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No-iteration mixed distributive laws

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We present the no-iteration version of a mixed distributive law of a monad over a comonad in a general 2-category. We present the simplifications that occur in the case of the 2-category **Cat**.

1. Introduction

Distributive laws for monads were introduced by Beck (1969). The mixed case, involving a monad and a comonad, was introduced by Van Osdol (1971), and it also appears in Burroni's paper (Burroni 1973). Taking Street's formal theory of monads (Street 1972) as foundation , Power and Watanabe (2002), give a study of the possible dualities for such a distributive law. Since the work of Moggi (1991), monads have become important in operational semantics, and thus, distributive laws have also been studied there (see for instance (Turi and Plotkin 1997)). More recently, in the context of bialgebras, these distributive laws have been given the name of entwining structures as in Brzeziński and Majid (1998).

In his Ph.D. thesis, Walters (1970) presented a no-iteration version of monads (such a presentation avoids the iteration of the endofunctor involved) and also of the categories of algebras for them, a work that came to the attention of the first named author of the present paper after the publication of Marmolejo and Wood (2010). The starting point of Marmolejo and Wood (2010) was Manes' no-iteration presentation of monads that appears in Manes (1976) without a name, but that was called a monad in an extension form in Manes (2003). The name they go by in computer science is *Kleisli triple* following Moggi (1991). Then, Marmolejo and Wood (2010) proceeded to the similar description of the algebras, and the corresponding simplifications of distributive laws between monads that such an approach produces. The present paper deals in the same spirit with mixed distributive laws.

In Section 2, we recall some of the relevant definitions and, for the convenience of the reader, we also recall some of the work done in Marmolejo and Wood (2010) regarding the no-iteration presentation of algebras and of distributive laws. As an illustration of how this works in practice, we include the examples of the product comonad (and its dual, the exceptions monad), and the streams comonad. In Section 3, we recall the definition

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of a mixed distributive law of a monad over a comonad in **Cat**, as well as the no-iteration definition of a monad and of its algebras. We also do this for comonads to fix notation. We then present the form that our main theorem takes in the particular case where the 2-category is **Cat**. As an example, we present what we believe is a new result concerning the product comonad determined by an object E, namely that the distributive laws of an arbitrary monad $\mathbb{S}$ over this comonad are in bijective correspondence with $\mathbb{S}$-algebra structures for E. We then observe, dually, that the same kind of result is true for the exceptions monad. For the next example, we consider the streams comonad and an arbitrary monad, and we observe that there is always a natural candidate for a distributive law of the monad over the comonad. We observe that in the case where the monad is the list monad, this candidate recovers an example given in Power and Watanabe (2002). We close the section with a slight generalization of the original example of Van Osdol (1971); the comonad of this example also appeared in Marmolejo (1998). The category of coalgebras for this comonad for a fixed topological space is equivalent, as Van Osdol shows, to the category of sheaves over the topological space. In Section 4, we work in an arbitrary 2-category, and we recall from Marmolejo and Wood (2010) the definition of a monad as an extension system, as well as the categories of algebras, and we prove the theorem for mixed distributive laws in its no-iteration mode.

It is worthwhile to mention that the 'bind' operation, which computer scientists often employ, is nothing but an exponential transpose of the extension operation used below, namely

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extension: (B --> SA) --> (SB --> SA)
bind:      SB --> ((B -->SA) --> SA)
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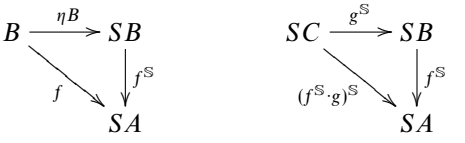
2. No-iteration monads and algebras

We first recall the no-iteration way to give a monad as in Manes (1976), but see also Walters (1970). As we mentioned in the introduction, Manes called these *monads in extension form* in Manes (2003); the name they go by in computer science is *Kleisli triples* following Moggi (1991). Thus, a monad $\mathbb{S} = (S, \eta, (\ )^{\mathbb{S}})$ on a category $\mathbf{C}$ consists of the following:

— a function $S : \mathsf{Ob}(\mathbf{C}) \to \mathsf{Ob}(\mathbf{C})$;
— for every $A \in \mathbf{C}$, an arrow $\eta A : A \to SA$ in $\mathbf{C}$,
— for every arrow $f : B \to SA$ in $\mathbf{C}$, an arrow $f^{\mathbb{S}} : SB \to SA$,

in such a way that

— for every $A \in \mathbf{C}$, $(\eta A)^{\mathbb{S}} = 1_{SA}$,
— for every $f : B \to SA$ and every $g : C \to SB$ in $\mathbf{C}$, the diagrams



commute.

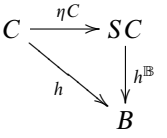
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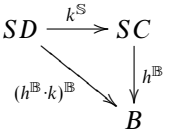
We recall that given an arrow $h : A \to B$ in $\mathbf{C}$, we obtain Sh as

$$Sh = (\eta B \cdot h)^{\mathbb{S}} : SA \to SB. \tag{1}$$

It is perhaps less known that there is a similar no-iteration description of the algebras, as given in Marmolejo and Wood (2010), but again see Walters (1970). The category of Eilenberg–Moore algebras $\mathbf{C}^{\mathbb{S}}$ can be described in the following manner:

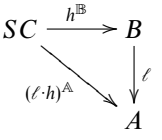
— objects are of the form $\mathbb{B} = (B, (\ )^{\mathbb{B}})$, where B is an object of $\mathbf{C}$, and every arrow $h : C \to B$ in $\mathbf{C}$ is assigned an arrow $h^{\mathbb{B}} : SC \to B$, in such a way that for every $h : C \to B$ and $k : D \to SC$, the diagrams





commute,

— a morphism $(B, (\ )^{\mathbb{B}}) \to (A, (\ )^{\mathbb{A}})$ is an arrow $\ell : B \to A$ in $\mathbf{C}$ such that for every $h : C \to B$, the diagram



commutes,

— composition is as in $\mathbf{C}$.

