A NEW DECISION METHOD FOR INTUITIONISTIC LOGIC BY 3-VALUED NON-DETERMINISTIC TRUTH-TABLES

RENATO R. LEME, MARCELO E. CONIGLIO, AND BRUNO LOPES

Abstract. Kurt Gödel proved that it is not possible to characterize intuitionistic propositional logic (*IPL*) by means of finite and deterministic truth-tables. After extending the same result with respect to non-deterministic matrices (Nmatrices), we provide a semantical characterization of *IPL* by means of a 3-valued Nmatrix with a restricted set of valuations. This structure allows to define an algorithm to delete unsound rows from the non-deterministic truth-tables generated for each formula, which constitutes a new and very simple decision procedure for *IPL*. This method can be seen as truth-tables in a broader sense, and a way to overcome Gödel's limiting result.

§1. Introduction. In 1932, Kurt Gödel established the uncharacterizability of intuitionistic propositional logic (*IPL*) with respect to finite-valued deterministic matrices [10]. He demonstrated that any finite deterministic matrix, strongly enough to validate every intuitionistic tautology, will also validate at least one intuitionistically invalid proposition. Later, in 1940, James Dugundji proved a similarly significant result by adapting Gödel's argument to show that the same holds true for modal logic $S5$ and all of its subsystems [8].

These limitations led to new interpretations for modal and intuitionistic systems, such as Saul Kripke's relational semantics, where the notion of *possible world* plays a central role. The tradition of using possible worlds to explain modalities dates back to Leibniz and is at the core of philosophical debates: Are possible worlds real or just helpful fiction? John Kearns challenged this tradition by defending that they are neither one nor the other [19, p. 86]. To support his position, he proposed, in 1981, a four-valued semantics for the modal systems *KT*, *S4*, and *S5* where the truth-value of a formula depends solely on the truth assignments of its subformulas. In Kearns' approach, the meaning of certain connectives is not entirely determined by their matrices, as some inputs can yield more than one output. This type of semantics was later formalized under the notion of *non-deterministic matrices* (Nmatrices, in short) [2]. To address this indeterminacy, instead of possible worlds, Kearns introduced a method for restricting the set of valuations, which was later referred to more generally as *level valuations* (see Section 4). In this approach, not all valuations in the Nmatrix are permitted. Instead, only a subset that meets specific conditions to ensure the validity of the necessitation rule is allowed; this subset is precisely the set of level valuations. Nmatrices with a restricted set of valuations lead to the notion

Received January 7, 2025.
2020 *Mathematics Subject Classification.* Primary 03B20, 03B25, 03B35, 03B45, 68V15.
Key words and phrases. intuitionism, non-determinism, truth-table, decision procedure.

© The Author(s), 2025. Published by Cambridge University Press on behalf of The Association for Symbolic Logic.
0022-4812/00/0000-0000
DOI:10.1017/jsl.2025.10174

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

of *Restricted non-deterministic matrices* or *Restricted Nmatrices* (RNmatrices, in short), central in the present article.

Unlike Kripke semantics, however, Kearns's level valuations do not draw a straightforward decision procedure for those logics. Until recently, it was unclear whether obtaining such a decision procedure from level valuations would be possible.

> (...) the interesting problem of determining whether or not a decision procedure can be obtained by means of level valuations remains open. [6]

In 2021, Lukas Grätz solved this problem for some modal systems (see [12]). He built upon Kearns's original RNmatrix 3-valued RNmatrices for *KT* and *S4* with two decidable notions of *partial level valuation*, one for *KT* and the other for *S4* and showed that both are analytic and co-analytic regarding the level valuations of the RNmatrix for the respective modal systems. To obtain the truth-table, one discards (by means of a simple algorithm) any valuation that does not fit the corresponding definition. Ultimately, the remaining partial valuations suffice to decide the formula's validity. Analyticity and co-analyticity ensure that partial level valuation semantics (which constitute the rows of the truth-tables) coincide with the corresponding RNmatrix semantics, which characterizes each logic.

