*Combinatorics, Probability and Computing* (2002) 11, 217–248. © 2002 Cambridge University Press
DOI: 10.1017/S0963548301005114 Printed in the United Kingdom

# Exact Expectations and Distributions for the Random Assignment Problem

SVEN ERICK ALM and GREGORY B. SORKIN

Department of Mathematics, Uppsala University,
S-751 06 Uppsala, Sweden
(e-mail: sea@math.uu.se)

Department of Mathematical Sciences,
IBM T. J. Watson Research Center,
Yorktown Heights, NY 10598, USA
(e-mail: sorkin@watson.ibm.com)

*Received 8 December 1999; revised 27 November 2001*

A generalization of the random assignment problem asks the expected cost of the minimum-cost matching of cardinality $k$ in a complete bipartite graph $K_{m,n}$, with independent random edge weights. With weights drawn from the exponential distribution with intensity 1, the answer has been conjectured to be

$$\sum_{i,j\geq 0,\ i+j<k} \frac{1}{(m-i)(n-j)}.$$

Here, we prove the conjecture for $k \leqslant 4$, $k = m = 5$, and $k = m = n = 6$, using a structured, automated proof technique that results in proofs with relatively few cases. The method yields not only the minimum assignment cost's expectation but the Laplace transform of its distribution as well. From the Laplace transform we compute the variance in these cases, and conjecture that, with $k = m = n \to \infty$, the variance is $2/n + O(\log n/n^2)$. We also include some asymptotic properties of the expectation and variance when $k$ is fixed.

## 1. Introduction

For a nonnegative $m \times n$ matrix $A$, given an integer $k$, let $A^\star$ be the smallest possible sum of $k$ elements of $A$ when no two elements share a row or a column; in other words, $A^\star$ is the cost of the minimum $k$-assignment of $A$.

Let $A^\star(k, m, n)$ be the random variable corresponding to the case when the elements of the $m \times n$ matrix $A$ are independent, identically distributed (i.i.d.) random variables (r.v.s) drawn from a given distribution. Specifically, we will use the distribution exp(1), where exp($\lambda$) is the exponential distribution with intensity $\lambda$, and density $\lambda e^{-\lambda x}$ over $x \geqslant 0$.

It was conjectured by Mezard and Parisi [18, 19] that $\mathbb{E}A^\star(n, n, n) \to \pi^2/6$. Later

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

Parisi [22] made the stronger conjecture that $\mathbb{E}A^\star(n,n,n) = \sum_{i=1}^{n} 1/i^2$, and most recently Coppersmith and Sorkin [7] made the following still stronger conjecture.

**Conjecture 1.1.** *A minimum-cost $k$-assignment, in an $K_{m,n}$ bipartite graph with edge weights independently chosen from the exponential distribution with intensity* 1*, has expected cost*

$$\mathbb{E}A^\star(k,m,n) = \sum_{i+j<k} \frac{1}{(m-i)(n-j)}.$$

In this paper we prove the last conjecture for $k \leqslant 4$, $k = m = 5$, and $k = m = n = 6$. In these cases, our proof method also yields the Laplace transform of the random variable $A^\star(k,m,n)$.

We note that there is a natural association between an $m \times n$ matrix $A$ and an edge-weighting of the complete bipartite graph $K_{m,n}$, in which case the minimum $k$-assignment can also be thought of as the minimum-cost cardinality-$k$ matching in $K_{m,n}$. Choosing some other underlying graph $(V,E)$ in lieu of $K_{m,n}$ gives a natural generalization of the following problem: characterize the cost $A^\star$ of a minimum $k$-matching for a random edge-weighting $A : E \to \mathbb{R}^{+,0}$ of $(V,E)$. Our results and notation will be specific to the $K_{m,n}$ case, but our methods (specifically Theorem 4.4) may be more widely applicable.

It is interesting to compare the situation for a minimum matching with that for a minimum spanning tree (MST). For a matching, the value $\mathbb{E}A^\star(k,m,n)$ appears to have the simple form given above, but both this and the limiting $\mathbb{E}A^\star(n,n,n) \to \zeta(2) = \sum_{i=1}^{\infty} i^{-2} = \pi^2/6$ are only conjectures. For a tree, it is known that the limiting expected cost of an MST in $K_n$ is $\zeta(3)$ (Frieze [10]), that the limiting expectation in $K_{n,n}$ is $2\zeta(3) = 2\sum_{i=1}^{\infty} i^{-3}$ [11], and that for any $r$-regular graph with some expansion properties, the expected MST cost is asymptotic (in $r$) to $\frac{n}{r}\zeta(3)$ [4]. For finite $n$ the calculations of the expected minimum cost of an MST on $K_n$ are also much simpler than are similar calculations for the assignment cost. On the other hand, the results for trees seem to offer no simple pattern analogous to Conjecture 1.1.

## 2. Linear-programming formulations

Implicitly or explicitly, linear-programming (LP) formulations underlie much of the past work on the random assignment problem, including the lower bounds derived by Lazarus [14, 15] and Olin [20] on $\lim_{n\to\infty} \mathbb{E}A^\star(n,n,n)$, and the upper bound of 2 derived by Karp [13] and generalized by [8].

Our method for computing $\mathbb{E}A^\star(k,m,n)$ does not appear to be subsumed in the LP view, but has some similarities with it. We therefore present the LP view quickly, so the interested reader can compare it with our approach but uninterested readers will not be harmed by skipping it.

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

For the classical, nonrandom assignment problem, that is, find the minimum cost $n$-assignment in a given $n \times n$ matrix $A$, the corresponding LP and its dual are well known:

| Primal | Dual |
|---|---|
| minimize $\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_{ij}$ | maximize $\sum_{i=1}^{m} u_i + \sum_{j=1}^{n} v_j$ |
| subject to | subject to |
| $x_{ij} \geqslant 0$ | $a_{ij} - u_i - v_j \geqslant 0$ |
| $\sum_j x_{ij} = 1 \ (i = 1 \ldots n)$ | |
| $\sum_i x_{ij} = 1 \ (j = 1 \ldots n)$ | |

As a special case of the transportation problem LP, 'basic' optimal solutions to this LP give integer values to the 'indicator' variables $x_{ij}$, so solving the LP indeed solves the original problem.

For a $k$-assignment in an $m \times n$ matrix $A$, as observed in [6], the LP and its dual are as shown below, and again basic optimal solutions to the primal are integral:

| Primal | Dual |
|---|---|
| minimize $\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_{ij}$ | maximize $kz - \sum_{i=1}^{m} u_i - \sum_{j=1}^{n} v_j$ |
| subject to | subject to |
| $x_{ij} \geqslant 0$ | $z \geqslant 0;\ u_i \geqslant 0;\ v_j \geqslant 0$ |
| $\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij} = k$ | $a_{ij} - z + u_i + v_j \geqslant 0$ |
| $\sum_j x_{ij} \leqslant 1$ | |
| $\sum_i x_{ij} \leqslant 1$ | |

## 3. Example: calculating $A^{\star}(2, m, n)$

We will begin with an intuitive derivation of $A^{\star}(2, m, n)$ to introduce the method we will use to calculate $A^{\star}(k, m, n)$ more generally.

First, some notation. Let $X(\lambda)$ denote an exponentially distributed variable with intensity lambda. Let $B(p)$ denote a Bernoulli random variable that is 1 with probability $p$ and 0 with probability $1 - p$. When several r.v.s $X$ and $B$ appear in an expression, unless otherwise indicated, they are independent. Write $Y \sim X$ to indicate that the r.v.s $X$ and $Y$ have the same distribution.

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

Order statistics and the lack of memory of the exponential distribution are simple and well known; everything we need is incorporated in the following property.

**Property 3.1.** *Let $(X_1, \ldots, X_n)$ be i.i.d. exponential r.v.s with intensity* 1*, of which the smallest is $X_i$. Then $i$ is uniform in $\{1 \ldots n\}$, $X_i$ is exponentially distributed with intensity $n$, the differences $X_j - X_i$, $j \neq i$, are i.i.d. exponential* (1)*, and these differences, $i$, and $X_i$ are all independent.*

**Proposition 3.2.** *The minimum 2-assignment in an $m \times n$ matrix $A$ of i.i.d. exponential r.v.s has cost $A^\star(2, m, n) \sim 2X(mn) + X((m-1)n) + B(1/n)X(m(n-1))$.*

**Proof.** First, select the smallest element of $A$; without loss of generality suppose it is $a_{11}$. Subtract it from all other elements of $A$ to yield $A'$. Subtracting $a_{11}$ from $A$ means it is subtracted twice from any 2-assignment $\sigma$, so $\text{cost}(A', \sigma) = \text{cost}(A, \sigma) - 2a_{11}$. In particular, this holds for the common minimum 2-assignment $\sigma^\star$ for $A$ and $A'$, so $A^\star = A'^\star + 2a_{11}$. Since $a_{11} \sim X(mn)$ and is independent of the values of $A'$, $A^\star \sim A'^\star + 2X(mn)$. Letting $A' = A - a_{11}$, $a'_{11} = 0$ but all other elements of $A'$ are i.i.d. exponential (1). We will proceed by introducing further 0s until a 2-assignment appears.

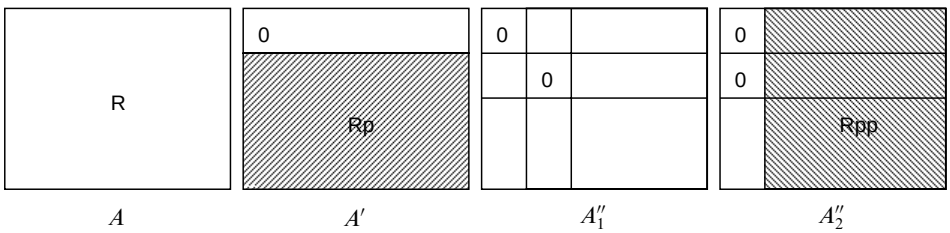



*Figure 1* Subtracting the minimum value of $A$ from all $A$ results in $A'$ with a single zero. Subtracting the minimum element of the region $R'$ within $A'$ from the same region gives a second zero, either in the pattern of $A''_1$ or $A''_2$. $A''_1$ contains a 2-assignment of 0s. $A''_2$ occurs with probability $1/n$, and subtracting its minimum over region $R''$ from the same region forces a 2-assignment of 0s

Let $R'$ be the 'region' consisting of all of $A'$ except its first row, that is, $R' = \{(i, j) : 2 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$; see Figure 1. Observe that every *minimal* 2-assignment $\sigma$ uses *exactly* one element from $R'$: any 2-assignment must have *at least* one element outside row 1, and on the other hand a cheapest 2-assignment cannot have *both* elements outside row 1, for then replacing one of its elements (the element in column 1, if there is one) with the 0 at $a'_{11}$ would give a cheaper assignment.

It follows that subtracting any value $c$ from the region $R'$ yields a matrix $A''$ with $A''^\star = A'^\star - c$. In particular, let $\min_{R'} a'_{ij}$ denote the smallest element in the region $R'$ of $A'$, and let

$$A'' = A' -_{R'} \min_{R'} a'_{ij}$$

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

be defined by subtracting this minimum value from all elements in the same region, that is,

$$a''_{ij} = \begin{cases} a'_{ij} - \min_{R'} a'_{ij}, & \text{for } (i,j) \in R', \\ a'_{ij}, & \text{for } (i,j) \notin R'. \end{cases}$$

Note that $\min_{R'} a'_{ij} \sim X((m-1)n)$, so $A''^{\star} \sim A'^{\star} - X((m-1)n)$. By Property 3.1, $\min_{R'} a'_{ij}$ is independent of $\min_{ij} a_{ij}$, so $A^{\star} \sim A''^{\star} + X(mn) + X((m-1)n)$, where the random variables $X$ and the elements of $A''$ are all independent.

