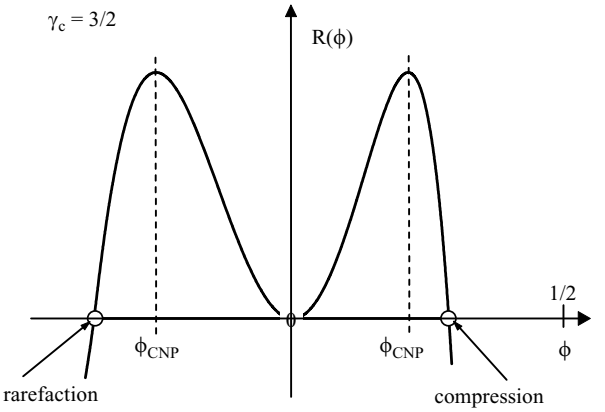


Figure 4. An example of the plasma momentum function $R(\phi)$ possessing both compressive and rarefactive points.

For the illustrative case ($\gamma_c = \frac{3}{2}$), Fig. 4 provides a plot of $R(\phi)$ for parameters in (M_{pc}, f) space appropriate to the existence of both compressive and rarefactive solitons.

4. Special cases of γ_c

4.1. The case $\gamma_c = 2$

The case $\gamma_c = \gamma_h = 2$ is particularly simple, with the electron expressions in $R(\phi)$ reducing to quadratic form. Hence it is easy to show that the upper limit on the collective Mach number, M_c, governing the existence of compressive solitons, which is given by equation (3.7), simplifies to

$$\frac{M_c^2}{4} \equiv \frac{M_{ph}^2(1-f) + fM_{pc}^2}{4} = 1. \tag{4.1a}$$

Hence, compressive solitons exist in the region

$$1 < M < 2, \tag{4.1b}$$

which, in the parameter space of (M_{pc}, M_{ph}), is the region bounded by the two ellipses as shown in Fig. 2. In this case the plasma momentum function becomes

$$R(\phi) = \sqrt{1-2\phi} - 1 + \phi + \frac{\phi^2}{2}M^2, \tag{4.2}$$

which shows explicitly that for $M > 1$, the CNP (at $R'(\phi) = 0$) is necessarily compressive, with the corresponding potential ϕ_{CNP} given by,

$$\phi_{CNP} = \frac{1}{4}\left[1 - \frac{4}{M^2} + \left(1 + \frac{8}{M^2}\right)^{1/2}\right], \quad M > 1. \tag{4.3}$$

In fact, the wave structure equation assumes a more transparent form if cast in terms of the proton speed u_p. Using normalized forms of (2.1) and (2.2) one can show that (3.2) simplifies to

$$\frac{d\phi}{dx} = -u_p\frac{du_p}{dx} = \pm(u_p - 1)\sqrt{\frac{M^2}{4}(u_p+1)^2 - 1}, \tag{4.4}$$

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which integrates in terms of elementary transcendental functions to provide the fully nonlinear counterpart of the sech^2 pulses associated with weakly nonlinear, Korteweg–de Vries (KdV), solitons. It is immediately clear that at the centre of the wave the protons undergo a compression of strength

$$u_{\mathrm{p,eq}} = \frac{2}{M} - 1, \tag{4.5}$$

which is accompanied by a potential hill of amplitude,

$$\phi_{\mathrm{m}} = \frac{2}{M}\left(1 - \frac{1}{M}\right), \tag{4.6}$$

in both of which equations M is confined by the inequality (4.1b). The signature of this ion-acoustic soliton is that of a potential hill accompanied by corresponding compressions in the protons and the electrons (3.1), which are slowed to

$$u_{\mathrm{e}j} = \frac{1}{\left(1 + M_{\mathrm{p}j}^2\phi_{\mathrm{m}}\right)} \quad (j = \mathrm{c}, \mathrm{h}) \tag{4.7}$$

at the centre of the wave. These properties are depicted in Fig. 5.

4.2. The 'Boltzmann', isothermal, case, $\gamma_{\mathrm{c}} = 1$

Although some of what has gone before also applies to the 'Boltzmann' case of isothermal electrons ($\gamma = 1$), a number of the expressions obtained are not valid for that special case.

