Math. Struct. in Comp. Science (2015), *vol.* 25, *pp.* 1466–1483. © Cambridge University Press 2014
doi:10.1017/S0960129513000236 First published online 10 November 2014

Generalized geometric theories and set-generated classes

PETER ACZEL[†], HAJIME ISHIHARA[‡], TAKAKO NEMOTO[‡] and YASUSHI SANGU[‡]

[†]*Schools of Mathematics and Computer Science, University of Manchester, Manchester, M13 9PL, United Kingdom*
[‡]*School of Information Science, Japan Advanced Institute of Science and Technology, Nomi, Ishikawa 923-1292, Japan*

Received 28 March 2012

We introduce infinitary propositional theories over a set and their models which are subsets of the set, and define a generalized geometric theory as an infinitary propositional theory of a special form. The main result is that *the class of models of a generalized geometric theory is set-generated*. Here, a class $\mathcal{X}$ of subsets of a set is set-generated if there exists a subset G of $\mathcal{X}$ such that for each $\alpha \in \mathcal{X}$, and finitely enumerable subset τ of α there exists a subset $\beta \in G$ such that $\tau \subseteq \beta \subseteq \alpha$. We show the main result in the constructive Zermelo–Fraenkel set theory (**CZF**) with an additional axiom, called the set generation axiom which is derivable in **CZF**, both from the relativized dependent choice scheme and from a regular extension axiom. We give some applications of the main result to algebra, topology and formal topology.

1. Introduction

The late Errett Bishop (Bishop 1967) developed his mathematics, Bishop's constructive mathematics, not based on the principles of classical logic such as the principle of excluded middle, but based on the principles of intuitionistic logic which is weaker than classical logic; see Troelstra and van Dalen (1988, Chapter 1) for some history of constructive mathematics, and see Bishop (1967); Bishop and Bridges (1985); Bridges and Richman (1987); Bridges and Vîţă (2006) and Mines *et al.* (1988) for the practice in Bishop's constructive mathematics.

After the publication of Bishop's book, Myhill (1975) developed a constructive set theory as a set theoretical foundation for Bishop's constructive mathematics, and Martin-Löf (1984) developed a predicative type theory, the Martin-Löf type theory, as a foundational approach to Bishop's constructive mathematics. Aczel (1977, 1982, 1986) introduced a constructive set theory, the constructive Zermelo–Fraenkel set theory (**CZF**), which has a quite natural interpretation into the Martin-Löf type theory, and hence it is a predicative theory without the power set axiom and the full separation axiom.

When developing mathematics in a predicative way, say in **CZF**, we are always faced with a difficulty in constructing an object as a *set*. Sometimes, we are able to find, case by case, constructions of objects as sets; see Crosilla *et al.* (2005) for a construction of

https://doi.org/10.1017/S0960129513000236 Published online by Cambridge University Press

an order completion, Berger *et al.* (2012) for a construction of a completion of a uniform space and Curi (2012) and its references for a construction of the set of continuous morphisms from a compact regular formal topology into a completely regular and set-presented formal topology and so on; see Sambin (1987, 2003, forthcoming) for formal topology. However, in general, we have to introduce some notion of an approximation, and deal with a class whose elements can be approximated by elements of a subset of the class.

Aczel (2006) introduced the notion of a *set-generated* directed complete partial order (dcpo) using some terminology from domain theory. A partially ordered class is a dcpo if each directed subset has a least upper bound, where a subset is *directed* if any pair of elements of the subset has an upper bound in the subset. A dcpo $\mathcal{X}$ is *set-generated* if there is a subset G of $\mathcal{X}$ such that, for each $a \in \mathcal{X}$, $\{x \in G \mid x \leqslant a\}$ is a directed subset whose least upper bound is a.

If we restrict our attention to a class $\mathcal{X}$ of subsets of a set with the inclusion $\subseteq$ as a partial order, then we may say that $\mathcal{X}$ is *set-generated* if there exists a subset G, called a *generating subset*, of $\mathcal{X}$ such that

$$\forall \alpha \in \mathcal{X} \forall \tau \in \mathrm{Fin}(\alpha) \exists \beta \in G[\tau \subseteq \beta \subseteq \alpha],$$

where $\mathrm{Fin}(\alpha)$ is the set of finitely enumerable subsets of α. As the reader has seen in the literature, say Aczel (2006) and Palmgren (2006) for the class of formal points of a set-presented formal topology, Ishihara and Palmgren (2006) and Palmgren (2005) for constructions of a quotient topology and a coequalizer in formal topology, and will see in Section 7, the notion of a set-generated class in this sense has played crucial roles in predicative constructive mathematics; see van den Berg (2013) for non-deterministic definitions and set-generated classes.

In this paper, after reviewing the predicative constructive set theory (**CZF**), we introduce infinitary propositional theories over a set S and their models which are subsets of S, and then introduce a *generalized geometric theory* and their *rank*. Our main result is the following.

The class of models of a generalized geometric theory of rank n is set-generated.

This will be proved by showing that (1) each theory of rank $n+1$ has a strongly conservative extension of rank $\max\{1, n\}$ (Proposition 4.3), (2) if the class of models of a strongly conservative extension of a theory is set-generated, then the class of models of the theory is set-generated (Proposition 4.4), and (3) the class of models of a theory of rank 1 is set-generated (in Theorem 4.5). We will prove (3) in **CZF** with an additional axiom, called the set generation axiom (SGA):

SGA: for each set S and each subset Z of $\mathrm{Fin}(S) \times \mathrm{Pow}(\mathrm{Pow}(S))$, the class

$$\mathcal{M}(Z) = \{\alpha \in \mathrm{Pow}(S) \mid \forall (\sigma, \Gamma) \in Z\, [\sigma \subseteq \alpha \Rightarrow \exists U \in \Gamma (U \subseteq \alpha)]\}$$

is set-generated.

We will show, in **CZF**, that the axiom SGA both follows from the *relativized dependent choice* (RDC) in Section 5 and also from a *regular extension axiom* (RRS_2-uREA) in Section 6, respectively. Finally, in Section 7, we will give some applications of the main

https://doi.org/10.1017/S0960129513000236 Published online by Cambridge University Press

result to algebra, topology and formal topology including some constructions mentioned above; see Ishihara and Kawai (2014) for applications in the categories of basic pairs and concrete spaces introduced by Sambin (2003, forthcoming).

2. The constructive set theory CZF

The constructive set theory **CZF**, founded by Aczel (1977, 1982, 1986), grew out of Myhill's constructive set theory (Myhill 1975) as a formal system for Bishop's constructive mathematics, and permits a quite natural interpretation in Martin-Löf type theory (Martin-Löf 1984).

Definition 2.1. The language of **CZF** contains variables for sets, a constant $\mathbf{N}$, and the binary predicates $=$ and $\in$. The axioms and rules are those of intuitionistic predicate logic with equality, and the following set theoretic axioms:

1. **Extensionality:** $\forall a \forall b(\forall x(x \in a \Leftrightarrow x \in b) \Rightarrow a = b)$.
2. **Pairing:** $\forall a \forall b \exists c \forall x(x \in c \Leftrightarrow x = a \vee x = b)$.
3. **Union:** $\forall a \exists b \forall x(x \in b \Leftrightarrow \exists y \in a(x \in y))$.
4. **Restricted separation:**
$$\forall a \exists b \forall x(x \in b \Leftrightarrow x \in a \wedge \varphi(x))$$
for every *restricted* formula $\varphi(x)$. Here, a formula $\varphi(x)$ is restricted, or Δ_0, if all the quantifiers occurring in it are bounded, i.e. of the form $\forall x \in c$ or $\exists x \in c$.
5. **Strong collection:**
$$\forall a(\forall x \in a \exists y \varphi(x, y) \Rightarrow \exists b(\forall x \in a \exists y \in b \varphi(x, y) \wedge \forall y \in b \exists x \in a \varphi(x, y)))$$
for every formula $\varphi(x, y)$.
6. **Subset collection:**
$$\forall a \forall b \exists c \forall u(\forall x \in a \exists y \in b \varphi(x, y, u) \Rightarrow \exists d \in c(\forall x \in a \exists y \in d \varphi(x, y, u) \wedge \forall y \in d \exists x \in a \varphi(x, y, u)))$$
for every formula $\varphi(x, y, u)$.
7. **Infinity:**
$$(\mathbf{N}1) \quad 0 \in \mathbf{N} \wedge \forall x(x \in \mathbf{N} \Rightarrow x + 1 \in \mathbf{N}),$$
$$(\mathbf{N}2) \quad \forall y(0 \in y \wedge \forall x(x \in y \Rightarrow x + 1 \in y) \Rightarrow \mathbf{N} \subseteq y),$$
where $x + 1$ is $x \cup \{x\}$, and 0 is the empty set $\varnothing = \{x \in \mathbf{N} \mid \bot\}$.
8. **$\in$-Induction:**
$$(\mathrm{IND}_{\in}) \quad \forall a(\forall x \in a \varphi(x) \Rightarrow \varphi(a)) \Rightarrow \forall a \varphi(a)$$
for every formula $\varphi(a)$.