We recall that we recover the usual $\mathbb{S}$-algebra structure as

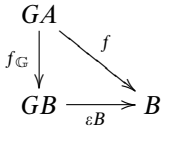
$$(1_B)^{\mathbb{B}} : SB \to B. \tag{2}$$

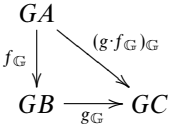
To fix notation we also write down the comonad case. Thus, a comonad $\mathbb{G} = (G, \varepsilon, (\ )_{\mathbb{G}})$ on a category $\mathbf{C}$ consists of

— a function $G : \mathsf{Ob}(\mathbf{C}) \to \mathsf{Ob}(\mathbf{C})$;
— for every $A \in \mathbf{C}$, an arrow $\varepsilon A : GA \to A$ in $\mathbf{C}$,
— for every arrow $f : GA \to B$ in $\mathbf{C}$, an arrow $f_{\mathbb{G}} : GA \to GB$,

in such a way that

— for every $A \in \mathbf{C}$, $(\varepsilon A)_{\mathbb{G}} = 1_{GA}$,
— for every $f : GA \to B$ and every $g : GB \to C$ in $\mathbf{C}$, the diagrams



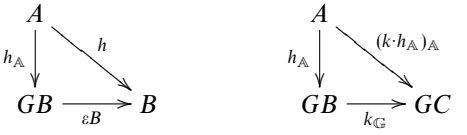


commute.

The category of Eilenberg–Moore coalgebras $\mathbf{C}_{\mathbb{G}}$ can be described as follows:

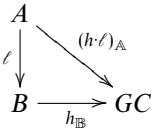
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— objects are of the form $\mathbb{A} = (A, (\ )_{\mathbb{A}})$, where A is an object of $\mathbf{C}$, and every arrow $h : A \to B$ in $\mathbf{C}$ is assigned an arrow $h_{\mathbb{A}} : A \to GB$, in such a way that for every $h : A \to B$ and $k : GB \to C$, the diagrams



commute,

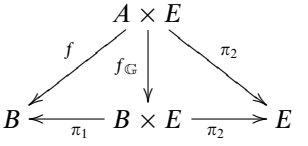
— a morphism $(A, (\ )_{\mathbb{A}}) \to (B, (\ )_{\mathbb{B}})$ is an arrow $\ell : A \to B$ in $\mathbf{C}$ such that for every $h : B \to C$, the diagram



commutes,

— composition is as in $\mathbf{C}$.

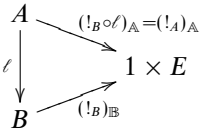
Example 2.1. Consider the product comonad $\mathbb{G}$ given by a fixed set E on **Set** (this comonad appears in Uustalu and Vene (2008) for example). We describe the structure of the comonad. For every A, $GA = A \times E$. The counit $\varepsilon A : A \times E \to A$ is the first projection. For any $f : GA = A \times E \to B$, $f_{\mathbb{G}}$ is $\langle f, \pi_2 \rangle$, that is, $f_{\mathbb{G}}$ is the only arrow that makes the following diagram:



commute (where the π's denote projections). We want to show, using the description of the coalgebras given above, that the category of coalgebras $\mathbf{Set}_{\mathbb{G}}$ for $\mathbb{G}$ is isomorphic to $\mathbf{Set}/E$ (another possible route is to apply Beck's monadicity theorem). Assume we have a $\mathbb{G}$-coalgebra $(A, (\ )_{\mathbb{A}})$. Now the unique morphism $!_A : A \to 1$ induces $(!_A)_{\mathbb{A}} : A \to 1 \times E$, and we define $\alpha : A \to E$ as the composite

$$\alpha = (A \xrightarrow{(!_A)_{\mathbb{A}}} 1 \times E \xrightarrow{\pi_2} E).$$

Observe that for any $h : A \to B$ we have that $(!_{B \times E})_{\mathbb{G}} \circ h_{\mathbb{A}} = (!_{B \times E} \circ h_{\mathbb{A}})_{\mathbb{A}} = (!_A)_{\mathbb{A}}$. Composing this with $\pi_2 : 1 \times E \to E$ we obtain that $\pi_2 \circ h_{\mathbb{A}} = \alpha$ (where now $\pi_2 : B \times E \to E$). Since $\pi_1 \circ h_{\mathbb{A}} = \varepsilon B \circ h_{\mathbb{A}} = h$, we have thus shown that α tells us what $(\ )_{\mathbb{A}}$ is: $h_{\mathbb{A}} = \langle h, \alpha \rangle : A \to B \times E$. If now $\ell : (A, (\ )_{\mathbb{A}}) \to (B, (\ )_{\mathbb{B}})$ is a morphism of coalgebras, then the diagram

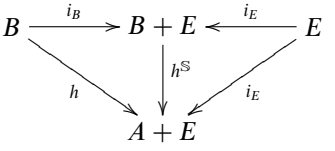


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commutes. Composing with $\pi_2 : 1 \times E \to E$ we obtain a morphism in $\mathbf{Set}/E$ from the induced arrow $A \to E$ to the induced arrow $B \to E$ defined above.

In the opposite direction, given any $\alpha : A \to E$, it is not hard to see that the formula $h_{\mathbb{A}} = \langle h, \alpha \rangle : A \to B \times E$ gives A a structure of a $\mathbb{G}$-coalgebra. And that for any morphism in $\mathbf{Set}/E$ we obtain a morphism of $\mathbb{G}$-coalgebras.

Example 2.2. Dual to the product comonad from Example 2.1 we have the exceptions monad (this can be found in Moggi (1991) for example). We again take a set E and define $SA = A + E$ for any $A \in \mathbf{Set}$; $\eta A : A \to A + E$ is the first inclusion and for $h : B \to A + E$, $h^{\mathbb{S}} : B + E \to A + E$ is the only map that makes the diagram



commute. We apply the same reasoning as in Example 2.1 and, mutatis mutandis, we obtain that the category of $\mathbb{S}$ algebras $\mathbf{Set}^{\mathbb{S}}$ is isomorphic to $E/\mathbf{Set}$.

Example 2.3. Consider the streams comonad $\mathbb{G}$ on $\mathbf{Set}$. That is, $A \mapsto A^\omega$ on objects, $\varepsilon A : A^\omega \to A$ is evaluation at zero, and for $h : A^\omega \to B$ we have $h_{\mathbb{G}} : A^\omega \to B^\omega$ given by

$$(h_{\mathbb{G}}(\varphi))(n) = h(\varphi \circ s^n),$$

for any $\varphi \in A^\omega$, $n \in \omega$, where $s : \omega \to \omega$ is the successor map.