With probability $\frac{n-1}{n}$ (the case $A'' = A''_1$ in Figure 1), $A''$ has its second zero outside column 1, in which case its minimum 2-assignment consists of the two zero elements, and $A''^{\star}_1 = 0$. Otherwise, with probability $1/n$ (the case $A'' = A''_2$), the second zero is also in column 1; without loss of generality say $a''_{21} = 0$. Here, the minimum 2-assignment of $A''_2$ must contain *exactly* one element outside column 1: any 2-assignment must have *at least* one element outside column 1, and on the other hand any one such element suffices, as it may be combined with whichever zero is not in its row to produce a 2-assignment. So, defining $R''$ to be the region of $A''_2$ consisting of all but the first column, $R'' = \{(i,j) : 1 \leqslant i \leqslant m,\ 2 \leqslant j \leqslant n\}$, and letting $A''' = A''_2 -_{R''} \min_{R''} a''_{ij}$, we have $A''^{\star}_2 \sim A'''^{\star} + X(m(n-1))$, while $A'''^{\star} = 0$.

We conclude that $A^{\star} \sim 2X(mn) + X((m-1)n) + B(1/n)X(m(n-1))$, where the various random variables $X$ and $B$ are all independent. In particular,

$$\mathbb{E}A^{\star} = \frac{2}{mn} + \frac{1}{(m-1)n} + \frac{1}{n}\frac{1}{m(n-1)} = \frac{1}{mn} + \frac{1}{(m-1)n} + \frac{1}{m(n-1)},$$

as in Conjecture 1.1. □

## 4. Calculating $A^{\star}(k,m,n)$

To fix some nomenclature, let integers $0 \leqslant k \leqslant m \leqslant n$ be given, along with a 'zero-subset' $Z \subseteq K_{m,n}$ of the edges of the $m \times n$ complete bipartite graph. Let $A$ be a nonnegative edge-weighted version of $K_{m,n}$, so that edge $(i,j)$ has weight $a_{(i,j)} \geqslant 0$. Say that $A$ is *consistent* with $Z$ if $a_{(i,j)} = 0$ whenever $(i,j) \in Z$. Define an (*i.i.d. exponential*) *random weighting* $A$ consistent with $Z$ to be a collection of i.i.d. exponential edge weights $a_{(i,j)}$ conditioned upon $a_{(i,j)} = 0$ whenever $(i,j) \in Z$. (Equivalently, fix $a_{(i,j)} = 0$ for all $(i,j) \in Z$, and choose the remaining weights $a_{(i,j)}$ as i.i.d. exponential r.v.s.)

**Problem 4.1. (Generalized random assignment)** *Given $k,m,n$ and $Z \subseteq K_{m,n}$, let $A$ be an i.i.d. exponential edge-weighting of $K_{m,n}$ consistent with $Z$. What is the distribution of $A^{\star}(k,m,n,Z)$, the cost of a minimum $k$-matching in $A$ and, in particular, what is its expectation $f(k,m,n,Z)$?*

**Lemma 4.2.** *Let $Z \subseteq K_{m,n}$ be any subset, and $M$ and $M'$ be two $k$-matchings in $K_{m,n}$ satisfying $M - Z \subseteq M' - Z$. Then, for any edge-weighting $A$ of $K_{m,n}$ consistent with $Z$, the cost of the matching $M$ in $A$ is less than or equal to that of $M'$. That is, defining $c_A(M) \equiv \sum_{ij \in M} a_{ij}$, $M - Z \subseteq M' - Z' \implies c_A(M) \leqslant c_A(M')$.*

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

**Proof.** Compared with $M$, the matching $M'$ omits some 0-cost edges, but adds some edges whose cost in general is positive. □

**Lemma 4.3.** *Given $k, m, n$, let $\mathscr{M}(k,m,n)$ be the collection of* all *$k$-matchings $M$ in $K_{m,n}$. Let $\mathscr{M}|Z$ be the collection of matchings 'reduced by $Z$' which are not proper supersets of any other matching reduced by $Z$:*

$$\mathscr{M}|Z \quad \equiv \quad \{M \mid (\exists M_1 \in \mathscr{M} : M_1 - Z = M) \text{ and } (\nexists M_2 \in \mathscr{M} : M_2 - Z \subsetneq M)\}.$$

*For any nonnegative edge-weighting $A$ of $K_{m,n}$, $\min_{M\in\mathscr{M}} c_A(M) = \min_{M\in\mathscr{M}|Z} c_A(M)$.*

We call $\mathscr{M}|Z$ the collection of 'potentially minimal matchings'.

**Theorem 4.4. (Main theorem)** *Given $k, m, n$, and a zero-set $Z \subseteq K_{m,n}$, let us choose $A^\star(k,m,n,Z)$ to be the (random) cost of a minimum $k$-assignment of a (random) matrix $A$ consistent with $Z$. Suppose there exists a subset $R \subseteq K_{m,n}$, with $R \cap Z = \emptyset$, and with the property that the $k$-matchings $M \in \mathscr{M}|Z$ have fixed intersection size $s \geqslant 0$ with $R$, that is,*

$$(\forall M \in \mathscr{M}|Z) \qquad |M \cap R| = s.$$

*Then*

$$A^\star(k,m,n,Z) \sim s \cdot X(|R|) + \sum_{(i,j)\in R} \chi(i,j) A^\star(k,m,n,Z \cup \{(i,j)\}), \tag{4.1}$$

*where $X(\lambda)$ denotes an exponential random variable with intensity $\lambda$ (mean $1/\lambda$), and $\chi(i,j)$ is an indicator variable taking on the value 1 for a single edge $(i,j)$ selected uniformly at random from $R$, and 0 for all others.*

**Proof.** Let $A$ be an edge-weighting of $K_{m,n}$ consistent with $Z$. Let $\Delta = \min_{(i,j)\in R} a_{(i,j)}$, and define $A'$ by

$$a'_{(i,j)} = \begin{cases} a_{(i,j)}, & \text{if } (i,j) \notin R, \\ a_{(i,j)} - \Delta, & \text{otherwise.} \end{cases} \tag{4.2}$$

Note that, for any fixed assignment $\sigma$, $\mathrm{cost}(A,\sigma) = s\Delta + \mathrm{cost}(A',\sigma)$, from which it follows that $A$ and $A'$ have a common minimizing assignment $\sigma^\star$, and that the cost of this assignment differs by $s\Delta$ in the two cases.

The random variable $\Delta$ is simply a minimum of $|R|$ i.i.d. exponential r.v.s, and has intensity $|R|$. With any of the $|R|$ random variables equally likely to be the minimum, when $\Delta$ is subtracted from each $a_{(i,j)}$ for $(i,j) \in R$, the zero-set $Z'$ of $A'$ is just the union of $Z$ and a random element of $R$, and the remaining elements of $A'$ (specifically those in $R$) remain i.i.d. exponential r.v.s. □

**Corollary 4.5.**

$$f(k,m,n,Z) = \frac{s}{|R|} + \frac{1}{|R|} \sum_{(i,j)\in R} f(k,m,n,Z \cup \{(i,j)\}).$$

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

Two special cases of the theorem are worth mentioning. First, when $Z = \emptyset$ and $R = K_{m,n}$, the conditions are satisfied with $s = k$, so $f(k,m,n) = \frac{k}{mn} + f(k,m,n,\{(1,1)\})$; that is, the expected cost of a $k$-assignment in a random matrix $A$ is equal to $k/mn$ plus the expected cost of a $k$-assignment in a random matrix $A'$ one of whose entries (by symmetry, it doesn't matter which) is fixed at 0. This is Lemma 25 of [7]. The second interesting case corresponds to Lemma 26 of [7]. There it is shown that, if there is an edge $(i,j) \notin Z$ which is incident to at least $k$ edges $(i,j)' \in Z$, then there is always a minimum $k$-assignment that excludes $(i,j)$, which is to say that $R = \{(i,j)\}$ satisfies our theorem's hypothesis, with $s = 0$.

The theorem is all we will need to calculate $f(k,m,n)$ for various parameter values. Beginning with $Z_0 = \emptyset$ and $f(k,m,n) = f(k,m,n,Z_0)$, to resolve any $f(k,m,n,Z)$ we will find a set $R$ having fixed intersection size $s$ with $Z$, so that, by Corollary 4.5,

$$f(k,m,n,Z) = \frac{s}{|R|} + \frac{1}{|R|} \sum_{(i,j)\in R} f(k,m,n,Z \cup \{(i,j)\}), \tag{4.3}$$

and we apply the same procedure to each of the new terms $f(k,m,n,Z')$ so generated. For any $Z$ that includes a $k$-matching, $f(k,m,n,Z) = 0$, and there is no need to expand further.

The procedure just described produces a directed expansion tree, with root $Z_0 = \emptyset$ and edges from each $Z$ to its 'children' $Z'$ given by (4.3). A single subgraph $Z \subseteq K_{m,n}$ may appear repeatedly in the tree, produced by different parents or produced several times by one parent (when different edges $(i,j) \in R$ are symmetrically equivalent). For greater efficiency, one may identify all nodes corresponding to isomorphic subgraphs $Z$. This dynamic programming approach produces a directed acyclic graph in lieu of the tree.

Before proceeding with our results, we mention a generalization of Lemma 26 of [7].

**Lemma 4.6.** *Suppose that a matrix $A$ contains a (typically nonzero) element $a_{11}$, $k_1$ zeros in row 1 and column 1, and $k_2$ independent zeros, all in different rows and columns from the $k_1$ earlier zeros and from one another. Then, for $k \leqslant k_1 + k_2$, any $k$-assignment in $A$ that includes $a_{11}$ can be replaced by an assignment of lesser or equal cost which avoids $a_{11}$. It follows that if $a_{11}$ is set to 0 to define a new matrix $B$, then $A^\star = B^\star$.*

**Proof.** We may conveniently assume that all elements not stated to be zero, $a_{11}$ included, are strictly nonzero; if in this case we can replace the $a_{11}$-including assignment with a strictly cheaper one, then in the general case the same replacement will give an assignment of lesser or equal cost. Consider a putatively minimal $k$-assignment $\sigma$ containing $a_{11}$.

Remove from $A$ all rows and columns that contain a zero in $\sigma$. We now have a $k'$-assignment $\sigma'$ of a smaller matrix $A'$, where $\sigma'$ contains no zeros. The new values $k_1'$ and $k_2'$ corresponding to $k_1$ and $k_2$ still satisfy $k' \leq k_1' + k_2'$. This is because $\sigma$ uses $a_{11}$, so a zero in $\sigma$ must be one of the $k_2$ zeros outside row 1 and column 1. Each of these zeros is independent of all other zeros, so removing the row and column containing it does not remove any other zero from $A$. Also, $\sigma'$ clearly still contains $a_{11}$. If there is an $a_{11}$-avoiding $k'$-assignment in $A'$ cheaper than $\sigma'$, it induces an $a_{11}$-avoiding $k$-assignment in $A$ cheaper

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

than $\sigma$. Dropping the primes, then, it suffices to prove the theorem in the case where $\sigma$ uses no zeros.

The proof is by contradiction. Suppose $\sigma$ is a minimum-cost assignment using $a_{11}$. Let $Z$ be the set of zeros of $A$, and let $M$ be the set of entries used by $\sigma$. Construct a bipartite graph $G$ with vertex classes $Z$ and $M$. Join $z \in Z$ to $m \in M$ if $m$ is in the same row or column as $z$. For every $z$, the degree $d_G(z) = 2$: $z$ must have an element of $M$ in the same row and another in the same column (or else we could use $z$ to replace an entry of $M$ and get a cheaper assignment), and there is at most one element of $M$ in any row or column. Therefore $|E(G)| = 2|Z|$. The graph $G$ cannot contain any cycles, since if $m_1 z_1 m_2 z_2 \dots m_r z_r$ is a cycle, then the $m_i$s span the same set of rows and columns as the $z_i$ and we can replace the $m_i$s with the $z_i$s to get a cheaper assignment. $G$ is therefore a forest (and contains at least one vertex, namely $a_{11} \in M$) and so $2|Z| = |E(G)| < |V(G)| = |Z| + |M|$. Therefore $k_1 + k_2 = |Z| < |M| = k$, a contradiction. □

## 5. Algorithmic issues

Implementing the program discussed above requires solving several problems. First, to take advantage of dynamic programming, we must determine when two zero-sets $Z$ and $Z'$ are equivalent, which means checking isomorphism of $Z$ and $Z'$ as subgraphs of $K_{m,n}$. Second, given a zero-set $Z$, how can we efficiently calculate the set $\mathscr{M}$ of potentially minimal matchings? Third, and most interesting, given $\mathscr{M}$, how can we find a set $R$ having constant intersection size with $\mathscr{M}$, and does such a set always exist? Finally, how can we perform the computations when $n$ is a parameter (not a fixed value), or when $m$ and $n$ are both parameters?