Constructing non-deterministic (i.e., row-branching) truth-tables in a finite-valued Nmatrix can be seen as a generalization of the standard (deterministic) truth-tables method for deciding validity in a given logic. Indeed, Nmatrices were formally introduced by A. Avron and I. Lev in [2] with the aim of overcoming Gödel–Dugundji's uncharacterizability results for some logics. The main purpose of Nmatrices was to obtain a decision procedure, analogous to finite-valued truth-tables, for logics that do not admit such a characterization. It is worth noting that Nmatrices were already considered in the literature before these authors. For instance, N. Rescher already defined an Nmatrix semantics in 1962, under the name of *quasi-truth-functional systems* (see [29]). Afterwards, and among other authors, this notion was independently proposed by Y. Ivlev for studying non-normal modal logics, starting in the 1970s (see, for instance, [14–17]), as well as J. Kearns (with restricted valuations).

In spite of its greater expressive power with respect to standard (deterministic) matrices, the generalization of the notion of truth-tables by means of Nmatrices has also limitations: some logics, such as da Costa's paraconsistent logic C_1, cannot be characterized by a single finite Nmatrix, as proved by Avron in [1, Theorem 11]. The same is true for *KT* and *S4*, as proved by Grätz in [12]. Moreover, in Theorem 3.5 below we will show that *IPL* also cannot be characterized by a single finite Nmatrix, extending so Gödel's theorem from 1932. These examples show that Gödel–Dugundji's results hold, for some logics, in a stronger sense, that is, w.r.t. Nmatrices. The key to overcoming this limitation is to consider *some* valuations in a suitable Nmatrix. This kind of semantical structures was formally introduced in [7], under the name of *restricted Nmatrices*, or RNmatrices, showing that each da Costa's logics C_n (for $n \geq 1$) can be characterized by a single (decidable) $(n+2)$-valued RNmatrix; in particular, validity in C_1 can be decided by a 3-valued RNmatrix by means of branching truth-tables (plus a simple criterion to delete inadequate rows).

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

Analogously, Grätz decidable 3-valued RNmatrices produce a decision procedure for *KT* and *S4* by means of 3-valued branching truth-tables (together with a simple method to delete unsound rows).

Finite-valued non-deterministic truth-tables, together with an algorithm to delete inadequate rows, are a decision procedure that can be seen as truth-tables in a broader sense, and is a way to overcome Gödel–Dugundji's limiting results, even for more complicated logics, such as *KT*, *S4*, C_1, and *IPL*. The aim of this article is, precisely, to obtain a new decision procedure for *IPL* based on this kind of generalized truth-tables.

Recall that, in 1933, Gödel proposed an interpretation of *IPL* into *S4* by means of a translation mapping t between their languages (see [11] and [9], page 301). In 1948, Tarski and McKinsey prove that Gödel's mapping t constitutes a faithful interpretation, that is: α is a theorem of *IPL* iff its translation $t(\alpha)$ is a theorem of *S4* [26]. Since Gödel translation t is computable, then, by composing it with any algorithm for deciding *S4*, a decision procedure for *IPL* is immediately obtained, defined by using a translation (from *IPL* to *S4*). In particular, by composing Gödel translation t with Grätz's algorithm for *S4*, a decision procedure is obtained for *IPL*, based on an RNmatrix for *S4* (as well as the translation t). The definition of a *pure* RNmatrix for *IPL* (i.e., without depending on Gödel and Grätz's constructions), which is also decidable by truth-tables, as discussed above, is a challenge that naturally arises.

A pure RNmatrix for *IPL* with such features is desirable for several reasons. First, it can lead to improvements in efficiency. As will become clear later, the computational cost of producing a truth-table is directly related to the number of subformulas, and the translated formulas are twice as large as the original intuitionistic formulas. The second reason is that a pure RNmatrix can explain the actual meaning of intuitionistic operators. As it will be argued below, the RNmatrix for *IPL* proposed here has significant differences when compared to the RNmatrix for *S4*, and those differences are motivated by philosophical considerations on the nature of *IPL* itself.

In this article, a pure (and decidable by non-deterministic truth-tables) RNmatrix for *IPL* will be constructed by abstracting the composition of Gödel's translation algorithm and Grätz's decision procedure for *S4*. This abstraction will provide a direct way to define the RNmatrix for *IPL*. Hence, a 3-valued Nmatrix, together with suitable notions of level valuations and partial level valuations (different, in nature, to the ones for modal logics) is introduced, and the soundness and completeness of the method are proved without relying on the results of Gödel [11] and Grätz [12].