For any $A \in \mathbf{Set}$, denote by $\mathrm{ev}_n : A^\omega \to A$ evaluation at n. Thus, $\varepsilon A = \mathrm{ev}_0$. Observe that for any $m \in \omega$ and any $\varphi \in A^\omega$

$$((\mathrm{ev}_n)_{\mathbb{G}}(\varphi))(m) = \mathrm{ev}_n(\varphi \circ s^m) = \varphi(s^m(n)) = \varphi(n + m) = (\varphi \circ s^n)(m),$$

thus $(\mathrm{ev}_n)_{\mathbb{G}} = (- \circ s^n) : A^\omega \to A^\omega$. It follows that $\mathrm{ev}_1 \circ (\mathrm{ev}_n)_{\mathbb{G}} = \mathrm{ev}_{n+1}$.

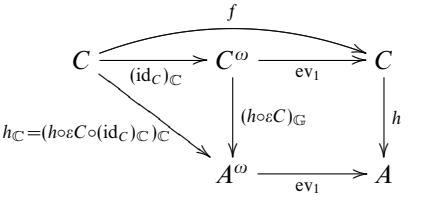
Assume now that $C \in \mathbf{Set}_{\mathbb{G}}$ is a coalgebra. We define $f : C \to C$ as the composite

$$f := (C \xrightarrow{(\mathrm{id}_C)_{\mathbb{C}}} C^\omega \xrightarrow{\mathrm{ev}_1} C).$$

We want to show that f determines the structure of C as a $\mathbb{G}$-coalgebra. Specifically, we want to show that for any $h : C \to A$ we have that $h_{\mathbb{C}} : C \to A^\omega$ is given by

$$(h_{\mathbb{C}}(c))(n) = h(f^n(c)).$$

In order to do this, we observe that the diagram



commutes. That is, for any $h : C \to A$ we have that

$$ev_1 \circ h_{\mathbb{C}} = h \circ f. \tag{3}$$

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We then observe that

$$\mathrm{ev}_2 \circ h_{\mathbb{C}} = \mathrm{ev}_1 \circ (\mathrm{ev}_1)_{\mathbb{G}} \circ h_{\mathbb{C}} = \mathrm{ev}_1 \circ (\mathrm{ev}_1 \circ h_{\mathbb{C}})_{\mathbb{C}} = \mathrm{ev}_1 \circ (h \circ f)_{\mathbb{C}} = h \circ f \circ f,$$

where the last equation follows from Equation (3) with $h \circ f$ in place of h. Next we have

$$\mathrm{ev}_3 \circ h_{\mathbb{C}} = \mathrm{ev}_1 \circ (\mathrm{ev}_2)_{\mathbb{G}} \circ h_{\mathbb{C}} = \mathrm{ev}_1 \circ (\mathrm{ev}_2 \circ h_{\mathbb{C}})_{\mathbb{C}} = \mathrm{ev}_1 \circ (h \circ f \circ f)_{\mathbb{C}} = h \circ f \circ f \circ f.$$

and so on. We conclude that

$$h_{\mathbb{C}}(c) = \langle h(f^n(c)) \rangle_{n=0}^{\infty}$$

for any $c \in C$. Thus f determines the structure of $\mathbb{G}$-coalgebra of C.

Conversely, given any set C and any endomorphism $f : C \to C$, we obtain a $\mathbb{G}$-coalgebra structure if, for any $h : C \to A$, we define $h_{\mathbb{C}} : C \to A^{\omega}$ such that

$$(h_{\mathbb{C}}(c))(n) = h(f^n(c)).$$

Dealing in a similar fashion with the coalgebra homomorphisms one is lead to conclude that the category of coalgebras $\mathbf{Set}_{\mathbb{G}}$ is isomorphic to the category of discrete dynamical systems in discrete time, that is, to the category whose objects are pairs (C, f) where $C \in \mathbf{Set}$ and $f : C \to C$ is a function, and whose morphisms $g : (C, f) \to (C', f')$ are functions that make the diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ C' & \xrightarrow[f']{} & C' \end{array}$$

commute.

This result is surely known, but since we were unable to find an appropriate reference in the literature we write it as a proposition.

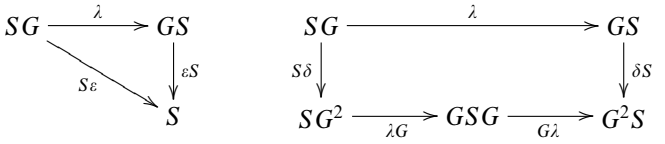
Proposition 2.1. The category of coalgebras for the streams comonad is isomorphic to the category of discrete dynamical systems in discrete time (as described above).

3. Mixed distributive laws in Cat

In this section, we consider a no-iteration distributive law of a monad $\mathbb{S}$ over a comonad $\mathbb{G}$. For no-iteration distributive laws between monads we refer the reader to Marmolejo and Wood (2010). Let $\mathbf{C}$ be a category. Recall (see Van Osdol (1971) or Burroni (1973)) that given a monad $\mathbb{S} = (S, \eta, \mu)$ and a comonad $\mathbb{G} = (G, \varepsilon, \delta)$, both on $\mathbf{C}$, a distributive law of the monad $\mathbb{S}$ over the comonad $\mathbb{G}$ consists of a natural transformation $\lambda : SG \to GS$ such that the diagrams

$$\begin{array}{ccc} G & & \\ {\scriptstyle \eta G}\downarrow & \searrow^{G\eta} & \\ SG & \xrightarrow[\lambda]{} & GS \end{array} \qquad \begin{array}{ccccc} S^2G & \xrightarrow{S\lambda} & SGS & \xrightarrow{\lambda S} & GS^2 \\ {\scriptstyle \mu G}\downarrow & & & & \downarrow{\scriptstyle G\mu} \\ SG & & \xrightarrow[\lambda]{} & & GS \end{array} \tag{4}$$

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commute.

We have the following equivalent way to give the data for a mixed distributive law in **Cat**.