Let a and b be sets. Using strong collection, the *cartesian product* $a \times b$ of a and b consisting of the ordered pairs $(x, y) = \{\{x\}, \{x, y\}\}$ with $x \in a$ and $y \in b$ can be introduced in **CZF**. A *relation* r between a and b is a subset of $a \times b$. A relation $r \subseteq a \times b$ is *total*

https://doi.org/10.1017/S0960129513000236 Published online by Cambridge University Press

(or is a *multivalued function*) if for every $x \in a$ there exists $y \in b$ such that $(x, y) \in r$. The class of total relations between a and b is denoted by $\mathrm{mv}(a, b)$, or more formally

$$r \in \mathrm{mv}(a, b) \Leftrightarrow r \subseteq a \times b \wedge \forall x \in a \exists y \in b((x, y) \in r).$$

A *function* from a to b is a total relation $f \subseteq a \times b$ such that for every $x \in a$ there is exactly one $y \in b$ with $(x, y) \in f$. The class of functions from a to b is denoted by b^a, or more formally

$$f \in b^a \Leftrightarrow f \in \mathrm{mv}(a, b) \wedge \forall x \in a \forall y, z \in b((x, y) \in f \wedge (x, z) \in f \Rightarrow y = z).$$

In **CZF**, we can prove

Fullness: $\forall a \forall b \exists c(c \subseteq \mathrm{mv}(a, b) \wedge \forall r \in \mathrm{mv}(a, b) \exists r_0 \in c(r_0 \subseteq r))$,

and, as a corollary, we see that b^a forms a set, that is

Exponentiation: $\forall a \forall b \exists c \forall f(f \in c \Leftrightarrow f \in b^a)$.

An *inductive definition* is a class Φ of ordered pairs. If Φ is an inductive definition and $(X, a) \in \Phi$, then we write $X/a \in \Phi$, and call X/a an *(inference) step* of Φ. A class $\mathcal{Y}$ is Φ-*closed* if $X \subseteq \mathcal{Y}$ implies $a \in \mathcal{Y}$ for each step X/a of Φ. We will use the following result (Aczel 1982; Aczel and Rathjen 2001) for the axiom system **CZF**; see Aczel and Rathjen (2001) for more details of **CZF**.

Theorem 2.2 (Class inductive definition theorem). For each inductive definition Φ, there is a smallest Φ-closed class $\mathcal{I}(\Phi)$.

We say that the class $\mathcal{I}(\Phi)$ is *inductively defined by* the inductive definition Φ.

In the following, capital and small letters $A, B, \ldots$ and $a, b, \ldots$ will denote sets, and calligraphic letters $\mathcal{A}, \mathcal{B}, \ldots$ will denote classes unless stated otherwise.

3. Infinitary propositional theories

Definition 3.1. Let S be a set. Then the class of S-*formulae* is defined inductively by the following clauses.

1. p_s is an S-formula for each $s \in S$;
2. if φ and ψ are S-formulae, then $(\varphi \to \psi)$ is an S-formula;
3. if U is a set of S-formulae, then $\bigwedge U$ and $\bigvee U$ are S-formulae.

If I is a set and φ_i is an S-formula for each $i \in I$, then we write

$$\textstyle\bigwedge_{i \in I} \varphi_i \equiv \bigwedge\{\varphi_i \mid i \in I\} \quad \text{and} \quad \bigvee_{i \in I} \varphi_i \equiv \bigvee\{\varphi_i \mid i \in I\}.$$

We denote the class of S-formulae by $\mathcal{F}^S$, and let $\mathcal{F}_0^S = \{p_s \mid s \in S\}$. We call a subset T of $\mathcal{F}^S$ an S-theory.

Remark 3.2. Note that, the class $\mathcal{F}^S$ is inductively defined by the steps: $\varnothing/p_s$ $(s \in S)$, $\{\varphi, \psi\}/(\varphi \to \psi)$, $U/\bigwedge U$ and $U/\bigvee U$ (U is a set). To represent this class in constructive set theory it is necessary to use some coding. For example, we can let $p_s \equiv (0, s)$, $(\varphi \to \psi) \equiv (1, (\varphi, \psi))$, $\bigwedge U \equiv (2, U)$ and $\bigvee U \equiv (3, U)$.

https://doi.org/10.1017/S0960129513000236 Published online by Cambridge University Press

Let $0 = \varnothing$, $1 = \{0\}$ and $\Omega = \mathrm{Pow}(1)$. For each formula φ of **CZF**, define the subclass $\langle\varphi\rangle$ of 1 by $\langle\varphi\rangle = \{x \in 1 \mid \varphi\}$, where the variable x is chosen not to be free in φ. For each class A let $!A \equiv (0 \in A)$. Note that $\varphi \leftrightarrow !\langle\varphi\rangle$, and if φ is a restricted formula, then $\langle\varphi\rangle \in \Omega$.

Proposition 3.3. For each $\alpha \in Pow(S)$ there is a unique class function $[\![-]\!]_\alpha : \mathcal{F}^S \to \Omega$ such that for $s \in S$, $\varphi, \psi \in \mathcal{F}^S$ and $U \in Pow(\mathcal{F}^S)$,

1. $[\![p_s]\!]_\alpha = \langle s \in \alpha\rangle$,
2. $[\![\varphi \to \psi]\!]_\alpha = \langle !{[\![\varphi]\!]_\alpha} \to !{[\![\psi]\!]_\alpha}\rangle$,
3. $[\![\bigwedge U]\!]_\alpha = \langle \forall \varphi \in U\ !{[\![\varphi]\!]_\alpha}\rangle$,
4. $[\![\bigvee U]\!]_\alpha = \langle \exists \varphi \in U\ !{[\![\varphi]\!]_\alpha}\rangle$.

Proof. Let Φ_α be the inductive definition having the following steps, for $s \in S$, $\varphi, \psi \in \mathcal{F}^S$, $U \in Pow(\mathcal{F}^S)$ and $F : U \to \Omega$.

1. $\varnothing / (p_s, \langle s \in \alpha\rangle)$,
2. $\{(\varphi, a), (\psi, b)\} / (\varphi \to \psi, \langle !a \to !b\rangle)$,
3. $F / (\bigwedge U, \langle \forall \varphi \in U\ !F(\varphi)\rangle)$,
4. $F / (\bigvee U, \langle \exists \varphi \in U\ !F(\varphi)\rangle)$.

Note that, $F = \{(\varphi, F(\varphi)) \mid \varphi \in U\}$. Since the class $\mathcal{F}^S \times \Omega$ is Φ_α-closed, the class $\mathcal{I}(\Phi_\alpha)$ is a subclass of $\mathcal{F}^S \times \Omega$. Moreover, since it is straightforward to see that the class $\{\varphi \mid \exists! a \in \Omega((\varphi, a) \in \mathcal{I}(\Phi_\alpha))\}$ is closed under the inductive definition for $\mathcal{F}^S$ in Remark 3.2, the class $\mathcal{I}(\Phi_\alpha)$ is the graph of the required function $[\![-]\!]_\alpha : \mathcal{F}^S \to \Omega$. □

Definition 3.4. For $\alpha \in Pow(S)$ and $\varphi \in \mathcal{F}^S$ let

$$\alpha \models \varphi \quad \text{iff} \quad !{[\![\varphi]\!]_\alpha}.$$

Proposition 3.5. For $\alpha \in Pow(S)$, $s \in S$, $\varphi, \psi \in \mathcal{F}^S$ and $U \in Pow(\mathcal{F}^S)$,

1. $\alpha \models p_s$ iff $s \in \alpha$,
2. $\alpha \models (\varphi \to \psi)$ iff $\alpha \models \varphi$ implies $\alpha \models \psi$,
3. $\alpha \models \bigwedge U$ iff $\alpha \models \varphi$ for all $\varphi \in U$,
4. $\alpha \models \bigvee U$ iff $\alpha \models \varphi$ for some $\varphi \in U$.

Proof. Routine. □

If T is an S-theory then, for each $\alpha \in Pow(S)$ let $\alpha \models T$ iff $\alpha \models \varphi$ for all $\varphi \in T$ and let $\mathfrak{M}(T) \equiv \{\alpha \in Pow(S) \mid \alpha \models T\}$.