Before addressing these questions in turn, let us summarize our results.

### 5.1. Results

For a number of triples $(k, m, n)$, we are able to compute both the expectation and the Laplace transform of $A^\star(k, m, n)$. For some values of $k$, arguments in Section 5.5 imply that the conclusions drawn from calculations for a particular choice of $m$ and $n$ can be generalized to arbitrary $m$ and $n$; these choices are used to construct Table 1.

In all instances, the conclusions are consistent with Conjecture 1.1.

For each choice of $(k, m, n)$, the 'cases' and 'dependencies' correspond to the vertices and edges of the directed acyclic graph described after (4.3). The number of cases and dependencies can vary slightly from run to run: for a given zero-set $Z$, when zeros are

*Table 1*

| (k,m,n) | #cases | #dependencies | time |
|---|---|---|---|
| (2,3,3) $\Longrightarrow$ (2,m,n) | 3 | 4 | 2 sec |
| (3,4,4) $\Longrightarrow$ (3,m,n) | 8 | 14 | 7 sec |
| (4,5,5) $\Longrightarrow$ (4,m,n) | 29 | 68 | 1 min |
| (5,5,6) $\Longrightarrow$ (5,5,n) | 68 | 156 | 4 min |
| (6,6,6) | 300 | 735 | 26 min |

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

inserted as in Theorem 4.4, there can be several choices for a region $R$, with different choices generating different subcases $Z'$, and possibly a different set of cases over the whole run. The numbers of cases shown in the table are for runs detailed in Appendix 2 and the larger appendix of [2], and are typical values. The number of case dependencies likewise can vary from run to run, and also depends on the degree of care taken to identify symmetric cases.

Run times shown are wall-clock time elapsed on a 233MHz PentiumII ThinkPad, running our APL2 code, and calling the IBM Optimization and Solution Library to solve linear programs described in Section 5.3.

### 5.2. Graph isomorphism

Isomorphism of bipartite graphs is no easier than that of general graphs, and no polynomial-time algorithm is known for this extensively studied problem. Moreover, what we really want to do is rapidly check whether we have seen the current zero-set $Z$ before; that is, to check whether an unlabelled graph belongs to a *list* of such graphs. Fortunately, there are practical heuristics for canonically labelling graphs. We implemented our own, but could have used McKay's `Nauty` package [17].

### 5.3. Minimal subsets

Calculation of the set $\mathscr{M}$ of all potentially minimal assignments $M$ can be done more or less by brute force. We begin with the set $\mathscr{M}_0$ of all $k$-assignments $M$. Since each assignment $M$ is determined by a choice of $k$ of the $m$ 'left' vertices (without regard to order), and then a choice of $k$ of the $n$ corresponding 'right' vertices (with order), $\mathscr{M} = C_k^m P_k^n$. Given a zero-set $Z$, it is straightforward to subtract $Z$ from each $M \in \mathscr{M}$.

From this collection of sets, we must retain only the minimal elements, discarding any set which properly contains another. This seems a rather natural, general problem: given a collection $\mathscr{M}$ of subsets of a ground set $U$, find all minimal elements of $\mathscr{M}$. Comparing every subset with every other defines an $O(|\mathscr{M}|^2|U|)$-time algorithm, but is there something more efficient?

In our case, where $|\mathscr{M}|$ is large but each $M \in \mathscr{M}$ has small cardinality, a method suggested by Coppersmith is more efficient. A set $M \in \mathscr{M}$ is redundant if and only if some proper subset $M' \subsetneq M$ is also an element of $\mathscr{M}$. This contorted view suggests the following algorithm. Generate all proper subsets $M'$ of all elements $M$ of $\mathscr{M}$, and lexicographically sort these along with the original sets $M$ themselves, keeping track of which ones are subsets and which are originals. If any 'original' set is identical to a 'subset', then the original set that generated the subset is not minimal and should be discarded. The run time of the algorithm is $O(k|\mathscr{M}|2^k \log(|\mathscr{M}|2^k))$, which in our instances really is preferable to $O(|\mathscr{M}|^2|U|)$. (One computation, $(k, m, n) = (5, 7, 7)$, gave $|\mathscr{M}| = 52{,}920$.)

### 5.4. Finding a set of constant intersection size

Given a collection $\mathscr{M}$ of subsets $M$ of a ground set $U$ (in our case $U$ is edges of $K_{m,n}$), how can we efficiently find a subset $R \subseteq U$, such that $|M \cap R|$ has the same value for any $M \in \mathscr{M}$? Or, if no such $R$ exists, how can we determine that?

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

It is possible simply to test every subset $R \subseteq U$. While this is within reason for the instances we require, it is neither elegant nor fast.

Another approach expresses the problem as an integer or linear program. For each element $u \in U$, let an indicator variable $x_u$ be 1 if $u$ is to be included in $R$, and 0 otherwise. Also, for each $u$, define an $|\mathcal{M}|$-vector $v_u$, with $v_u(M) = 1$ if $u \in M$, and 0 otherwise. Note that $\sum_{u \in U} x_u v_u$ is a vector whose $M$th entry is $|M \cap R|$, so the desired condition is that for some fixed 'size' $s$, $\sum_{u \in U} x_u v_u = s\mathbf{1}$, where $\mathbf{1}$ is the all-ones $|\mathcal{M}|$-vector. For a given value of $s$, this is a set of linear constraints, and may be thought of as a linear integer program whose cost function is irrelevant. (The trivial solution $x_u \equiv 0$ is not of interest, but a solution with $s = 0$ produced by a non-empty $R = U - \cup_{M \in \mathcal{M}} M$ does allow us to introduce 0s without affecting the assignment cost at all.)

If there is no nontrivial $s = 0$ solution, try to solve the integer program (IP) either directly or, as we have done, by relaxing it to a linear program (LP). If the LP has a feasible solution $\mathbf{x}$ for some value of $s$, $s = s_0$ say, then it also has a feasible solution for $s = 1$, namely $\frac{1}{s_0}\mathbf{x}$. Thus, once we have checked the case $s = 0$, if the LP for $s = 1$ is infeasible, no solution $R$ exists. On the other hand, a solution to the $s = 1$ LP gives a solution for $R$ if it happens to be 0–1 integral, or is so when multiplied by a value $s \leqslant k$; or we can hope for an integral solution to the LP for some other values $s \leqslant k$.

In practice, solving a linear program proved to be an effective method of finding a set of constant intersection size, or proving that no such set exists.

This leaves one question: Given a zero-set $Z$, does there always exist a suitable set $R$ allowing us to continue expanding? Unfortunately, the answer is 'no', and the points of failure are precisely what limits our proofs of Conjecture 1.1 to $k \leqslant 4$, $k = m = 5$, and $k = m = n = 6$. For example, our run for $A^\star(5,6,6)$ produced five bad configurations (up to isomorphism); see Figure 2.

```
.00...   000...   000...   000...   0000..
0.....   0.....   0.....   0.....   0.....
0.....   0.....   0.....   0.....   0.....
......   ......   ...0..   0..0..   0..0..
......   ......   ......   ......   ......
......   ......   ......   ......   ......
```

*Figure 2* Bad cases for 556: dots indicate nonzero elements

We have not exploited it, but a generalization is worth mentioning. Consider for example the case $(k, m, n) = (5, 6, 6)$ in which zeros have been inserted to give the configuration

```
a11 0 0 . . .
 0  X X X X X
 0  X X X X X
 .  X X X X X
 .  X X X X X
 .  X X X X X
```

If the region shown by Xs is given weights (indicator variable values) of 1 as usual, $a_{11}$ is given weight $-1$, and remaining entries are given weight 0, then every minimal assignment has the same total weight: most use 3 elements from the X region, while a few use 4 of the Xs and also use $a_{11}$, giving weight $4 \cdot 1 + 1 \cdot (-1) = 3$. If any value $x$ is subtracted from all the entries with weight 1, and added to the entry with weight $-1$, every minimal

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

assignment's cost is decreased by exactly $3x$. Choosing $x$ to be the minimum of the entries with weight 1, as usual, introduces a new 0 into the matrix; it also complicates matters by causing $a_{11}$ to become the sum of two exponentials, but it does leave all the other entries either 0 or simple exponentials, and it does leave *all* entries of the matrix independent.

In fact, we can make some use of *any* set of weight values, including fractions as well as negative values. Define $x = \min_{w_{ij}>0} a_{ij}/w_{ij}$; note that $x$ is an exponential with intensity $\sum_{w_{ij}>0} w_{ij}$. Subtracting $w_{ij}x$ from each $a_{ij}$ (which for $w_{ij} < 0$ means increasing $a_{ij}$) has various effects: the matrix element responsible for the value $x$ becomes 0; other matrix elements with $w_{ij} > 0$ remain unconditioned exponentials; matrix elements of 0 with $w_{ij} < 0$ all take on the value $x$, and are thus exponential but (perfectly) correlated; and nonzero matrix elements with $w_{ij} < 0$ become the sum of exponentials, and in addition are correlated. The situation is potentially tractable if there is only one negative value $w_{ij}$, especially if it coincides with a 0 element of $A$.

Limited attempts to apply this method to $A^\star(5,6,6)$ were somewhat promising, but it is not at all clear that all cases will be completely resolved, and some effort would be required to find out.

### 5.5. Parametrizing *m* and *n*

The techniques described compute the minimum-cost random variable for *fixed* values $k$, $m$, and $n$. How then do we use the methods, for example, to prove that Conjecture 1.1 holds for $k = 4$ and *all* values $m, n \geqslant 4$? We exploit the following simple lemma.

**Lemma 5.1.** *Given $k$, $m$, $n$, and a zero-set $Z \subseteq K_{m,n}$, let $A$ be a matrix of i.i.d. exponential random variables consistent with $Z$. If $Z$ contains a matching $M$ of cardinality $k_0$, then any potentially minimal assignment $\sigma$ uses exactly $k-k_0$ elements in rows (respectively, columns) not spanned by $M$.*

**Proof.** It is evident that $\sigma$ must use *at least* $k - k_0$ elements in rows outside $M$, since there are only $k_0$ rows within $M$, and no two elements can share a row. The opposite direction of the inequality follows from a simple counting argument.

Each (0-weight) element of $M$ must be 'accounted for': either used in $\sigma$ or having a 'blocking' element of $\sigma$ in its row and another in its column. Let $a$ be the number of elements in $M \cap \sigma$, $b$ the number of other elements of $\sigma$ within the rows spanned by $M$, and $c$ the number of remaining elements of $\sigma$. Since every element of $M$ is either used or has a blocking element in its row, $a + b \geqslant k_0$. Counting all elements in $\sigma$, $a + b + c = k$. Subtracting, the number of rows outside $M$ occupied by $\sigma$ is $c \leqslant k - k_0$. The result for columns follows symmetrically. □

During our construction, rows with no 0s are indistinguishable from one another, so as long as there are at least $k-k_0$ of them (with $k_0$ the cardinality of a largest 0-matching in $Z$), we may parametrize the total number of rows $m$, and count in how many ways any case can occur. In the example discussed above, for instance, the number of rows spanned by the region $R$ of Xs is $2 + (m - 3)$: the two rows that also include 0s and the following

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

all-but-3 rows that do not. It is clear that, had we started with any $m \geqslant 6$, at the same stage $R$ would have spanned $m-3$ zero-free rows. The $n$ columns can be treated similarly, and in this way we can use calculations for a single pair $m, n$ to give results for any equal or larger values; here, for example, it is clear that the total number of elements in the region $R$ (25, in the $6 \times 6$ case) would in general be $2 \cdot 2 + 2(n-3) + (m-3)2 + (m-3)(n-3)$, or $(m-1)(n-1)$.