Theorem 3.1. Given a monad $\mathbb{S}$ and a comonad $\mathbb{G}$ on the same category $\mathbf{C}$, a distributive law of $\mathbb{S}$ over $\mathbb{G}$ can equivalently be given by the following data:

— for every $A \in \mathbf{C}$, an $\mathbb{S}$-algebra $(GSA, (\ )^\lambda)$,
— for every $A \in \mathbf{C}$, a $\mathbb{G}$-coalgebra $(SGA, (\ )_\lambda)$,

in such a way that

— for every $t : GA \to SB$ in $\mathbf{C}$, $(t_{\mathbb{G}})^\lambda = (t^{\mathbb{S}})_\lambda$,
— for every $h : B \to SA$, $G(h^{\mathbb{S}}) : (GSB, (\ )^\lambda) \to (GSA, (\ )^\lambda)$ is a morphism of $\mathbb{S}$-algebras,
— for every $k : GB \to A$, $S(k_{\mathbb{G}}) : (SGB, (\ )_\lambda) \to (SGB, (\ )_\lambda)$ is a morphism of $\mathbb{G}$-coalgebras.

Under the conditions of this theorem, the usual distributive law $\lambda : SG \to GS$ is obtained as $\lambda A = (G\eta A)^\lambda$ or as $\lambda A = (S\varepsilon A)_\lambda$. In the opposite direction, given $\lambda : SG \to GS$ a usual distributive law, and $p : X \to GSA$ and $q : SGA \to Y$ we define

$$p^\lambda = (SX \xrightarrow{Sp} SGSA \xrightarrow{\lambda SA} GS^2A \xrightarrow{G\mu A} GSA)$$

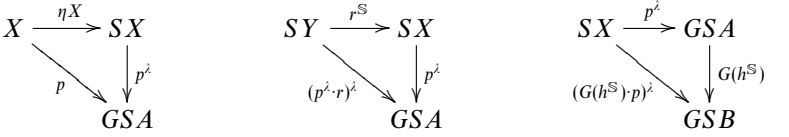
and

$$q_\lambda = (SGA \xrightarrow{S\delta A} SG^2A \xrightarrow{\lambda GA} GSGA \xrightarrow{Gq} GY).$$

As we mentioned in the introduction, we give the proof for monads in 2-categories in the following section, and we leave the particular case for **Cat** to the reader. We observe that the condition $(t_{\mathbb{G}})^\lambda = (t^{\mathbb{S}})_\lambda$ can be replaced by the condition that for every A

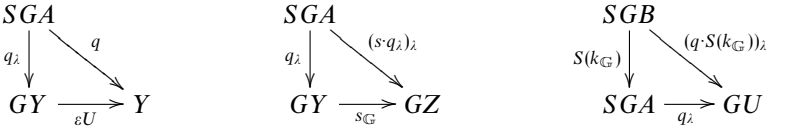
$$(G\eta A)^\lambda = (S\varepsilon A)_\lambda.$$

We do spell out in elementary terms what the theorem says. A distributive law of the monad $\mathbb{S}$ over the comonad $\mathbb{G}$ is given by assigning, for every arrow $p : X \to GSA$ in $\mathbf{C}$ an arrow $p^\lambda : SX \to GSA$, for every arrow $q : SGA \to Y$ in $\mathbf{C}$ an arrow $q_\lambda : SGA \to GY$, in such a way that for every $t : GB \to SA$, $(t_{\mathbb{G}})^\lambda = (t^{\mathbb{S}})_\lambda$, and for arrows $h : A \to SB$, $r : Y \to SX$, $k : GB \to A$ and $s : GY \to Z$ in $\mathbf{C}$, the diagrams



(5)

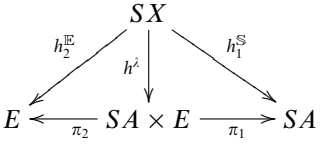
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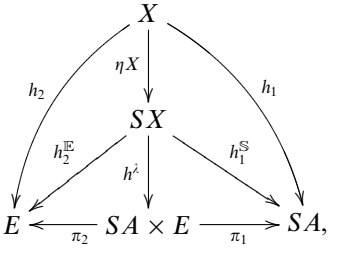
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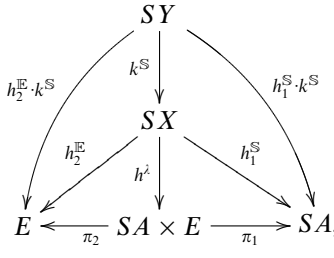
Example 3.1. Let $\mathbb{S}$ be any monad on **Set** and let $\mathbb{G}$ be the product comonad considered in Example 2.1. We first show that any $\mathbb{S}$-algebra structure, $(E, (\ )^{\mathbb{E}})$, on the set E induces a distributive law of $\mathbb{S}$ over $\mathbb{G}$. We induce the structure following Theorem 3.1. So we first give a $\mathbb{S}$-algebra structure to $GSA = SA \times E$ as follows. Given $h = \langle h_1, h_2 \rangle : X \to SA \times E$ we define h^{λ} as the unique arrow that makes the diagram

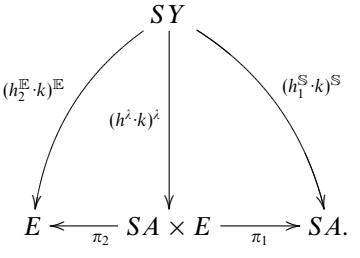


commute, that is, $h^{\lambda} = \langle h_1^{\mathbb{S}}, h_2^{\mathbb{E}} \rangle$. The first equation in Equation (5) is given by the following commutative diagram:



(where the upper-left triangle commutes because $(E, (\ )^{\mathbb{E}})$ is an $\mathbb{S}$-algebra and upper triangle on the right commutes because $\mathbb{S}$ is a monad) that is, $h^{\lambda} \cdot \eta X = \langle h_1, h_2 \rangle = h$. To prove the second equation in Equation (5) we first observe that, given $k : Y \to SX$, we have that $h^{\lambda} \cdot k = \langle h_1^{\mathbb{S}} \cdot k, h_2^{\mathbb{E}} \cdot k \rangle$, and we consider the commutative diagrams





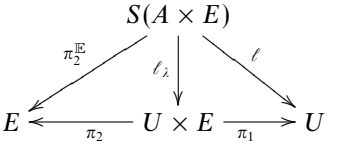
Since $(h_2^{\mathbb{E}} \cdot k)^{\mathbb{E}} = h_2^{\mathbb{E}} \cdot k^{\mathbb{S}}$ and $(h_1^{\mathbb{S}} \cdot k)^{\mathbb{S}} = h_1^{\mathbb{S}} \cdot k^{\mathbb{S}}$, we conclude that $(h^{\lambda} \cdot k)^{\lambda} = h^{\lambda} \cdot k^{\mathbb{S}}$. Thus $(GSA, (\ )^{\lambda})$ is indeed an $\mathbb{S}$-algebra.