The article is organized as follows. Section 2 introduces the central notions that will be used, such as the notion of Nmatrices, RNmatrices, and their valuation functions. Section 3 extends Gödel's uncharacterizability theorem for *IPL* to the non-deterministic case. Section 4 presents Kearns's Nmatrices, and the notion of level valuation for *S4* is detailed. Section 5 presents Grätz's RNmatrix for *S4*. Section 6 expands the RNmatrix for *S4* with conjunction and disjunction and introduces Gödel's box-translation. Section 7 introduces the RNmatrix for *IPL* and proves its soundness and completeness regarding a new notion of level valuation for *IPL*. Section 8 presents the truth-table method based on the RNmatrix, and

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

some examples are discussed. Section 9 briefly considers some aspects of the computational complexity of the method. Finally, Section 10 discusses some related works and possibilities of future research.

§2. Preliminaries: Logics, Nmatrices, and RNmatrices. Along this article, a *propositional signature* is a denumerable family $\Theta = (\Theta_n)_{n\geq 0}$ of pairwise disjoint sets. Elements in Θ_n are called *n-ary connectives*. An *algebra over* Θ is a pair $\mathcal{A} = \langle A, \mathcal{O}\rangle$ such that A is a non-empty set called the *universe* or *domain* of $\mathcal{A}$ and, $\mathcal{O}$ is a function assigning, to each n-ary connective $\#$ of Θ, a function $\mathcal{O}(\#) : A^n \to A$. The (absolutely) free algebra over Θ generated by a denumerable set $\mathcal{P} = \{p_0, p_1, ...\}$ of propositional variables is called *the algebra of formulas over* Θ, and will be denoted by $For(\Theta)$.

A *logic* is a pair $\mathbf{L} = \langle For, \vdash_{\mathbf{L}}\rangle$ where For is the set of formulas and $\vdash_{\mathbf{L}}$ is the consequence relation. A logic $\mathbf{L}$ is said to be *Tarskian* if it satisfies the following: (1) if $\varphi \in \Gamma$ then $\Gamma \vdash_{\mathbf{L}} \varphi$; (2) if $\Gamma \vdash_{\mathbf{L}} \varphi$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash_{\mathbf{L}} \varphi$; and (3) if $\Delta \vdash_{\mathbf{L}} \varphi$ and $\Gamma \vdash_{\mathbf{L}} \psi$ for every $\psi \in \Delta$ then $\Gamma \vdash_{\mathbf{L}} \varphi$. A Tarskian logic $\mathbf{L}$ is *finitary* if it satisfies: (4) if $\Gamma \vdash_{\mathbf{L}} \varphi$ then there exists a finite subset Γ_0 of Γ such that $\Gamma_0 \vdash_{\mathbf{L}} \varphi$.

A very natural way to characterize (Tarskian) logics is by means of logical matrices, which formalize the idea of truth-tables.

DEFINITION 2.1 (Logical matrices). Let Θ be a signature. A *logical matrix* over Θ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ where

1. $\mathcal{V}$ is a non-empty set (of truth-values);
2. $\mathcal{D} \subseteq \mathcal{V}$ is a non-empty set (of designated values);
3. $\mathcal{O}$ is a function assigning, to each $\# \in \Theta_n$, a truth function $\tilde{\#} : \mathcal{V}^n \to \mathcal{V}$ (that is, $\mathcal{A}_{\mathcal{M}} = \langle \mathcal{V}, \mathcal{O}\rangle$ is an algebra over Θ).

DEFINITION 2.2. A valuation over a matrix $\mathcal{M}$ is a homomorphism $v : For(\Theta) \to \mathcal{A}_{\mathcal{M}}$. That is, for every $\# \in \Theta_n$ and for every $\varphi_1, \ldots, \varphi_n \in For(\Theta)$: $v(\#(\varphi_1, \ldots, \varphi_n)) = \mathcal{O}(\#)(v(\varphi_1), \ldots, v(\varphi_n))$. The set of valuations over $\mathcal{M}$ is denoted by $Val(\mathcal{M})$. We say that φ is a semantical consequence of Γ w.r.t. $\mathcal{M}$, denoted by $\Gamma \models_{\mathcal{M}} \varphi$ if, for every $v \in Val(\mathcal{M})$: $v(\varphi) \in \mathcal{D}$ whenever $v(\gamma) \in \mathcal{D}$ for every $\gamma \in \Gamma$. We write $\models_{\mathcal{M}} \varphi$ instead of $\emptyset \models_{\mathcal{M}} \varphi$.

