J. Appl. Prob. **56,** 57–75 (2019)
*doi:*10.1017/jpr.2019.6
© *Applied Probability Trust* 2019

EXACT SIMULATION OF GENERALISED VERVAAT PERPETUITIES

ANGELOS DASSIOS* AND
YAN QU,* ** *London School of Economics and Political Science*
JIA WEI LIM,*** *University of Bristol*

Abstract

We consider a generalised Vervaat perpetuity of the form $X = Y_1W_1 + Y_2W_1W_2 + \cdots$, where $W_i \sim \mathcal{U}^{1/t}$ and $(Y_i)_{i\geq 0}$ is an independent and identically distributed sequence of random variables independent from $(W_i)_{i\geq 0}$. Based on a distributional decomposition technique, we propose a novel method for exactly simulating the generalised Vervaat perpetuity. The general framework relies on the exact simulation of the truncated gamma process, which we develop using a marked renewal representation for its paths. Furthermore, a special case arises when $Y_i = 1$, and X has the generalised Dickman distribution, for which we present an exact simulation algorithm using the marked renewal approach. In particular, this new algorithm is much faster than existing algorithms illustrated in Chi (2012), Cloud and Huber (2017), Devroye and Fawzi (2010), and Fill and Huber (2010), as well as being applicable to the general payments case. Examples and numerical analysis are provided to demonstrate the accuracy and effectiveness of our method.

Keywords: Vervaat perpetuity; Dickman distribution; truncated gamma process; exact simulation; marked renewal process

2010 Mathematics Subject Classification: Primary 60G51
Secondary 60K15; 65C05

1. Introduction

A stochastic perpetuity is a random variable which takes the form

$$X = Y_1W_1 + Y_2W_1W_2 + Y_3W_1W_2W_3 + \cdots + Y_nW_1W_2\cdots W_n + \cdots, \tag{1.1}$$

where the W_i and Y_i are independent and identically distributed (i.i.d.) random variables independent of each other. Perpetuities arise in a diverse range of fields, such as economics [11], insurance mathematics [16], astrophysics [3], analytic number theory [6], and in the analysis of Hoare's selection algorithm [15]. We refer the reader to [10] and [19] for a huge variety of references and examples of applications, ranging from the brightness of the Milky Way to ARCH processes of financial modelling. In the context of economics, the quantity Y_i represents a random cashflow at time i and W_i represents the discount rate from time i to time $i-1$. The random variable X then represents the net present value of the stochastic cashflows.

Received 9 October 2017; revision received 19 November 2018.
* Postal address: Department of Statistics, London School of Economics and Political Science, Houghton Street, London WC2A 2AE, UK.
** Email address: y.qu3@lse.ac.uk
*** Postal address: Department of Mathematics, University of Bristol, Senate House, Tyndall Avenue, Bristol BS8 1TH, UK.

https://doi.org/10.1017/jpr.2019.6 Published online by Cambridge University Press

When $W_i \overset{\mathrm{D}}{=} U^{1/t}$ for $U \sim \text{Uniform}\,(0, 1)$, the random variable X is known as a generalised Vervaat perpetuity. It was shown in [19] that the series converges if and only if $\mathbb{E}(\log^+ |Y_i|) < \infty$, and it satisfies the distributional identity

$$X \overset{\mathrm{D}}{=} W(X + Y), \tag{1.2}$$

where W and Y represent random variables distributed like W_i and Y_i, respectively. The Vervaat perpetuity has been much studied. Vervaat [19] showed that the random variable X is infinitely divisible, and obtained its Lévy–Khintchine representation. This is the basis of our simulation framework, and, furthermore, we show that, under the mild assumption that the Laplace transform of Y exists, X turns out to be a Lévy process with Lévy measure proportional to $\mathbb{P}(Y > y)/y$. There is also no loss of generality to consider only the case when $Y > 0$, as it can be shown that a two-sided perpetuity can be split into the difference of two one-sided perpetuities. Goldie and Grübel [13] and Grübel and Rösler [14] studied the tails of these perpetuities. Random variables of this type also appear as the limit of shot noise processes [17], [18].

In this paper we are interested in exactly simulating a generalised Vervaat perpetuity. In the literature much attention has been paid to the special case when $Y_i = 1$ almost surely, in which X has the generalised Dickman distribution [1], [9]. Even so, explicit expressions for the distribution of X are difficult to obtain. The first simulation method is based on a density approximation given in [7]. Later, Devroye and Fawzi [8] and Fill and Huber [12] used a dominated coupling from the past procedure for Markov chains. Cloud and Huber [5] proposed an improvement by running both an upper and lower bound on the Markov chain. Chi [4] suggested an alternative approach based on a rejection sampling procedure. However, these algorithms are efficient only for the Dickman distribution, where the parameter $t = 1$, and their speed slows down considerably when t becomes larger than 1.

We develop a faster and more efficient simulation algorithm for the generalised Dickman distribution with parameter t. Since it is viewed here as a Lévy process over the range of parameters t, with Lévy measure $\nu(\mathrm{d}y) = (1/y)\mathbf{1}_{\{y<1\}}\,\mathrm{d}y$, we thus call it the Dickman process. The paths of the Dickman process can be characterised by a marked renewal process where the renewal intervals are the hitting times of level 1, and the jumps are the overshoots at this time. By simulating its path along each hitting time of level 1, we obtain a simulation algorithm for the Dickman process. Similar to our method, the approach in [4] also utilises the truncated density of the perpetuity. However, the advantage of our algorithm is that it is based on a marked renewal process, which not only provides a pathwise probabilistic interpretation of the corresponding Lévy process, but also means that the incremental random variables we need to simulate are i.i.d. pairs, allowing us to employ vectorisation to reduce the total computation time. This is not the case for the algorithm suggested in [4], where the incremental variables depend on the previous variables. We provide comparisons of the simulation speed of our algorithm with the algorithms suggested in [4], [5], [8], and [12]. We found that our method outperforms the other algorithms and even more substantially when t is large.

Blanchet and Sigman [2] proposed a modification of the dominated coupling from the past algorithm to sample from generalised Vervaat perpetuities. The algorithm imposes similar regularity conditions to ours on the density of Y_i. However, it also requires Y_i to have a heavy tail. In this paper we develop a framework for simulating from generalised Vervaat perpetuities. In particular, we prove a distributional decomposition result, which states that a generalised Vervaat perpetuity X can be represented as the sum of a gamma or truncated gamma process

https://doi.org/10.1017/jpr.2019.6 Published online by Cambridge University Press

and independent compound Poisson processes. Thus, it can be sampled by generating each of these components separately and adding them up. The condition we need is

$$\lim_{y \downarrow 0} \frac{\mathbb{P}(Y < y)}{y} < \infty, \tag{1.3}$$

which holds for a wide range of distributions, such as the gamma (shape parameter larger than 1), Pareto, Weibull (light tailed), beta, and normal distributions as well as all discrete distributions (to avoid technicalities 0 is not a possible value). Whenever a density exists, it should possess a finite derivative at 0.

Using the marked renewal representation, we develop an exact simulation algorithm for the truncated gamma process, which we define to be the Lévy process with Lévy measure

$$\nu(\mathrm{d}y) = \frac{\mathrm{e}^{-\mu y}}{y} \mathbf{1}_{\{y<b\}} \, \mathrm{d}y$$

for some truncation level b. The truncated gamma process can be obtained from the gamma process by restricting the size of its jumps to be bounded from above by b. It is also a generalised Vervaat perpetuity with $Y_i \sim V \wedge 1$, where $V \sim \mathrm{Exp}(\mu)$. This algorithm is important, as it forms the basis of other algorithms for more general distributions of Y_i. We provide specific examples to demonstrate the methodology, and present numerical results and graphical illustrations for the cases when Y_i has a gamma distribution, and when Y_i has a normal distribution.

This paper is organised as follows. In Section 2 we give a representation of the generalised Vervaat perpetuities as gamma-related processes. We also prove that a two-sided perpetuity is the difference of two positive perpetuities. In Section 3 we study the specific cases of a Dickman process and a truncated gamma process. The marked renewal path representation is described and the exact simulation algorithms developed. Numerical analysis and comparison with existing algorithms are provided. In Section 4 we prove the distributional decomposition result, which enables us to exactly simulate a generalised Vervaat perpetuity. Numerical examples are also given to demonstrate the accuracy and efficiency of our algorithms.

2. Generalised Vervaat perpetuities

We obtain the following distributional decomposition result for a positive perpetuity $X > 0$.

