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Nonlinear dispersive instabilities in Kelvin–Helmholtz MHD flows

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The nonlinear evolution of Kelvin–Helmholtz instability is examined in 2+1 dimensions in the context of magnetohydrodynamics. When the velocity difference U is less than the critical velocity U_c, the equation governing the amplitude evolves into a self-focusing singularity. The self-focusing of waves predominates at short wavelengths, is directionally dependent, and also depends sensitively on the strength of the applied magnetic field. The minimum velocity that allows the existence of self-focusing increases with increasing magnetic field strength. The explosive instability at the second-harmonic resonance is also investigated.

1. Introduction

The linear theory governing the Kelvin–Helmholtz instability in magnetohydrodynamics (MHD) when two superposed fluids move with uniform speeds was developed by Chandrashekhar (1961). This instability, which arises as a consequence of a relative drift velocity of two fluids along the surface of discontinuity, has great relevance to various physical phenomena such as cometary tails and the magnetospheric boundary. The nonlinear aspects of such an instability in ideal fluids were investigated by Drazin (1970), Nayfeh and Saric (1972) and Weissman (1979). In MHD, Kant and Malik (1982) and Malik and Singh (1985) studied the nonlinear evolution of wave packets for such a type of instability. They reported that for subcritical MHD flows where the velocity difference U is smaller than the critical velocity U_c, the dynamical equation yielding the amplitude is a Schrödinger equation with cubic nonlinearity. Kant and Malik (1982) also showed that a wave train of constant amplitude is unstable against modulations in the presence of a magnetic field. Using inverse-scattering transforms, Zakharov and Shabat (1972) obtained the solution of the nonlinear Schrödinger equation in 1+1 dimensions. Their analysis reveals that smooth initial data lead to localized soliton solutions, and is related to the terminal state of the modulational instability.

For postcritical values of U, the amplitude evolution for supercritical MHD flows is described by a nonlinear Klein–Gordon equation. From this equation, Malik and Singh (1985) derived kink, antikink and solitary-wave solutions. If one neglects the spatial variation in amplitude then the Klein–Gordon equation gives rise to exponential-decay, subcritical, supercritical and explosive

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instabilities. Recently, considerable effort has been devoted to the phenomenon of self-focusing and collapse: in nonlinear optics by Zakharov and Synakh (1975), in plasma physics by Zakharov (1972) and in two-layer Kelvin–Helmholtz flows by Gibbon and McGuinness (1980). When self-focusing occurs in a medium, the wave field becomes singular at a point called the focus and the amplitude of the wave packet becomes infinite, leading to turbulent bursts. This effect may be incorporated in studying the envelope properties of (2+1)-dimensional wave packet. The resulting Schrödinger equation in two space dimension yields a solution that evolves from smooth data and becomes singular at a point in space after a finite time. In this paper, we shall examine the Kelvin–Helmholtz magnetohydrodynamic flow in the subcritical regime, when the velocity difference U is less than the critical velocity U_c.

We shall also investigate the phenomenon of nonlinear internal resonance for the case where the fundamental and second-harmonic mode both travel with the same phase velocity when $U \geqslant U_m$. Here U_m is the velocity at the marginal curve. Such a resonance arises owing to the occurrence of a zero divisor in the representation of second-order solution of the Kelvin–Helmholtz flow. This type of resonance was analysed for gravity waves by Wilton (1915), McGoldrick (1970) and Simmons (1969), for streaming flows by Murakami (1988) and for ferrofluids by Malik and Singh (1989). In this presentation, we plan to study the internal resonance when the frequencies and wavenumbers of two interacting waves satisfy the conditions $\omega_2 = 2\omega_1$ and $k_2 = 2k_1$. In order to incorporate this effect, we use a special scaling to determine the nonlinear interaction coefficients. The solution is then obtained by the method of multiple scales. We derive the dynamical equations governing the amplitudes of the fundamental mode and the second-harmonic mode. Analytical solutions of these dynamical equations are not possible. These equations are therefore integrated numerically by employing the fourth-order Runge–Kutta method. The computed solution thus arrived at has eight degrees of freedom, and furnishes a super-exponential instability at finite time.