As mentioned in the Introduction, the formal notion of Nmatrices was introduced by Avron and Lev in [2] as a way to overcome some Gödel–Dugundji's uncharacterizability results. Finite-valued Nmatrices which are analytical (i.e., each partial valuation can be extended to a full valuation) provide a decision procedure for some logics which are uncharacterizable by a single finite-valued logic, such as some paraconsistent logics (see, for instance, [1]).

DEFINITION 2.3 (Nmatrices, [2]). Let Θ be a signature. An Nmatrix over Θ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ where

1. $\mathcal{V}$ is a non-empty set (of truth-values);
2. $\mathcal{D} \subseteq \mathcal{V}$ is a non-empty set (of designated values);
3. $\mathcal{O}$ is a function assigning, to each $\# \in \Theta_n$, a (non-deterministic) truth function $\mathcal{O}(\#) : \mathcal{V}^n \to (\wp(\mathcal{V})) \setminus \emptyset$ (that is, $\mathcal{A}_{\mathcal{M}} = \langle \mathcal{V}, \mathcal{O}\rangle$ is a non-deterministic algebra over Θ).

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

DEFINITION 2.4 (Valuations on Nmatrices, [2]). A valuation over an Nmatrix $\mathcal{M}$ is a function $v : For(\Theta) \to \mathcal{V}$ such that, for every $\# \in \Theta_n$ and for every $\varphi_1, \ldots, \varphi_n \in For(\Theta)$, $v(\#(\varphi_1, \ldots, \varphi_n)) \in \mathcal{O}(\#)(v(\varphi_1), \ldots, v(\varphi_n))$. The set of valuations over $\mathcal{M}$ is denoted by $Val^{nd}(\mathcal{M})$. The notion of semantical consequence w.r.t. an Nmatrix $\mathcal{M}$, as well as the notation, will be analogous to the ones in Definition 2.2, but now w.r.t. valuations over the Nmatrix $\mathcal{M}$.

However, as mentioned before, certain logics are still uncharacterizable by finite-valued Nmatrices, such as da Costa's paraconsistent logic C_1 (see [1]), modal logics *KT* and *S4* (see [12]), or *IPL* (see Theorem 3.5 below). In these cases, finite-valued and decidable RNmatrices, which are analytical and co-analytical, can overcome this limitation.

DEFINITION 2.5 (RNmatrices, [7]). Let Θ be a signature. An RNmatrix over Θ is a pair $\mathcal{R} = \langle \mathcal{M}, \mathcal{F} \rangle$, where $\mathcal{M}$ is an Nmatrix over Θ and $\mathcal{F} \subseteq Val^{nd}(\mathcal{M})$. The notion of semantical consequence w.r.t. an RNmatrix $\mathcal{R} = \langle \mathcal{M}, \mathcal{F} \rangle$, as well as the notation, will be analogous to the ones in Definition 2.2, but now w.r.t. valuations in $\mathcal{F}$.

It is worth noting that the logic generated by an RNmatrix is indeed Tarskian.

§3. Gödel's uncharacterizability theorem updated. As mentioned in the Introduction, Gödel proved in 1932 that *IPL* cannot be characterized by a single finite-valued (deterministic) logical matrix. It is a natural question to determine whether *IPL* could be characterized by means of a single finite-valued Nmatrix. As we shall show in the sequel, it is easy to adapt the proof by Gödel in [10] in order to see that the answer to the question above is negative.

DEFINITION 3.1. Consider the signature $\Omega = (\{\neg\}, \{\wedge, \vee, \to\})$ for *IPL*. The set of intuitionistic formulas will be denoted by $For(\Omega)$.