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From Example 2.1 we know that giving a structure of $\mathbb{G}$-coalgebra over $SGA = S(A \times E)$ is equivalent to giving a morphism $S(A \times E) \to E$. So we take the morphism

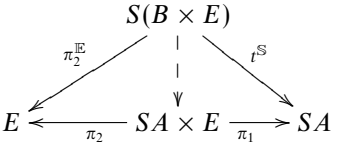
$$S(A \times E) \xrightarrow{\pi_2^{\mathbb{E}}} E,$$

where $\pi_2 : A \times E \to E$ is the projection. This means that for $\ell : SGA \to U$ we have that ℓ_λ is the unique arrow that makes



commute. We immediately obtain that the first two diagrams in Equation (6) commute.

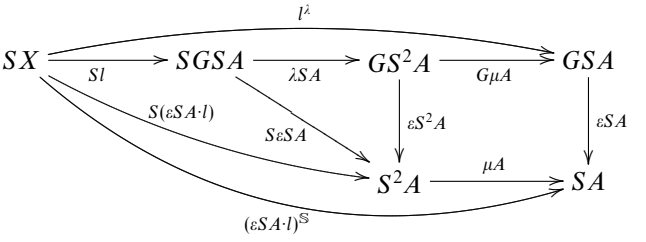
Given $t : GB \to SA$ we have that $(t_{\mathbb{G}})^\lambda$ and $(t^{\mathbb{S}})_\lambda : SGB \to GSA$ both make the diagram



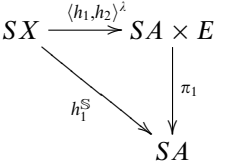
commute, and thus are equal. The third equation in Equations (5) and (6) are left to the reader. Thus we obtain the distributive law of the monad $\mathbb{S}$ over the comonad $\mathbb{G}$:

$$\lambda A = (G\eta A)^\lambda = (S\varepsilon A)_\lambda = \langle S\pi_1, \pi_2^{\mathbb{E}} \rangle : S(A \times E) \to SA \times E.$$

We show next that this is the only possible way to produce such a distributive law. Assume we have a distributive law λ of $\mathbb{S}$ over $\mathbb{G}$. Observe that, for every $l : X \to GSA$, we have the commutative diagram



which says that $\varepsilon SA : (GSA, (\ )^\lambda) \to (SA, (\ )^{\mathbb{S}})$ is a morphism of $\mathbb{S}$-algebras. In the particular case at hand, l is of the form $\langle h_1, h_2 \rangle : X \to SA \times E$ and the above diagram becomes

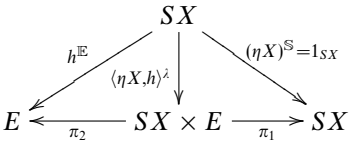


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For an arrow $h : X \to E$ we define $h^{\mathbb{E}}$ as the composite

$$h^{\mathbb{E}} = (SX \xrightarrow{\langle \eta X, h\rangle^{\lambda}} SX \times E \xrightarrow{\pi_2} E) .$$

Thus the diagram



(7)

commutes. We show that $(E, (\ )^{\mathbb{E}})$ is an $\mathbb{S}$-algebra. Indeed

$$h^{\mathbb{E}} \cdot \eta X = \pi_2 \cdot \langle \eta X, h\rangle^{\lambda} \cdot \eta X = \pi_2 \cdot \langle \eta X, h\rangle = h,$$

and, for $k : Y \to SX$,

$$\begin{aligned}
h^{\mathbb{E}} \cdot k^{\mathbb{S}} &= \pi_2 \cdot \langle \eta X, h\rangle^{\lambda} \cdot k^{\mathbb{S}} && \text{definition of } (\ )^{\mathbb{E}} \\
&= \pi_2 \cdot (\langle \eta X, h\rangle^{\lambda} \cdot k)^{\lambda} && (\ )^{\lambda} \text{ is an } \mathbb{S}\text{-algebra} \\
&= \pi_2 \cdot \langle k, h^{\mathbb{E}} \cdot k\rangle^{\lambda} && \text{diagram (7) preceded by } k \\
&= \pi_2 \cdot (k^{\mathbb{S}} \times E \cdot \langle \eta Y, h^{\mathbb{E}} \cdot k\rangle)^{\lambda} \\
&= \pi_2 \cdot (G(k^{\mathbb{S}}) \cdot \langle \eta Y, h^{\mathbb{E}} \cdot k\rangle)^{\lambda} \\
&= \pi_2 \cdot G(k^{\mathbb{S}}) \cdot \langle \eta Y, h^{\mathbb{E}} \cdot k\rangle^{\lambda} && G(k^{\mathbb{S}}) \text{ morphism of } \mathbb{S}\text{-algebras} \\
&= \pi_2 \cdot \langle \eta Y, h^{\mathbb{E}} \cdot k\rangle^{\lambda} \\
&= (h^{\mathbb{E}} \cdot k)^{\mathbb{E}} && \text{definition of } (\ )^{\mathbb{E}}.
\end{aligned}$$

To show that λ is the distributive law induced by $(E, (\ )^{\mathbb{E}})$ we must show that for any $\langle h_1, h_2\rangle : X \to SA \times E$ we have that

$$\langle h_1, h_2\rangle^{\lambda} = \langle h_1^{\mathbb{S}}, h_2^{\mathbb{E}}\rangle.$$

Since we already know that $\pi_1 \cdot \langle h_1, h_2\rangle^{\lambda} = h_1^{\mathbb{S}}$ we are left with showing that $\pi_2 \cdot \langle h_1, h_2\rangle^{\lambda} = h_2^{\mathbb{E}}$. But

$$\pi_2 \cdot \langle h_1, h_2\rangle^{\lambda} = \pi_2 \cdot (G(h_1^{\mathbb{S}}) \cdot \langle \eta_X, h_2\rangle)^{\lambda} = \pi_2 \cdot G(h_1^{\mathbb{S}}) \cdot \langle \eta_X, h_2\rangle^{\lambda} = \pi_2 \cdot \langle \eta_X, h_2\rangle^{\lambda} = h_2^{\mathbb{E}}.$$

Observe that we have not used any of the special properties of **Set** above. Therefore, the theorem that we obtain is more general.