2. Formulation

We consider two superposed fluids separated with the interface $z = 0$. The fluids are semi-infinite, inviscid, incompressible and perfectly conducting. The fluid with density ρ_1 occupies the half-space $z < 0$ whereas $z > 0$ is taken by the fluid with density ρ_2. The upper and lower fluids are moving with uniform velocities $U^{(1)}$ and $U^{(2)}$ along the x axis. The system is subjected to a uniform tangential magnetic field $\mathbf{H}(H, 0, 0)$ permeating both fluids. The system is also under the influence of gravitational force. The basic equations describing this MHD flow are

$$\frac{d\mathbf{u}^{(i)}}{dt} = -\boldsymbol{\nabla}\,\Pi^{(i)} + (\mathbf{h}^{(i)}\cdot\boldsymbol{\nabla})\,\mathbf{h}^{(i)} + \mathbf{g}, \tag{1}$$

$$\frac{\partial \mathbf{h}^{(i)}}{\partial t} = (\mathbf{h}^{(i)}\cdot\boldsymbol{\nabla})\,\mathbf{u}^{(i)} - (\mathbf{u}^{(i)}\cdot\boldsymbol{\nabla})\,\mathbf{h}^{(i)}, \tag{2}$$

$$\boldsymbol{\nabla}\cdot\mathbf{u}^{(i)} = 0, \qquad \boldsymbol{\nabla}\cdot\mathbf{h}^{(i)} = 0, \tag{3}$$

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where $i = 1$ and 2 correspond to the lower and upper regions respectively. The fluid velocity is $\mathbf{u}^{(i)} = (u^{(i)}, v^{(i)}, w^{(i)})$, the acceleration due to gravity $\mathbf{g} = (0, 0, -g)$, the magnetic field is $\mathbf{h}^{(i)}$ and the total pressure is $\Pi^{(i)} = p^{(i)}/\rho_i + \frac{1}{2}|h^{(i)}|^2$.

The boundary conditions at the free interface $z = \eta(x, y, z)$ are

$$\frac{\partial \eta}{\partial t} + (\mathbf{u}^{(i)} \cdot \boldsymbol{\nabla})\, \eta = w^{(i)}, \tag{4}$$

$$\mathbf{h}^{(1)} \cdot \mathbf{n}^{(1)} = \mathbf{h}^{(2)} \cdot \mathbf{n}^{(2)}, \tag{5}$$

where $\mathbf{n}$ is the outward-drawn normal and η denotes the elevation of the free surface above the unperturbed level. The continuity of normal stress at the perturbed surface $z = \eta(x, y, z)$ requires that

$$\begin{aligned}&\rho_1\, \Pi^{(1)} - \rho_2\, \Pi^{(2)} - (\rho_1 - \rho_2)\, g\eta \\ &= T\left\{\frac{\partial^2 \eta}{\partial x^2}\left[1 + \left(\frac{\partial \eta}{\partial y}\right)^2\right] + \frac{\partial^2 \eta}{\partial y^2}\left[1 + \left(\frac{\partial \eta}{\partial x}\right)^2\right] - 2\frac{\partial^2 \eta}{\partial x\, \partial y}\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}\right\}\left[1 + \left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2\right]^{-3/2},\end{aligned} \tag{6}$$

where T is the surface tension.

3. Nonlinear focusing

We plan to study the propagation of nonlinear wave packets of small but finite amplitude propagating in the principal direction along the x axis with modulation allowed in both the x and y directions.

Following Kant and Malik (1982) (see also Pusri and Malik 1995), and using the method of multiple scales, (1)–(6) yield the linear dispersion relation

$$D(\omega, k) = -(\alpha_1\, \sigma_1^2 + \alpha_2\, \sigma_2^2) + k^2 V_A^2 + k^3 + k = 0, \tag{7}$$

where

$$\alpha_1 = \frac{\rho_1}{\rho_1 + \rho_2}, \qquad \alpha_2 = \frac{\rho_2}{\rho_1 + \rho_2}, \qquad V_A^2 = \frac{\mu H^2}{\rho_1 + \rho_2},$$

$$\sigma_i = lU^{(i)} - \omega, \qquad k^2 = l^2 + m^2, \qquad i = 1, 2. \tag{8}$$