Proposition 3.6. Let $\iota : S \to S'$. Then there is a unique class function $(-)^\iota : \mathcal{F}^S \to \mathcal{F}^{S'}$ such that, for $s \in S$, $\varphi, \psi \in \mathcal{F}^S$ and $U \in Pow(\mathcal{F}^S)$,

1. $p_s^\iota \equiv p_{\iota(s)}$;
2. $(\varphi \to \psi)^\iota \equiv (\varphi^\iota \to \psi^\iota)$;
3. $(\bigwedge U)^\iota \equiv \bigwedge U^\iota$;
4. $(\bigvee U)^\iota \equiv \bigvee U^\iota$,

where $U^\iota = \{\varphi^\iota \mid \varphi \in U\}$.

https://doi.org/10.1017/S0960129513000236 Published online by Cambridge University Press

Proof. As in the proof of Proposition 3.3. □

Proposition 3.7. Let $\iota : S \to S'$, $\alpha' \in \mathrm{Pow}(S')$ and $\alpha \equiv \iota^{-1}(\alpha')$. Then

$$\alpha \models \varphi \;\Leftrightarrow\; \alpha' \models \varphi^{\iota}$$

for each $\varphi \in \mathcal{F}^S$, and hence

$$\alpha \models T \;\Leftrightarrow\; \alpha' \models T^{\iota}$$

for each S-theory T.

Proof. Let $[\![-]\!]_{\alpha} : \mathcal{F}^S \to \Omega$ and $[\![-]\!]_{\alpha'} : \mathcal{F}^{S'} \to \Omega$ be class functions defined as in the proof of Proposition 3.3. Then, the class

$$\{\varphi \in \mathcal{F}^S \mid \alpha \models \varphi \;\Leftrightarrow\; \alpha' \models \varphi^{\iota}\}$$

is closed under the inductive definition for $\mathcal{F}^S$ in Remark 3.2. □

Let T be an S-theory and let φ be an S-formula. Then, we write $T \models \varphi$ if $\alpha \models \varphi$ for each $\alpha \in \mathfrak{M}(T)$.

Definition 3.8. Let $\iota : S \to S'$.

1. An S'-theory T' is an *ι-extension* of T if $T \models \varphi$ implies $T' \models \varphi^{\iota}$ for each $\varphi \in \mathcal{F}^S$.
2. An ι-extension T' of T is a *conservative* ι-extension if $T' \models \varphi^{\iota}$ implies $T \models \varphi$ for each $\varphi \in \mathcal{F}^S$, and it is a *strongly conservative* ι-extension if for each $\alpha \in \mathfrak{M}(T)$ there exists $\alpha' \in \mathfrak{M}(T')$ such that $\alpha = \iota^{-1}(\alpha')$.

Note that, if T' is a strongly conservative ι-extension of T, then it is a conservative ι-extension, by Proposition 3.7.

Proposition 3.9. Let $\iota : S \to S'$ and let T and T' be an S-theory and S'-theory, respectively. Then the following are equivalent.

1. T' is an ι-extension of T;
2. $T' \models \varphi^{\iota}$ for each $\varphi \in T$;
3. $\iota^{-1}(\alpha') \in \mathfrak{M}(T)$ for each $\alpha' \in \mathfrak{M}(T')$.

Proof. (1) $\Rightarrow$ (2): Suppose that T' is an ι-extension of T. If $\varphi \in T$, then $T \models \varphi$, and hence $T' \models \varphi^{\iota}$.

(2) $\Rightarrow$ (3): Suppose that $\alpha' \in \mathfrak{M}(T')$. Then $\alpha' \models \varphi^{\iota}$ for each $\varphi \in T$, by (2), and hence $\iota^{-1}(\alpha') \models \varphi$, by Proposition 3.7. Therefore, $\iota^{-1}(\alpha') \in \mathfrak{M}(T)$.

(3) $\Rightarrow$ (1): Suppose that $T \models \varphi$ and $\alpha' \in \mathfrak{M}(T')$. Then $\iota^{-1}(\alpha') \in \mathfrak{M}(T)$, by (3), and hence $\iota^{-1}(\alpha') \models \varphi$. Therefore, $\alpha' \models \varphi^{\iota}$, by Proposition 3.7. Thus, $T' \models \varphi^{\iota}$. □

4. Generalized geometric theories

Let σ be a finitely enumerable subset of $\mathcal{F}_0^S$, and let Γ be a set of S-theories. Then define

$$\sigma \multimap \Gamma \equiv (\bigwedge \sigma \to \bigvee_{U \in \Gamma} \bigwedge U).$$

https://doi.org/10.1017/S0960129513000236 Published online by Cambridge University Press

Note that, $\alpha \models (\sigma \multimap \Gamma)$ if and only if $\alpha \models \sigma$ implies $\alpha \models U$ for some $U \in \Gamma$. For a subclass $\mathcal{C}$ of $\mathcal{F}^S$, let

$$\Delta_0^S(\mathcal{C}) = \{\sigma \multimap \Gamma \mid \sigma \in \mathrm{Fin}(\mathcal{F}_0^S), \Gamma \in \mathrm{Pow}(\mathrm{Pow}(\mathcal{C}))\},$$

and let

$$\Delta^S(\mathcal{C}) = (\mathcal{F}_0^S \cap \mathcal{C}) \cup \Delta_0^S(\mathcal{C}).$$

Definition 4.1. For $t \in S$ and $\varphi \in \Delta^S(\mathcal{F}^S)$ let

$$(t \sqsupset \varphi) \;\equiv\; (\sigma_\varphi \cup \{p_t\} \;\multimap\; \Gamma_\varphi),$$

where σ_φ and Γ_φ are defined as follows. If $\varphi \equiv p_s \in \mathcal{F}_0^S$ then $\sigma_\varphi \equiv \varnothing$ and $\Gamma_\varphi \equiv \{\{p_s\}\}$ and if $\varphi = (\sigma \multimap \Gamma) \in \Delta_0^S(\mathcal{F}^S)$ then $\sigma_\varphi \equiv \sigma$ and $\Gamma_\varphi \equiv \Gamma$.

Lemma 4.2. Let $\mathcal{C}$ be a subclass of $\mathcal{F}^S$ and let $t \in S$. If $\varphi \in \Delta^S(\mathcal{C})$ then $(t \sqsupset \varphi) \in \Delta_0^S(\mathcal{C})$ such that, for $\alpha \in Pow(S)$,

$$\alpha \models (t \sqsupset \varphi) \quad \text{iff} \quad t \in \alpha \text{ implies } \alpha \models \varphi.$$

Proof. Let $\varphi \in \Delta^S(\mathcal{C})$. Then, as $\Gamma_\varphi \in Pow(Pow(\mathcal{C}))$,

$$(t \sqsupset \varphi) \equiv (\sigma_\varphi \cup \{p_t\} \multimap \Gamma_\varphi) \in \Delta_0^S(\mathcal{C}).$$

Note that, when $\varphi \in \mathcal{F}_0^S \cap \mathcal{C}$ then

$$\alpha \models (\sigma_\varphi \multimap \Gamma_\varphi) \;\text{ iff }\; \alpha \models (\varnothing \multimap \{\{\varphi\}\}) \;\text{ iff }\; \alpha \models \varphi.$$

So, for $\varphi \in \Delta^S(\mathcal{C})$,

$$\begin{aligned} \alpha \models (t \sqsupset \varphi) \;\; &\text{iff } \alpha \models (\sigma_\varphi \cup \{p_t\}) \text{ implies } \alpha \models U \text{ for some } U \in \Gamma_\varphi \\ &\text{iff } \alpha \models p_t \text{ implies } \alpha \models (\sigma_\varphi \multimap \Gamma_\varphi) \\ &\text{iff } t \in \alpha \text{ implies } \alpha \models \varphi. \end{aligned}$$

□

For $n \in \mathbf{N}$, define a class $\mathcal{G}_n^S$ by

$$\begin{aligned} \mathcal{G}_0^S &= \mathcal{F}_0^S; \\ \mathcal{G}_{n+1}^S &= \Delta^S(\mathcal{G}_n^S). \end{aligned}$$

It is straightforward to see that $\mathcal{F}_0^S$ is a subclass of each $\mathcal{G}_n^S$ so that, for each n, $\mathcal{G}_{n+1}^S = \mathcal{F}_0^S \cup \Delta_0^S(\mathcal{G}_n^S)$ and $\mathcal{G}_n^S$ is a subclass of $\mathcal{G}_{n+1}^S$. An S-theory T is a *generalized geometric theory of rank n over S* if $T \subseteq \mathcal{G}_n^S$.

Theorem 4.3. Let T be a generalized geometric theory of rank $n+1$ over a set S. Then there exists a set S', a mapping $\iota : S \to S'$, and a generalized geometric theory T' of rank $n' = \max\{1, n\}$ over S' which is a strongly conservative ι-extension of T.