Theorem. Let **A** be a category with finite products, $\mathbb{G}$ be the product comonad on **A** with respect to an object E of **A**, and let $\mathbb{S}$ be any monad on **A**. There is a bijective correspondence between distributive laws of $\mathbb{S}$ over $\mathbb{G}$ and $\mathbb{S}$-algebra structures for E.

Example 3.2. Everything we said in Example 3.1 for the product monad can be (dually) said for the exceptions monad. So again, mutatis mutandis, we obtain

Theorem. Let **A** be a category with finite coproducts, $\mathbb{S}$ be the exceptions monad on **A** with respect to an object E of **A**, and let $\mathbb{G}$ be any comonad on **A**. There is a bijective correspondence between distributive laws of $\mathbb{S}$ over $\mathbb{G}$ and $\mathbb{G}$-coalgebra structures for E.

Example 3.3. Take $\mathbb{G}$ to be the streams comonad of Example 2.3, and take $\mathbb{S}$ to be an arbitrary monad on **Set**. For every $A \in$ **Set** we have that A^{ω} has its canonical $\mathbb{G}$-coalgebra

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structure, that according to Example 2.3 corresponds an endomorphism

$$f_A : A^\omega \longrightarrow A^\omega.$$

It is not hard to see that $f_A(\varphi) = \varphi \circ s$ for every $\varphi \in A^\omega$, where $s : \omega \to \omega$ is again the successor map. Thus, $SGA = S(A^\omega)$ immediately has the $\mathbb{G}$-coalgebra structure

$$S(f_A) : S(A^\omega) \longrightarrow S(A^\omega).$$

That is, for $\ell : S(A^\omega) \to U$ we have that $\ell_\lambda : S(A^\omega) \to U^\omega$ given by $(\ell_\lambda(x))(n) = \ell(S(f_A)^n(x))$ satisfies the first two conditions of Equation (6). Furthermore, observe that for every $v : B^\omega \to A$, the diagram

$$\begin{array}{ccc} S(B^\omega) & \xrightarrow{S(f_B)} & S(B^\omega) \\ {\scriptstyle S(v_{\mathbb{G}})}\downarrow & & \downarrow{\scriptstyle S(v_{\mathbb{G}})} \\ S(A^\omega) & \xrightarrow[S(f_A)]{} & S(A^\omega) \end{array}$$

commutes (since the diagram without the functor S commutes). Thus, $S(v_{\mathbb{G}})$ is a morphism of $\mathbb{G}$-coalgebras and the third diagram from Equation (6) commutes. We obtain a candidate for a distributive law of $\mathbb{S}$ over $\mathbb{G}$ as

$$(S\varepsilon A)_\lambda : S(A^\omega) \to (SA)^\omega.$$

It is not hard to see that for an $x \in S(A^\omega)$ we have that

$$(S\varepsilon A)_\lambda(x) = \langle S(\mathrm{ev}_0)(x), S(\mathrm{ev}_1)(x), \ldots, S(\mathrm{ev}_n)(x), \ldots \rangle.$$

To check that this is a distributive law we would need to verify that the conditions of Equation (5) are satisfied when for any $h : X \to (SA)^\omega$ we define h^λ as the composite

$$SX \xrightarrow{Sh} S((SA)^\omega) \xrightarrow{(S\varepsilon SA)_\lambda} (SSA)^\omega \xrightarrow{(\mu A)^\omega} (SA)^\omega.$$

We verify that this is the case when $\mathbb{S}$ is the list monad or free monoid monad, and we recover the distributive law of $\mathbb{S}$ over $\mathbb{G}$ given in Power and Watanabe (2002). Thus for any set A, $SA = \{[a_1, \ldots, a_m] | m \in \mathbb{N}, a_i \in A, i = 1, \ldots, m\}$ is the set of finite lists, $\eta A : A \to SA$ is such that $\eta A(a) = [a]$, and for $f : B \to SA$, $f^{\mathbb{S}} : SB \to SA$ is such that $f^{\mathbb{S}}([b_1, \ldots, b_m]) = f(b_1) \sim \cdots \sim f(b_m)$, where $\sim$ denotes concatenation. Observe that, in this case, an object in $S(A^\omega)$ is of the form $[\langle a_n^1 \rangle_n, \ldots, \langle a_n^m \rangle_n]$, and then $(S\varepsilon A)_\lambda([\langle a_n^1 \rangle_n, \ldots, \langle a_n^m \rangle_n])$ is

$$\langle [a_0^1, \ldots, a_0^m], [a_1^1, \ldots, a_1^m], [a_2^1, \ldots, a_2^m], [a_3^1, \ldots, a_3^m], \ldots \rangle.$$

Now, the calculation of the above recipe for h^λ yields the formula

$$h^\lambda[x_1, \ldots, x_m] = \langle hx_1(n) \sim \cdots \sim hx_m(n) \rangle_{n \in \mathbb{N}}.$$

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That is, if we write $hx_1, \ldots, hx_m$ vertically as follows:

$$\begin{array}{cccc} hx_1(0) & hx_2(0) & \cdots & hx_m(0) \\ hx_1(1) & hx_2(1) & \cdots & hx_m(1) \\ hx_1(2) & hx_2(2) & \cdots & hx_m(2) \\ \vdots & \vdots & \vdots & \vdots \\ hx_1(n) & hx_2(n) & \cdots & hx_m(n) \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

then the nth term of the string $h^{\lambda}[x_1, \ldots, x_m]$ is obtained by listing the entries on the nth row. It is not hard to show that the diagrams of Equation (5) commute in this case. Also, $(S\varepsilon A)_{\lambda} = (G\eta A)^{\lambda}$ for every $A \in \mathbf{Set}$ follows by an easy calculation.

Example 3.4. This example is a slight generalization of the original one given in Van Osdol (1971), and the comonad part was already considered in Marmolejo (1998). Van Osdol shows that in 'good categories for sheaf theory' (**Set** for example, but many other categories as well) the category of coalgebras for this comonad determined by a topological space is equivalent to the category of sheaves over the space.