Here, all the perturbed quantities are normalized with respect to the characteristic length $[T/(\rho_1 - \rho_2)\, g]^{1/2}$ and the characteristic speed $[(\rho_1 - \rho_2)\, gT/(\rho_1 + \rho_2)]^{1/2}$. The quantities l and m are the components of the wave vector along the x and y directions respectively. Equation (7) furnishes the critical velocity U_c at $k = k_c = 1$:

$$U_c = (\alpha_1\, \alpha_2)^{-1/2}\, (2 + V_A^2)^{1/2}. \tag{9}$$

In the dispersion relation (7), we observe the magnetic field having a stabilizing influence on the wave motion. Since we are interested primarily in studying the amplitude modulation of the progressive wave, we assume that $|U^{(1)} - U^{(2)}| < U_c$ and proceed to obtain the second-order solution. This second-order solution is uniformly valid subject to the solvability condition

$$\frac{\partial A}{\partial t} + v_l \frac{\partial A}{\partial x_1} + v_m \frac{\partial A}{\partial y_1} = 0. \tag{10}$$

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The quantities x_1 and y_1 are the fast space scales, and v_l and v_m, which represent group velocities along the x and the y directions respectively, are given by

$$v_l = [2(\alpha_1 U^{(1)}\sigma_1 + \alpha_2 U^{(2)}\sigma_2) - k^{-3/2}(3k^2+1) - 21V_A^2][2(\alpha_1\sigma_1+\alpha_2\sigma_2)]^{-1}, \quad (11)$$

$$v_m = -\frac{m}{k}(3k^2+1)[2(\alpha_1\sigma_1+\alpha_2\sigma_2)]^{-1}. \quad (12)$$

It is interesting to remark here that, unlike ordinary gravity waves, for Kelvin–Helmholtz flows the group velocities along the x and y directions are not parallel to the wave velocities. Furthermore, the asymptotic expansions break down for $k^2 = \frac{1}{2}$, which corresponds to a second-harmonic resonance. At resonance, the fundamental and second-harmonic modes propagate with the same phase speed, resulting in energy transfer from one mode to another. We shall deal with this case in the subsequent section. In this section, we avoid such a resonance and assume that $k^2 \neq \frac{1}{2}$. With a view to describing the amplitude modulation of the progressive waves at the surface of the fluid, we need to proceed to the third-order solution, which yields the following nonlinear equation for the evolution of wave packets in $2+1$ dimensions:

$$2i\left(\frac{\partial A}{\partial t_2} + v_l\frac{\partial A}{\partial x_2} + v_m\frac{\partial A}{\partial y_2}\right) + P_1\frac{\partial^2 A}{\partial x_1^2} + 2P_2\frac{\partial^2 A}{\partial x_1\partial y_1} + P_3\frac{\partial^2 A}{\partial y_1^2} = Q_1|A|^2 A, \quad (13)$$

where

$$P_1 = (\alpha_1\sigma_1+\alpha_2\sigma_2)^{-1}\{2[\alpha_1(U^{(1)})^2 + \alpha_2(U^{(2)})^2] - k^{-1}(3k^2+1) + l^2k^{-3}(1-3k^2) - 4v_l(\alpha_1 U^{(1)} + \alpha_2 U^{(2)}) - 2v_l^2 - 2V_A^2\}, \quad (14a)$$

$$P_2 = (\alpha_1\sigma_1+\alpha_2\sigma_2)^{-1}\left[\frac{lm}{k^3}(1-3k^2) - 2(\alpha_1 U^{(1)} + \alpha_2 U^{(2)})\,v_l - 2v_l\,v_m\right], \quad (14b)$$

$$P_3 = (\alpha_1\sigma_1+\alpha_2\sigma_2)^{-1}\left[\frac{m^2}{k^3}(1-3k^2) - \frac{1}{k}(3k^2+1)\right], \quad (14c)$$

$$Q_1 = \tfrac{1}{2}[4(2k^2-1)^{-1}(\alpha_1\sigma_1^2 - \alpha_2\sigma_2^2 - k^2V_A^2) - 4k^2 - k^4]. \quad (14d)$$

The partial differential equation (13) is elliptic or hyperbolic, depending on the sign of $P = (P_2^2 - P_1P_3)$.