Theorem 2.1. *Consider the generalised Vervaat perpetuity X with $Y_i > 0$. We define the Laplace transforms $\hat{f}(\beta) = \mathbb{E}(\exp(-\beta X))$ and $\hat{g}(\beta) = \mathbb{E}(\exp(-\beta Y_i))$. Then we have*

$$\hat{f}(\beta) = \exp\Big(-t \int_0^\infty \frac{1 - \mathrm{e}^{-\beta y}}{y} \mathbb{P}(Y > y) \, \mathrm{d}y\Big). \tag{2.1}$$

Hence, $X \overset{\mathrm{D}}{=} Z_t$, where Z_t is a Lévy process with Lévy measure

$$\nu(\mathrm{d}y) = \frac{1}{y} \mathbb{P}(Y > y) \, \mathrm{d}y. \tag{2.2}$$

Proof. For $W = U^{1/t}$, the density of W is tw^{t-1}. Then, from (1.2), we have

$$\hat{f}(\beta) = \int_0^1 \hat{f}(\beta w) \hat{g}(\beta w) t w^{t-1} \, \mathrm{d}w,$$

https://doi.org/10.1017/jpr.2019.6 Published online by Cambridge University Press

and, therefore,

$$\beta^t \hat{f}(\beta) = \int_0^\beta \hat{f}(w)\hat{g}(w) t w^{t-1}\,\mathrm{d}w.$$

Differentiating with respect to β on both sides, we have

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\hat{f}(\beta) = -t\frac{1-\hat{g}(\beta)}{\beta}\hat{f}(\beta).$$

Solving the first-order differential equation with respect to β, we obtain

$$\hat{f}(\beta) = \exp\Big(-t\int_0^\beta \frac{1-\hat{g}(u)}{u}\,\mathrm{d}u\Big) = \exp\Big(-t\int_0^\infty \frac{1-\mathrm{e}^{-\beta y}}{y}\mathbb{P}(Y>y)\,\mathrm{d}y\Big),$$

and (2.1) follows. Thus, $X \overset{\mathrm{D}}{=} Z_t$, where Z_t has Lévy measure defined in (2.2). □

We can see that Z_t is a gamma-related Lévy process with Lévy measure density proportional to $\mathbb{P}(Y>y)y^{-1}$. For a two-sided perpetuity, where Y takes values in the whole real line, the following result shows that it is the difference of two positive perpetuities and, thus, it suffices to study only positive perpetuities.

Corollary 2.1. *Suppose now that* $\mathbb{P}(Y<0)>0$. *Then*

$$\hat{f}(\beta) = \exp\Big(-t\Big(p\int_0^\infty \frac{1-\mathrm{e}^{-\beta y}}{y}\mathbb{P}(Y>y \mid Y>0)\,\mathrm{d}y + (1-p)\int_0^\infty \frac{1-\mathrm{e}^{\beta y}}{y}\mathbb{P}(Y<-y \mid Y<0)\,\mathrm{d}y\Big)\Big),$$

where $p = \mathbb{P}(Y>0)$.

Proof. For two-sided Y, the Laplace transform of the perpetuity X satisfies

$$\begin{aligned}
&\exp\Big(-t\int_0^\beta \frac{1-\hat{g}(u)}{u}\,\mathrm{d}u\Big)\\
&\quad= \exp\Big(-t\Big(\int_0^\beta\int_0^\infty \frac{1-\mathrm{e}^{-uy}}{u}\mathbb{P}(Y\in\mathrm{d}y)\,\mathrm{d}u + \int_0^\beta\int_0^\infty \frac{1-\mathrm{e}^{uy}}{u}\mathbb{P}(Y\in\mathrm{d}(-y))\,\mathrm{d}u\Big)\Big)\\
&\quad= \exp\Big(-t\Big(\int_0^\beta\int_0^\infty \mathrm{e}^{-uy}\mathbb{P}(Y>y)\,\mathrm{d}y\,\mathrm{d}u + \int_0^\beta\int_0^\infty \mathrm{e}^{uy}\mathbb{P}(Y<-y)\,\mathrm{d}y\,\mathrm{d}u\Big)\Big)\\
&\quad= \exp\Big(-t\Big(\int_0^\infty \frac{1-\mathrm{e}^{-\beta y}}{y}\mathbb{P}(Y>y)\,\mathrm{d}y + \int_0^\infty \frac{1-\mathrm{e}^{\beta y}}{y}\mathbb{P}(Y<-y)\,\mathrm{d}y\Big)\Big)\\
&\quad= \exp\Big(-t\Big(p\int_0^\infty \frac{1-\mathrm{e}^{-\beta y}}{y}\mathbb{P}(Y>y \mid Y>0)\,\mathrm{d}y\\
&\qquad + (1-p)\int_0^\infty \frac{1-\mathrm{e}^{\beta y}}{y}\mathbb{P}(Y<-y \mid Y<0)\,\mathrm{d}y\Big)\Big),
\end{aligned}$$

which is easily seen as the difference of two positive perpetuities, one with $W_i \overset{\mathrm{D}}{=} \mathcal{U}^{1/pt}$ and $Y_i \overset{\mathrm{D}}{=} Y \mid Y>0$, and the other with $W_i \overset{\mathrm{D}}{=} \mathcal{U}^{1/(1-p)t}$ and $Y_i \overset{\mathrm{D}}{=} -Y \mid Y<0$. □

Our aim is to simulate the generalised Vervaat perpetuity X for a given distribution of Y_i. In the simplest case when $Y_i = 1$ almost surely, X is known as the Vervaat perpetuity and it reduces

https://doi.org/10.1017/jpr.2019.6 Published online by Cambridge University Press

to the Dickman process. When $Y_i \sim V$, where $V \sim \text{Exp}(\mu)$, we obtain the gamma process with parameter μ, and when $Y_i \sim V \wedge 1$, we obtain the truncated gamma process. Making use of a marked renewal representation for the paths, we develop fast and exact simulation algorithms for these perpetuities. These cases are important as they form the basis of the other algorithms. We will then give other examples to demonstrate how our methodology can be adapted to general random variables Y_i. Our method works for any Y_i satisfying (1.3).

3. Generalised Dickman distribution and the truncated gamma process

When $Y \sim \text{Exp}(\mu)$, we have $\mathbb{P}(Y > y) = \mathrm{e}^{-\mu y}$, and, thus,

$$\hat{f}(\beta) = \exp\Big(- t \int_0^\infty (1 - \mathrm{e}^{-\beta y}) \frac{1}{y} \mathrm{e}^{-\mu y} \, \mathrm{d}y \Big),$$

which is the Laplace transform of a gamma process with parameter μ. This can be simulated easily by sampling from the gamma distribution.

When $Y = 1$ almost surely, we have $\mathbb{P}(Y > y) = \mathbf{1}_{\{y<1\}}$, and, thus,

$$\hat{f}(\beta) = \exp\Big(- t \int_0^1 \frac{1 - \mathrm{e}^{-\beta y}}{y} \, \mathrm{d}y \Big),$$

which is the Laplace transform of a generalised Dickman distribution with parameter t. This is also a Lévy process Z_t with Lévy measure $(1/y)\mathbf{1}_{\{y<1\}}$, and, hence, we call it the Dickman process.

When $Y = V \wedge 1$, where $V \sim \text{Exp}(\mu)$, we have $\mathbb{P}(Y > y) = \mathrm{e}^{-\mu y} \mathbf{1}_{\{y<1\}}$, and, thus,

$$\hat{f}(\beta) = \exp\Big(- t \int_0^1 (1 - \mathrm{e}^{-\beta y}) \frac{\mathrm{e}^{-\mu y}}{y} \, \mathrm{d}y \Big),$$

which is the Laplace transform of a truncated gamma process with parameter μ. The truncated gamma process refers to the gamma process with truncated Lévy measure

$$\frac{\mathrm{e}^{-\mu y}}{y} \mathbf{1}_{\{y<1\}}.$$

To simulate the Dickman process and the truncated gamma process, we utilise a marked renewal process representation of its sample paths. We now describe the sample paths of truncated subordinators using a marked renewal process.

3.1. The marked renewal process

The paths of the truncated subordinator Z_t can be characterised by a marked renewal process. First, we define a sequence of hitting times $T_1, T_2, T_3, \ldots$, and defining $S_i = \sum_{j=1}^{i} T_j$, let

$$T_i := \inf\{t > S_{i-1} \mid Z_t > Z_{S_{i-1}} + 1\}, \qquad i = 2, 3, \ldots. \tag{3.1}$$

These $T_1, T_2, \ldots$ are treated as the holding times for the renewal process. We further define $M_1, M_2, \ldots$ to be the overshoots at time $S_1, S_2, \ldots$, i.e.

$$M_i := Z_{S_i} - Z_{S_{i-1}} - 1.$$

Hence, at time S_i the process will increase by $(1 + M_i)$ units for all i. Since the process Z_t has independent and stationary increments, each pair of (T_i, M_i) are i.i.d. with joint density given

https://doi.org/10.1017/jpr.2019.6 Published online by Cambridge University Press

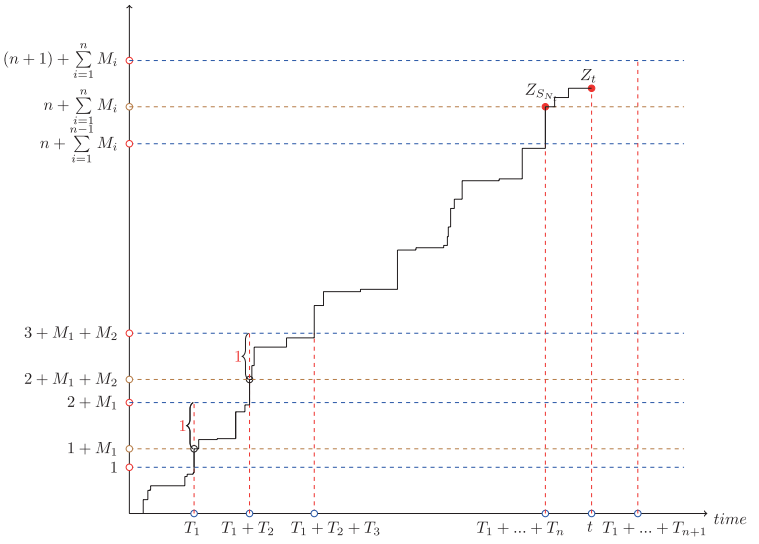


FIGURE 1: Graphical illustration of a sample path of Z_t.

in Theorem 3.1. In addition, M_i will be bounded by 0 and 1 for all i as the jump sizes of the process are restricted by 1. The value of the process at time S_n will be $Z_{S_n} = \sum_{i=1}^{n} (1 + M_i)$. The position of the process Z_t at time t can therefore be expressed as the sum of a marked renewal process with a random variable,

$$Z_t = \sum_{i=1}^{N_t} (1 + M_i) + (Z_t - Z_{S_{N_t}}), \tag{3.2}$$

where $N_t = \sum_{i=1}^{\infty} \mathbf{1}_{\{S_i \le t\}}$ is the renewal process determined via (3.1) such that $S_{N_t} \le t < S_{N_t+1}$. We have

$$Z_t - Z_{S_{N_t}} \overset{\mathrm{D}}{=} Z_{t-S_{N_t}} \mid S_{N_t+1} > t \overset{\mathrm{D}}{=} Z_t \mid T > t,$$

where the last equality is due to the strong Markov property of Z_t.