If the calculations were performed with values $m_c$ and $n_c$, how do we prove the resulting formulas' validity for *smaller* values of $m$ (*i.e.*, $k \leqslant m < m_c$) and $n$? Since there are only finitely many of these, we could of course check all of them, but there is a better way.

The values $m$ and $n$ show up in our calculations only when they describe rectangular regions: either a 'good' region $R$ or a subregion leading on to further cases.

The only danger is that some of the configurations we considered for $m_c$ and $n_c$ do not exist for the smaller $m$ or $n$. The calculations consider 'good' regions $R$ and the subregions leading on to further cases, and it is clear that reducing $m$ or $n$ by 1 can shrink any dimension of any of these regions by at most 1. As $m$ (or $n$) decreases, then, if any region is to disappear, it must first shrink to size 0; this means that one of the terms in our probability expressions $\frac{(m-a)(n-b)}{(m-c)(n-d)}$ must become 0. Thus we are safe for any values $m \geqslant \max\{a, c+1\}$ and $n \geqslant \max\{b, d+1\}$. Inspection of the tables in Appendix 11 in [2] shows that this gets us all the way down to $m, n = k$.

## 6. Laplace transforms

By using the main theorem (Theorem 4.4) and the technique of the previous section, we can also compute the Laplace transform of the distribution of the minimum cost, and through this the variance of the distribution.

The structure of the next few sections is as follows. In this section we introduce basic properties of Laplace transforms generally, and the Laplace transform $\psi(k, m, n, t)$ of $A^\star(k, m, n)$ particularly. We introduce a function $S(k, m, n, t)$ which is quite close to $\psi$ and is more tractable. In Section 7 we show that the expectation of $S(k, m, n, t)$ would satisfy our Conjecture 1.1, and thus is equal to the expectation of $\psi$ wherever we can calculate the latter. Furthermore, the variances of $S$ and $\psi$ are close, in the cases where we can calculate $\psi$. In Section 8 we study asymptotics of the 'benchmark' distribution $S$ and of $A^\star$ itself. For $k$ fixed, both can be analysed completely. For $k = m = n \to \infty$, the variance of $S(n, n, n, t)$ is of order $2/n$; we conjecture that $\mathrm{Var}(A^\star(n, n, n))$ has the same asymptotic behaviour, and we provide suggestive lower and upper bounds.

Recall that the Laplace transform of a random variable $X$ is $L(X, t) = \mathbb{E}e^{-tX}$ and that the Laplace transform has the following properties.

$$L(aX, t) = L(X, at) \text{ and } L(a + X, t) = e^{at} L(X, t).$$

For independent random variables $X$ and $Y$, $L(X + Y, t) = L(X, t)L(Y, t)$.

Given random variables $X_1, \ldots, X_s$ and probabilities $p_1, \ldots, p_s$, define the random variable $I$ to take on value $i$ with probability $p_i$. (This is as in (4.1), with slightly different

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

notation.) Then

$$L(X_I) = \sum_i p_i L(X_i, t).$$

Note the special cases $L(0,t) = 1$ and so $L(B(p)X, t) = (1-p) + pL(X,t)$.

Moments are given by

$$\mathbb{E}(X^r) = (-1)^r L^{(r)}(X, 0),$$

so that $\mathbb{E}(X) = -L'(X,0)$ and $\mathrm{Var}(X) = L''(X,0) - (L'(X,0))^2$.

To decide if a function $\varphi(t)$ is a Laplace transform, we can use the following theorem; see for example [9].

**Theorem 6.1.** *A function $\varphi(t)$, $t > 0$, is the Laplace transform of a probability distribution if and only if it is completely monotone and $\varphi(0) = 1$.*

**Definition.** A function $\varphi(t)$, $t > 0$, is *completely monotone* if it possesses derivatives $\varphi^{(r)}$ of all orders and $(-1)^r \varphi^{(r)}(t) \geq 0$ for $t > 0$.

As before, let $X(a)$ denote an exponential random variable with intensity $a$ (mean $1/a$), and let $g(a,t) = a/(a+t)$ be the Laplace transform of the exponential distribution with intensity $a$. Further, let $\psi(k,m,n,t)$ be the Laplace transform of $A^\star(k,m,n)$.

**Remark.** With $k = 1$, the minimum assignment cost's distribution and its Laplace transform are given by

$$A^\star(1,m,n) \sim X(mn) \text{ and}$$
$$\psi(1,m,n,t) = g(mn,t). \tag{6.1}$$

Similar results for $k = 2$ follow after a simple lemma based on an observation of Janson's [12].

**Lemma 6.2.** *For any positive integers $a$ and $n$, $X(n) \sim X(n+a) + B(a/(n+a))X(n)$, where $B$ and the $X$s denote Bernoulli and exponential random variables, all independent.*

**Proof.** Consider a collection of $n+a$ 'exponential alarm clocks', each with intensity 1. One of the first $n$ alarm clocks rings at a time $X(n)$, the left-hand side above. The right-hand side gives another characterization of the same time. Wait until the first of all $n+a$ clocks rings. If this was not one of the first $n$ clocks (that is, with probability $a/(n+a)$), then (by the memoryless property of the exponential) the additional time until one of the first $n$ rings is again $X(n)$. □

Janson [12] noticed the following.

**Proposition 6.3.** *With $k = 2$, the minimum assignment cost's distribution and its Laplace transform are given by*

$$A^\star(2,m,n) + A^\star(1,m,n) \sim 2A'^\star(1,m,n) + A^\star(1,m-1,n) + A^\star(1,m,n-1), \tag{6.2}$$

$$\psi(2,m,n,t) = \frac{\psi(1,m,n,2t)}{\psi(1,m,n,t)} \cdot \psi(1,m-1,n,t) \cdot \psi(1,m,n-1,t), \tag{6.3}$$

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

*where in* (6.2) *the variables on the left-hand side and those on the right-hand side are independent, respectively, and* $A'^{\star}(1,m,n)$ *has the same distribution as* $A^{\star}(1,m,n)$.

**Proof.** As in Proposition 3.2,

$$A^{\star}(2,m,n) \sim 2X(mn) + X((m-1)n) + B(1/n)X(m(n-1)).$$

Using the preceding lemma,

$$\begin{aligned} A^{\star}(2,m,n) &+ X(mn) \\ &\sim 2X(mn) + X((m-1)n) + [B(1/n)X(m(n-1)) + X(mn)] \\ &\sim 2X(mn) + X((m-1)n) + X(m(n-1)). \end{aligned}$$

The second part of the proposition follows by taking Laplace transforms. □

Janson conjectured that the pattern of (6.1) and (6.3) would continue; to be more precise we first define some functions.

**Definition.** Define a sequence of functions, $h_k(t)$, $k = 0, 1, 2, 3, \ldots$, by

$$\begin{aligned} h_0(t) &\equiv 1 \\ h_1(t) &= g(mn,t) \\ h_2(t) &= g(mn,2t)\cdot g((m-1)n,t)\cdot g(m(n-1),t) \\ h_3(t) &= g(mn,3t)\cdot g((m-1)n,2t)\cdot g(m(n-1),2t) \\ &\quad \cdot g((m-2)n,t)\cdot g((m-1)(n-1),t)\cdot g(m(n-2),t) \\ &\vdots \\ h_k(t) &= \prod_{i,j\geqslant 0,\ i+j<k} g((m-i)(n-j),(k-i-j)t) \\ &\vdots \end{aligned}$$

and define

$$S(k,m,n,t) = \frac{h_k(t)}{h_{k-1}(t)} \text{ for } k \geqslant 1. \tag{6.4}$$

Then (6.1) and (6.3) are equivalent to

$$\psi(1,m,n,t) = S(1,m,n,t) \text{ and} \tag{6.5}$$

$$\psi(2,m,n,t) = S(2,m,n,t). \tag{6.6}$$

Janson's conjecture was the following [12].

**Conjecture 6.4. (Janson)** $\psi(k,m,n,t) = S(k,m,n,t)$ *for all* $1 \leqslant k \leqslant m \leqslant n$.

Unfortunately, the conjecture fails already at $k = 3$. However, the functions $S(k,m,n,t)$ may nevertheless be useful: they are tractable, and provide a useful point of comparison for $\psi(k,m,n,t)$.

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

**Proposition 6.5.** *$S(k,m,n,t)$ is the Laplace transform of a probability distribution.*

**Proof.** As $S(k,m,n,0) = \frac{h_k(0)}{h_{k-1}(0)} = 1$, by Theorem 6.1, it remains to show that $S(k,m,n,t)$ is completely monotone. Rewrite, using $g(a,t) = a/(a+t)$,

$$\begin{aligned} S(k,m,n,t) =& \frac{h_k(t)}{h_{k-1}(t)} \\ =& \frac{g(mn,kt)}{g(mn,(k-1)t)} \cdot \frac{g((m-1)n,(k-1)t)}{g((m-1)n,(k-2)t)} \cdot \frac{g(m(n-1),(k-1)t)}{g(m(n-1),(k-2)t)} \\ & \cdots \frac{g((m+2-k)n,2t)}{g((m+2-k)n,t)} \cdots \frac{g(m(n+2-k),2t)}{g(m(n+2-k),t)} \\ & \cdot g((m+1-k)n,t) \cdots g(m(n+1-k),t). \end{aligned}$$

Thus, $S(k,m,n,t)$ is a product of $g((m+1-k)n,t) \cdots g(m(n+1-k),t)$, and functions of the type $k(t) = \frac{g(a,lt)}{g(a,(l-1)t)}$, $a > 0$, and $l \geq 2$. As a product of completely monotone functions is completely monotone, and $g(a,t)$ is a Laplace transform and thus completely monotone, it only remains to show that the functions $k(t)$ are also completely monotone. Now,

$$k(t) = \frac{g(a,lt)}{g(a,(l-1)t)} = \frac{a+(l-1)t}{a+lt} = 1 - \frac{t}{a+lt} \geq 0.$$

Further, $k'(t) = \frac{-a}{(a+lt)^2}$ so that $-k'(t) = \frac{a}{(a+lt)^2} = \frac{a}{a+lt} \cdot \frac{1}{a+lt}$, a product of Laplace transforms. Thus $k'(t)$ is completely monotone, and so is $k(t)$. □

## 7. Exact distributional results

Let $V(k,m,n) = \mathrm{Var}(A^\star(k,m,n))$. The expectation and the variance can be calculated using the Laplace transform by

$$\begin{aligned} f(k,m,n) &= \mathbb{E}(A^\star(k,m,n)) = -\psi'(k,m,n,0), \\ V(k,m,n) &= \psi''(k,m,n,0) - f^2(k,m,n). \end{aligned}$$

Let $E_S(k,m,n)$ and $V_S(k,m,n)$ denote the mean and the variance of a random variable with Laplace transform $S(k,m,n,t)$ as defined in (6.4).

**Conjecture 7.1.** *For all $1 \leqslant k \leqslant m \leqslant n$, $f(k,m,n) = E_S(k,m,n)$, and $V(k,m,n) \leqslant V_S(k,m,n)$.*

Before we prove the conjecture in various cases, let us compute the mean and variance of $S(k,m,n,t)$.

**Proposition 7.2.** *With $E_S$ and $V_S$ as above,*

$$E_S(k,m,n) = \sum_{i,j\geq 0,\ i+j<k} \frac{1}{(m-i)(n-j)}, \text{ and} \tag{7.1}$$

$$V_S(k,m,n) = \sum_{i,j\geq 0,\ i+j<k} \frac{2k-1-2i-2j}{(m-i)^2(n-j)^2} = \sum_{r=0}^{k-1} \sum_{i=0}^{r} \frac{2k-1-2r}{(m-i)^2(n-r+i)^2}. \tag{7.2}$$

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

**Proof.** Let $Y(k,m,n)$ be a random variable with Laplace transform $S(k,m,n,t)$, and $X(a)$ be an exponentially distributed random variable with Laplace transform $g(a,t) = \frac{a}{a+t}$.