Proof. Let T be an S-theory with $T \subseteq \mathcal{G}_{n+1}^S$. Then $T = P \cup Q$ with $P \subseteq \mathcal{F}_0^S \cap \mathcal{G}_n^S$ and $Q \subseteq \Delta_0^S(\mathcal{G}_n^S)$. Let $Q' = \{(\varphi, U, \theta) \mid \varphi \in Q, U \in \Gamma_\varphi, \theta \in U\}$, and, if $q \in Q'$, let $q = (\varphi_q, U_q, \theta_q)$.

https://doi.org/10.1017/S0960129513000236 Published online by Cambridge University Press

Let $S' = S + Q' = (\{0\} \times S) \cup (\{1\} \times Q')$, and define $\iota : S \to S'$ by $\iota(s) = (0, s)$ for $s \in S$. Let $Q^* = \{(1,q) \sqsupset \theta^\iota_q \mid q \in Q'\} \subseteq \mathcal{G}^{S'}_{n'}$, and let

$$\widetilde{Q} = \{\sigma^\iota_\varphi \multimap \Gamma'_\varphi \mid \varphi \in Q\} \subseteq \mathcal{G}^{S'}_1.$$

Here, for $\varphi \in Q$,

$$\Gamma'_\varphi = \{U'_\varphi \mid U \in \Gamma_\varphi\} \in Pow(Pow(\mathcal{F}^{S'}_0)),$$

where, for $U \in \Gamma_\varphi$, $U'_\varphi = \{p_{(1,(\varphi,U,\theta))} \mid \theta \in U\} \in Pow(\mathcal{F}^{S'}_0)$. Finally, define a generalized geometric theory T' over S' of rank n' by

$$T' = P^\iota \cup Q^* \cup \widetilde{Q}.$$

Then, we shall show that T' is a strongly conservative ι-extension of T.

First, we show that T' is an ι-extension of T using Proposition 3.9; i.e. we show that if $\alpha' \in \mathrm{Pow}(S')$ such that $\alpha' \models T'$ and $\alpha \equiv \iota^{-1}(\alpha')$ then

$$\alpha' \models T' \quad \text{implies} \quad \alpha \models T.$$

So let $\alpha' \models T'$; i.e. (i) $\alpha' \models P^\iota$, (ii) $\alpha' \models Q^*$ and (iii) $\alpha' \models \widetilde{Q}$.

We have $\alpha \models P$, by (i) and Proposition 3.7. So it only remains to show that $\alpha \models Q$. If $\varphi = (\sigma_\varphi \multimap \Gamma_\varphi) \in Q$ and $\alpha \models \sigma_\varphi$ we must show that $\alpha \models U$ for some $U \in \Gamma_\varphi$. Since $\alpha' \models \sigma^\iota_\varphi$, by (iii),

$$\alpha' \models \sigma^\iota_\varphi \multimap \Gamma'_\varphi,$$

so that there exists $U \in \Gamma_\varphi$ such that $\alpha' \models U'_\varphi$.

We must show that $\alpha \models U$. So let $\theta \in U$. Then, since $q = (\varphi, V, \theta) \in Q'$ and $p_{(1,q)} \in U'_\varphi$, we have $\alpha' \models p_{(1,q)}$, and hence $(1,q) \in \alpha'$. Since $(1,q) \sqsupset \theta^\iota \in Q^*$, we have $\alpha' \models (1,q) \sqsupset \theta^\iota$, by (ii), and hence $\alpha' \models \theta^\iota$, by Lemma 4.2, and so $\alpha \models \theta$ by Proposition 3.7. Thus, we have $\alpha \models U$. So we have shown that T' is an ι-extension of T.

Next, we show that T' is a strongly conservative ι-extension of T. To this end, let $\alpha \in \mathrm{Pow}(S)$ such that $\alpha \models T$. Define a subset α' of S' by

$$\alpha' = \alpha + \{q \in Q' \mid \alpha \models \theta_q\}.$$

We show that $\alpha' \models T'$. Clearly $\alpha = \iota^{-1}(\alpha')$, and, since $\alpha \models P$, we have $\alpha' \models P^\iota$, by Proposition 3.7. Let $q \in Q'$. If $(1,q) \in \alpha'$, then $\alpha \models \theta_q$, and hence $\alpha' \models \theta^\iota_q$, by Proposition 3.7. Therefore, $\alpha' \models (1,q) \sqsupset \theta^\iota_q$, by Lemma 4.2. Thus, $\alpha' \models Q^*$. Let $\varphi \in Q$ and suppose that $\alpha' \models \sigma^\iota_\varphi$. Then, $\alpha \models \sigma_\varphi$, and, since $\alpha \models \varphi$, there exists $U \in \Gamma_\varphi$ such that $\alpha \models U$. Therefore, $\alpha' \models U'_\varphi$, and so

$$\alpha' \models \sigma^\iota_\varphi \multimap \Gamma'_\varphi.$$

Thus, $\alpha' \models \widetilde{Q}$. As $T' \equiv P^\iota \cup Q^* \cup \widetilde{Q}$ we are done. □

Proposition 4.4. Let $\iota : S \to S'$, and let T and T' be generalized geometric theories over S and S', respectively, such that T' is a strongly conservative ι-extension of T. If the class $\mathfrak{M}(T')$ is set-generated, then the class $\mathfrak{M}(T)$ is set-generated.

Proof. Suppose that $\mathfrak{M}(T')$ is set-generated. Then there exists a generating subset G of $\mathfrak{M}(T')$. Let $H = \{\iota^{-1}(\alpha') \mid \alpha' \in G\}$. Then for each $\alpha \in \mathfrak{M}(T)$ there exists $\alpha' \in \mathfrak{M}(T')$ such

https://doi.org/10.1017/S0960129513000236 Published online by Cambridge University Press

that $\alpha = \iota^{-1}(\alpha')$. Hence for each $\tau \in \mathrm{Fin}(\alpha)$, since $\iota(\tau) \in \mathrm{Fin}(\alpha')$, there exists $\beta' \in G$ such that $\iota(\tau) \subseteq \beta' \subseteq \alpha'$, and therefore $\tau \subseteq \iota^{-1}(\beta') \subseteq \iota^{-1}(\alpha') = \alpha$. Thus, H is a generating subset of $\mathfrak{M}(T)$. □

Theorem 4.5. Assume SGA. Then the class $\mathfrak{M}(T)$ of models of a generalized geometric theory T of rank n is set-generated.

Proof. It is enough to show that the class of models of a generalized geometric theory of rank 1 is set-generated, by Theorem 4.3 and Proposition 4.4. Let T be a generalized geometric theory over a set S of rank 1. Then, since $\alpha \models p_s$ if and only if $\alpha \models \varnothing \multimap \{\{p_s\}\}$ for each $\alpha \in \mathrm{Pow}(S)$, we may assume without loss of generality that $T \subseteq \Delta_0^S(\mathcal{F}_0^S)$. For each $\varphi \equiv \sigma_\varphi \multimap \Gamma_\varphi \in T$, let $\sigma = \{s \in S \mid p_s \in \sigma_\varphi\}$, and let $\Gamma = \{\{s \in S \mid p_s \in U\} \mid U \in \Gamma_\varphi\}$. Then, we have $\mathcal{M}(Z) = \mathfrak{M}(T)$, and hence $\mathfrak{M}(T)$ is set-generated, by SGA. □

In the following two sections, we will show that SGA can be proved in **CZF** + RDC, where RDC is the *relativized dependent choice*, and can also be proved in **CZF** + RRS_2-uREA, where RRS_2-uREA is a *regular extension axiom*.

5. The relativized dependent choice

The RDC is stated as follows.

RDC: if $\forall x \in \mathcal{A} \exists y \in \mathcal{A}((x, y) \in \mathcal{R})$ and $b_0 \in \mathcal{A}$, then there exists a function $f : \mathbf{N} \to \mathcal{A}$ such that $f(0) = b_0$ and $\forall n \in \mathbf{N}((f(n), f(n+1)) \in \mathcal{R})$,

where $\mathcal{A}$ is a *class* and $\mathcal{R}$ is a class relation. Note that, RDC clearly implies the *dependent choice* (DC):

DC: if $\forall x \in A \exists y \in A((x, y) \in \mathcal{R})$ and $b_0 \in A$, then there exists a function $f : \mathbf{N} \to A$ such that $f(0) = b_0$ and $\forall n \in \mathbf{N}((f(n), f(n+1)) \in \mathcal{R})$,

where A is a *set* and $\mathcal{R}$ is a class relation.

In this section, we will give a proof of the following theorem.

Theorem 5.1. RDC implies SGA.

Before proving the theorem, we show the following small lemma which will be used in the proof.

Lemma 5.2. Let a, b and R be sets, and let

$$r \in \mathrm{mv}_R(a, b) \Leftrightarrow r \in \mathrm{mv}(a, b) \wedge r \subseteq R,$$
$$\mathrm{Full}_R(a, b, c) \Leftrightarrow c \subseteq \mathrm{mv}_R(a, b) \wedge \forall r \in \mathrm{mv}_R(a, b) \exists r_0 \in c(r_0 \subseteq r).$$

Then there exists a set c such that $\mathrm{Full}_R(a, b, c)$.