We illustrate the marked renewal framework graphically in Figure 1. The sum in the first part of (3.2) represents the position of the truncated process at time S_{N_t} just before it reaches t. The second term in (3.2) represents the final movement of the process within time $t - N_t$. Thus, Z_t can be simulated by generating pairs of hitting times and overshoots (T_i, M_i), stopping when the sum of T_i that have been generated, say S_{N_t+1}, becomes larger than the input t. We then generate the final part $\{Z_t \mid T > t\}$ and return the entire sum (3.2).

3.2. Hitting times and overshoots

Now we obtain analytically the first hitting times and overshoots of the truncated gamma process and the Dickman process.

Theorem 3.1. *Let T be the first hitting time of level 1 of a truncated gamma process Z_t with $\mu \ge 0$, and let M be the overshoot at this time. Then the joint distribution of (T, M) is given by*

$$f_{T,M}(t, m) = \int_m^1 \frac{\mathrm{e}^{\Gamma(0,\mu)t}\mu^t}{\Gamma(t)} y^{t-1} \frac{\mathrm{e}^{-\mu(1+m)}}{1+m-y}\,\mathrm{d}y \quad \textit{for } \mu > 0, \tag{3.3}$$

https://doi.org/10.1017/jpr.2019.6 Published online by Cambridge University Press

where $t \in (0, \infty)$, $m \in (0, 1)$. For the Dickman process, the joint density of (T, M) is the limit of (3.3) as $\mu \to 0$,

$$f_{T,M}(t, m) = \int_m^1 \frac{\mathrm{e}^{-\gamma t}}{\Gamma(t)} \frac{y^{t-1}}{1+m-y}\,\mathrm{d}y, \tag{3.4}$$

where γ is the Euler–Mascheroni constant, $t \in (0, \infty)$, and $m \in (0, 1)$.

Proof. Let $f(x;t)$ denote the density of Z_t. For $\mu > 0$, the density of Z_t restricted to $(0, 1)$ can be derived through its Laplace transform. We have

$$\begin{aligned}
f(x;t) &= \mathcal{L}^{-1}\{\mathbb{E}[\mathrm{e}^{-\beta Z_t}]\}\mathbf{1}_{\{0<x<1\}} \\
&= \mathcal{L}^{-1}\Big\{ \exp\Big(-t\int_0^\infty (1-\mathrm{e}^{-\beta s})s^{-1}\mathrm{e}^{-\mu s}\,\mathrm{d}s\Big) \exp\Big(t\int_1^\infty (1-\mathrm{e}^{-\beta s})s^{-1}\mathrm{e}^{-\mu s}\,\mathrm{d}s\Big)\Big\} \\
&\quad \times \mathbf{1}_{\{0<x<1\}} \\
&= \mathcal{L}^{-1}\Big\{\Big(1+\frac{\beta}{\mu}\Big)^{-t} \exp\Big(t\int_1^\infty s^{-1}\mathrm{e}^{-\mu s}\,\mathrm{d}s\Big) \sum_{k=0}^\infty \frac{(-t)^k}{k!}\Big(\int_1^\infty \mathrm{e}^{-\beta s}s^{-1}\mathrm{e}^{-\mu s}\,\mathrm{d}s\Big)^k\Big\} \\
&\quad \times \mathbf{1}_{\{0<x<1\}} \\
&= \mathcal{L}^{-1}\Big\{\mathrm{e}^{t\Gamma(0,\mu)}\Big(1+\frac{\beta}{\mu}\Big)^{-t} \\
&\quad + \mathrm{e}^{t\Gamma(0,\mu)}\Big(1+\frac{\beta}{\mu}\Big)^{-t} \sum_{k=1}^\infty \frac{(-t)^k}{k!}\Big(\int_1^\infty \mathrm{e}^{-\beta s}s^{-1}\mathrm{e}^{-\mu s}\,\mathrm{d}s\Big)^k\Big\}\mathbf{1}_{\{0<x<1\}} \\
&= \mathcal{L}^{-1}\Big\{\mathrm{e}^{t\Gamma(0,\mu)}\Big(1+\frac{\beta}{\mu}\Big)^{-t}\Big\}\mathbf{1}_{\{0<x<1\}} \\
&= \frac{\mathrm{e}^{\Gamma(0,\mu)t}\mu^t}{\Gamma(t)} x^{t-1}\mathrm{e}^{-\mu x}\mathbf{1}_{\{0<x<1\}}.
\end{aligned} \tag{3.5}$$

Using the strong Markov property of Lévy processes, we have

$$\begin{aligned}
\mathbb{P}(T \in \mathrm{d}t, M > m) &= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{P}(Z_{t-\varepsilon} \le 1, Z_t > 1+m) \\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^1 \mathbb{P}(Z_{t-\varepsilon} \in \mathrm{d}y)\mathbb{P}(Z_\varepsilon > 1+m-y) \\
&= \int_0^1 \frac{\mathrm{e}^{\Gamma(0,\mu)t}\mu^t}{\Gamma(t)} y^{t-1}\mathrm{e}^{-\mu y}\int_{1+m-y}^\infty z^{-1}\mathrm{e}^{-\mu z}\,\mathrm{d}z\,\mathrm{d}y.
\end{aligned} \tag{3.6}$$

Differentiating (3.6) with respect to m, the joint density of (T, M) in (3.3) directly follows. For the Dickman process, we take the limit as $\mu \to 0$. The density of Z_t within $(0, 1)$ is obtained by taking the limit of (3.5), i.e.

$$f(x, t) = \lim_{\mu\to 0} \frac{\mathrm{e}^{[\Gamma(0,\mu)+\log(\mu)]t}}{\Gamma(t)} x^{t-1}\mathrm{e}^{-\mu x} = \frac{\mathrm{e}^{-\gamma t}}{\Gamma(t)} x^{t-1}, \qquad 0 < x < 1, \tag{3.7}$$

with γ being the Euler–Mascheroni constant. Hence, the associated distribution of (T, M) directly follows from (3.4). □

https://doi.org/10.1017/jpr.2019.6 Published online by Cambridge University Press

Given that the first passage time of Z_t hitting level 1 is greater than t, the distribution of the truncated process Z_t is characterised via its density within (0, 1). The details are as follows.

Theorem 3.2. *The density of* $Z_t \mid T > t$ *is given by*

$$f_{Z_t|T>t}(x;t) = \begin{cases} \dfrac{\mu^t}{\Gamma(t,\,\mu)} x^{t-1} \mathrm{e}^{-\mu x}, & \mu > 0, \\ t x^{t-1}, & \mu = 0, \end{cases} \tag{3.8}$$

where $0 < x < 1$.

Proof. We know that

$$Z_t \mid T > t \overset{\mathrm{D}}{=} Z_t \mid Z_t < 1,$$

since $\{T > t\}$ is equivalent to $\{Z_t < 1\}$. The density therefore satisfies

$$f_{Z_t \mid T>t}(x;t) = f_{Z_t \mid Z_t<1}(x;t) = \frac{f(x;t)}{\int_0^1 f(x;t)\,\mathrm{d}x}.$$

Thus, the density immediately follows from (3.8) . □

3.3. Exact simulation of the generalised Dickman distribution

Based on the marked renewal framework and choosing suitable acceptance–rejection (A/R) envelopes for $f_{T,M}(t, m)$ and $f_{Z_t \mid T>t}(x;t)$, we have the following simulation algorithm for the Dickman process, or, equivalently, the generalised Dickman distribution.

Algorithm 3.1. (Simulation of the Dickman process (generalised Dickman distribution).)