When P is negative, we have an elliptic case, for which the appropriate transformations are

$$\zeta_1 = \frac{x_1}{P_1^{1/2}}, \qquad \eta_1 = \left(P_3 - \frac{P_2^2}{P_1}\right)^{-1/2}\left(y_1 - \frac{P_2}{P_1}x_1\right). \quad (15)$$

Under such a transformation, (13) now reduces to a standard two-dimensional nonlinear Schrödinger equation. On rewriting (13) in the group-velocity reference frame, we obtain the elliptic equation

$$2i\frac{\partial A}{\partial\tau} + \frac{\partial^2 A}{\partial\zeta_1^2} + \frac{\partial^2 A}{\partial\eta_1^2} = Q(l, m, U, V_A^2)\,|A|^2 A, \quad (16)$$

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where

$$t_2 = \tau, \qquad Q = \frac{Q_1}{P_1}[(\alpha_1\sigma_1 + \alpha_2\sigma_2)\rho_1]^{-1}. \tag{17}$$

Equation (16) is the standard nonlinear Schrödinger equation in $2+1$ dimensions. For the hyperbolic case, P is positive. We introduce the transformations

$$\zeta_1 = \frac{x_1}{(P_1)^{1/2}}, \qquad \eta_1 = \left(\frac{P_2^2}{P_1^2} - P_3\right)^{-1/2}\left(\frac{P_2}{P_1}x_1 - y_1\right). \tag{18}$$

Proceeding as before, we get

$$2i\frac{\partial A}{\partial \tau} + \frac{\partial^2 A}{\partial \zeta_1^2} - \frac{\partial^2 A}{\partial \eta_1^2} = Q|A|^2 A. \tag{19}$$

Equations (16) and (19) can be expressed in the form

$$2i\frac{\partial A}{\partial \tau} + \Delta_1\frac{\partial^2 A}{\partial \zeta_1^2} + \Delta_2\frac{\partial^2 A}{\partial \eta_1^2} = Q|A|^2 A, \tag{20}$$

with $\Delta_1 = 1$, and

$$\Delta_2 = \begin{cases} 1 & \text{when } P_2^2 - P_1P_3 < 0, \\ -1 & \text{when } P_2^2 - P_1P_3 > 0. \end{cases} \tag{21}$$

We shall now discuss the stability of the solution of (20). Following Zakharov (1972), we define two integrals of motion for the elliptic case $P < 0$:

$$N = \iint |A|^2\, d\zeta_1\, d\eta_1, \tag{22}$$

$$H = \frac{1}{2}\iint\left(\Delta_1\left|\frac{\partial A}{\partial \zeta_1}\right|^2 + \Delta_2\left|\frac{\partial A}{\partial \eta_1}\right|^2 + \frac{Q}{2}|A|^2\right)d\zeta_1\, d\eta_1, \tag{23}$$

where N is the wave action and H is the Hamiltonian. Use of (22)–(23) in (20) yields the virial equation

$$\frac{\partial^2 I}{\partial \tau^2} = 4H. \tag{24}$$

Here I, the moment of inertia of the wave packet, is defined as

$$I = \iint(\zeta_1^2 + \eta_1^2)\,|A|^2\, d\zeta_1\, d\eta_1. \tag{25}$$

Equation (24) when integrated gives

$$\langle I\rangle = \frac{2\tau^2 H}{N} + C_1\tau + C_2, \tag{26}$$

where the average value of the inertia is given by

$$\langle I\rangle = \frac{1}{N}\iint(\zeta_1^2 + \eta_1^2)\,|A|^2\, d\zeta_1\, d\eta_1. \tag{27}$$

The constants C_1 and C_2 are to be determined from the initial conditions. For focusing or collapse to occur in a finite time, we require that $\langle I\rangle$ tend to zero,