Proof. By Fullness, there exists a set d such that $d \subseteq \mathrm{mv}(a, b)$ and $\forall r \in \mathrm{mv}(a, b) \exists r_0 \in d(r_0 \subseteq r)$. Let $c = \{r \in d \mid r \subseteq R\}$, by Restricted Separation. Then, $c \subseteq \mathrm{mv}_R(a, b)$. For each $r \in \mathrm{mv}_R(a, b)$, since $r \in \mathrm{mv}(a, b)$, there exists $r_0 \in d$ such that $r_0 \subseteq r$, and therefore, since $r_0 \subseteq r \subseteq R$, we have $r_0 \in c$. □

https://doi.org/10.1017/S0960129513000236 Published online by Cambridge University Press

Let S be a set, and let Z be a subset of $\mathrm{Fin}(S) \times \mathrm{Pow}(\mathrm{Pow}(S))$. For $\alpha \in \mathrm{Pow}(S)$, let $Z_\alpha = \{(\sigma, \Gamma) \in Z \mid \sigma \subseteq \alpha\}$.

Let $b = \bigcup_{(\sigma,\Gamma)\in Z} \Gamma$, let $R = \{((\sigma, \Gamma), U) \in Z \times b \mid U \in \Gamma)\}$, and define a class $\mathcal{V}$ by

$$\mathcal{V} = \{(\alpha, c) \mid \alpha \in \mathrm{Pow}(S) \wedge \mathrm{Full}_R(Z_\alpha, b, c)\}.$$

Proposition 5.3. There exists a set $V \subseteq \mathcal{V}$ such that

1. $\forall \tau \in \mathrm{Fin}(S) \exists c((\tau, c) \in V)$,
2. $\forall (\alpha, c) \in V \forall r \in c \exists (\alpha', c') \in V(\alpha \cup \bigcup \mathrm{ran}(r) = \alpha')$.

Proof. Let $\mathcal{R}$ be a class relation defined by

$$(X, Y) \in \mathcal{R} \Leftrightarrow \forall (\alpha, c) \in X \forall r \in c \exists (\alpha', c') \in Y(\alpha \cup \bigcup \mathrm{ran}(r) = \alpha').$$

We show that $\forall X \in \mathrm{Pow}(\mathcal{V}) \exists Y \in \mathrm{Pow}(\mathcal{V})((X, Y) \in \mathcal{R})$. To this end, suppose that X is a set with $X \subseteq \mathcal{V}$. Then for each $(\alpha, c) \in X$ and $r \in c$, letting $\alpha' = \alpha \cup \bigcup \mathrm{ran}(r)$ and constructing a set c' such that $\mathrm{Full}_R(Z_{\alpha'}, b, c')$, by Lemma 5.2, we have $(\alpha', c') \in \mathcal{V}$. Therefore,

$$\forall ((\alpha, c), r) \in \sum_{(\alpha,c)\in X} c \exists (\alpha', c') \in \mathcal{V}(\alpha \cup \bigcup \mathrm{ran}(r) = \alpha'),$$

and hence there exists a set $Y \subseteq \mathcal{V}$ such that

$$\forall ((\alpha, c), r) \in \sum_{(\alpha,c)\in X} c \exists (\alpha', c') \in Y(\alpha \cup \bigcup \mathrm{ran}(r) = \alpha'),$$

by Strong Collection. Clearly, we have $(X, Y) \in \mathcal{R}$.

Since $\forall \tau \in \mathrm{Fin}(S) \exists c \mathrm{Full}_R(Z_\tau, b, c)$, by Lemma 5.2, there exists a set $X_0 \subseteq \mathcal{V}$ such that $\forall \tau \in \mathrm{Fin}(S) \exists c((\tau, c) \in X_0)$ by Strong Collection. Applying RDC to $\forall X \in \mathrm{Pow}(\mathcal{V}) \exists Y \in \mathrm{Pow}(\mathcal{V})((X, Y) \in \mathcal{R})$ and X_0, we have a function $f : \mathbf{N} \to \mathrm{Pow}(\mathcal{V})$ such that $f(0) = X_0$ and

$$\forall n \in \mathbf{N}[(f(n), f(n + 1)) \in \mathcal{R}].$$

Let $V = \bigcup_{n\in\mathbf{N}} f(n)$. Then it is straightforward to see (1) and (2). □

Using Exponentiation, Restricted Separation and Strong Collection, define sets B and G by

$$B = \{\langle(\alpha_n, c_n)\rangle_{n\in\mathbf{N}} \in V^{\mathbf{N}} \mid \forall n \in \mathbf{N} \exists r \in c_n(\alpha_n \cup \bigcup \mathrm{ran}(r) = \alpha_{n+1}))\},$$
$$G = \{\bigcup_{n\in\mathbf{N}} \alpha_n \mid \langle(\alpha_n, c_n)\rangle_{n\in\mathbf{N}} \in B\}.$$

Proposition 5.4. G is a subset of $\mathcal{M}(Z)$.

Proof. Let $\alpha \in G$. Then there exists $\langle(\alpha_n, c_n)\rangle_{n\in\mathbf{N}} \in B$ such that $\alpha = \bigcup_{n\in\mathbf{N}} \alpha_n$. Suppose that $(\sigma, \Gamma) \in Z$ and $\sigma \subseteq \alpha$. Then, since $\sigma \in \mathrm{Fin}(S)$ and $\alpha_n \subseteq \alpha_{n+1}$ for each $n \in \mathbf{N}$, there exist $m \in \mathbf{N}$ and $r \in c_m$ such that $\sigma \subseteq \alpha_m$ and $\alpha_m \cup \bigcup \mathrm{ran}(r) = \alpha_{m+1}$. Therefore, since $r \in \mathrm{mv}_R(Z_{\alpha_m}, b)$ and $(\sigma, \Gamma) \in Z_{\alpha_m}$, there exists $U \in b$ such that $U \in \Gamma$, and hence $U \subseteq \bigcup \mathrm{ran}(r) \subseteq \alpha_{m+1} \subseteq \alpha$. Thus, $\alpha \in \mathcal{M}(Z)$. □

Proposition 5.5. Let $\gamma \in \mathcal{M}(Z)$, and let $\tau \in \mathrm{Fin}(\gamma)$. Then there exists $\beta \in G$ such that $\tau \subseteq \beta \subseteq \gamma$.

https://doi.org/10.1017/S0960129513000236 Published online by Cambridge University Press

Proof. Let $V_\gamma = \{(\alpha, c) \in V \mid \alpha \subseteq \gamma\}$. We show that

$$\forall (\alpha, c) \in V_\gamma \exists (\alpha', c') \in V_\gamma \exists r_0 \in c(\alpha \cup \bigcup \mathrm{ran}(r_0) = \alpha').$$

To this end, suppose that $(\alpha, c) \in V_\gamma$. Define a set

$$r = \{((\sigma, \Gamma), U) \in Z_\alpha \times b \mid U \in \Gamma \wedge U \subseteq \gamma\},$$

by Restricted Separation. Then, since $Z_\alpha \subseteq Z_\gamma$ and $\gamma \in \mathcal{M}(Z)$, for each $(\sigma, \Gamma) \in Z_\alpha$ there exists $U \in \Gamma$ such that $U \subseteq \gamma$. Therefore, $r \in \mathrm{mv}_R(Z_\alpha, b)$, and so there exists $r_0 \in c$ such that $r_0 \subseteq r$. Note that, $\bigcup \mathrm{ran}(r_0) \subseteq \bigcup \mathrm{ran}(r) \subseteq \gamma$. Then there exists $(\alpha', c') \in V$ such that $\alpha' = \alpha \cup \bigcup \mathrm{ran}(r_0) \subseteq \gamma$, by Proposition 5.3 (2). Thus $(\alpha', c') \in V_\gamma$.

By Proposition 5.3 (1), there exists c such that $(\tau, c) \in V_\gamma$. Applying DC, we have a function $h : n \mapsto (\alpha_n, c_n)$ with domain $\mathbf{N}$ and range V_γ such that $(\alpha_0, c_0) = (\tau, c)$ and $\forall n \in \mathbf{N} \exists r_0 \in c_n(\alpha_n \cup \bigcup \mathrm{ran}(r_0) = \alpha_{n+1})$. Therefore, since $h \in B$, we have $\beta = \bigcup_{n \in \mathbf{N}} \alpha_n \in G$ and $\tau \subseteq \beta \subseteq \gamma$. □

Remark 5.6. We can prove Proposition 5.3 using the *relation reflection scheme* (RRS) of Aczel (2008) which is weaker than RDC.

RRS: for a class $\mathcal{A}$ and a class relation $\mathcal{R}$ such that $\forall x \in \mathcal{A} \exists y \in \mathcal{A}((x, y) \in \mathcal{R})$, if A is a subset of $\mathcal{A}$, then there is a subset B of $\mathcal{A}$ such that $A \subseteq B$ and $\forall x \in B \exists y \in B((x, y) \in \mathcal{R})$.

However, in the proof of Proposition 5.5, we have invoked DC which, together with RRS, implies RDC; see Aczel (2008, Theorem 2.4).

6. A regular extension axiom

A set A is *regular* if it is *transitive*, i.e. $a \subseteq A$ for each $a \in A$, and for each $a \in A$ and $R \in \mathrm{mv}(a, A)$ there exists $b \in A$ such that

$$\forall x \in a \exists y \in b((x, y) \in R) \wedge \forall y \in b \exists x \in a((x, y) \in R).$$

A set A is *union-closed* if $\bigcup a \in A$ for each $a \in A$.