The definition of $S(k,m,n,t)$ in (6.4) gives the following representation (compare with (6.2)):

$$\begin{aligned} Y(k,m,n) + (k-1)\cdot X(mn) + (k-2)\cdot X(m(n-1)) + (k-2)\cdot X((m-1)n) \\ + \cdots + X(m(n+2-k)) + \cdots + X((m+2-k)n) \\ = k\cdot X'(mn) + (k-1)\cdot X'(m(n-1)) + (k-1)\cdot X'((m-1)n) \\ + \cdots + X'(m(n+1-k)) + \cdots + X'((m+1-k)n), \end{aligned}$$

where the variables on the left-hand side and the variables on the right-hand side are independent, respectively, and $X'(a)$ has the same distribution as $X(a)$.

This gives

$$\begin{aligned} &E_S(k,m,n) \\ &\quad + \frac{k-1}{mn} + \frac{k-2}{m(n-1)} + \frac{k-2}{(m-1)n} + \cdots + \frac{1}{m(n+2-k)} + \cdots + \frac{1}{(m+2-k)n} \\ &\quad = \frac{k}{mn} + \frac{k-1}{m(n-1)} + \frac{k-1}{(m-1)n} + \cdots + \frac{1}{m(n+1-k)} + \cdots + \frac{1}{(m+1-k)n}, \end{aligned}$$

so that

$$E_S(k,m,n) = \frac{1}{mn} + \frac{1}{m(n-1)} + \frac{1}{(m-1)n} + \cdots + \frac{1}{m(n+1-k)} + \cdots + \frac{1}{(m+1-k)n}.$$

Similarly for the variance,

$$\begin{aligned} V_S(k,m,n) + \frac{(k-1)^2}{m^2n^2} + \frac{(k-2)^2}{m^2(n-1)^2} + \frac{(k-2)^2}{(m-1)^2n^2} + \cdots \\ + \frac{1}{m^2(n+2-k)^2} + \cdots + \frac{1}{(m+2-k)^2n^2} \\ = \frac{k^2}{m^2n^2} + \frac{(k-1)^2}{m^2(n-1)^2} + \frac{(k-1)^2}{(m-1)^2n^2} + \cdots \\ + \frac{1}{m^2(n+1-k)^2} + \cdots + \frac{1}{(m+1-k)^2n^2}, \end{aligned}$$

so that

$$\begin{aligned} V_S(k,m,n) &= \frac{2k-1}{m^2n^2} + \frac{2k-3}{m^2(n-1)^2} + \frac{2k-3}{(m-1)^2n^2} + \cdots \\ &\quad + \frac{1}{m^2(n+1-k)^2} + \cdots + \frac{1}{(m+1-k)^2n^2} \\ &= \sum_{r=0}^{k-1}\sum_{i=0}^{r} \frac{2k-1-2r}{(m-i)^2(n-r+i)^2}. \end{aligned}$$

□

**Remark.** The expression (7.1) for $E_S(k,m,n)$ is identical to the expression for $f(k,m,n)$ in Conjecture 1.1.

We now return to a partial proof of Conjecture 7.1.

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

**Proposition 7.3.** *Conjecture 7.1 holds for* $k \leq 4$, $k = m = 5$, *and* $k = m = n = 6$.

**Proof.** Equality of $f(k,m,n)$ and $E_S(k,m,n)$ is equivalent to Conjecture 1.1. For $k \leqslant 4$, $k = m = 5$, and $k = m = n = 6$, it follows from the main theorem (Theorem 4.4) and the calculations represented in Appendix 2 (covering $k \leqslant 3$), and the larger appendix of [2] (covering $k \leqslant 4$, $k = m = 5$, and $k = m = n = 6$).

In the same cases, and by the same method, we compute $V(k,m,n) = \psi''(k,m,n,0) - (\psi'(k,m,n,0))^2$. By equality of the expectations for $\psi$ and $S$, $V(k,m,n) - V_S(k,m,n) = \psi''(k,m,n,0) - S''(k,m,n,0)$. With the first two equalities below coming immediately from (6.5) and (6.6), and the rest from straightforward calculation, we have

$$\begin{aligned}
V_S(1,m,n) - V(1,m,n) &= 0,\\
V_S(2,m,n) - V(2,m,n) &= 0,\\
\Delta_3(t) = S(3,m,n,t) - \psi(3,m,n,t) &= \frac{t^2 \cdot h_3(t)}{m^2n^2(m-1)(n-1)},\\
V_S(3,m,n) - V(3,m,n) = \Delta_3''(0) &= \frac{2 \cdot h_3(0)}{m^2n^2(m-1)(n-1)}\\
&= \frac{2}{m^2n^2(m-1)(n-1)}\\
&> 0 \text{ for all } m,n \geqslant 3,\\
\Delta_4(t) = S(4,m,n,t) - \psi(4,m,n,t) &= \frac{t^2 \cdot h_4(t) \cdot a(m,n,t)}{m^3n^3(m-1)^2(n-1)^2(m-2)(n-2)},
\end{aligned}$$

where

$$\begin{aligned}
a(m,n,t) &= 22 \cdot t^3 + t^2 \cdot (44mn - 28(m+n) - 8)\\
&\quad + t \cdot (28m^2n^2 - 37mn(m+n) + 12(m^2+n^2) - 17mn + 4(m+n) - 8)\\
&\quad + 6m^3n^3 - 13m^2n^2(m+n) + 8mn(m^2+n^2) + 24m^2n^2\\
&\quad - 14mn(m+n) + 8mn,\\
V_S(4,m,n) - V(4,m,n) = \Delta_4''(0) &= \frac{2 \cdot h_4(0) \cdot a(m,n,0)}{m^3n^3(m-1)^2(n-1)^2(m-2)(n-2)},
\end{aligned}$$

and

$$\begin{aligned}
a(m,n,0) &= mn \cdot \{6m^2n^2 - 13mn(m+n) + 8(m^2+n^2) + 24mn - 14(m+n) + 8\}\\
&= mn \cdot \{6(m-4)^2(n-4)^2 + 35(m-4)^2(n-4) + 35(m-4)(n-4)^2\\
&\quad + 52(m-4)^2 + 52(n-4)^2 + 200(m-4)(n-4) + 290(m-4)\\
&\quad + 290(n-4) + 408\}\\
&> 408 \text{ for all } m,n \geqslant 4,
\end{aligned}$$

so that $V_S(4,m,n) - V(4,m,n) > 0$ for all $m,n \geq 4$.

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

Further,

$$V_S(5,5,n) - V(5,5,n) = \frac{1370(n-5)^4 + 18015(n-5)^3 + 88036(n-5)^2 + 189177(n-5) + 150786}{1800n^2(n-1)^2(n-2)^2(n-3)^2}$$
$$> 0 \quad \text{for all } n \geq 5; \text{ and}$$
$$V_S(6,6,6) - V(6,6,6) = \frac{20063}{1620000} > 0. \qquad \square$$

## 8. Asymptotic results

In all cases calculated above, $V_S$ gives an upper bound for $V$, with a difference that is $O(\frac{1}{m^3n^3})$ as $m,n \to \infty$, that is, of a lesser order than $V_S$ and $V$, which are both of order $O(\frac{1}{m^2n^2})$. This leads us to the following conjecture.

**Conjecture 8.1.** *For all $k \leq m \leq n$, $V(k,m,n) \leqslant V_S(k,m,n)$ and $V_S(k,m,n) - V(k,m,n) = O(\frac{1}{m^3n^3})$ as $m,n \to \infty$.*

Regardless of this conjecture's truth, in some cases of interest we can certainly relate the asymptotic expectation and variance of $A^\star(k,m,n)$ to those of $S(k,m,n,t)$. The next proposition describes the asymptotics for $S$. Theorem 8.3 proves that, for $k$ fixed, the expectation and variance of $A^\star$ have the same asymptotic behaviour as those for $S$. For $k = m = n \to \infty$, the variance of $A^\star$ is considered in Conjecture 8.4, with corresponding upper and lower bounds given by Theorems 8.5 and 8.6.

**Proposition 8.2.**

(i) *For fixed $k,m$, asymptotically as $n \to \infty$,*

$$E_S(k,m,n) = \frac{1}{n} \cdot \sum_{i=0}^{k-1} \frac{k-i}{m-i} + O\left(\frac{1}{n^2}\right), \tag{8.1}$$

$$V_S(k,m,n) = \frac{1}{n^2} \cdot \sum_{i=0}^{k-1} \frac{(k-i)^2}{(m-i)^2} + O\left(\frac{1}{n^3}\right). \tag{8.2}$$

*In particular,*

$$E_S(k,k,n) = \frac{k}{n} + O\left(\frac{1}{n^2}\right),$$
$$V_S(k,k,n) = \frac{k}{n^2} + O\left(\frac{1}{n^3}\right).$$

(ii) *For fixed $k$, asymptotically as $m,n \to \infty$,*

$$E_S(k,m,n) = \frac{\binom{k+1}{2}}{mn} + O\left(\frac{m+n}{m^2n^2}\right), \tag{8.3}$$

$$V_S(k,m,n) = \frac{k \cdot (k+1) \cdot (2k+1)/6}{m^2n^2} + O\left(\frac{m+n}{m^3n^3}\right). \tag{8.4}$$

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

*In particular,*

$$E_S(k,n,n) = \frac{\binom{k+1}{2}}{n^2} + O\left(\frac{1}{n^3}\right),$$

$$V_S(k,n,n) = \frac{k\cdot(k+1)\cdot(2k+1)/6}{n^4} + O\left(\frac{1}{n^5}\right).$$

(iii)
$$E_S(n,n,n) = \sum_{i=1}^{n} \frac{1}{i^2}, \tag{8.5}$$

$$V_S(n,n,n) = \frac{2}{n} + O\left(\frac{\log n}{n^2}\right) \text{ as } n \to \infty. \tag{8.6}$$

**Proof.** The proof is by straightforward but tedious manipulation of the sums of Proposition 7.2, and is therefore deferred to Appendix 1. □

In the next theorem we show that the first two parts of Proposition 8.2 in fact hold for the assignment problem itself.

**Theorem 8.3.**

(i) *For fixed $k, m$, asymptotically as $n \to \infty$,*

$$f(k,m,n) = \frac{1}{n}\cdot\sum_{i=0}^{k-1}\frac{k-i}{m-i} + O\left(\frac{1}{n^2}\right), \tag{8.7}$$

$$V(k,m,n) = \frac{1}{n^2}\cdot\sum_{i=0}^{k-1}\frac{(k-i)^2}{(m-i)^2} + O\left(\frac{1}{n^3}\right). \tag{8.8}$$

*In particular,*

$$f(k,k,n) = \frac{k}{n} + O\left(\frac{1}{n^2}\right),$$

$$V(k,k,n) = \frac{k}{n^2} + O\left(\frac{1}{n^3}\right).$$

(ii) *For fixed $k$, asymptotically as $m, n \to \infty$,*

$$f(k,m,n) = \frac{\binom{k+1}{2}}{mn} + O\left(\frac{m+n}{m^2n^2}\right), \tag{8.9}$$

$$V(k,m,n) = \frac{k\cdot(k+1)\cdot(2k+1)/6}{m^2n^2} + O\left(\frac{m+n}{m^3n^3}\right). \tag{8.10}$$

*In particular,*

$$f(k,n,n) = \frac{\binom{k+1}{2}}{n^2} + O\left(\frac{1}{n^3}\right),$$

$$V(k,n,n) = \frac{k\cdot(k+1)\cdot(2k+1)/6}{n^4} + O\left(\frac{1}{n^5}\right).$$

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

**Proof.** A lower bound for the optimal assignment is given by the $k$ cheapest row minima, while an upper bound is given by the greedy algorithm which, at each step, picks the smallest remaining element and eliminates its row and column. Thus,

$$A^\star(k,m,n) \geqslant L = \sum_{i=1}^{k} X_i$$
$$\leqslant U = \sum_{i=1}^{k} Y_i,$$

where $X_1, X_2, \ldots, X_k$ are independent exponentially distributed random variables with means $\frac{k}{mn}, \frac{k-1}{(m-1)n}, \ldots, \frac{1}{(m+1-k)n}$ and $Y_1, Y_2, \ldots, Y_k$ are independent exponentially distributed random variables with means $\frac{k}{mn}, \frac{k-1}{(m-1)(n-1)}, \ldots, \frac{1}{(m+1-k)(n+1-k)}$. This gives, as $n \to \infty$,

$$\mathbb{E}(L) = \sum_{i=1}^{k} \mathbb{E}(X_k) = \sum_{i=0}^{k-1} \frac{k-i}{m-i} \cdot \frac{1}{n},$$
$$\mathbb{E}(U) = \sum_{i=1}^{k} \mathbb{E}(Y_k) = \sum_{i=0}^{k-1} \frac{k-i}{(m-i)(n-i)} = \sum_{i=0}^{k-1} \frac{k-i}{m-i} \cdot \left(\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right),$$

so that

$$f(k,m,n) = \sum_{i=0}^{k-1} \frac{k-i}{m-i} \cdot \left(\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right). \tag{8.11}$$

This proves (8.7) and, combined with (A1.1) of Appendix 1, also proves (8.9).