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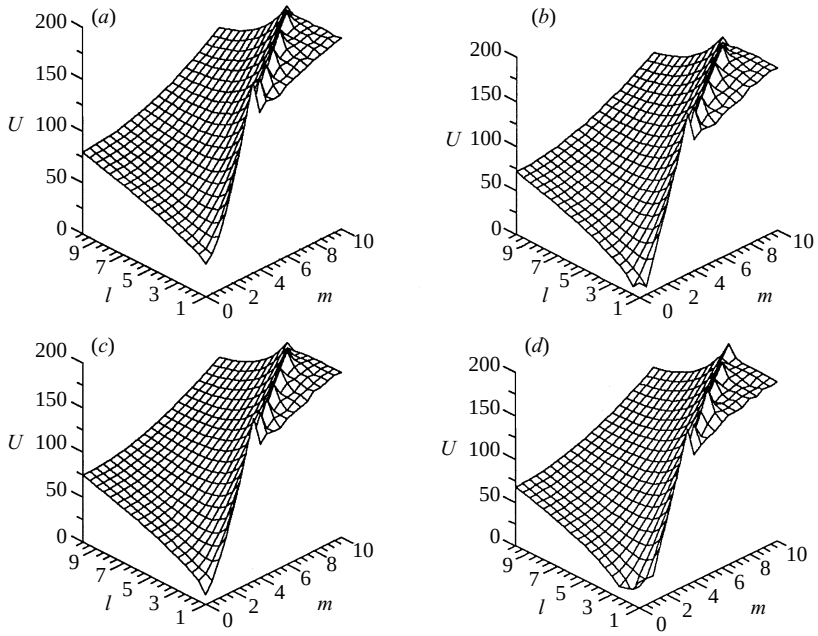


Figure 1. A plot of minimum velocity difference U against l and m when $\rho_1 = 1.1 \times 10^3$ kg m^{-3} and $\rho_2 = 1.25 \times 10^3$ kg m^{-3}, with different values of V_A^2, showing self-focusing: (*a*) $V_A^2 = 0$; (*b*) 1.0; (*c*) 1.5; (*d*) 2.0.

implying thereby $Q < 0$. This demands that $|A| \to 0$ everywhere except at the focus. Total wave collapse will take place provided the blow-up singularity does not occur at times earlier than the collapse predicted by the virial theorem. Rasmussen and Rypdal (1986) and Berge (1994) have shown that the virial theorem cannot be used to predict the time of blow up, and it fails to give information about whether or not $\langle I \rangle$ is concentrated at the singular point. In an actual physical case, the dissipative effects will inhibit the earlier collapse from occurring. However, Berge (1994) has shown that collapse will not take place in the hyperbolic two-dimensional case. It is interesting to note here that the corresponding equation (23) for the Hamiltonian H does not contain any nonlinear term for the hyperbolic case. We should like to point out here that for collapse to occur in finite time, the group-velocity rate P and the nonlinear interaction coefficient Q must change sign as the quantities ρ, l, m and V_A are varied.

The numerical search reveals the existence of a region for nonlinear focusing when wavenumbers $k^2 > k_{\rm res}^2 = \frac{1}{2}$. Figures 1–3 depict three-dimensional plots for the minimum value of U versus the wavenumbers l and m at which such a phenomenon takes place for the elliptic case only. The graphs show that the focusing condition is sensitive to the Alfvén velocity, and the velocity at which this condition is satisfied increases with increasing magnetic field strength and density ratio. Moreover, the focusing has directional dependence, and is significant at shorter wavelengths.

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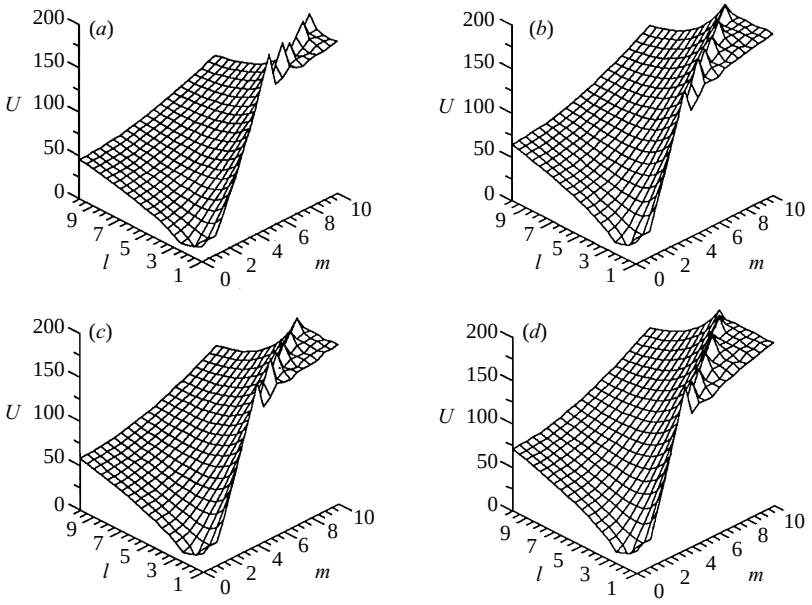