The union-closed regular extension axiom is stated as follows.

uREA: every set is a subset of a union-closed regular set.

A regular set A is RRS_2-*regular* if for each $A' \subseteq A$ and $R \in \mathrm{mv}(A' \times A', A')$, if $a_0 \in A'$, then there exists $A_0 \in A$ such that $a_0 \in A_0 \subseteq A'$ and $\forall x, y \in A_0 \exists z \in A_0(((x, y), z) \in R)$.

We will use the following strong form of the union-closed regular extension axiom to prove our result.

RRS_2-uREA: every set is a subset of a union-closed RRS_2-regular set.

Proposition 6.1. uREA and DC imply RRS_2-uREA.

Proof. Let S be a set. Then, by uREA, there exists a union-closed regular set A such that $\{\mathbf{N}\} \cup S \subseteq A$. Note that, $a \times a \in A$ for each $a \in A$. We show that A is RRS_2-regular. Suppose that $A' \subseteq A$ and $R \in \mathrm{mv}(A' \times A', A')$. Let $A_{A'} = \{a \in A \mid a \subseteq A'\}$, and let $a \in A_{A'}$.

https://doi.org/10.1017/S0960129513000236 Published online by Cambridge University Press

Then, $\forall(x,y) \in a \times a \exists z \in A(z \in A' \wedge ((x,y),z) \in R)$, and therefore, since $a \times a \in A$ and A is regular, there exists $b \in A$ such that

$$\forall(x,y) \in a \times a \exists z \in b(z \in A' \wedge ((x,y),z) \in R)$$

and

$$\forall z \in b \exists(x,y) \in a \times a(z \in A' \wedge ((x,y),z) \in R).$$

Since $2 \in A$, $a, b \in A$, and A is regular and union closed, we have $c = a \cup b \in A$, and so $c \in A_{A'}$. Hence,

$$\forall a \in A_{A'} \exists c \in A_{A'}[a \subseteq c \wedge \forall(x,y) \in a \times a \exists z \in c(((x,y),z) \in R)].$$

Let $a_0 \in A'$. Then $\{a_0\} \in A$, and hence $\{a_0\} \in A_{A'}$. Therefore, by DC, there exists a function $f : \mathbf{N} \to A_{A'}$ such that $f(0) = \{a_0\}$, and $f(n) \subseteq f(n+1)$ and $\forall(x,y) \in f(n) \times f(n) \exists z \in f(n+1)(((x,y),z) \in R)$ for each $n \in \mathbf{N}$. Letting $A_0 = \bigcup\{f(n) \mid n \in \mathbf{N}\}$, since A is regular and union closed, we have $a_0 \in A_0 \in A$. If $x, y \in A_0$, then there exists $n \in \mathbf{N}$ such that $x, y \in f(n)$, and hence there exists $z \in f(n+1) \subseteq A_0$. Therefore A_0 satisfies the desired condition. □

In the following, we will give a proof of the following theorem.

Theorem 6.2. RRS_2-uREA implies SGA.

Let S be a set, and let Z be a subset of $\mathrm{Fin}(S) \times \mathrm{Pow}(\mathrm{Pow}(S))$. For $\alpha \in \mathrm{Pow}(S)$, let $Z_\alpha = \{(\sigma,\Gamma) \in Z \mid \sigma \subseteq \alpha\}$, and note that, since $\sigma \in \mathrm{Fin}(S)$ for each $(\sigma,\Gamma) \in Z$, we have $Z_\alpha = \bigcup_{\tau \in \mathrm{Fin}(\alpha)} Z_\tau$. Let A be a union-closed RRS_2-regular set containing $\{\mathbf{N}, S\} \cup \{Z_\tau \mid \tau \in \mathrm{Fin}(S)\} \cup \{\Gamma \mid (\sigma,\Gamma) \in Z\}$, and let

$$G = \{\alpha \in A \mid \alpha \in \mathcal{M}(Z)\}.$$

Then, G is a set by Restricted Separation.

Proposition 6.3. Let $\gamma \in \mathcal{M}(Z)$, and let $\tau \in \mathrm{Fin}(\gamma)$. Then there exists $\beta \in G$ such that $\tau \subseteq \beta \subseteq \gamma$.

Proof. Let $A_\gamma = \{\alpha \in A \mid \alpha \subseteq \gamma\}$, and define a binary relation R on A_γ by

$$R = \{(\alpha,\alpha') \mid \forall(\sigma,\Gamma) \in Z_\alpha \exists U \in \Gamma(U \subseteq \alpha'), \alpha \subseteq \alpha'\}.$$

Let $\alpha \in A_\gamma$. Then, $\mathrm{Fin}(\alpha) \in A$, by (Aczel 2006, Lemma 49), and

$$\forall \tau \in \mathrm{Fin}(\alpha) \exists y \in A(y = Z_\tau).$$

Since A is regular, there exists $C \in A$ such that

$$\forall \tau \in \mathrm{Fin}(\alpha) \exists y \in C(y = Z_\tau) \wedge \forall y \in C \exists \tau \in \mathrm{Fin}(\alpha)(y = Z_\tau).$$

Since $C = \{Z_\tau \mid \tau \in \mathrm{Fin}(\alpha)\}$ and A is union-closed, we have $Z_\alpha = \bigcup C \in A$, and therefore, since $Z_\alpha \subseteq Z_\gamma$ and $\gamma \in \mathcal{M}(Z)$, we have

$$\forall(\sigma,\Gamma) \in Z_\alpha \exists U \in A(U \in \Gamma \wedge U \subseteq \gamma).$$

https://doi.org/10.1017/S0960129513000236 Published online by Cambridge University Press

Since A is regular, there exists $D \in A$ such that

$$\forall(\sigma,\Gamma) \in Z_\alpha \exists U \in D(U \in \Gamma \wedge U \subseteq \gamma) \wedge \forall U \in D \exists(\sigma,\Gamma) \in Z_\alpha(U \in \Gamma \wedge U \subseteq \gamma).$$

Therefore, since A is union-closed, we have $\delta = \bigcup D \in A_\gamma$ and, since $\{\alpha,\delta\} \in A$ and A is union-closed, we have $\alpha' = \alpha \cup \delta \in A$. Hence, we have $\alpha' \in A_\gamma$ and $(\alpha,\alpha') \in R$. Thus, R is a total binary relation on $A_\gamma \subseteq A$.

Define a relation R' between $A_\gamma \times A_\gamma$ and A_γ by

$$R' = \{((\alpha,\alpha'),\eta) \mid (\alpha \cup \alpha',\eta) \in R\}.$$

Then for each $\alpha,\alpha' \in A_\gamma$, since $\{\alpha,\alpha'\} \in A$ and A is union-closed, we have $\alpha \cup \alpha' \in A$, and hence $\alpha \cup \alpha' \in A_\gamma$. Since R is a total relation on A_γ, we have $R' \in \mathrm{mv}(A_\gamma \times A_\gamma, A_\gamma)$. Let $\tau \in \mathrm{Fin}(\gamma)$. Then, since $\tau \in \mathrm{Fin}(S)$, we have $\tau \in A$, and hence $\tau \in A_\gamma$. Therefore by RRS_2-uREA, there exists $A_0 \in A$ such that $\tau \in A_0 \subseteq A_\gamma$ and $\forall\alpha,\alpha' \in A_0 \exists\eta \in A_0(((\alpha,\alpha'),\eta) \in R')$. Letting $\beta = \bigcup A_0$, since A is union-closed, we have $\tau \subseteq \beta \in A$. Assume that $(\sigma,\Gamma) \in Z$ and $\sigma \subseteq \beta$. Then, since $\sigma \in \mathrm{Fin}(\beta)$, there exists $\alpha \in A_0$ such that $\sigma \subseteq \alpha$, and hence there exists $\alpha' \in A_0$ such that

$$\forall(\sigma',\Gamma') \in Z_\alpha \exists U \in \Gamma'(U \subseteq \alpha').$$

Therefore, since $(\sigma,\Gamma) \in Z_\alpha$, there exists $U \in \Gamma$ such that $U \subseteq \alpha' \subseteq \beta$. Thus, $\beta \in \mathcal{M}(Z)$, and so $\beta \in G$. □

7. Applications

In this section, we will give some applications of the results in the previous sections to algebra, topology and formal topology. Before giving these applications, we show the following property of a set-generated class.

Proposition 7.1. Let $\mathcal{X}$ be a class of inhabited subsets of a set S, and let $\mathrm{Min}(\mathcal{X})$ be a class of minimal elements of $\mathcal{X}$, that is, $\mathrm{Min}(\mathcal{X}) = \{x \in \mathcal{X} \mid \forall y \in \mathcal{X}(y \subseteq x \Rightarrow y = x)\}$. If $\mathcal{X}$ is set-generated, then $\mathrm{Min}(\mathcal{X})$ is a set.