1. *Set* $S = 0$, $\gamma = -digamma(1)$.
2. *Generate* (T, M) *via the following steps.*
   (a) *Generate* $U_1 \sim \mathcal{U}(0, 1)$, *and set* $T = -\frac{1}{0.8} \log U_1$.
   (b) *Generate* $Y \sim Beta(T, 0.5)$, $U_2 \sim \mathcal{U}(0, 1)$, *and set*
   $$M = Y - 1 + (1 - Y) \exp\left(-U_2 \log(1 - Y)\right).$$
   (c) *Generate* $V \sim \mathcal{U}(0, 1)$, *and if*
   $$V \leq \frac{1}{2.35} \frac{\Gamma(0.5)\mathrm{e}^{-(\gamma-0.8)T}}{0.8\Gamma(T + 0.5)} \frac{(-\log(1 - Y))}{(1 - Y)^{-0.5}}, \tag{3.9}$$
   then accept (T, M). *Otherwise, reject this candidate and go back to step 2(a).*
3. *If* $T > t$, *continue to step 4. Otherwise, set* $S = S + 1 + M$, $t = t - T$, *and go back to step 2.*
4. *Generate* $U \sim \mathcal{U}(0, 1)$ *and set* $X = U^{1/t}$.
5. *Return* $S + X$.

Proof. For the joint density $f_{T,M}(t, m)$, we simulate (T, M, Y) jointly from the integral in (3.4),

$$f(t, m, y) = \frac{\mathrm{e}^{-\gamma t} y^{t-1}}{\Gamma(t)} \frac{1}{1 + m - y}$$

https://doi.org/10.1017/jpr.2019.6 Published online by Cambridge University Press

for $t \in (0, \infty)$, $0 < m < y < 1$. The A/R envelope chosen is

$$g(t, m, y) = \sigma \mathrm{e}^{-\sigma t} \frac{\Gamma(t+\eta)}{\Gamma(t)\Gamma(\eta)} y^{t-1}(1-y)^{\eta-1} \frac{1/(1+m-y)}{(-\log(1-y))}, \tag{3.10}$$

where $\sigma = 0.8$ and $\eta = 0.5$ are chosen numerically via two-dimensional optimisation. Furthermore, $f_{Z_t \mid T>t}(x;t)$ can be simulated directly via inverse transformation resulting in step 4. □

3.4. Exact simulation of the truncated gamma process

For the truncated gamma process, we could generate a Dickman process and apply a final A/R scheme. However, the algorithm is not very efficient, especially for large μ as the acceptance rate is exponentially decreasing in μ. Thus, we propose the following algorithm based on the marked renewal framework to sample a truncated gamma process.

Algorithm 3.2. (Simulation of the truncated gamma process (marked renewal approach).)

1. *Set* $S = 0$.
2. *Generate* (T, M) *via the following steps.*
   (a) *Numerically minimise*

$$C(\vartheta, \delta) = \frac{\Gamma(\delta)\mathrm{e}^{-\mu}}{\vartheta(1-\delta)e} \times \frac{\mathrm{e}^{\Gamma(0,\mu)\exp(\Gamma(0,\mu)+\vartheta+\log(\mu))} \mu^{\exp(\Gamma(0,\mu)+\vartheta+\log(\mu))} \mathrm{e}^{\vartheta \exp(\Gamma(0,\mu)+\vartheta+\log(\mu))}}{\Gamma(\exp(\Gamma(0,\mu)+\vartheta+\log(\mu))+\delta)},$$

   record the optimal value ϑ^* *and* δ^*, *and set* $C = C(\vartheta^*, \delta^*)$.
   (b) *Generate* $U_1 \sim \mathcal{U}(0, 1)$ *and set* $T = -\log(U_1)/\vartheta^*$.
   (c) *Generate* $Y \sim Beta(T, \delta^*)$, $U_2 \sim \mathcal{U}(0, 1)$, *and set*

$$M = Y - 1 + (1-Y)\exp(-U_2 \log(1-Y)).$$

   (d) *Generate* $V_1 \sim \mathcal{U}(0, 1)$. *If*

$$V_1 \le \frac{1}{C} \frac{\Gamma(\delta^*)}{\vartheta^*} \frac{\mathrm{e}^{\Gamma(0,\mu)T} \mu^T \mathrm{e}^{\vartheta^* T}}{\Gamma(T+\delta^*)} \frac{-\log(1-Y)}{(1-Y)^{\delta^*-1}} \mathrm{e}^{-\mu(1+M)}, \tag{3.11}$$

   then accept (T, M). *Otherwise, reject this candidate and go back to step 2(b).*
3. *If* $T > t$, *continue to step 4. Otherwise, set* $S = S + 1 + M$, $t = t - T$, *and go back to step 2.*
4. *Generate X via the following steps.*
   (a) *Generate* $U_3 \sim \mathcal{U}(0, 1)$ *and set* $X = U_3^{1/t}$.
   (b) *Generate* $V_2 \sim \mathcal{U}(0, 1)$. *If*

$$V_2 \le \exp(-\mu X),$$

   then accept X. Otherwise, reject this candidate and go back to step 4(a).
5. *Return* $S + X$.

https://doi.org/10.1017/jpr.2019.6 Published online by Cambridge University Press

Proof. For the joint density $f_{T,M}(t, m)$, we simulate (T, M, Y) jointly from the integral in (3.3),

$$f(t, m, y) = \frac{\mathrm{e}^{\Gamma(0,\mu)t}\mu^t}{\Gamma(t)} y^{t-1} \frac{\mathrm{e}^{-\mu(1+m)}}{1+m-y},$$

where $t \in (0, \infty)$, $0 < m < y < 1$. We choose the envelope

$$g(t, m, y) = \vartheta \mathrm{e}^{-\vartheta t} \frac{\Gamma(t+\delta)}{\Gamma(t)\Gamma(\delta)} y^{t-1}(1-y)^{\delta-1} \frac{1/(1+m-y)}{[-\log(1-y)]}.$$

The ratio of the densities is bounded by

$$\frac{f(t, m, y)}{g(t, m, y)} \leq \frac{\Gamma(\delta)\mathrm{e}^{-\mu}}{\alpha(1-\delta)e} \frac{\mathrm{e}^{\Gamma(0,\mu)t^*}\mu^{t^*}\mathrm{e}^{\alpha t^*}}{\Gamma(t^*+\delta)} := C(\vartheta, \delta), \tag{3.12}$$

which we obtained by maximising the ratio with respect to t,

$$t^* = \exp\left(\Gamma(0, \mu) + \alpha + \log(\mu) + \mathcal{O}\left(\frac{1}{t^*}\right)\right) \approx \exp(\Gamma(0, \mu) + \alpha + \log(\mu)).$$

We then minimise $C(\vartheta, \delta)$ with respect to ϑ and δ via two-dimensional numerical optimisation. Finally, $\{Z_t \mid T > t\}$ can be simulated directly via an A/R scheme by choosing an envelope with density

$$g(x;t) = \frac{x^{t-1}}{t} \mathbf{1}_{\{0<x<1\}}.$$

□

3.5. Numerical results

In this section we present some numerical results of the exact simulation methods developed in Algorithms 3.1 and 3.2. Numerical validation and tests for our simulation algorithms are all based on the true mean. The associated errors are reported by the difference, the percentage errors, and *root mean square error* (RMSE) (calculated as $\mathrm{RMSE} = \sqrt{\mathrm{SE}^2 + \mathrm{Bias}^2}$). The detailed numerical results are reported in Table 1. We can see that each algorithm can achieve a very high level of accuracy, which is reflected by the difference in the theoretical mean and associated percentage errors. The simulation time is measured by the elapsed CPU time in seconds. We can see that both algorithms perform well. In particular, for Algorithm 3.2, the simulation time to generate the truncated gamma process is not highly influenced by μ.

For the generalised Dickman distribution, we implement a comparison between Algorithm 3.1 against the algorithms suggested by Devroye and Fawzi [8], Fill and Huber [12], Chi [4], and Cloud and Huber [5]. The detailed numerical results based on different parameter settings against t are reported in Table 2 and the associated log-log plots for the RMSE against the CPU time are presented in Figure 2. In addition, we also compared the computation time to generate 100 000 samples using these five algorithms for different values of t, the details of which are presented in Table 3. We can see that sampling based on Algorithm 3.1 is much faster than algorithms in Devroye and Fawzi [8], Fill and Huber [12], Chi [4], and Cloud and Huber [5]. For instance, when $t = 3$, Algorithm 3.1 is four times faster than Devroye and Fawzi [8], 50 times faster than Fill and Huber [12], 25 times faster than Chi [4], and

https://doi.org/10.1017/jpr.2019.6 Published online by Cambridge University Press

TABLE 1: Comparisons between the true means and the associated simulation results for Algorithms 3.1 and 3.2 for various t and μ.