Figure 2. A plot of minimum velocity difference U against l and m for different values of ρ_1, with $\rho_2 = 1.25 \times 10^3$ kg m^{-3} and $V_A^2 = 1.5$, showing self-focusing: (*a*) $\rho_1 = 0.5 \times 10^3$ kg m^{-3}; (*b*) 0.8×10^3 kg m^{-3}; (*c*) 1.0×10^3 kg m^{-3}; (*d*) 1.2×10^3 kg m^{-3}.

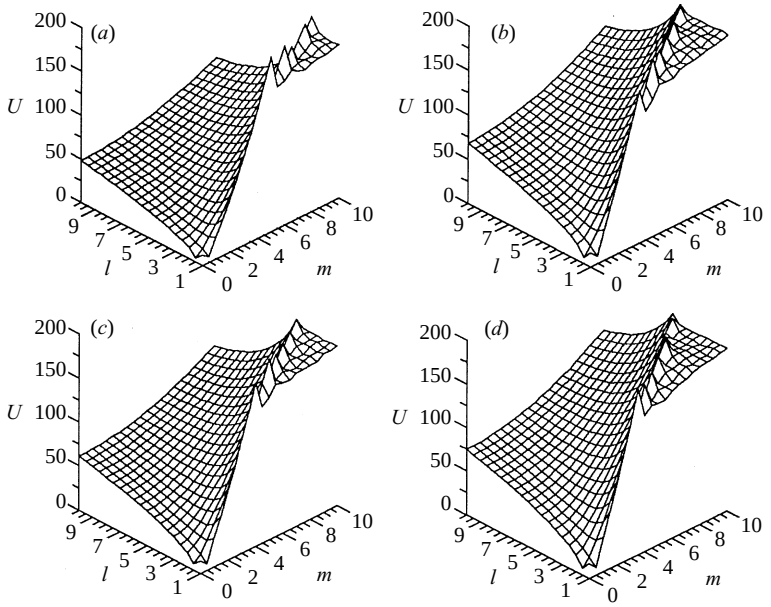


Figure 3. A plot of minimum velocity difference U against l and m for different values of ρ_1, with $\rho_2 = 1.25 \times 10^3$ kg m^{-3} and $V_A^2 = 2.0$, showing self-focusing: (*a*) $\rho_1 = 0.5 \times 10^3$ kg m^{-3}; (*b*) 0.8×10^3 kg m^{-3}; (*c*) 1.0×10^3 kg m^{-3}; (*d*) 1.2×10^3 kg m^{-3}.

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4. Second-harmonic resonance and explosive stability

In this section, we determine a uniform asymptotic solution valid near the second-harmonic resonance wavenumber $k^2 = \frac{1}{2}$. We consider the nonlinear resonant interaction between two marginally unstable modes of small but finite amplitude, and introduce a small parameter $\Delta = U/U_M - 1$. This parameter measures the departure of the system from the marginal value U_M, which is obtained from the linear dispersion relation by invoking the conditions $\omega_2 = 2\omega_1 + O(\Delta)$ and $k_2 = 2k_1 + O(\Delta)$. We assume that the entire band of unstable waves are present in the initial conditions. We shall investigate the stability of the wave packet in the neighbourhood of the neutral surface. We use the method of multiple scales to discuss harmonic resonance in the unstable region. The scaling used in this paper is, however, different from that of Nayfeh and Saric (1972), where the analysis was confined to a stable region only – and that for ideal fluids. To get a significant interaction in the unstable region at the second-harmonic resonance, the proper balance between the parameters Δ and ϵ is obtained as $\Delta = O(\epsilon^2)$. The various scales in this case are $x_n = \epsilon^n x$ and $t_n = \epsilon^n t$. On substituting these scales into (1)–(6), the linear dispersion relation for the fundamental and second harmonic mode is derived as

$$D(\omega, k, \mu, V_A^2) = \alpha_{1n}\,\sigma_1^2 + \alpha_{2n}\,\sigma_2^2 - k_n - k_n^3 - k_n\,V_A^2 = 0, \tag{28}$$

where

$$\sigma_{1n} = k_n\,U^{(1)} - \omega_n, \qquad \sigma_{2n} = k_n\,U^{(2)} - \omega_n, \qquad n = 1, 2.$$