Proof. Let G be a generating subset of $\mathcal{X}$. Assume that $x \in \mathrm{Min}(\mathcal{X})$. Then, since x is inhabited, there exists $s \in S$ such that $s \in x$, and therefore there exists $z \in G$ such that $\{s\} \in z \subseteq x$. Since x is minimal elements of $\mathcal{X}$, we have $x = z \in G$. Thus $\mathrm{Min}(\mathcal{X}) \subseteq \mathrm{Min}(G)$. Conversely, assume that $x \in \mathrm{Min}(G)$. Then for each $y \in \mathcal{X}$ with $y \subseteq x$, since there is $s \in S$ with $s \in y$, there exists $z \in G$ such that $\{s\} \in z \subseteq y \subseteq x$, and hence $z = y = x$, and hence $x \in \mathrm{Min}(\mathcal{X})$. Therefore, $\mathrm{Min}(G) \subseteq \mathrm{Min}(\mathcal{X})$. Clearly, $\mathrm{Min}(\mathcal{X}) = \mathrm{Min}(G)$ is a set, by Restricted Separation. □

7.1. *Algebra and topology*

Let $(R,+,\cdot,0,1)$ be a commutative ring. Then a subset $\mathfrak{p}$ of R is a prime ideal if

1. $0 \in \mathfrak{p}$;
2. $a,b \in \mathfrak{p} \Rightarrow a - b \in \mathfrak{p}$;
3. $a \in R, b \in \mathfrak{p} \Rightarrow a \cdot b \in \mathfrak{p}$;

https://doi.org/10.1017/S0960129513000236 Published online by Cambridge University Press

4. $a \cdot b \in \mathfrak{p} \Rightarrow a \in \mathfrak{p} \vee b \in \mathfrak{p}$;
5. $1 \notin \mathfrak{p}$.

Proposition 7.2. Assume SGA. Then, the class of prime ideals of a commutative ring is set-generated.

Proof. Let $(R, +, \cdot, 0, 1)$ be a commutative ring. Then a subset of R is a prime ideal if and only if it is a model of the following generalized geometric theory over R of rank 1:

$$\{p_0\} \cup \{\bigwedge\{p_a, p_b\} \to p_{a-b} \mid a, b \in R\} \cup \{p_b \to p_{a\cdot b} \mid a, b \in R\}$$
$$\cup \{p_{a\cdot b} \to \bigvee_{x\in\{a,b\}} p_x \mid a, b \in R\} \cup \{\neg p_1\}.$$

Hence, the class of prime ideals of $(R, +, \cdot, 0, 1)$ is set-generated, by Theorem 4.5. □

Corollary 7.3. Assume SGA. Then, the class of minimal prime ideals of a commutative ring forms a set.

Proof. By Proposition 7.2 and Proposition 7.1. □

Similarly, we are able to show that the class of ideals of a commutative ring is set-generated, and the class of minimal non-trivial ideals of a commutative ring forms a set.

A *neighbourhood space* (Bishop 1967) is a pair (X, τ) consisting of a set X and a subset τ of $\mathrm{Pow}(X)$ such that

1. $\forall x \in X \exists U \in \tau(x \in U)$,
2. $\forall x \in X \forall U, V \in \tau[x \in U \cap V \Rightarrow \exists W \in \tau(x \in W \subseteq U \cap V)]$.

We say that τ is an *open base* on X. A subset A of X is *open* if for each $x \in A$ there exists $U \in \tau$ such that $x \in U \subseteq A$. A function f between neighbourhood spaces (X, τ) and (Y, σ) is *continuous* if $f^{-1}(V)$ is open for each $V \in \sigma$.

Let $\{(X_i, \tau_i) \mid i \in I\}$ be a family of neighbourhood spaces, and let $\{f_i : X_i \to X \mid i \in I\}$ be a family of functions. Then an open base τ on the set X is *final* for the family $\{f_i \mid i \in I\}$ if for any neighbourhood space (Y, σ) and any function $g : X \to Y$, g is continuous if and only if $g \circ f_i : X_i \to Y$ is continuous for each $i \in I$.

Proposition 7.4 (Theorem 4.3 of Ishihara and Palmgren (2006)). Assume SGA. Let $\{(X_i, \tau_i) \mid i \in I\}$ be a family of neighbourhood spaces, and let $\{f_i : X_i \to X \mid i \in I\}$ be a family of functions. Then there exists a final open base on the set X for the family $\{f_i \mid i \in I\}$.

Proof. Note that, a subset U of X is a final open if and only if $f_i^{-1}(U)$ is open in (X_i, τ_i) for each $i \in I$. Hence, a subset of X is a final open if and only if it is a model of the following generalized geometric theory over X of rank 1:

$$\{p_{f_i(x)} \to \bigvee_{V\in\{W\in\tau_i \mid x\in W\}} \bigwedge_{y\in V} p_{f_i(y)} \mid x \in X_i, i \in I\}.$$

Therefore, the class of final opens on X is set-generated by Theorem 4.5, and its generating set forms a final open base. □

https://doi.org/10.1017/S0960129513000236 Published online by Cambridge University Press

7.2. *Formal topology*

In this subsection, we will give some applications of our main results in formal topology developed by Sambin (1987, 2003, forthcoming).

A *formal topology* $(S, \leqslant, \lhd)$ is a preordered set $(S, \leqslant)$ equipped with a class relation $\lhd \subseteq S \times \mathrm{Pow}(S)$ such that

1. $a \in U \Rightarrow a \lhd U$;
2. $a \lhd U$ and $\forall c \in U(c \lhd V) \Rightarrow a \lhd V$;
3. $a \lhd U$ and $a \lhd V \Rightarrow a \lhd \downarrow U \cap \downarrow V$;
4. $a \leqslant b \Rightarrow a \lhd \{b\}$,

where $\downarrow U = \{a \in S \mid \exists b \in U(a \leqslant b)\}$.

A formal topology $(S, \leqslant, \lhd)$ is *set-presented* if there exists a family of subsets $C(a, i)$ $(i \in I(a), a \in S)$ of S, called a *set-presentation* of $(S, \leqslant, \lhd)$, such that

$$a \lhd U \Leftrightarrow \exists i \in I(a)(C(a, i) \subseteq U).$$

Let $(S, \leqslant, \lhd)$ be a formal topology. Then a *formal point* of $(S, \leqslant, \lhd)$ is a subset $\alpha \subseteq S$ such that

1. α is inhabited;
2. $a, b \in \alpha \Rightarrow (\downarrow a \cap \downarrow b) \between \alpha$;
3. $a \in \alpha$ and $a \lhd U \Rightarrow U \between \alpha$,

where $\downarrow a = \downarrow \{a\}$ and $U \between V$ stands for $\exists a(a \in U \cap V)$. Note that, if $(S, \leqslant, \lhd)$ is set-presented with a set-presentation $C(a, i)$ $(i \in I(a), a \in S)$, then the last condition is equivalent to

$$\forall i \in I(a)[a \in \alpha \Rightarrow C(a, i) \between \alpha].$$

Proposition 7.5 (Theorem 4.3 of Palmgren (2006)). Assume SGA. Then, the class of formal points of a set-presented formal topology is set-generated.

Proof. Let $(S, \leqslant, \lhd)$ be a set-presented formal topology with a set-presentation $C(a, i)$ $(i \in I(a), a \in S)$. Then a subset of S is a formal point if and only if it is a model of the following generalized geometric theory over S of rank 1:

$$\{\textstyle\bigvee_{a \in S} p_a\} \cup \{\bigwedge\{p_a, p_b\} \to \bigvee_{c \in \downarrow a \cap \downarrow b} p_c \mid a, b \in S\}$$
$$\cup \{p_a \to \textstyle\bigvee_{b \in C(a,i)} p_b \mid i \in I(a), a \in S\}.$$

Hence, the class of formal points of $(S, \leqslant, \lhd)$ is set-generated, by Theorem 4.5. □

Corollary 7.6. Assume SGA. Then, the class of minimal formal points of a set-presented formal topology forms a set.

Proof. By Proposition 7.5 and Proposition 7.1. □

A formal topology $(S, \leqslant, \lhd)$ is T_1 if $\alpha \subseteq \beta \Rightarrow \alpha = \beta$ for each formal points α and β.

Corollary 7.7 (Corollary 4.4 of Palmgren (2006)). Assume SGA. Then, the class of formal points of a set-presented T_1 formal topology is a set.

https://doi.org/10.1017/S0960129513000236 Published online by Cambridge University Press

Proof. By Corollary 7.6. □

A *continuous morphism* from a formal topology $(S, \leqslant, \lhd)$ into a formal topology $(T, \leqslant', \lhd')$ is a relation $r \subseteq S \times T$ such that

1. $a\ r\ b$ and $b \lhd' V \Rightarrow a \lhd r^{-1}(V)$;
2. $a \lhd r^{-1}(T)$;
3. $a\ r\ b$ and $a\ r\ c \Rightarrow a \lhd r^{-1}(\downarrow b \cap \downarrow c)$.