Parameter setting	Paths	True	Sim	Diff	Error (%)	RMSE	Time
$t=0.5,\ \mu=0$	1000	0.5000	0.4879	−0.0121	−2.47	0.0155	0.11
	4000	0.5000	0.5057	0.0057	1.12	0.0079	0.13
	16 000	0.5000	0.5011	0.0011	0.22	0.0040	0.38
	64 000	0.5000	0.5008	0.0008	0.16	0.0020	1.09
	256 000	0.5000	0.5002	0.0002	0.05	0.0010	4.35
	1 024 000	0.5000	0.5009	0.0009	0.19	0.0005	16.63
$t=2.5,\ \mu=0$	1000	2.5000	2.4421	−0.0579	−2.37	0.0343	0.08
	4000	2.5000	2.5148	0.0148	0.59	0.0178	0.11
	16 000	2.5000	2.5073	0.0073	0.29	0.0089	0.36
	64 000	2.5000	2.4975	−0.0025	−0.10	0.0044	1.17
	256 000	2.5000	2.5058	0.0058	0.23	0.0022	4.43
	1 024 000	2.5000	2.5003	0.0003	0.01	0.0011	17.19
$t=1,\ \mu=0.5$	1000	0.7869	0.7890	0.0020	0.25	0.0183	0.14
	4000	0.7869	0.7909	0.0039	0.49	0.0095	0.28
	16 000	0.7869	0.7809	−0.0060	−0.76	0.0047	0.66
	64 000	0.7869	0.7891	0.0021	0.26	0.0024	2.12
	256 000	0.7869	0.7855	−0.0014	−0.18	0.0012	8.38
	1 024 000	0.7869	0.7865	−0.0004	−0.05	0.0006	33.09
$t=1,\ \mu=1$	1000	0.6321	0.6426	0.0105	1.63	0.0165	0.19
	4000	0.6321	0.6318	−0.0003	−0.04	0.0081	0.28
	16 000	0.6321	0.6338	0.0017	0.26	0.0041	0.83
	64 000	0.6321	0.6304	−0.0017	−0.27	0.0020	2.18
	256 000	0.6321	0.6324	0.0002	0.03	0.0010	8.55
	1 024 000	0.6321	0.6327	0.0005	0.08	0.0005	31.48
$t=3,\ \mu=0.5$	1000	2.3608	2.3699	0.0091	0.38	0.0326	0.17
	4000	2.3608	2.3431	−0.0177	−0.75	0.0163	0.30
	16 000	2.3608	2.3731	0.0123	0.51	0.0083	0.75
	64 000	2.3608	2.3629	0.0020	0.08	0.0041	2.12
	256 000	2.3608	2.3608	0.0000	0.00	0.0021	8.76
	1 024 000	2.3608	2.3579	−0.0029	−0.12	0.0010	32.33
$t=3,\ \mu=1$	1000	1.8964	1.9056	0.0092	0.48	0.0289	0.16
	4000	1.8964	1.8995	0.0031	0.16	0.0139	0.17
	16 000	1.8964	1.8964	0.0000	0.00	0.0070	0.50
	64 000	1.8964	1.8965	0.0002	0.01	0.0035	1.94
	256 000	1.8964	1.8977	0.0013	0.06	0.0018	6.98
	1 024 000	1.8964	1.8960	−0.0003	−0.01	0.0009	31.60

seven times faster than Cloud and Huber [5]. Compared with the other algorithms, the main advantage of Algorithm 3.1 is that the incremental random variables $(T_i, M_i)_{i\geq 0}$ are i.i.d. pairs, and this means that we can employ vectorisation to reduce total computation time, especially for large values of t. Although the expected simulation time is unbounded for Algorithm 3.1, the simulation idea based on the marked renewal approach still outperforms the dominated coupling approach and rejection sampling approach, and even more so when t is large.

4. More general Vervaat perpetuities

In this section we present a general method for simulating Vervaat perpetuities for more general random variables Y. This method is applicable to a wide range of distributions, and we give several examples to demonstrate this. We have the following result, which decomposes

https://doi.org/10.1017/jpr.2019.6 Published online by Cambridge University Press

https://doi.org/10.1017/jpr.2019.6 Published online by Cambridge University Press

TABLE 2: Comparisons between Algorithm 3.1 and the algorithms suggested in Devroye and Fawzi [8], Fill and Huber [12], Chi [4], and Cloud and Huber [5] for $t = 1$ and $t = 3$.

	Paths	True	Sim	Diff	Error (%)	RMSE	Time	True	Sim	Diff	Error (%)	RMSE	Time
		$t = 1$						$t = 3$					
Algorithm 3.1	1000	1.0000	0.9813	−0.0187	−1.91	0.0218	0.07	3.0000	3.0031	0.0031	0.10	0.0379	0.09
	4000	1.0000	1.0057	0.0057	0.57	0.0114	0.17	3.0000	3.0030	0.0030	0.10	0.0192	0.14
	16 000	1.0000	1.0043	0.0043	0.43	0.0056	0.28	3.0000	2.9979	−0.0021	−0.07	0.0096	0.41
	64 000	1.0000	0.9997	−0.0003	−0.03	0.0028	0.97	3.0000	2.9967	−0.0033	−0.11	0.0048	1.19
	256 000	1.0000	0.9985	−0.0015	−0.15	0.0014	4.30	3.0000	3.0041	0.0041	0.14	0.0024	4.51
	1 024 000	1.0000	0.9994	−0.0006	−0.06	0.0007	16.69	3.0000	2.9978	−0.0022	−0.07%	0.0012	18.84
Devroye and Fawzi [8]	1000	1.0000	0.9923	−0.0077	−0.77	0.0221	0.03	3.0000	2.9731	−0.0269	−0.90	0.0038	0.14
	4000	1.0000	0.9977	−0.0023	−0.22	0.0113	0.13	3.0000	3.0006	0.0006	0.02	0.0192	0.61
	16 000	1.0000	1.0011	0.0011	0.11	0.0056	0.57	3.0000	2.9935	−0.0065	−0.21	0.0097	2.00
	64 000	1.0000	1.0057	0.0057	0.56	0.0028	1.95	3.0000	2.9951	−0.0049	−0.16	0.0048	5.54
	256 000	1.0000	1.0034	0.0034	0.33	0.0014	7.99	3.0000	3.0014	0.0014	0.04	0.0024	23.93
	1 024 000	1.0000	0.9993	−0.0007	−0.07	0.0007	37.51	3.0000	3.0014	0.0014	0.04	0.0012	101.61
Fill and Huber [12]	4000	1.0000	0.9851	−0.0149	−1.50	0.0027	0.10	3.0000	2.9516	−0.0484	−1.64	0.0398	1.31
	4000	1.0000	0.9977	−0.0023	−0.23	0.0113	0.28	3.0000	2.9444	−0.0556	−1.88	0.0195	4.56
	16 000	1.0000	1.0004	0.0004	0.04	0.0056	1.09	3.0000	3.0175	0.0175	0.58	0.0099	18.69
	64 000	1.0000	0.9968	−0.0032	−0.32	0.0028	2.67	3.0000	2.9995	−0.0005	−0.02	0.0049	75.23
	256 000	1.0000	0.9977	−0.0023	−0.23	0.0013	11.14	3.0000	2.9969	−0.0031	−0.10	0.0025	325.37
	1 024 000	1.0000	0.9994	−0.0006	−0.06	0.0006	57.37	3.0000	3.0131	0.0131	0.43	0.0012	1358.24
Chi [4]	1000	1.0000	1.0070	0.0070	0.70	0.0233	0.33	3.0000	3.0109	0.0109	0.36	0.0392	0.62
	4000	1.0000	1.0100	0.0100	0.99	0.0115	0.78	3.0000	3.0208	0.0208	0.69	0.0197	2.19
	16 000	1.0000	0.9955	−0.0045	−0.45	0.0057	2.83	3.0000	3.0107	0.0107	0.35	0.0099	8.44
	64 000	1.0000	1.0026	0.0026	0.26	0.0029	12.24	3.0000	3.0027	0.0027	0.09	0.0050	31.16
	256 000	1.0000	1.0018	0.0018	0.18	0.0014	46.27	3.0000	3.0016	0.0016	0.05	0.0025	132.28
	1 024 000	1.0000	1.0036	0.0036	0.36	0.0007	193.00	3.0000	2.9978	−0.0002	−0.01%	0.0012	514.12
Cloud and Huber [5]	1000	1.0000	0.9832	−0.0168	−1.71	0.0220	0.09	3.0000	2.9376	−0.0624	−2.12	0.0379	0.28
	4000	1.0000	1.0153	0.0153	1.50	0.0113	0.33	3.0000	2.9986	−0.0014	−0.05	0.0195	0.92
	16 000	1.0000	1.0059	0.0059	0.58	0.0056	0.84	3.0000	3.0108	0.0108	0.36	0.0096	2.31
	64 000	1.0000	1.0076	0.0076	0.75	0.0028	3.07	3.0000	3.0014	0.0014	0.05	0.0048	8.61
	256 000	1.0000	1.0013	0.0013	0.13	0.0014	12.51	3.0000	3.0021	0.0021	0.07	0.0024	36.47
	1 024 000	1.0000	1.0017	0.0017	0.17	0.0007	58.19	3.0000	3.0038	0.0038	0.13	0.0012	155.93

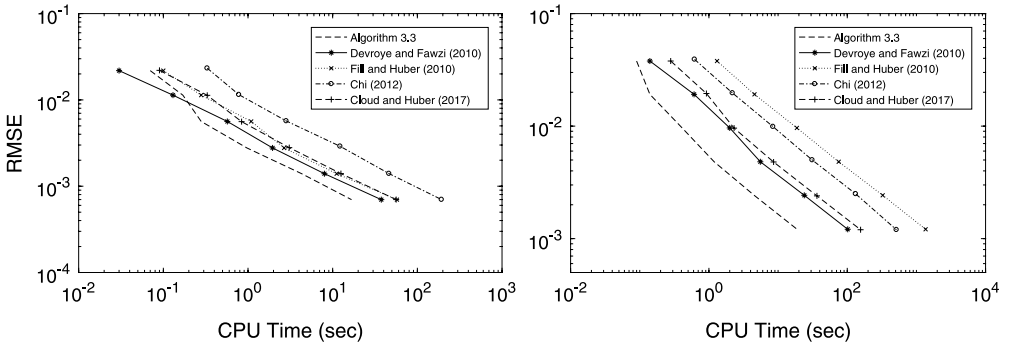


FIGURE 2: Convergence analysis of Algorithm 3.1 and the algorithms suggested in Chi [4], Cloud and Huber [5], Devroye and Fawzi [8], and Fill and Huber [12] for $t = 1$ and $t = 3$, respectively, with associated detailed numerical results reported in Table 2.