In the case of isothermal electrons ($\gamma_{\mathrm{c}} = \gamma_{\mathrm{h}} = 1$,) the plasma momentum function takes the form

$$R(\phi) = \sqrt{1 - 2\phi} - 1 + \frac{(1-f)}{M_{\mathrm{ph}}^2}\left[e^{M_{\mathrm{ph}}^2\phi} - 1\right] + \frac{f}{M_{\mathrm{pc}}^2}\left[e^{M_{\mathrm{pc}}^2\phi} - 1\right]. \tag{4.8}$$

Thus, the upper limit on the collective Mach number, defined by $R(\frac{1}{2}) = 0$, for compressive solitons, is now given by

$$\frac{(1-f)}{M_{\mathrm{ph}}^2}\left[e^{\frac{1}{2}M_{\mathrm{ph}}^2} - 1\right] + \frac{f}{M_{\mathrm{pc}}^2}\left[e^{\frac{1}{2}M_{\mathrm{pc}}^2} - 1\right] = 1. \tag{4.9}$$

Note that if there is only one electron species (either $f = 0, 1$), this reduces to the classical result

$$e^{\frac{1}{2}M_c^2} = 1 + M_{\mathrm{c}}^2. \tag{4.10}$$

Thus, as is well known, compressive solitons exist in a simple plasma only in the region $1 < M < M_{\mathrm{c}} \simeq 1.6$.

In a plasma with two electron components, for given f, (4.9) defines an 'elliptical' curve in $(M_{\mathrm{ph}}, M_{\mathrm{pc}})$ space and compressive solitons exist in the region bounded by the two 'ellipses'.

In the case of 'super-hot' electrons, $M_{\mathrm{ph}} \to 0$, the upper critical Mach number for compressive solitons satisfies

$$e^{\frac{1}{2}M_{\mathrm{pc}}^2} = 1 + \frac{M_{\mathrm{pc}}^2}{2}\left(1 + \frac{1}{f}\right), \tag{4.11a}$$

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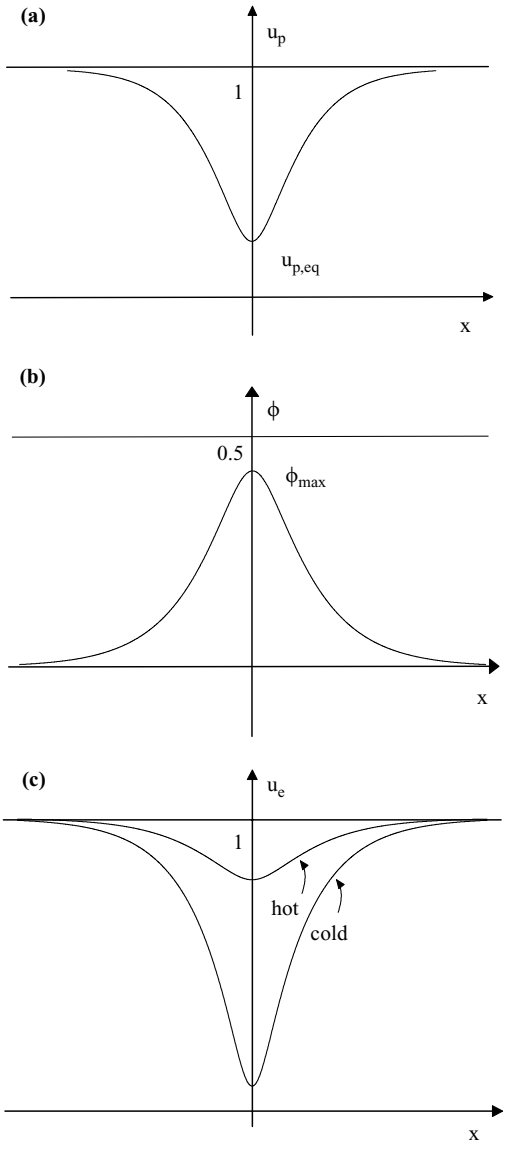


Figure 5. Signature of a compressive ion-acoustic soliton in a plasma with Boltzmann electrons.

and the soliton condition is, as before,

$$M^2 > M_s^2 = \frac{1}{f}. \tag{4.11b}$$

In this isothermal case the upper limit restriction M_r^2 for rarefactive solitons (3.11) falls away and, subject to the soliton condition $M > M_s$, there is always a rarefactive, as well as a compressive, CNP, thereby allowing the formation of both rarefactive and compressive ion-acoustic solitons in the regime shown in Fig. 6.