Let X be a topological space, and let $\mathbf{A}$ be a category with products and filtered colimits. Then we can define the following comonad $\mathbb{G}$ on $\mathbf{A}^{|X|}$, where $|X|$ is the underlying set of X. Define $G : \mathsf{Ob}(\mathbf{A}^{|X|}) \to \mathsf{Ob}(\mathbf{A}^{|X|})$ such that for an object $A = \langle A_x \rangle_{x \in X} \in \mathbf{A}^{|X|}$, $GA = \langle \varinjlim_{U \ni x} \prod_{y \in U} A_y \rangle_{x \in X}$; $\varepsilon A : GA \to A$ is such that the x-entry of it makes the diagram

$$\begin{array}{ccc} \varinjlim_{U \ni x} \prod_{y \in U} A_y & \xrightarrow{(\varepsilon A)_x} & A_x \\ {\scriptstyle \iota_U^x}\uparrow & \nearrow_{\pi_x} & \\ \prod_{y \in U} A_y & & \end{array}$$

commute; and given $f = \langle f_x \rangle_{x \in X} : GA \to B$, we have that $(f_{\mathbb{G}})_x$ makes the diagram

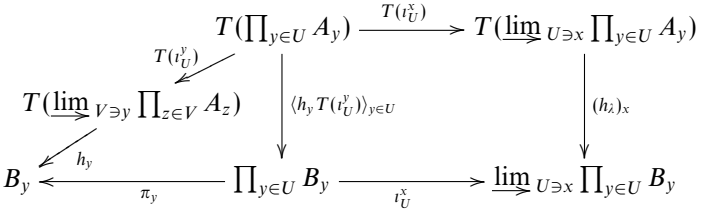
$$\begin{array}{ccccc} & & \prod_{y \in U} A_y & \xrightarrow{\iota_U^x} & \varinjlim_{U \ni x} \prod_{y \in U} A_y \\ & {\scriptstyle \iota_U^y} \swarrow & \downarrow {\scriptstyle \langle f_y \iota_U^y \rangle_{y \in U}} & & \downarrow {\scriptstyle (f_{\mathbb{G}})_x} \\ \varinjlim_{V \ni y} \prod_{z \in V} A_z & & & & \\ {\scriptstyle f_y}\swarrow & & & & \\ B_y & \xleftarrow[\pi_y]{} & \prod_{y \in U} B_y & \xrightarrow[\iota_U^x]{} & \varinjlim_{U \ni x} \prod_{y \in U} B_y \end{array}$$

commute for every open U with $x \in U$, and every $y \in U$. It is not hard to verify that these definitions satisfy the conditions given above to define a comonad.

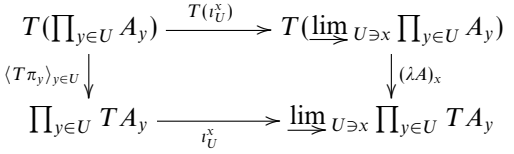
Assume now that we have a functor $T : \mathbf{A} \to \mathbf{A}$ that preserves filtered colimits, and consider the functor $S := T^{|X|} : \mathbf{A}^{|X|} \to \mathbf{A}^{|X|}$. For every $A \in \mathbf{A}^{|X|}$, we give SGA the following structure of $\mathbb{G}$-coalgebra. Given $h : SGA \to B$, $h_{\lambda} : SGA \to GB$ is such that for

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every $x \in X$, every open U with $x \in U$, and every $y \in U$ the diagram

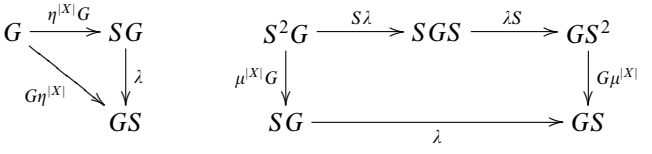


commutes, using the fact that T preserves filtered colimits. It is not hard to see that the conditions corresponding to Equation (6) are satisfied. Thus we obtain a natural transformation $\lambda : SG \to GS$ given at A by $(S\varepsilon A)_\lambda$. Thus, $(\lambda A)_x$ makes the diagram



commute.

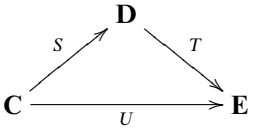
If furthermore the functor T is part of a monad $\mathbb{T} = (T, \eta, \mu)$, then we have the induced monad $\mathbb{S} = (S = T^{|X|}, \eta^{|X|}, \mu^{|X|})$ on $\mathbf{A}^{|X|}$. Then it is not hard to see that the diagrams



commute, giving us that the conditions (5) are satisfied. Thus λ is a distributive law of the monad $\mathbb{T}$ over the comonad $\mathbb{G}$.

4. Mixed distributive laws in 2-categories

This section present a no-iteration version of a mixed distributive law of a monad over a comonad in a general 2-category, and show that it is equivalent to the usual description of such laws. We recall first, from Marmolejo and Wood (2010), the equivalent way to give a monad on an object $\mathbf{C}$ on a 2-category $\mathcal{K}$ in terms of extension systems. Take three 1-cells



in $\mathcal{K}$ (where we are not assuming that the diagram commutes). Then a *pasting operator* $(\ )^\sharp : \mathcal{K}(-, \mathbf{D})(1, S) \to \mathcal{K}(-, \mathbf{E})(T, U)$ consists of a family of morphisms

$$(\ )^\sharp_{D,C} : \mathcal{K}(\mathbf{X}, \mathbf{D})(D, SC) \to \mathcal{K}(\mathbf{X}, \mathbf{E})(TD, UC),$$

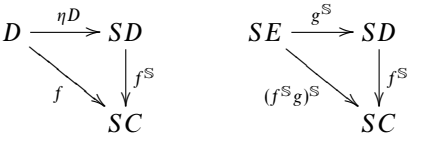
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one for each $\mathbf{C} \xleftarrow{C} \mathbf{X} \xrightarrow{D} \mathbf{D}$ in $\mathcal{K}$, that respects whiskering and blistering (this means that for $X : \mathbf{Y} \to \mathbf{X}$, $f : D \to SC : \mathbf{X} \to \mathbf{D}$ and $d : D' \to D : \mathbf{X} \to \mathbf{D}$ we have $(fX)^\sharp = f^\sharp X$ and $(f \cdot d)^\sharp = f^\sharp \cdot Td$).