For the variances, we get, as $n \to \infty$,

$$\mathrm{Var}(L) = \sum_{i=1}^{k} \mathrm{Var}(X_k) = \sum_{i=0}^{k-1} \frac{(k-i)^2}{(m-i)^2} \cdot \frac{1}{n^2},$$
$$\mathrm{Var}(U) = \sum_{i=1}^{k} \mathrm{Var}(Y_k) = \sum_{i=0}^{k-1} \frac{(k-i)^2}{(m-i)^2(n-i)^2} = \sum_{i=0}^{k-1} \frac{(k-i)^2}{(m-i)^2} \cdot \left(\frac{1}{n^2} + O\left(\frac{1}{n^3}\right)\right),$$

$$V(k,m,n) = \mathbb{E}(A^2(k,m,n)) - f^2(k,m,n) \leqslant \mathbb{E}(U^2) - (E(L))^2$$
$$\geqslant \mathbb{E}(L^2) - (E(U))^2.$$

Here,

$$\begin{aligned}
\mathbb{E}(L^2) - (\mathbb{E}(U))^2 &= \mathrm{Var}(L) + (\mathbb{E}(L))^2 - (\mathbb{E}(U))^2 \\
&= \mathrm{Var}(L) + (\mathbb{E}(L) + \mathbb{E}(U)) \cdot (\mathbb{E}(L) - \mathbb{E}(U)) \\
&= \sum_{i=0}^{k-1} \frac{(k-i)^2}{(m-i)^2} \cdot \frac{1}{n^2} + O\left(\sum_{i=0}^{k-1} \frac{k-i}{m-i} \cdot \frac{1}{n} \cdot \sum_{i=0}^{k-1} \frac{k-i}{m-i} \cdot \frac{1}{n^2}\right) \\
&= \sum_{i=0}^{k-1} \frac{(k-i)^2}{(m-i)^2} \cdot \frac{1}{n^2} + O\left(\left(\sum_{i=0}^{k-1} \frac{k-i}{m-i}\right)^2 \cdot \frac{1}{n^3}\right)
\end{aligned}$$

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

and similarly,

$$\begin{aligned}\mathbb{E}(U^2) - (\mathbb{E}(L))^2 &= \mathrm{Var}(U) + (\mathbb{E}(U))^2 - (\mathbb{E}(L))^2 \\ &= \mathrm{Var}(U) + (\mathbb{E}(U) + \mathbb{E}(L)) \cdot (\mathbb{E}(U) - \mathbb{E}(L)) \\ &= \sum_{i=0}^{k-1} \frac{(k-i)^2}{(m-i)^2} \cdot \left( \frac{1}{n^2} + O\left(\frac{1}{n^3}\right)\right) + O\left( \left( \sum_{i=0}^{k-1} \frac{k-i}{m-i} \right)^2 \cdot \frac{1}{n^3} \right) \\ &= \sum_{i=0}^{k-1} \frac{(k-i)^2}{(m-i)^2} \cdot \frac{1}{n^2} + O\left( \left( \sum_{i=0}^{k-1} \frac{(k-i)^2}{(m-i)^2} + \left[ \sum_{i=0}^{k-1} \frac{k-i}{m-i} \right]^2 \right) \frac{1}{n^3} \right),\end{aligned}$$

which proves (8.8) and, combined with (A1.1) and (A1.2) of Appendix 1, also proves (8.10). □

In the third part of Proposition 8.2, (8.5) corresponds to Parisi's conjecture that $f(n,n,n) = \sum_{i=1}^{n} 1/i^2$ [22], and we conjecture that (8.6) also holds for the assignment problem.

**Conjecture 8.4.** $V(n,n,n) = \dfrac{2}{n} + O\left(\dfrac{\log n}{n^2}\right)$.

The best-known upper bound for the variance $V(n,n,n)$ is by Talagrand [23].

**Theorem 8.5. (Talagrand)** $V(n,n,n) = O\left(\dfrac{1}{n} \cdot \dfrac{(\log n)^4}{(\log\log n)^2}\right)$.

We are not aware of any lower bounds for the variance, but it is easy to get one of the same order as our conjectured value.

**Theorem 8.6.** $V(n,n,n) \geqslant \dfrac{1}{n}$.

**Proof.** Let $X_1, X_2, \ldots, X_n$ be the row minima of $A(n,n,n)$. Then, $A^\star(n,n,n)$ can be written as

$$A^\star(n,n,n) = \sum_{i=1}^{n} X_i + R, \tag{8.12}$$

where $X_1, X_2, \ldots, X_n$ are independent and exponentially distributed with intensity $n$, $R \geqslant 0$ is the assignment cost of the 'residual' matrix after the $X_i$s are subtracted from their respective rows, and (by the memoryless property of the exponential), $R$ is independent of the collection of $X_i$s. ($R$ will depend on the columns in which the minimum elements

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

fall, but not on their values.) This gives

$$V(n,n,n) = \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) + \mathrm{Var}(R) \geqslant \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = n \cdot \frac{1}{n^2} = \frac{1}{n}. \qquad \square$$

We can get a better lower bound by adapting Lazarus's proof that $\liminf A^\star(n,n,n) \geqslant 1 + 1/e$ [15].

**Theorem 8.7.** $V(n,n,n) \geq \left(1 + \frac{1}{e}\right) \cdot \frac{1}{n} + O\left(\frac{1}{n} \cdot \sqrt{\frac{\log n}{n}}\right)$.

**Proof.** Represent $A^\star$ as

$$A^\star(n,n,n) = \sum_{i=1}^{n} X_i + \sum_{i=1}^{N} Y_i + R', \tag{8.13}$$

where $N$ is the number of columns containing no row minima, $Y_1, Y_2, \ldots, Y_N$ (the column minima in the columns with no row minima) are independent and each is exponentially distributed with intensity $n$, and $R' \geqslant 0$ is the assignment cost of the 'residual' matrix.

Writing $N = \sum_{i=1}^{n} I_i$, where $I_i$ is the indicator of the event that column $i$ contains no row minima, we easily get

$$\mathbb{E}(N) = n \cdot \left(1 - \frac{1}{n}\right)^n < \frac{n}{e}$$
$$> \frac{n-1}{e}.$$

Although $N$ and $R'$ are dependent, the idea can be made to work. By an application of Azuma's inequality (see [3, Theorem 7.5.1]),[1] for all $\varepsilon > 0$,

$$P(|N - \mathbb{E}(N)| > \varepsilon n) < 2 \cdot \exp\{-\varepsilon^2 n/2\}. \tag{8.14}$$

For any $\varepsilon_n > 0$ (a good choice will be specified at the end of the proof), let $n_0 = \lfloor \mathbb{E}(N) - n \cdot \varepsilon_n \rfloor$ where, as above, $N$ is the number of columns containing no row minima, and $Y_1, Y_2, \ldots, Y_N$ are the minima of these columns. Further, let $Z_1, Z_2, \ldots, Z_n$ be i.i.d. exponential random variables with intensity $n$, and define

$$V_i = \begin{cases} Y_i, & \text{if} \quad i \leq N, \\ Z_i, & \text{if} \quad i > N. \end{cases}$$

Then we can rewrite (8.13) as

$$A^\star(n,n,n) = \sum_{i=1}^{n} X_i + \sum_{i=1}^{n_0} V_i + R_1 - R_2, \tag{8.15}$$

1 Our use here corrects a small error in the book's statement.

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

where

$$R_1 = R' + \sum_{i=n_0+1}^{N} Y_i \cdot I\{N > n_0\},$$
$$R_2 = \sum_{i=N+1}^{n_0} Z_i \cdot I\{N < n_0\}.$$

The only dependencies in (8.15) are between $\sum_{i=1}^{n_0} V_i$ and $R_2$, and between $R_1$ and $R_2$, so that

$$\begin{aligned} V(n,n,n) &= \mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) + \mathrm{Var}\left(\sum_{i=1}^{n_0} V_i\right) + \mathrm{Var}(R_1 - R_2) \\ &\quad - 2 \cdot \mathrm{Cov}\left(\sum_{i=1}^{n_0} V_i,\ R_2\right) \\ &\geqslant (n + n_0) \cdot \frac{1}{n^2} - 2 \cdot \mathrm{Cov}\left(\sum_{i=1}^{n_0} V_i,\ R_2\right). \end{aligned} \tag{8.16}$$

Now,

$$\mathrm{Cov}\left(\sum_{i=1}^{n_0} V_i,\ R_2\right) = \sum_{i=1}^{n_0} \mathrm{Cov}(V_i, R_2),$$

and

$$\begin{aligned} \mathrm{Cov}(V_i, R_2) &= \mathbb{E}\left(\left(V_i - \frac{1}{n}\right) \cdot R_2\right) \\ &= \sum_{k=0}^{n_0-1} P(N = k) \cdot \mathbb{E}\left(\left(V_i - \frac{1}{n}\right) \cdot \sum_{j=k+1}^{n_0} V_j\right) \\ &= \sum_{k=0}^{n_0-1} P(N = k) \cdot \mathrm{Var}(V_i) \cdot I\{i > k\} \\ &= \frac{1}{n^2} \cdot P(N < i), \end{aligned}$$

so that

$$0 \leqslant \mathrm{Cov}\left(\sum_{i=1}^{n_0} V_i,\ R_2\right) = \frac{1}{n^2} \cdot \sum_{i=1}^{n_0} P(N < i) \leqslant \frac{n_0}{n^2} \cdot P(N < n_0) \leqslant \frac{1}{n} \cdot P(N < n_0).$$

By (8.14),

$$\begin{aligned} P(N < n_0) &\leqslant P\left(N < \mathbb{E}(N) - n\varepsilon_n\right) \\ &\leqslant P(|N - \mathbb{E}(N)| > n\varepsilon_n) \\ &\leqslant 2 \cdot \exp\{-n \cdot \varepsilon_n^2/2\}. \end{aligned}$$

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

Inserting the estimates into (8.16) gives

$$V(n,n,n) > \frac{1}{n^2}\cdot(n+n_0) + \exp\{-n\cdot\varepsilon_n^2/2\}\cdot O\left(\frac{1}{n}\right)$$
$$= \frac{1}{n}\cdot\left(1+\frac{1}{e}\right) - \frac{\varepsilon_n}{n} + O\left(\frac{1}{n^2}\right) + \exp\{-n\cdot\varepsilon_n^2/2\}\cdot O\left(\frac{1}{n}\right).$$

Choosing $\varepsilon_n = \sqrt{\frac{\log n}{n}}$ completes the proof. □

In the same way that we improved the lower bound of $1/n$ for the variance to $(1+\frac{1}{e})\cdot\frac{1}{n}\cdot(1+o(1))$, we might hope to improve it to $1.47\cdot\frac{1}{n}\cdot(1+o(1))$ or even $1.51\cdot\frac{1}{n}\cdot(1+o(1))$ by modifying Olin's proofs of corresponding lower bounds on the expectation [20]. However, it is questionable whether this approach will work, since serious difficulties are created by the covariance between quantities added to (as well as subtracted from) the matrix in the course of Olin's method, and the assignment cost of the 'residual' matrix thus generated.