TABLE 3: Comparisons of the CPU times in seconds for the algorithms suggested in Chi [4], Cloud and Huber [5], Devroye and Fawzi [8], and Fill and Huber [12], with Algorithm 3.1 for 100 000 replications.

Input t	0.60	0.80	1.00	1.20	1.40	1.60	1.80	2.00	2.20	2.40	2.60	2.80	3.00	3.20	3.40
Algorithm 3.1	1.81	1.68	1.60	1.71	1.79	1.82	1.83	1.78	2.03	1.72	1.77	1.83	2.02	1.83	1.95
Devroye and Fawzi [8]	2.23	3.19	3.94	5.66	5.47	5.23	5.31	5.93	8.48	7.22	7.51	7.97	8.47	10.88	11.50
Fill and Huber [12]	2.53	3.14	4.12	5.68	7.34	10.33	13.64	18.28	20.76	36.78	51.22	70.23	100.75	140.91	200.04
Chi [4]	6.28	9.62	18.52	21.25	23.53	27.45	31.80	34.78	36.92	38.41	41.57	55.31	50.54	55.83	59.23
Cloud and Huber [5]	3.09	3.93	4.71	5.66	6.71	7.57	9.70	11.16	11.82	13.45	13.72	14.71	15.54	16.99	17.25

the Vervaat perpetuity X into the sum of a gamma process (or truncated gamma process) and compound Poisson processes. For simplicity, we will let Y have a continuous density g.

Lemma 4.1. *Consider a perpetuity X defined in (1.1), with $Y \in \mathbb{R}^+$. If the density of Y satisfies (1.3) then there exists $k \in \mathbb{R}^+$ and $b \in \mathbb{R}^+$ such that $\mathbb{P}(Y > y) \geq \mathrm{e}^{-ky}$ for all $y < b$. We then have the following two cases.*

Case 1. *There exists $k \in \mathbb{R}^+$ such that $\mathbb{P}(Y > y) \geq \mathrm{e}^{-ky}$ for all $y \in \mathbb{R}^+$. Then we have*

$$B = \int_0^\infty \frac{\mathbb{P}(Y > y) - \mathrm{e}^{-ky}}{y} \, \mathrm{d}y < \infty, \tag{4.1}$$

and, thus,

$$X \stackrel{\mathrm{D}}{=} \Gamma_t + \sum_{i=1}^{N_t} J_i, \tag{4.2}$$

where

- *Γ_t is a gamma process such that $\Gamma_t \sim \Gamma(t, k)$;*
- *$\sum_{i=1}^{N_t} J_i$ is a compound Poisson process such that*
  - *N_t is a Poisson process with rate tB;*
  - *$\{J_i\}_{i=1,2,\ldots}$ are i.i.d. jumps with density*

$$g_{J_i}(y) = \frac{\mathbb{P}(Y \geq y) - \mathrm{e}^{-ky}}{By}, \qquad y \in \mathbb{R}^+.$$

https://doi.org/10.1017/jpr.2019.6 Published online by Cambridge University Press

Case 2. *There only exists* $k \in \mathbb{R}^+$ *such that* $\mathbb{P}(Y > y) \geq \mathrm{e}^{-ky}$ *for* $y < b$. *Then we have*

$$D = \int_0^b \frac{\mathbb{P}(Y > y) - \mathrm{e}^{-ky}}{y}\,\mathrm{d}y < \infty, \tag{4.3}$$

and, thus,

$$X \overset{\mathrm{D}}{=} Z_t + \sum_{i=1}^{N_t^1} J_i^{(1)} + \sum_{k=1}^{N_t^2} J_k^{(2)}, \tag{4.4}$$

where

- Z_t *is a truncated Gamma process with Lévy measure*

$$\nu(\mathrm{d}z) = z^{-1}\mathrm{e}^{-kz}\mathbf{1}_{\{0<z<b\}}\,\mathrm{d}z;$$

- $\sum_{i=1}^{N_t^1} J_i^{(1)}$ *is a compound Poisson process such that*
  - N_t^1 *is a Poisson process with rate* tD;
  - $\{J_i^{(1)}\}_{i=1,2,\ldots}$ *are i.i.d. jumps with density*

$$g_{J_i^{(1)}}(y) = \frac{\mathbb{P}(Y \geq y) - \mathrm{e}^{-ky}}{Dy}, \qquad y \in (0, b);$$

- $\sum_{k=1}^{N_t^2} J_k^{(2)}$ *is a compound Poisson process such that*
  - N_t^2 *is a Poisson process with rate* tE, *where*

$$E = \int_b^\infty \frac{\mathbb{P}(Y \geq y)}{y}\,\mathrm{d}y, \tag{4.5}$$

  - $\{J_k^{(2)}\}_{i=1,2,\ldots}$ *are i.i.d. jumps with density*

$$g_{J_k^{(2)}}(y) = \frac{\mathbb{P}(Y \geq y)}{Ey}, \qquad y \in (b, \infty).$$

Proof. If the density of Y satisfies (1.3) then $g(0) < \infty$. Hence, the survival function of Y, $\mathbb{P}(Y > y)$, decays at an exponential rate or slower at 0. This implies that there exists a $k \in \mathbb{R}^+$ such that $\mathbb{P}(Y > y) \geq \mathrm{e}^{-ky}$ for all $y < b$ for some $b \in \mathbb{R}^+$.

Case 1. If, furthermore, there exists such $k \in \mathbb{R}^+$ so that $\mathbb{P}(Y > y) \geq \mathrm{e}^{-ky}$ holds for all $y \in \mathbb{R}^+$, then, for B defined as in (4.1), it follows that the integral B is finite since

$$\int_0^\infty \frac{\mathbb{P}(Y > y) - \mathrm{e}^{-ky}}{y}\,\mathrm{d}y = \int_0^1 \frac{\mathbb{P}(Y > y) - \mathrm{e}^{-ky}}{y}\,\mathrm{d}y + \int_1^\infty \frac{\mathbb{P}(Y > y) - \mathrm{e}^{-ky}}{y}\,\mathrm{d}y.$$

The second term is finite since we have assumed that the mean of Y exists, and we also have

$$\begin{aligned}\int_0^1 \frac{\mathbb{P}(Y > y) - \mathrm{e}^{-ky}}{y}\,\mathrm{d}y &= [\log y(\mathbb{P}(Y > y) - \mathrm{e}^{-ky})]_0^1 + \int_0^1 \log y\left(g(y) - \frac{\mathrm{e}^{-ky}}{k}\right)\mathrm{d}y \\ &= \int_0^1 \log y\left(g(y) - \frac{\mathrm{e}^{-ky}}{k}\right)\mathrm{d}y \\ &< \infty,\end{aligned}$$

https://doi.org/10.1017/jpr.2019.6 Published online by Cambridge University Press

since $g(0) < \infty$. Hence, the Laplace transform of X can be expressed as

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{-\beta X}] &= \exp\Big(-t\int_0^\infty (1-\mathrm{e}^{-\beta y})\frac{\mathrm{e}^{-ky}}{y}\,\mathrm{d}y\Big)\\ &\quad\times\exp\Big(-tB\int_0^\infty (1-\mathrm{e}^{-\beta y})\frac{\mathbb{P}(Y>y)-\mathrm{e}^{-ky}}{By}\,\mathrm{d}y\Big)\\ &= \mathbb{E}[\mathrm{e}^{-\beta\Gamma_t}]\mathbb{E}[\mathrm{e}^{-\beta\sum_{i=1}^{N_t}J_i}],\end{aligned}$$

and we have the distributional decomposition result given in (4.2).

Case 2. In the case where we only have $\mathbb{P}(Y > y) \geq \mathrm{e}^{-ky}$ for $y < b$ for some $b \in \mathbb{R}^+$, we define the integral D as in (4.3) and E as in (4.5). An argument similar to that above shows that they are finite, and we thus have

$$\begin{aligned}\mathbb{E}[\mathrm{e}^{-\beta X}] &= \exp\Big(-t\int_0^b (1-\mathrm{e}^{-\beta y})\frac{\mathbb{P}(Y>y)}{y}\,\mathrm{d}y\Big)\\ &\quad\times\exp\Big(-t\int_b^\infty (1-\mathrm{e}^{-\beta y})\frac{\mathbb{P}(Y>y)}{y}\,\mathrm{d}y\Big)\\ &= \exp\Big(-t\int_0^b (1-\mathrm{e}^{-\beta y})\frac{\mathrm{e}^{-ky}}{y}\,\mathrm{d}y\Big)\\ &\quad\times\exp\Big(-tD\int_0^b (1-\mathrm{e}^{-\beta y})\frac{\mathbb{P}(Y>y)-\mathrm{e}^{-ky}}{Dy}\,\mathrm{d}y\Big)\\ &\quad\times\exp\Big(-tE\int_b^\infty (1-\mathrm{e}^{-\beta y})\frac{\mathbb{P}(Y>y)}{Ey}\,\mathrm{d}y\Big)\\ &= \mathbb{E}[\mathrm{e}^{-\beta Z_t}]\mathbb{E}[\mathrm{e}^{-\beta\sum_{i=1}^{N_t^1}J_i^{(1)}}]\mathbb{E}[\mathrm{e}^{-\beta\sum_{k=1}^{N_t^2}J_k^{(2)}}].\end{aligned}$$

The distributional decomposition result (4.4) thus follows. □

We now have a general method for exactly simulating generalised Vervaat perpetuities for a wide range of distributions for Y. In particular, case 2 is useful when the distribution of Y has finite support, or when Y has a tail which decays faster than exponential decay, in which case there does not exist $k \in \mathbb{R}^+$ such that $\mathbb{P}(Y > y) \geq \mathrm{e}^{-ky}$ for all y. In the rest of this section, we give some examples to demonstrate the methodology.