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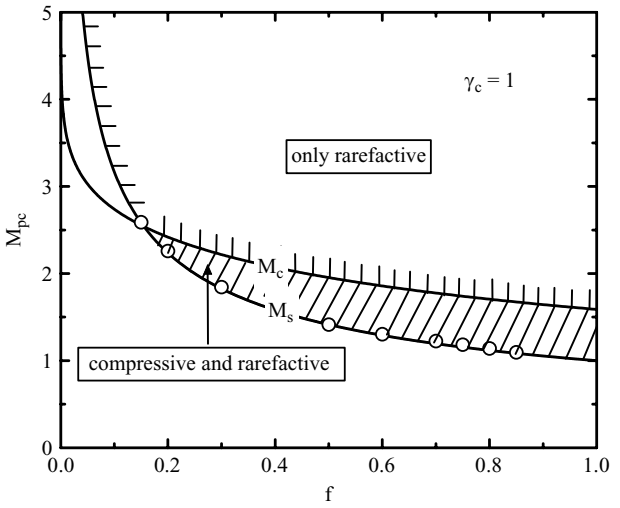


Figure 6. Regions in the parameter space $(M_{\rm pc}, f)$ in which the compressive and rarefactive solitons may exist for $\gamma_c = 1$. The open circles correspond to the minimum values of $M_{\rm pc}$ for given values of f in (4.12), which yield the rarefactive equilibrium condition (see text).

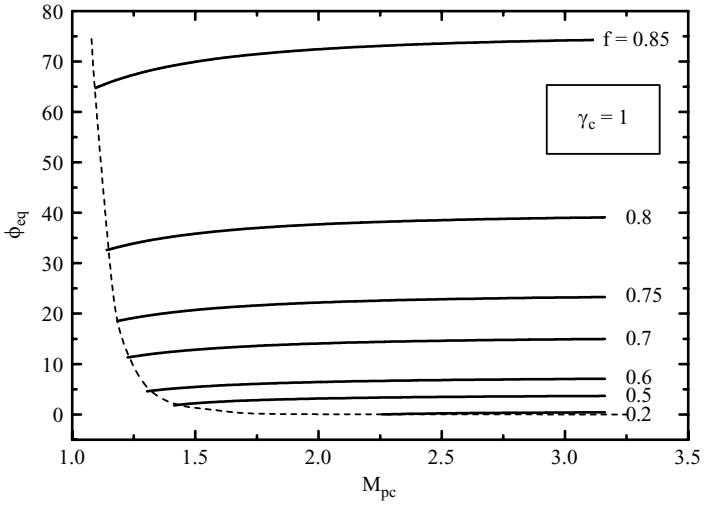


Figure 7. Curves for $\phi_{\rm eq}(M_{\rm pc})$, for various values of f, which satisfy the rarefactive equilibrium condition $R(\phi) = 0$ (4.12). The dashed curve is the loci of the points $\phi_{\rm eq}(M_{\rm pc})$, for each f, at which $M_{\rm pc}$ has its least value.

This has already been pointed out by, for example, Baboolal et al. (1990) and others in their numerical studies.

Assuming that the hot electrons are sufficiently hot to justify an expansion in the product $\phi M_{\rm ph}^2$, the plasma momentum function may be approximated by

$$R(\phi) = \sqrt{1-2\phi} - 1 + (1-f)\phi + \frac{f}{M_{\rm pc}^2}\left(e^{M_{\rm pc}^2\phi} - 1\right). \tag{4.12}$$

Curves for $\phi_{\rm eq}(M_{\rm pc})$, for various values of f, which satisfy the rarefactive equilibrium condition $(R(\phi) = 0)$ using (4.12), are shown in Fig. 7. The loci of the points

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$\phi_{\rm eq}(M_{\rm pc})$ for each f at which $M_{\rm pc}$ has its least value (indicated by the dashed curve in Fig. 7) are plotted in Fig. 6 (open circles). These points fall close to the curve $M_{\rm s}$, thereby justifying the expansion approximation used in (4.12) in this regime. It should be noted that this approximation may be less reliable for $\phi \simeq 30$ and above.