Conjecture 8.4, that $V(n,n,n) = \frac{2}{n}(1+o(1))$, may be compared against numerical results of Pardalos and Ramakrishnan [21]. Among other things, they calculate the sample standard deviation for 200 instances each of problems with $n = 1{,}000$, 2,000, 3,000, 4,000, 5,000, and 10,000. If it were the case that the variance were precisely $c/n$, their values would give the following estimates of $c$ at the various values of $n$: $c = 1.369$, 1.458, 1.083, 1.156, 1.445, 1.210. This would not suggest that $c$ is 2, but it is also slightly at odds with the lower bound $c \geqslant 1+1/e$, and the estimates of the variances may be suffering from the small sample sizes. For $n = 40$, with 1644 samples, we got a mean of 1.62 and a standard deviation of 0.2082, giving an estimate $c = 1.734$.

## 9. Conclusions and news

Our method supplies a simple principle for computing the minimum assignment cost's expectation, and the Laplace transform of its distribution, in a number of cases. The general veracity of the conjecture $\mathbb{E}A^\star(k,m,n) = \sum_{i+j<k} 1/((m-i)(n-j))$ remains an open question.

Calculating the variance by our method suggests a new conjecture, that $\mathrm{Var}(n,n,n) = 2/n + O(\log n/n^2)$; a lower bound is $\mathrm{Var}(n,n,n) \geqslant (1+1/e)/n + O(n^{-3/2}\sqrt{\log n})$.

We also show good asymptotic bounds on the expectation and variance of the minimum cost, in the easy-to-analyse cases where $k$ is fixed, $n\to\infty$, and $m$ is either fixed or tends to infinity.

Since this paper was submitted, there have been several exciting developments. A conjectured answer to our Problem 4.1, the expected cost $f(k,m,n,Z)$, was given by Linusson and Wästlund [16]. They use powerful new methods to extend ours, and prove the Coppersmith–Sorkin conjecture for $k \leqslant 6$ and for $k = m = n = 7$ ($n = 7$ in Parisi's conjecture). This also helps to assuage a niggling doubt that, for the larger values of $k$ where our proof method fails because of the more complex cases that arise, the conjecture might be false for the same reason. Buck, Chan, and Robbins [5] extend our

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

work to consider the case where the random variables are independent and exponentially distributed but have different rates.

Finally, using entirely different techniques, Aldous [1] has now proved Parisi's original conjecture: the limiting cost of an assignment is $\pi^2/6$. Aldous also elucidates other aspects of the problem, such as confirming a conjecture of Olin [20]: the limiting probability that a minimum assignment uses a given row's $i$th-smallest element is $2^{-i}$.

**Acknowledgements**

We are very grateful to Svante Janson and Don Coppersmith for stimulating conversations and good ideas. Svante first noticed that the Laplace transform of the cost was easy (for him!) to compute when $k = 2$, opening a new avenue. Don suggested the trick for selecting minimal subsets efficiently. Sorkin remains grateful to Boris Pittel for his enthusiastic telling of the random assignment problem. We are also grateful to a thorough referee whose suggestions resulted in the substantial shortening of several proofs, in particular that of Lemma 4.6.

## Appendix 1: Proof of Proposition 8.2

This appendix contains a proof of equations (8.1–8.6) of Proposition 8.2. The derivations are abbreviated, but a full exposition may be found in [2].

**Proof.** By (7.1),

$$E_S(k,m,n) = \sum_{i,j\geq 0,\ i+j<k} \frac{1}{(m-i)(n-j)} = \sum_{r=0}^{k-1}\sum_{i=0}^{r} \frac{1}{(m-i)(n-r+i)}.$$

As $n \to \infty$,

$$E_S(k,m,n) = \sum_{r=0}^{k-1}\sum_{i=0}^{r} \frac{1}{m-i}\left(\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) = \sum_{i=0}^{k-1} \frac{k-i}{m-i}\left(\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right),$$

which proves (8.1).

If also $m \to \infty$,

$$\sum_{i=0}^{k-1} \frac{k-i}{m-i} = \sum_{i=0}^{k-1}(k-i)\cdot\left(\frac{1}{m} + O\left(\frac{1}{m^2}\right)\right) = \frac{\binom{k+1}{2}}{m} + O\left(\frac{1}{m^2}\right), \tag{A1.1}$$

so that

$$E_S(k,m,n) = \left(\frac{\binom{k+1}{2}}{m} + O\left(\frac{1}{m^2}\right)\right)\cdot\left(\frac{1}{n} + O\left(\frac{1}{n^2}\right)\right) = \frac{\binom{k+1}{2}}{mn} + O\left(\frac{m+n}{m^2n^2}\right),$$

which proves (8.3).

(8.5) is proved in [7].

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

By (7.2),

$$V_S(k,m,n) = \sum_{r=0}^{k-1}\sum_{i=0}^{r} \frac{2k-1-2r}{(m-i)^2(n-r+i)^2} = \frac{1}{n^2}\cdot\sum_{i=0}^{k-1}\frac{(k-i)^2}{(m-i)^2} + O\left(\frac{1}{n^3}\right),$$

which proves (8.2).

If also $m \to \infty$,

$$\sum_{i=0}^{k-1}\frac{(k-i)^2}{(m-i)^2} = \frac{1}{m^2}\sum_{i=0}^{k-1}(k-i)^2 + O\left(\frac{1}{m^3}\right) = \frac{k\cdot(k+1)\cdot(2k+1)/6}{m^2} + O\left(\frac{1}{m^3}\right), \qquad \text{(A1.2)}$$

so that

$$V_S(k,m,n) = \frac{k\cdot(k+1)\cdot(2k+1)/6}{m^2n^2} + O\left(\frac{m+n}{m^3n^3}\right),$$

which proves (8.4).

It remains to show (8.6), that is,

$$V_S(n,n,n) = \frac{2}{n} + O\left(\frac{\log n}{n^2}\right) \text{ as } n\to\infty, \text{ where, rewriting (7.2),}$$

$$V_S(n,n,n) = \sum_{r=0}^{n-1}\sum_{i=0}^{r}\frac{2n-1-2r}{(n-i)^2(n-r+i)^2} = \sum_{r=1}^{n}(2r-1)\sum_{i=0}^{n-r}\frac{1}{(n-i)^2(r+i)^2}.$$

We decompose into partial fractions,

$$\frac{1}{(n-i)^2(r+i)^2} = \frac{1}{(n-i)^2(n+r)^2} + \frac{2}{(n-i)(n+r)^3} + \frac{1}{(i+r)^2(n+r)^2} + \frac{2}{(i+r)(n+r)^3},$$

so that

$$\sum_{i=0}^{n-r}\frac{1}{(n-i)^2(r+i)^2} = \frac{2}{(n+r)^2}\sum_{i=r}^{n}\frac{1}{i^2} + \frac{4}{(n+r)^3}\sum_{i=r}^{n}\frac{1}{i}.$$

Approximating by integrals, we find that

$$V_S(n,n,n) \leqslant \sum_{r=1}^{n}(2r-1)\cdot\left(\frac{2}{(n+r)^2}\cdot\left(\frac{1}{r^2}+\frac{n-r}{r\cdot n}\right) + \frac{4}{(n+r)^3}\cdot\left(\frac{1}{r}+\log\frac{n}{r}\right)\right),$$

$$V_S(n,n,n) \geqslant \sum_{r=1}^{n}(2r-1)\cdot\left(\frac{2}{(n+r)^2}\cdot\left(\frac{1}{n^2}+\frac{n-r}{r\cdot n}\right) + \frac{4}{(n+r)^3}\cdot\left(\frac{1}{n}+\log\frac{n}{r}\right)\right),$$

which we write as

$$V_S(n,n,n) \leqslant A+B+C+D, \qquad V_S(n,n,n) \geqslant A+B,$$

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

where

$$A = \frac{2}{n} \cdot \sum_{r=1}^{n} \frac{(2r-1)\cdot(n-r)}{r\cdot(n+r)^2}, \qquad B = 4 \cdot \sum_{r=1}^{n} \frac{(2r-1)\cdot(\log n - \log r)}{(n+r)^3},$$

$$C = 2 \cdot \sum_{r=1}^{n} \frac{2r-1}{r^2\cdot(n+r)^2} = O\left(\frac{\log n}{n^2}\right), \qquad D = 4 \cdot \sum_{r=1}^{n} \frac{2r-1}{r\cdot(n+r)^3} = O\left(\frac{1}{n^2}\right).$$

This gives $V_S(n,n,n) = A + B + O(\frac{\log n}{n^2})$ and it remains to estimate $A$ and $B$.

Decomposing the summand of $A$ into partial fractions, we get

$$\frac{(2r-1)\cdot(n-r)}{r\cdot(n+r)^2} = \frac{2\cdot(2n+1)}{(n+r)^2} - \frac{2n-1}{n\cdot(n+r)} - \frac{1}{n\cdot r},$$

so that $A = A_1 - A_2 - A_3$, where

$$A_1 = \frac{4\cdot(2n+1)}{n} \sum_{r=1}^{n} \frac{1}{(n+r)^2}, \qquad A_2 = \frac{2\cdot(2n-1)}{n^2} \sum_{r=1}^{n} \frac{1}{n+r},$$

$$A_3 = \frac{2}{n^2} \sum_{r=1}^{n} \frac{1}{r} = O\left(\frac{\log n}{n^2}\right).$$

Here, again using integral-derived bounds,

$$\sum_{r=1}^{n} \frac{1}{(n+r)^2} \leqslant \frac{1}{2n} + O\left(\frac{1}{n^2}\right), \qquad \sum_{r=1}^{n} \frac{1}{(n+r)^2} \geqslant \frac{1}{2n} + O\left(\frac{1}{n^2}\right),$$

$$\sum_{r=1}^{n} \frac{1}{n+r} \leqslant \log 2, \qquad \sum_{r=1}^{n} \frac{1}{n+r} \geqslant \log 2 + O\left(\frac{1}{n}\right),$$

which gives

$$A_1 = \frac{4}{n} + O\left(\frac{1}{n^2}\right), \quad A_2 = \frac{4\log 2}{n} + O\left(\frac{1}{n^2}\right), \text{ and } \quad A = \frac{4 - 4\log 2}{n} + O\left(\frac{\log n}{n^2}\right).$$

Similarly, we decompose the summand of $B$ into partial fractions,

$$\begin{aligned} B &= 4 \cdot \sum_{r=1}^{n} \frac{(2r-1)\cdot(\log n - \log r)}{(n+r)^3} \\ &= 8 \cdot \sum_{r=1}^{n} \frac{\log n - \log r}{(n+r)^2} - 4\cdot(2n+1)\cdot \sum_{r=1}^{n} \frac{\log n - \log r}{(n+r)^3} \\ &= B_1 - B_2, \end{aligned}$$

and estimate the sums

$$\frac{\log 2}{n} + O\left(\frac{\log n}{n^2}\right) \leqslant \sum_{r=1}^{n} \frac{\log n - \log r}{(n+r)^2} \leqslant \frac{\log 2}{n} + O\left(\frac{\log n}{n^2}\right),$$

$$\frac{\log 2}{2n^2} + \frac{1}{4n^2} + O\left(\frac{\log n}{n^3}\right) \leqslant \sum_{r=1}^{n} \frac{\log n - \log r}{(n+r)^3} \leqslant \frac{\log 2}{2n^2} + \frac{1}{4n^2} + O\left(\frac{\log n}{n^3}\right),$$

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

so that

$$B_1 = 8 \cdot \frac{\log 2}{n} + O\left(\frac{\log n}{n^2}\right), \qquad B_2 = \frac{4\log 2 + 2}{n} + O\left(\frac{\log n}{n^2}\right), \quad \text{and}$$

$$B = \frac{4\log 2 - 2}{n} + O\left(\frac{\log n}{n^2}\right).$$

Combining the estimates, we finally get

$$V_S(n,n,n) = A + B + O\left(\frac{\log n}{n^2}\right) = \frac{2}{n} + O\left(\frac{\log n}{n^2}\right). \qquad \square$$

**Remark.** Numerical calculations indicate that the true result may be

$$V_S(n,n,n) = \frac{2}{n} + O\left(\frac{1}{n^2}\right).$$

## Appendix 2: Results

In this section we show the detailed calculations for $k = 2$ and for $k = 3$. Similar tables for $k = 4$ and for $k = m = 5$ are given in [2], but are too voluminous to include here; the tables for $k = m = n = 6$ are larger still.