Definition 4.1. An *extension operator* on an object $\mathbf{C}$ of $\mathcal{K}$ consists of an arrow $S : \mathbf{C} \to \mathbf{C}$, a 2-cell $\eta : 1_{\mathbf{C}} \to S$ and a pasting operator

$$(\ )^{\mathbb{S}} : \mathcal{K}(-, \mathbf{C})(1, S) \to \mathcal{K}(-, \mathbf{C})(S, S),$$

subject to the condition $\eta^{\mathbb{S}} = 1_S$, and the commutativity of the diagrams



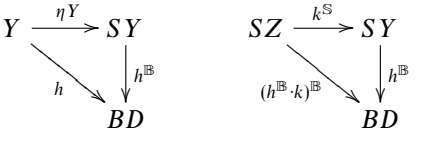
for every $C, D, E : \mathbf{X} \to \mathbf{C}$, and $f : D \to SC$ and $g : E \to SD$.

It is a result in Marmolejo and Wood (2010) that extension operators on $\mathbf{C}$ and monads on $\mathbf{C}$ are in bijective correspondence.

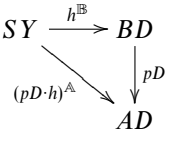
We also recall the equivalent way to define algebras. Given an extension operator $(S, \eta, (\ )^{\mathbb{S}})$ on an object $\mathbf{C}$ of $\mathcal{K}$, and an object $\mathbf{X}$ of $\mathcal{K}$, the category of algebras for $(S, \eta, (\ )^{\mathbb{S}})$ based on $\mathbf{X}$ is denoted by $\mathcal{K}(\mathbf{X}, (\mathbf{C}, (S, \eta, (\ )^{\mathbb{S}})))$ and it is defined as follows: Its objects are of the form $\mathbb{B} = (B, (\ )^{\mathbb{B}})$ where $B : \mathbf{X} \to \mathbf{C}$ and

$$(\ )^{\mathbb{B}} : \mathcal{K}(-, \mathbf{C})(1, B) \to \mathcal{K}(-, \mathbf{C})(S, B),$$

is a pasting operator such that for every $h : Y \to BD : \mathbf{Y} \to \mathbf{C}$ and $k : Z \to SY : \mathbf{Y} \to \mathbf{C}$ the diagrams



commute. A morphism $p : (B, (\ )^{\mathbb{B}}) \to (A, (\ )^{\mathbb{A}})$ is a 2-cell $p : B \to A$ such that for every $h : Y \to BD$ the diagram



commutes. It is also shown in Marmolejo and Wood (2010) that $\mathcal{K}(\mathbf{X}, (\mathbf{C}, (S, \eta, (\ )^{\mathbb{S}})))$ is isomorphic to the usual category of algebras for the monad based on $\mathbf{X}$ as defined in Street (1972) or Marmolejo (1999).

We proceed now to the description of mixed distributive laws via extension systems similar to the description given for distributive laws for monads in Marmolejo and Wood (2010). Let us first recall from Van Osdol (1971) the definition of a mixed distributive law.

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Definition 4.2. Given a monad $\mathbb{S} = (S, \eta, \mu)$ and a comonad $\mathbb{G} = (G, \varepsilon, \delta)$ on the same object $\mathbf{C}$ of a 2-category $\mathcal{K}$, a mixed distributive law consists of a 2-cell $\lambda : SG \to GS$ such that the diagrams in (4) commute.

Now the promised description of mixed distributive laws.

Theorem 4.1. Let $\mathbb{S} = (S, \eta, (\ )^{\mathbb{S}})$ be a monad on $\mathbf{C}$ and $\mathbb{G} = (G, \varepsilon, (\ )_{\mathbb{G}})$ a comonad on $\mathbf{C}$ in the 2-category $\mathcal{K}$. To give a mixed distributive law of $\mathbb{S}$ over $\mathbb{G}$, is equivalently to give the following data:

i. an $\mathbb{S}$-algebra $(GS, (\ )^{\lambda})$;
ii. an $\mathbb{G}$-coalgebra $(SG, (\ )_{\lambda})$;

in such a way that

a. for every $A, B : \mathbf{X} \to \mathbf{C}$, $k : B \to SA$, $Gk^{\mathbb{S}} : (GSB, (\ )^{\lambda}) \to (GSA, (\ )^{\lambda})$ is a 1-cell of $\mathbb{S}$-algebras;
b. for every $A, B : \mathbf{X} \to \mathbf{C}$, $\ell : GB \to A$, $S\ell_{\mathbb{G}} : (SGB, (\ )_{\lambda}) \to (SGA, (\ )_{\lambda})$ is a 1-cell of $\mathbb{G}$-coalgebras;
c. $(G\eta)^{\lambda} = (S\varepsilon)_{\lambda} : SG \to GS$.

The usual distributive law is obtain by either formula in (c), whereas, given a usual mixed distributive law $\lambda : SG \to GS$, the algebra and coalgebra structures are given as

$$h^{\lambda} = (SX \xrightarrow{Sh} SGSD \xrightarrow{\lambda SD} GS^2D \xrightarrow{G\mu D} GSD),$$

for $h : X \to GSD$, and

$$k_{\lambda} = (SGE \xrightarrow{S\delta E} SG^2E \xrightarrow{\lambda GE} GSGE \xrightarrow{Gk} GY),$$

for $k : SGE \to Y$, respectively.

Proof. The top two diagrams in Definition 4.2 mean that we have a 1-cell $(G, \lambda) : \mathbb{S} \to \mathbb{S}$ in the 2-category $\mathrm{Mnd}(\mathcal{K})$. According to Theorem 4.3 in Marmolejo and Wood (2010), this is equivalent to (i) such that (a). On the other hand, the bottom two diagrams in Definition 4.2 mean that we have a 1-cell $(S, \lambda) : \mathbb{G} \to \mathbb{G}$ in the 2-category $\mathrm{Mnd}(\mathcal{K}^{\mathrm{co}})$. Again, by Theorem 4.3 in Marmolejo and Wood (2010), this is equivalent to (ii) such that (b). Condition (c) tells us that we obtain the same $\lambda : SG \to GS$ from the $\mathbb{S}$-algebra structure on GS and from the $\mathbb{G}$-coalgebra structure on SG, since the former is given by $(G\eta)^{\lambda}$ and the latter by $(S\varepsilon)_{\lambda}$. □

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