DEFINITION 3.2 (Gödel [10]). For $n \geq 1$ let G_n be the formula expressing that "there are at most n truth-values":

$$G_n := \bigvee_{1 \leq i < j \leq n+1} (p_i \to p_j) \wedge (p_j \to p_i).$$

LEMMA 3.3 (Gödel [10]). *None of the formulas G_n is provable in IPL.*

LEMMA 3.4. *If $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a finite Nmatrix such that $\mathcal{V}$ has exactly $n \geq 1$ elements and $\mathcal{M}$ models IPL, then the formula G_n is valid in $\mathcal{M}$.*

PROOF. For each connective $\# \in \Omega$ let us denote $\mathcal{O}(\#)$ by $\bar{\#}$. Since $\mathcal{M}$ models *IPL* then, for every $a, b \in \mathcal{V}$: (i) $a \,\bar{\vee}\, b \subseteq \mathcal{D}$ if either $a \in \mathcal{D}$ or $b \in \mathcal{D}$; (ii) $a \,\bar{\wedge}\, b \subseteq \mathcal{D}$ if both $a \in \mathcal{D}$ and $b \in \mathcal{D}$; and (iii) $a \,\bar{\to}\, a \subseteq \mathcal{D}$ (since $\varphi \to \varphi$ is a theorem of *IPL*). Given that $\mathcal{V}$ has exactly $n \geq 1$ elements, if v is a valuation over $\mathcal{M}$ then, by the pigeonhole principle, there exists $1 \leq i < j \leq n+1$ such that $v(p_i) = v(p_j)$. Let $a := v(p_i)$. By consideration (iii) above, $v(p_i \to p_j) \in a \,\bar{\to}\, a \subseteq \mathcal{D}$, hence $v(p_i \to p_j) \in \mathcal{D}$. Analogously, $v(p_j \to p_i) \in \mathcal{D}$ and so, by (ii), $v((p_i \to p_j) \wedge (p_j \to p_i)) \in \mathcal{D}$. Thus, from the fact that G_n is a disjunction where $(p_i \to p_j) \wedge (p_j \to p_i)$ is one of its disjuncts, repeated application of (i) implies $v(G_n) \in \mathcal{D}$. ⊣

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

	$\neg^{Ke}$
T	$\{F\}$
t	$\{f\}$
f	$\{t\}$
F	$\{F\}$

	$\Box^{Ke}$
T	$\{T\}$
t	$\{f,F\}$
f	$\{f,F\}$
F	$\{F\}$

$\vee^{Ke}$	T	t	f	F
T	$\{T\}$	$\{T\}$	$\{T\}$	$\{T\}$
t	$\{T\}$	$\{T,t\}$	$\{T,t\}$	$\{t\}$
f	$\{T\}$	$\{T,t\}$	$\{f\}$	$\{f\}$
F	$\{T\}$	$\{t\}$	$\{f\}$	$\{F\}$

FIGURE 1. Tables for $\mathcal{M}_{Ke}$.

THEOREM 3.5. *The logic IPL cannot be characterized by a single finite-valued Nmatrix.*

PROOF. Suppose that $\mathcal{M}$ is a finite Nmatrix with exactly $n \geq 1$ elements such that $\mathcal{M}$ models IPL. Suppose, by absurdum, that $\mathcal{M}$ characterizes IPL. Hence, every formula valid in $\mathcal{M}$ can be proved in IPL. By Lemma 3.4, the formula G_n is valid in $\mathcal{M}$. By hypothesis, G_n must be provable in IPL. But this contradicts Lemma 3.3. This shows that no finite Nmatrix $\mathcal{M}$ can characterize IPL. ⊣

§4. Kearns's characterization of *S4* by RNmatrices. Regarding semantics for normal modal logic, Kripke-style semantics are the usual choice. However, there are other attractive alternatives, such as algebraic semantics. In the 1980s, Kearns proposed in [19] an RNmatrix-based semantics for modal logics *KT*, *S4*, and *S5*. He introduced the level valuations as the set $\mathcal{F}$ of valuations to be used in his RNmatrix. According to him,

> [It] is simpler than the standard [Kripke's] account in virtue of having dispensed with possible worlds and their relations. I also think that my account is philosophically preferable to the standard account for having done this. For I do not think there are such things as possible worlds, or even that they constitute a useful fiction. (Kearns, [19, p. 86])

Figure 1 reconstructs Kearns's Nmatrix for *S4* over the signature $\Sigma_{Ke} = (\{\Box, \neg\}, \{\vee\})$, where the truth-values are: T (necessarily true), t (contingently true), f (contingently false), and F (necessarily false). In this Nmatrix, the set of designated values is given by $\mathcal{D} = \{T, t\}$.