### A2.1. Results for $k = 2$, $m = n = 3$

By the argument of Section 5.5, calculations for $k = 2$, $m = n = 3$ suffice to deduce results for $k = 2$ and any $m$ and $n$; the cases are depicted for $m = n = 3$, but their labels are phrased in terms of the general parameters.

Each configuration – consisting of edge subsets $Z$ and $R$ of a bipartite graph $K_{m,n}$ – is represented by an $m \times n$ array with 0s marking the set $Z$, Xs marking the set $R$, and dots elsewhere.

When Lemma 26 of [7] allows replacement of entire rows or columns of entries with 0s, we show the matrix without these rows or columns. We are then looking for a smaller-cardinality assignment in a smaller matrix. For example, case 3 for $k = 2$ has a column of 0s suppressed, to become an $m \times (n-1)$ matrix; the shrinkage by 1 implies that the corresponding assignment size sought is also reduced by 1 to $k - 1$.

#### Cases

Each case is labelled as follows. First comes the case number in bold. Case number 1 is the original problem of finding a $k$-matching in an $m \times n$ matrix. The next line lists the dimensions of the case in terms of the original dimensions, followed by the size $s$ of the intersection of any potentially minimal matching with the region marked by Xs. The third line, in square brackets, lists the subcases generated by setting various Xs to 0. The probabilities of these subcases are not indicated here, but can be read off from the 'case dependencies' in the next section.

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

| **1** | **2** | **3** |
|---|---|---|
| $m, n, 2$ | $m, n, 1$ | $m, n-1, 1$ |
| [2] | [0 3] | [0] |
| XXX | 0 . . | XX |
| XXX | XXX | XX |
| XXX | XXX | XX |

**Case dependencies**

The probability with which each indexed configuration is reduced to any of its children depends only on the number of Xs whose selection would produce that child as opposed to some other. This in turn determines the algebraic calculations shown. $E_1$ is the expectation of interest (for $K_{m,n}$ and empty zero-set), and $E_i$ the expectation of the minimum-cost assignment of the configuration with index $i$.

$$
\begin{aligned}
E_0 &:= 0 \\
E_1 &= \frac{2}{(m-0)(n-0)} + \frac{(m-0)(n-0)}{(m-0)(n-0)} E_2 \\
E_2 &= \frac{1}{(m-1)(n-0)} + \frac{(m-1)(n-1)}{(m-1)(n-0)} E_0 + \frac{(m-1)(1)}{(m-1)(n-0)} E_3 \\
E_3 &= \frac{1}{(m-0)(n-1)} + \frac{(m-0)(n-1)}{(m-0)(n-1)} E_0
\end{aligned}
$$

The sequences of calculations for the Laplace transforms are a trivial modification of those for the mean. The following equations – in one-to-one correspondence with the preceding ones – describe the Laplace transform of the configurations' minimum-cost assignments. Because the Laplace transform equations are easy transformations of those for the mean, we will omit them in cases beyond $k = 2$. Unfortunately, in contrast to the situation for the expectation, in cases beyond $k = 2$ the Laplace transforms seem to have no compact, elegant expression.

$$
\begin{aligned}
g(a, t) &\equiv a/(a+t) \\
L_0(t) &\equiv 1 \\
L_1(t) &= g((m-0)(n-0)/2, t) \cdot \left( \frac{(m-0)(n-0)}{(m-0)(n-0)} \cdot L_2(t) \right) \\
L_2(t) &= g((m-1)(n-0)/1, t) \cdot \left( \frac{(m-1)(n-1)}{(m-1)(n-0)} \cdot L_0(t) + \frac{(m-1)1}{(m-1)(n-0)} \cdot L_3(t) \right) \\
L_3(t) &= g((m-0)(n-1)/1, t) \cdot \left( \frac{(m-0)(n-1)}{(m-0)(n-1)} \cdot L_0(t) \right)
\end{aligned}
$$

From the Laplace transform of a random variable, it is straightforward to calculate its variance as well as its mean; see Section 7 for details.

### A2.2. Results for $k = 3$, $m = n = 4$

Calculations for $k = 3$, $m = n = 4$ suffice to deduce results for $k = 3$ and any $m$ and $n$.

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

**Cases**

| **1** | **2** | **3** | **4** |
|---|---|---|---|
| $m, n, 3$ | $m, n, 2$ | $m, n, 2$ | $m, n, 1$ |
| [2] | [3 6] | [4 7] | [0 5 7] |
| XXXX | 0 . . . | 0XXX | 00 . . |
| XXXX | XXXX | 0XXX | 0 . . . |
| XXXX | XXXX | .XXX | XXXX |
| XXXX | XXXX | .XXX | XXXX |

| **5** | **6** | **7** | **8** |
|---|---|---|---|
| $m, n-2, 1$ | $m, n, 1$ | $m, n-1, 1$ | $m-1, n-1, 1$ |
| [0] | [0 7] | [0 8] | [0] |
| XX | 0 . . . | 0XX | XXX |
| XX | . 0 . . | .XX | XXX |
| XX | XXXX | .XX | XXX |
| XX | XXXX | .XX | |

**Case dependencies**

$$E_0 := 0$$

$$E_1 = \frac{3}{(m-0)(n-0)} + \frac{(m-0)(n-0)}{(m-0)(n-0)}E_2$$

$$E_2 = \frac{2}{(m-1)(n-0)} + \frac{(m-1)(1)}{(m-1)(n-0)}E_3 + \frac{(m-1)(n-1)}{(m-1)(n-0)}E_6$$

$$E_3 = \frac{2}{(m-0)(n-1)} + \frac{(2)(n-1)}{(m-0)(n-1)}E_4 + \frac{(m-2)(n-1)}{(m-0)(n-1)}E_7$$

$$E_4 = \frac{1}{(m-2)(n-0)} + \frac{(m-2)(1)}{(m-2)(n-0)}E_5 + \frac{(m-2)(1)}{(m-2)(n-0)}E_7 + \frac{(m-2)(n-2)}{(m-2)(n-0)}E_0$$

$$E_5 = \frac{1}{(m-0)(n-2)} + \frac{(m-0)(n-2)}{(m-0)(n-2)}E_0$$

$$E_6 = \frac{1}{(m-2)(n-0)} + 2\frac{(m-2)(1)}{(m-2)(n-0)}E_7 + \frac{(m-2)(n-2)}{(m-2)(n-0)}E_0$$

$$E_7 = \frac{1}{(m-0)(n-2)} + \frac{(1)(n-2)}{(m-0)(n-2)}E_8 + \frac{(m-1)(n-2)}{(m-0)(n-2)}E_0$$

$$E_8 = \frac{1}{(m-1)(n-1)} + \frac{(m-1)(n-1)}{(m-1)(n-1)}E_0$$

### A2.3. Further cases

Complete descriptions of the cases for $k = 4$ and for $k = m = 5$ can be found in [2], while the cases for $k = m = n = 6$ were too voluminous even for presentation there.

One pattern established by the data for the cases shown above fails to hold more generally. In the examples above, a good region is always a submatrix, one of whose dimensions is equal to the corresponding dimension of the matrix in which it occurs. That is, the region is either a union of full rows or of full columns, which is at least plausibly consistent with an interpretation in terms of LP duality (see Section 2). But with $k = 4$ and $Z = \{(1,1),(1,2),(1,3),(2,1),(3,1)\}$, a good region consists of the matrix but for its

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

first row and column – a submatrix, both of whose dimensions are one smaller than that of the matrix in which it lies – seeming to defy such an interpretation. However, it was always the case that our good regions were submatrices, and this was not a precondition of our finding them.

## References


[1] Aldous, D. (2000) The $\zeta(2)$ limit in the random assignment problem. *Random. Struct. Alg.* **18** 381–418.

[2] Alm, S. E. and Sorkin, G. B. (1999) Exact expectations and distributions for the random assignment problem. Technical Report 1999:27, University of Uppsala Department of Mathematics Report, Uppsala, Sweden. Accessible at `http://www.math.uu.se/~sea/assign.ps`.

[3] Alon, N. and Spencer, J. H. (1992) *The Probabilistic Method*, Wiley.

[4] Beveridge, A., Frieze, A. and McDiarmid, C. (1998) Random minimum length spanning trees in regular graphs. *Combinatorica* **18** 311–333.

[5] Buck, M. W., Chan, C. S. and Robbins, D. P. (2000) On the expected value of the minimum assignment. Mathematics arXiv, `http://arxiv.org/ps/math/0004175v2`.

[6] Coppersmith, D. and Sorkin, G. B. (2002) On the expected incremental cost of a minimum assignment. In *Contemporary Mathematics* (B. Bollobás, ed.), Vol. 10 of *Bolyai Society Mathematical Studies*, Springer. To appear.

[7] Coppersmith, D. and Sorkin, G. B. (1999) Constructive bounds and exact expectations for the random assignment problem. *Random Struct. Alg.* **15** 133–144.

[8] Dyer, M. E., Frieze, A. M. and McDiarmid, C. J. H. (1986) On linear programs with random costs. *Mathematical Programming* **35** 3–16.

[9] Feller, W. (1971) *An Introduction to Probability Theory and its Applications*, Vol. 2, 2nd edn, Wiley, New York.

[10] Frieze, A. M. (1985) On the value of a random minimum spanning tree problem. *Discrete Appl. Math.* **10** 47–56.

[11] Frieze, A. M. and McDiarmid, C. J. H. (1989) On random minimum length spanning trees. *Combinatorica* **9** 363–374.

[12] Janson, S. (1999) Personal communication.

[13] Karp, R. M. (1987) An upper bound on the expected cost of an optimal assignment. In *Discrete Algorithms and Complexity: Proc. Japan–US Joint Seminar* (D. S. Johnson *et al.*, eds), Vol. 15 of *Perspectives in Computing*, Academic Press, pp. 1–4.

[14] Lazarus, A. J. (1979) The assignment problem with uniform (0,1) cost matrix. Master's thesis, Department of Mathematics, Princeton University, Princeton, NJ.

[15] Lazarus, A. J. (1993) Certain expected values in the random assignment problem. *Oper. Res. Lett.* **14** 207–214.

[16] Linusson, S. and Wästlund, J. (2000) A generalization of the random assignment problem. Mathematics arXiv, `http://arxiv.org/ps/math/0006146`.

[17] McKay, B. D. (1978) Computing automorphisms and canonical labellings of graphs. *Combinatorial Mathematics* **686** 223–232. See also `http://cs.anu.edu.au/~bdm/`.

[18] Mézard, M. and Parisi, G. (1985) Replicas and optimization. *J. Physique Lettres* **46** 771–778.

[19] Mézard, M. and Parisi, G. (1987) On the solution of the random link matching problems. *J. Physique Lettres* **48** 1451–1459.

[20] Olin, B. (1992) Asymptotic properties of random assignment problems. PhD thesis, Kungl. Tekniska Högskolan, Stockholm, Sweden.

[21] Pardalos, P. M. and Ramakrishnan, K. G. (1993) On the expected optimal value of random assignment problems: Experimental results and open questions. *Comput. Optim. Appl.* **2** 261–271.


https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press

[22] Parisi, G. (1998) A conjecture on random bipartite matching. Physics e-Print archive: http://xxx.lanl.gov/ps/cond-mat/9801176.
[23] Talagrand, M. (1995) Concentration of measure and isoperimetric inequalities in product spaces. *Inst. Hautes Études Sci. Publ. Math.* **81** 73–205.

https://doi.org/10.1017/S0963548302005114 Published online by Cambridge University Press