The neutral point of instability occurs when $k^2 = \frac{1}{2}$, and the critical velocity U is a minimum at k on the marginal curve. This minimum value is

$$U_m(k_1) = U_m(k_2) = \frac{k_1 + k_1^3 + k_1^2\,V_A^2}{\alpha_1\,\alpha_2\,k_1^2}, \qquad \text{where} \quad k_1^2 = k_{\text{res}}^2 = \tfrac{1}{2}. \tag{29}$$

The hydromagnetic system is unstable for $U > U_m$. The system bifurcates into a new unstable state when nonlinear terms are included in the theory. The type of scalings used in this paper imply that the nonlinear coupling at the second-harmonic resonance is expected to play a dominant role in transferring energy from the mean flow.

We shall confine ourselves to the case where $U = U_m(1 \pm \epsilon^2)$, where ϵ denotes the detuning parameter. To derive the equation for amplitude, we solve the third-order problem for $U = U_m$ and $k_{\text{res}}^2 = \frac{1}{2}$. For solution to be uniformly valid, the nonsecularity condition is

$$\sum_{n=1}^{2} (-2\alpha_1\,\alpha_{1n} - 2\alpha_2\,\alpha_{2n})\frac{\partial A_n}{\partial t_1} + (-2\alpha_1\,\alpha_{1n}\,U^{(1)} - 2\alpha_2\,\alpha_{2n}\,U^{(2)} + 2k_n + V_A^2)\frac{\partial A_n}{\partial t_1} = 0. \tag{30}$$

Equation (30) may now be expressed as

$$\frac{\partial D}{\partial \omega_n}\frac{\partial A_n}{\partial t_1} - \frac{\partial D}{\partial k_n}\frac{\partial A_n}{\partial x_1} = 0, \qquad n = 1, 2. \tag{31}$$

On the marginal surface, $U = U_m$ and $k_m = k_{\text{res}}$, which gives $\partial D/\partial \omega_n = 0$, so

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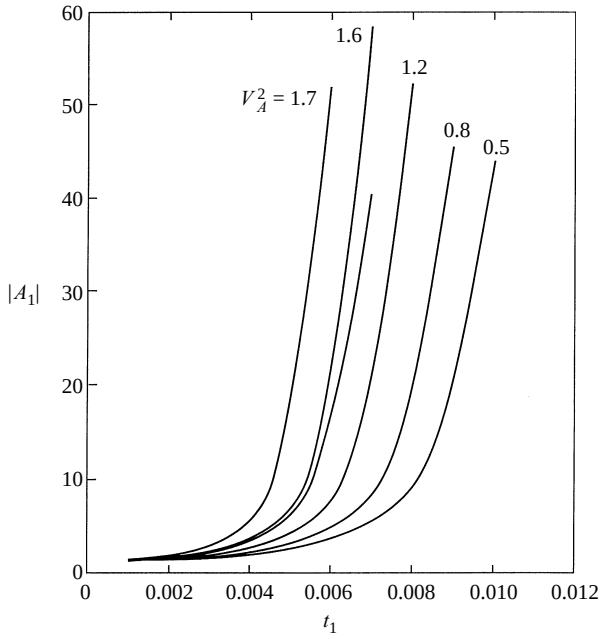


Figure 4. A sketch of the fundamental mode $|A_1|$ versus time for different values of V_A^2, with $\rho_1 = 2.0 \times 10^3$ kg m^{-3}, $\rho_2 = 1.25 \times 10^3$ kg m^{-3} and $\Delta = 0.08$.