Note that, if $(S, \leqslant, \lhd)$ and $(T, \leqslant', \lhd')$ are set-presented with set-presentations $C(a, i)$ $(i \in I(a), a \in S)$ and $D(b, j)$ $(j \in J(b), b \in T)$, respectively, then these conditions are respectively equivalent to

1. $\forall j \in J(b)[a\ r\ b \Rightarrow \exists i \in I(a) \forall a' \in C(a, i) \exists b' \in D(b, j)(a'\ r\ b')]$;
2. $\exists i \in I(a) \forall a' \in C(a, i) \exists b \in T(a'\ r\ b)$;
3. $a\ r\ b$ and $a\ r\ c \Rightarrow \exists i \in I(a) \forall a' \in C(a, i) \exists d \in \downarrow b \cap \downarrow c (a'\ r\ d)$.

Proposition 7.8. Assume SGA. Then, the class of continuous morphisms between set-presented formal topologies is set-generated.

Proof. Let $(S, \leqslant, \lhd)$ and $(T, \leqslant', \lhd')$ be set-presented formal topologies with set-presentations $C(a, i)$ $(i \in I(a), a \in S)$ and $D(b, j)$ $(j \in J(b), b \in T)$, respectively. Then a subset of $S \times T$ is a continuous morphism if and only if it is a model of the following generalized geometric theory over $S \times T$ of rank 2:

$$\begin{aligned}
&\{p_{(a,b)} \to \textstyle\bigvee_{i \in I(a)} \bigwedge_{a' \in C(a,i)} \bigvee_{b' \in D(b,j)} p_{(a',b')} \mid j \in J(b), a \in S, b \in T\} \\
&\cup \{\textstyle\bigvee_{i \in I(a)} \bigwedge_{a' \in C(a,i)} \bigvee_{b \in T} p_{(a',b)} \mid a \in S\} \\
&\cup \{\textstyle\bigwedge\{p_{(a,b)}, p_{(a,c)}\} \to \bigvee_{i \in I(a)} \bigwedge_{a' \in C(a,i)} \bigvee_{d \in \downarrow b \cap \downarrow c} p_{(a',d)} \mid a \in S, b, c \in T\}.
\end{aligned}$$

Hence, the class of continuous morphisms between $(S, \leqslant, \lhd)$ and $(T, \leqslant', \lhd')$ is set-generated, by Theorem 4.5. □

As the final application, we show the following proposition which is the crucial step in the construction, given by Palmgren (2005), of coequalizers in the category of set-presented formal topologies; see Ishihara and Kawai (2014) for applications of our main result in the categories of basic pairs and concrete spaces introduced by Sambin (2003, forthcoming).

Proposition 7.9 (Lemma 2–4 of Palmgren (2005)). Assume SGA. If r and s are continuous morphisms between a set-presented formal topology $(S, \leqslant, \lhd)$ and a formal topology $(T, \leqslant', \lhd')$, then the class $\mathcal{C}$ of subsets of T defined by

$$\mathcal{C} = \{U \in \mathrm{Pow}(T) \mid \forall a \in S(a \lhd r^{-1}(U) \Leftrightarrow a \lhd s^{-1}(U))\}$$

is set-generated.

Proof. Let $C(a, i)$ $(i \in I(a), a \in S)$ be a set-presentation of $(S, \leqslant, \lhd)$. Then, since

$$\begin{aligned}
&\forall a \in S(a \lhd r^{-1}(U) \Rightarrow a \lhd s^{-1}(U)) \\
&\Leftrightarrow \forall a \in S(a \in r^{-1}(U) \Rightarrow a \lhd s^{-1}(U)) \\
&\Leftrightarrow \forall b \in T[b \in U \Rightarrow \forall a \in r^{-1}(b) \exists i \in I(a) \forall a' \in C(a, i) \exists b' \in s(a')(b' \in U)],
\end{aligned}$$

https://doi.org/10.1017/S0960129513000236 Published online by Cambridge University Press

a subset of T is in the class $\mathcal{C}$ if and only if it is a model of the following generalized geometric theory over T of rank 3:

$$\{p_b \to \bigwedge_{a\in r^{-1}(b)} \bigvee_{i\in I(a)} \bigwedge_{a'\in C(a,i)} \bigvee_{b'\in s(a')} p_{b'} \mid b \in T\}$$
$$\cup \{p_b \to \bigwedge_{a\in s^{-1}(b)} \bigvee_{i\in I(a)} \bigwedge_{a'\in C(a,i)} \bigvee_{b'\in r(a')} p_{b'} \mid b \in T\}.$$

Hence, the class $\mathcal{C}$ is set-generated, by Theorem 4.5. □

Acknowledgements

The second author is grateful to Erik Palmgren and Benno van den Berg for discussions on an early draft of this paper during his stay at the Institut Mittag-Leffler in November 2009, and to the Japan Society for the Promotion of Science (Grant-in-Aid for Scientific Research (C) No. 19500012 and No. 23540130) for partly supporting the research. The first author is grateful to the Cambridge Newton Institute for providing an excellent working environment for his work on the later drafts of this paper, carried out during his stay at the Institute in the period January to March, 2012. The authors wish to thank Peter Schuster for suggesting to them some applications of our main result. They also thank the referees for comments and suggestions.

References

Aczel, P. (1978) The type theoretic interpretation of constructive set theory. In: Macintyre, A., Pacholski, L. and Paris, J. (eds.) *Logic Colloquium '77*, North-Holland, Amsterdam 55–66.

Aczel, P. (1982) The type theoretic interpretation of constructive set theory: Choice principles. In: Troelstra, A.S. and van Dalen, D. (eds.) *The L.E.J. Brouwer Centenary Symposium*, North-Holland, Amsterdam 1–40.

Aczel, P. (1986) The type theoretic interpretation of constructive set theory: Inductive definitions. In: Marcus, R. B. *et al.* (eds.) *Logic, Methodology, and Philosophy of Science VII*, North-Holland, Amsterdam 17–49.

Aczel, P. (2006) Aspects of general topology in constructive set theory. *Annals of Pure and Applied Logic* **137** 3–29.

Aczel, P. (2008) The relation reflection scheme. *MLQ Mathematical Logic Quarterly* **54** 5–11.

Aczel, P. and Rathjen, M. (2001) Notes on constructive set theory. *Technical Report No. 40*, Institut Mittag-Leffler, The Royal Swedish Academy of Sciences.

Berger, J., Ishihara, H., Palmgren, E. and Schuster, P. (2012) A predicative completion of a uniform space. *Annals of Pure and Applied Logic* **163** 975–980.

Bishop, E. (1967) *Foundations of Constructive Mathematics*, McGraw-Hill, New York.

Bishop, E. and Bridges, D. (1985) *Constructive Analysis*, Springer, Berlin.

Bridges, D. and Richman, F. (1987)*Varieties of Constructive Mathematics*, Cambridge University Press, Cambridge.

Bridges, D. and Vîţă, L. (2006) *Techniques of Constructive Analysis*, Springer, New York.

Crosilla, L., Ishihara, H. and Schuster, P. (2005) On constructing completions. *Journal of Symbolic Logic* **70** 969–978.

Curi, G. (2012) Topological inductive definitions. *Annals Pure Applied Logic* **163** 1471–1483.

Ishihara, H. and Kawai, T. (2014) Completeness and cocompleteness of the categories of basic pairs and concrete spaces. *Mathematical Structures in Computer Science*. to appear.

https://doi.org/10.1017/S0960129513000236 Published online by Cambridge University Press

Ishihara, H. and Palmgren, E. (2006) Quotient topologies in constructive set theory and type theory. *Annals of Pure and Applied Logic* **141** 257–265.

Martin-Löf, P. (1984) *Intuitionistic Type Theory*, Studies in Proof Theory volume 1, Bibliopolis, Naples.

Mines, R., Richman, F. and Ruitenburg, W. (1988) *A Course in Constructive Algebra*, Springer-Verlag, New York.

Myhill, J. (1975) Constructive set theory. *Journal of Symbolic Logic* **40** 347–382.

Palmgren, E. (2005) Quotient spaces and coequalisers in formal topology. *Journal of Universal Computer Science* **11** 1996–2007.

Palmgren, E. (2006) Maximal and partial points in formal spaces. *Annals of Pure and Applied Logic* **137** 291–298.

Sambin, G. (1987) Intuitionistic formal spaces – a first communication. In: Skordev, D. (ed.) *Mathematical Logic and its Applications*, Plenum, New York 187–204.

Sambin, G. (2003) Some points in formal topology. *Theoretical Computer Science* **305** 347–408.

Sambin, G. *The Basic Picture and Positive Topology*, Clarendon Press, Oxford, forthcoming.

Troelstra, A. S. and van Dalen, D. (1988) *Constructivism in Mathematics*, volume I and II, North-Holland, Amsterdam.

van den Berg, B. (2013) Non-deterministic inductive definitions. *Archive for Mathematical Logic* **52** 113–135.

https://doi.org/10.1017/S0960129513000236 Published online by Cambridge University Press