Example 1. Let $Y \sim \text{Pareto}(\alpha, \sigma)$. From Theorem 2.1, the Laplace transform of X is

$$\mathbb{E}[\mathrm{e}^{-\beta X}] = \exp\Big(-t\int_0^\sigma \frac{1-\mathrm{e}^{-\beta y}}{y}\,\mathrm{d}y\Big)\exp\Big(-t\int_\sigma^\infty \frac{1-\mathrm{e}^{-\beta y}}{y}\Big(\frac{\sigma}{y}\Big)^\alpha\,\mathrm{d}y\Big).$$

Hence, $X \overset{\mathrm{D}}{=} Z_t + \text{CP}$, where $Z_t \sim$ Dickman process and CP is a compound Poisson process, independent of each other. The Dickman process can be generated using Algorithm 3.1, and the compound Poisson process can be easily generated via an A/R scheme.

Example 2. Let $Y \sim \text{Gamma}(\alpha, \beta)$. Since $g(0) < \infty$ for all $\alpha \geq 1$, we can simulate all gamma perpetuities with $\alpha \geq 1$ and all values of β. Furthermore, for all $\alpha \geq 1$, there exists a k such that $\mathbb{P}(Y > y) \geq \mathrm{e}^{-ky}$ for all y; hence, it falls under case 1 of Lemma 4.1. Thus, the gamma

https://doi.org/10.1017/jpr.2019.6 Published online by Cambridge University Press

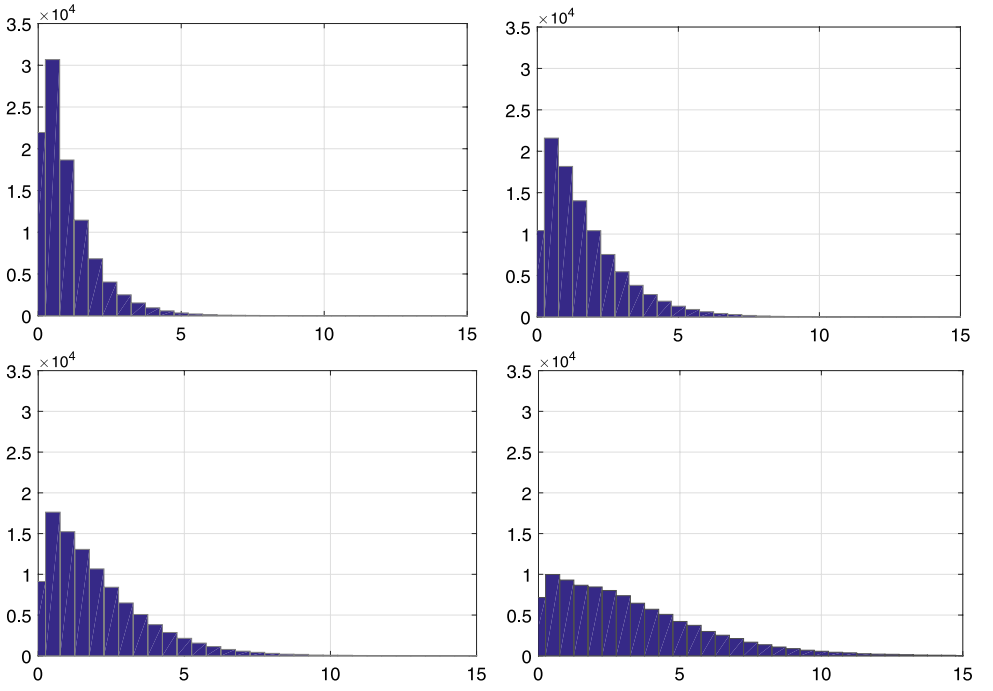


FIGURE 3: Histograms of perpetuity X with $Y \sim \Gamma(\alpha, \beta)$ under the parameter setting $\alpha = 1, 1.5, 2, 2.5$ and $\beta = 1$.

perpetuity X can be split into $X \overset{\mathrm{D}}{=} Z_t + \mathrm{CP}$, where Z_t is a gamma process and CP is a compound Poisson process independent of Z_t by Lemma 4.1. In Figure 3 we show the histograms of the Vervaat perpetuity X for $Y \sim \Gamma(\alpha, \beta)$ with different shape parameters α.

Example 3. Let $Y \sim \mathrm{Weibull}(\kappa, \lambda)$. Since $g(0) < \infty$ for all $\kappa \geq 1$, we can simulate all Weibull perpetuities with $\kappa \geq 1$ and all values of λ. Furthermore, there only exists a k such that $\mathbb{P}(Y > y) \geq \mathrm{e}^{-ky}$ for $y \in (0, b)$ for some $b \in \mathbb{R}^+$. Hence, the perpetuity falls into case 2 of Lemma 4.1, and can be split into $X \overset{\mathrm{D}}{=} Z_t + \mathrm{CP}_1 + \mathrm{CP}_2$, where Z_t is a truncated gamma process, and CP_1 and CP_2 are compound Poisson processes, independent of each other and Z_t. The truncated gamma process can be generated using Algorithm 3.2 and the two compound Poisson processes can be easily generated via A/R schemes.

Example 4. Let $Y \sim \mathrm{Beta}(\alpha, \beta)$. Since $g(0) < \infty$ for all $\alpha \geq 1$, we can simulate all beta perpetuities with $\alpha \geq 1$ and all values of β. Furthermore, since the distribution has a finite support, there only exists a k such that $\mathbb{P}(Y > y) \geq \mathrm{e}^{-ky}$ for $y \in (0, b)$ for some $b \in \mathbb{R}^+$. Hence, the perpetuity falls into case 2 of Lemma 4.1, and the Lévy measure needs to be truncated. The beta perpetuity can be split into $X \overset{\mathrm{D}}{=} Z_t + \mathrm{CP}_1 + \mathrm{CP}_2$ according to Lemma 4.1, where Z_t is a truncated gamma process, and CP_1 and CP_2 are two compound Poisson processes, independent of each other and Z_t.

https://doi.org/10.1017/jpr.2019.6 Published online by Cambridge University Press

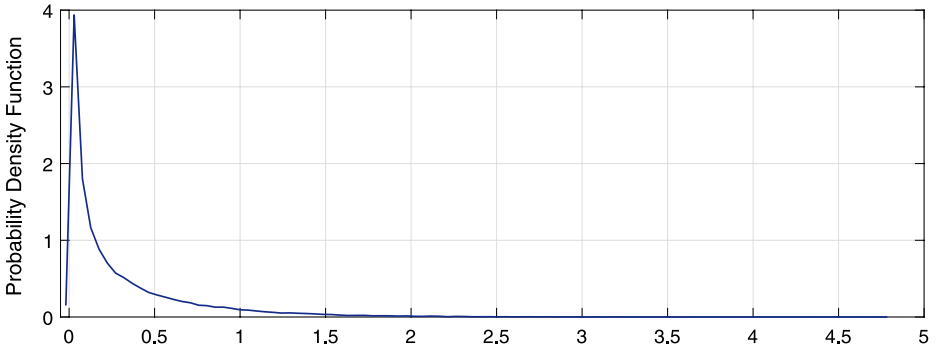


FIGURE 4: Probability density function of perpetuity X_i with $Y_i \sim$ half-normal(0, 1).

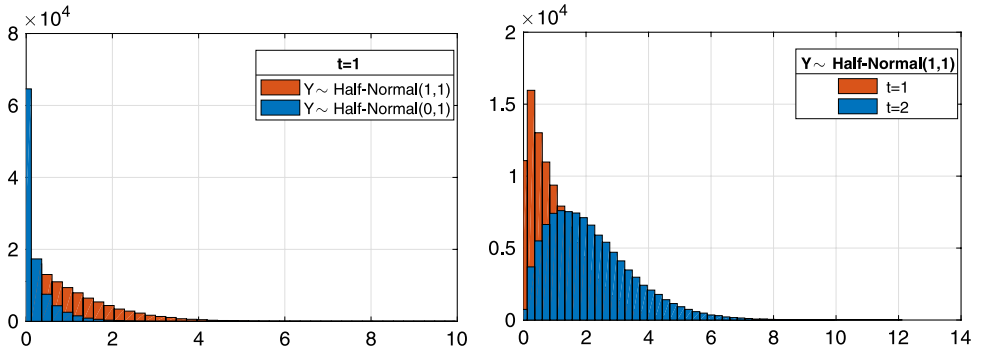


FIGURE 5: Histograms of perpetuity X_i with $Y_i \sim$ half-normal(0, 1) and $Y_i \sim$ half-normal(1, 1) with $t = 1, 2$.