that (31) yields $\partial D/\partial k_n \neq 0$. This implies that

$$\frac{\partial A_n}{\partial x_1} = 0, \qquad n = 1, 2. \tag{32}$$

We now proceed to $O(\epsilon^4)$. As before, we substitute the second- and third-order solutions into the equations for the fourth order. Since the homogeneous part of the fourth order has a non-trivial solution, the inhomogeneous problem has a solution if and only if the inhomogeneous part is orthogonal to every solution of the adjoint homogeneous problem. Thus the dynamical equations describing the modulation of the amplitudes of the fundamental and second harmonic modes are

$$\frac{\partial^2 A_1}{\partial t_1^2} + \frac{1}{\rho_1 + \rho_2} \frac{\partial A_1}{\partial x_2} \left[\frac{\partial D}{\partial k_1}\right]_{k_1 = k_m} = 2\alpha_1 \alpha_2 k_1^2 U_m \Delta A_1 + J\sigma A_2 \overline{A_1} \exp(i\Gamma), \tag{33}$$

$$\frac{\partial^2 A_2}{\partial t_1^2} + \frac{1}{\rho_1 + \rho_2} \frac{\partial A_2}{\partial x_2} \left[\frac{\partial D}{\partial k_2}\right]_{k_2 = k_m} = 4\alpha_1 \alpha_2 k_2^2 U_m \Delta A_2 + J\sigma A_1^2 \exp(-i\Gamma), \tag{34}$$

where

$$J = 2k_1^3[\alpha_1 \alpha_2 U_m^2(\alpha_2 - \alpha_1) + V_A^2].$$

The system of equations (33) and (34) does not have analytical solutions. For simplicity, we ignore the spatial variation and consider the case of perfect resonance, i.e. $\Gamma = 0$. We then integrate these equations numerically using the fourth-order Runge–Kutta method. The Figs 4 and 5 show graphs of the amplitudes $|A_1|$ and $|A_2|$ versus time t_1 for various values of the Alfvén velocity

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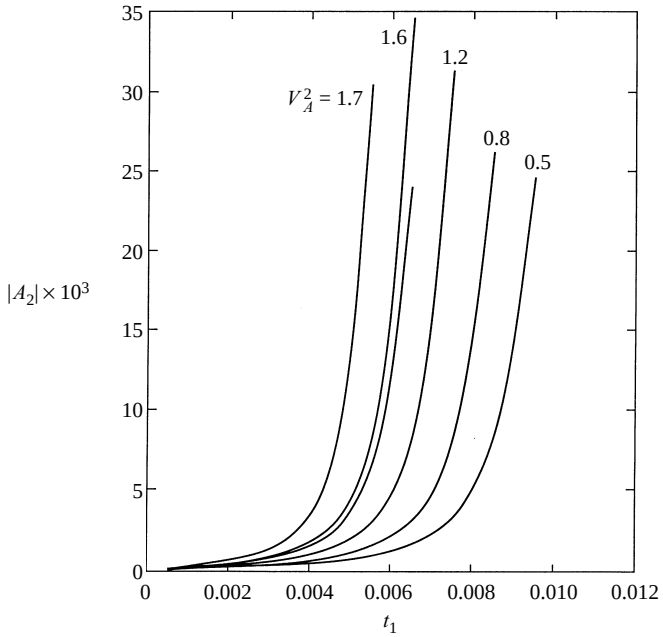


Figure 5. A graph of the second-harmonic mode $|A_2|$ versus time for different values of V_A^2, with $\rho_1 = 2.0 \times 10^3$ kg m^{-3} and $\rho_2 = 1.25 \times 10^3$ kg m^{-3}.

V_A, keeping the parameter $\Delta = 0.08$. We have taken the initial data as $\mathrm{Re}(A_1) = 1$, $\mathrm{Im}(A_1) = 0.5$, $\mathrm{Re}(A_2) = -0.5$ and $\mathrm{Im}(A_2) = -1.5$. The initial speeds of A_1 and A_2 are taken to be zero. Figures 4 and 5 show that the amplitudes $|A|$ of the fundamental and second-harmonic modes become unbounded at finite time, thus giving rise to an explosive instability at the second-harmonic resonance. The magnetic field has a destabilizing influence on such an instability, delaying the time τ_B at which the explosive instability sets in. We observe similar behaviour with different sets of initial data. In a real physical situation, this type of growth will be checked by dissipative mechanisms like viscosity. At time $\tau = \tau_B$, the asymptotic method developed in this paper breaks down. As the global properties of (33) and (34) are difficult to investigate analytically, we have neglected the spatial variation of the amplitude. The spatial modulation will affect the time development of the instability. Whether it will lead to the suppression of divergence or chaos is an open question. Preliminary numerical results with periodic boundary conditions do point towards the redistribution of energy in various modes, but no firm conclusions can be drawn.

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