5. Summary and discussion

Our analysis shows that it is possible to construct rarefactive, as well as compressive, ion-acoustic solitons in a plasma containing two electron populations, one 'hot' and the other 'cool', but both of which are 'subsonic' in the sense that their thermal pressures dominate over their dynamic pressures. The key to the existence of both types of soliton lies in recognizing which conditions favour the formation of a CNP in the transition. This point, where the electric stresses maximize, must be reached before the centre of the wave. In both cases the necessary condition (3.5) is a collective supersonic condition which ensures that the plasma momentum function has a double positive zero at the beginning of the wave transition, and is also equivalent to the linear stationary waves of the system being of an evanescent character.

For compressive solitons there exists an upper limit to the collective Mach number, which, in the case studied here of cold protons, is given by the requirement that the plasma momentum function does actually possess a simple zero in the compressive range $0 < \phi < \frac{1}{2}$, where the upper limit corresponds to an infinite proton compression. It is this feature that always ensures the existence of a CNP in the compressive regime, because although the proton number density initially increases more slowly than the sum of the electron number densities, the proton density ultimately overtakes the electron density. This condition $R(\frac{1}{2}) = 0$ yields an elliptical curve in the Mach number space $(M_{\rm ph}, M_{\rm pc})$, for given parameters $(\gamma_{\rm c}, \gamma_{\rm h}, f)$, and compressive solitons exist in the annulus between this curve and that defined by the necessary condition. The neat analytic solution for the special case of $\gamma_{\rm c,h} = 2$, given in Sec. 4, embodies the essential properties of compressive ion-acoustic solitons.

In seeking conditions that are favourable for the formation of a rarefactive charge neutral point we note that for a general value of γ_j the electron populations exhibit a classical 'atmospheric lid' at which their densities drop to zero at a (negative) potential $\phi_{\rm L} = -1/((\gamma_j - 1)M_{{\rm p}j}^2)$, which may be regarded as the polytropic scale height. It is necessary, therefore, that the hot electron scale height be sufficiently large (implying $M_{\rm ph}^2$ sufficiently small) to approximately balance the rather gentle decay ($\propto 1/\sqrt{-\phi}$) of the proton population before the cool electron population experiences its 'lid' at $\phi_{\rm L}$. In addition to this condition on the existence of a charge neutral point, it is further necessary that the total plasma momentum function $R(\phi)$ should vanish for a value of potential such that $|\phi| \leqslant |\phi_{\rm L}|$. This is a more restrictive condition than the requirement of the existence of a negative charge neutral point. In the case of extremely hot electrons ($M_{\rm ph} \to 0$) we have obtained the parameter regime, in $(M_{\rm pc}, f)$ space, in which rarefactive solitons may be formed. The various parameter regimes in which the two types of solitons may exist are shown in Figs. 3 (for $\gamma_{\rm c} = \frac{3}{2}$) and 6 (for $\gamma_{\rm c} = 1$).

It is, at first sight, surprising that the highly subsonic hot electron approximation carried out here is entirely analogous to the case of ion-acoustic solitons in a

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bi-ion plasma, in which one of the ion species is massive and negatively charged (see McKenzie et al. (2004b)). Although the massive negative ions undergo imperceptible motions in this latter case (and their density is very nearly constant), they nevertheless contribute finite momentum, through their dynamic pressure, to the total plasma momentum function. Similarly, in the case studied here of very hot electrons, although undergoing imperceptible squeezing and rarefying, they contribute to the total momentum through the change in their thermal pressure. It is these conditions (in this very hot electron approximation) which permit the existence of rarefactive ion-acoustic solitons in the parameter regime shown in Figs 3 and 6.

A more general analysis is underway, extending these results to non-zero values of $M_{\rm ph}$, as well as seeking limits associated with double layers. In particular, a critical Mach number value for $M_{\rm ph}$, above which rarefactive solitons cannot be constructed, has been found.

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