Example 5. Let $Y \sim \text{Normal}(\mu, \sigma^2)$. The normal perpetuity X is two sided. According to Corollary 2.1, one could sample independent X_1 and X_2, each of which are half-normal perpetuities, and the difference $X_1 - X_2$ is the realisation of the two-sided perpetuity X.

For a half-normal perpetuity X_i, where $Y_i \sim$ half-normal(μ, σ), there only exists a k such that $\mathbb{P}(Y > y) \geq \mathrm{e}^{-ky}$ for $y \in (0, b)$ for some $b \in \mathbb{R}^+$. Hence, the perpetuity X_i falls into case 2 and can be split into a truncated gamma process and two compound Poisson processes according to Lemma 4.1. The truncated gamma process can be generated via Algorithm 3.2 and the two compound Poisson processes can be easily generated via A/R schemes. In Figure 4 we illustrate a sample density plot for the perpetuity X_i with $Y_i \sim$ half-normal(0, 1). In Figure 5 we compare the distribution of X_i under different parameter settings.

Now we can simulate the normal perpetuity X by generating two independent half-normal perpetuities X_1 and X_2 and taking the difference $X \overset{\mathrm{D}}{=} X_1 - X_2$. In Figures 6 and 7 we demonstrate the distribution behaviour of the two-sided perpetuity X via its histogram and density plot. We can also obtain other variations of the normal perpetuities by introducing a new variable $B \sim \text{Bernoulli}(p)$ and setting $\hat{X} \overset{\mathrm{D}}{=} pX_1 - (1-p)X_2$. When $p = \frac{1}{2}$ and $X_i \sim$ Normal(0, 1), this is identical to the Normal(0, 1) perpetuity. In Figure 8 we illustrate the differences of the perpetuity $\hat{X}$ and X in terms of their histograms.

https://doi.org/10.1017/jpr.2019.6 Published online by Cambridge University Press

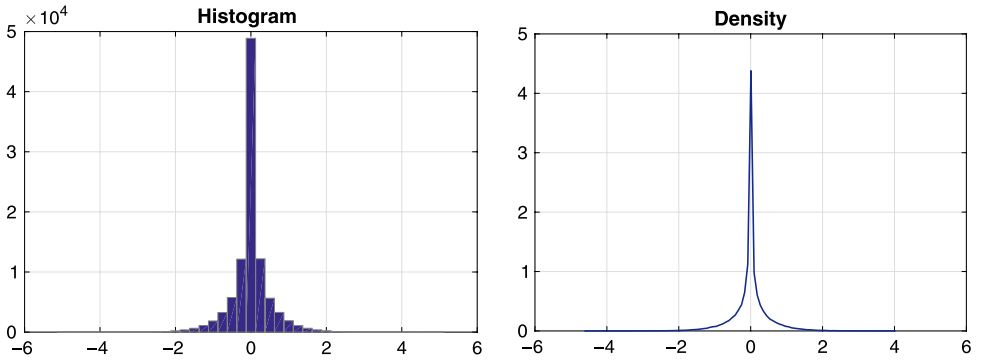


FIGURE 6: Histogram and density plot of perpetuity X with $Y \sim$ Normal(0, 1).

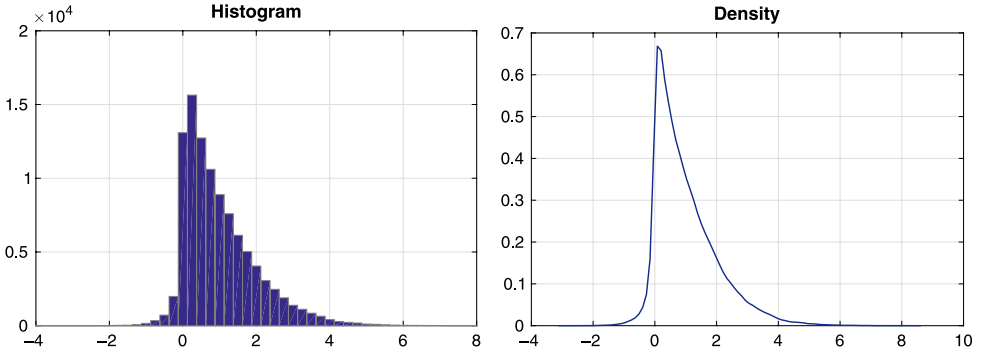


FIGURE 7: Histogram and density plot of perpetuity X with $Y \sim$ Normal(1, 1).

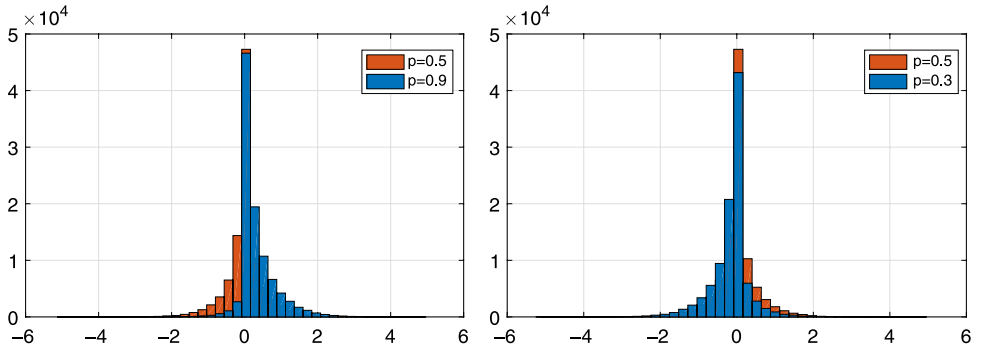


FIGURE 8: Histogram of perpetuity $\hat{X} = BX_1 - (1 - B)X_2$ with $B \sim$ Bernoulli(p) under parameter setting $p = 0.3, 0.5, 0.9$.

Acknowledgements

The authors would like to thank the Associated Editor and two reviewers for many constructive comments that greatly improved the presentation of our results.

https://doi.org/10.1017/jpr.2019.6 Published online by Cambridge University Press

References

[1] ARRATIA, R. (1998). On the central role of scale invariant Poisson processes on $(0, \infty)$. In *Microsurveys in Discrete Probability*, eds D. Aldous and J. Propp, American Mathematical Society, pp. 21–41.

[2] BLANCHET, J. H. AND SIGMAN, K. (2011). On exact sampling of stochastic perpetuities. In *Probability, Analysis and Number Theory* (J. Appl. Prob. Spec. Vol. **48A**), eds C. M. Goldie and A. Mijatović, Applied Probability Trust, sheffield, pp. 165–182.

[3] CHANDRASEKHAR, S. AND MÜNCH, G. (1950). The theory of the fluctuations in brightness of the Milky Way, I and II. *Astrophys. J.* **112**, 380–398.

[4] CHI, Z. (2012). On exact sampling of nonnegative infinitely divisible random variables. *Anal. Appl. Prob.* **44**, 842–873.

[5] CLOUD, K. AND HUBER, M. (2017). Fast perfect simulation of Vervaat perpetuities. *J. Complexity* **42**, 19–30.

[6] DE BRUIJN, N. (1951). The asymptotic behaviour of a function occurring in the theory of primes. *J. Indian Math. Soc.* **15**, 20–32.

[7] DEVROYE, L. (2001). Simulating perpetuities. *Methodol. Comput. Appl. Probab.* **3**, 97–115.

[8] DEVROYE, L. AND FAWZI, O. (2010). Simulating the Dickman distribution. *Statist. Probab. Lett.* **80**, 242–247.

[9] DICKMAN, K. (1930). On the frequency of numbers containing prime factors of a certain relative magnitude. *Arkiv for Matematik, Astronomi och Fysik* **22**, 1–14.

[10] EMBRECHTS, P. AND GOLDIE, C. M. (1994). Perpetuities and random equations. In *Asymptotic Statistics*, eds P. Mandl and M. Hušková, Springer, pp. 75–86.

[11] EMBRECHTS, P., KLÜPPELBERG, C. AND MIKOSCH, T. (1996). *Modelling Extremal Events.* Springer.

[12] FILL, J. AND HUBER, M. (2010). Perfect simulation of Vervaat perpetuities. *Electron. J. Prob.* **15**, 96–109.

[13] GOLDIE, C. M. AND GRÜBEL, R. (1996). Perpetuities with thin tails. *Anal. Appl. Prob.* **28**, 463–480.

[14] GRÜBEL, R. AND RÖSLER, U. (1996). Asymptotic distribution theory for Hoare's selection algorithm. Anal. Appl. Prob. **28**, 252–269.

[15] MAHMOUD, H.M., MODARRES, R. AND SMYTHE, R.T. (1995). Analysis of Quickselect: an algorithm for order statistics. *RAIRO Theoret. Inform. Appl.* **29**, 255–276.

[16] NYRHINEN, H. (2001). Finite and infinite time ruin probabilities in a stochastic economics environment. *Stoch. Proc. Appl.* **92**, 265–285.

[17] TAKÁCS, L. (1954). On secondary processes generated by a Poisson process and their applications in physics. *Acta Math. Hungar.* **5**, 203–236.

[18] TAKÁCS, L. (1955). On stochastic processes generated by a Poisson process and their applications in physics. *Acta Math. Hungar.* **6**, 363–380.

[19] VERVAAT, W. (1979). On a stochastic difference equation and a representation of non–negative infinitely divisible random variables. *Anal. Appl. Prob.* **11**, 750–783.

https://doi.org/10.1017/jpr.2019.6 Published online by Cambridge University Press