Kearns proved that, in order to fully characterize *S4* using $\mathcal{M}_{Ke}$, it is sufficient to introduce the notion of *level valuations*, which are defined as follows.

DEFINITION 4.1. (Kearns's level valuations for *S4*)

Let $Val^{nd}(\mathcal{M}_{Ke})$ be the set of valuations over $\mathcal{M}_{Ke}$. We define the set of nth-level valuations $\mathcal{L}_k^{Ke}$, where $n \in \omega$, as follows:

i) $\mathcal{L}_0^{Ke} = Val^{nd}(\mathcal{M}_{Ke})$;
ii) $\mathcal{L}_{n+1}^{Ke} = \{v \in \mathcal{L}_n^{Ke} \mid \forall \alpha \in For(\Sigma_{Ke}), \mathcal{L}_n^{Ke}(\alpha) \subseteq \{T, t\} \Rightarrow v(\alpha) = T\}$,

where $\mathcal{L}_n^{Ke}(\alpha) = \{v(\alpha) \mid v \in \mathcal{L}_n^{Ke}\}$.

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

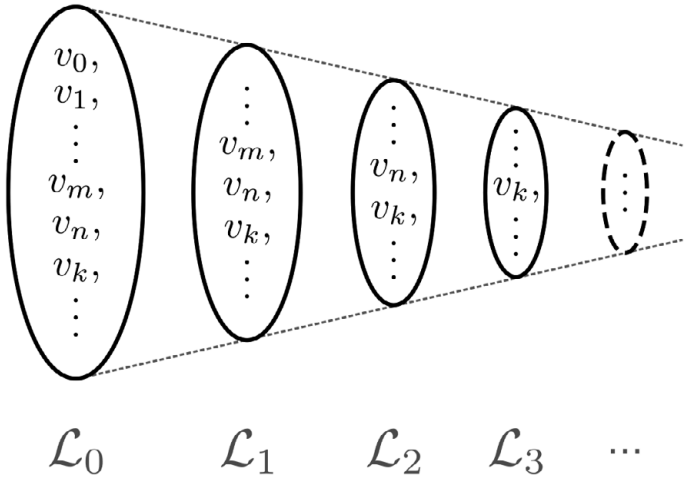


FIGURE 2. Assume that $Val^{nd}(\mathcal{M}_{Ke}) = \{v_0, v_1, \ldots, v_m, v_n, v_k, \ldots\}$. We remove a valuation v in level $\mathcal{L}_{n+1}$ iff there is some α such that $v(\alpha) = t$ and $w(\alpha) \in \{T, t\}$ for every $w \in \mathcal{L}_n$.

The set of level valuations in $\mathcal{M}_{Ke}$ is the intersection of the sets $\mathcal{L}_n^{Ke}$, for $n \geq 0$:

$$\mathcal{L}_{Ke} = \bigcap_{n \geq 0}^{\infty} \mathcal{L}_n^{Ke}. \tag{1}$$

The way in which level valuations work can be appreciated in Figure 2.

DEFINITION 4.2 (Kearns's RNmatrix for *S4*). The RNmatrix for *S4* defined by Kearns is the pair $\mathcal{R}(\mathcal{M}_{Ke}) = \langle \mathcal{M}_{Ke}, \mathcal{L}_{Ke} \rangle$.

THEOREM 4.3 [19]. *For every* Γ *and* φ *over* Σ_{Ke}: $\Gamma \vdash_{S4} \varphi$ *iff* $\Gamma \models_{\mathcal{R}(\mathcal{M}_{Ke})} \varphi$.

§5. Grätz's decision procedure for *S4*. Kearns proved that his Nmatrix for *S4* with level valuations is sound and complete regarding the logic *S4*. However, as Section 4 explains, level valuations are defined as the intersection of an infinite family of subsets of valuation functions. How do we reproduce this procedure using finite steps? While it is true that level valuation semantics characterize *S4*, it is not straightforward to decide the validity of a given formula using Kearns's tables.

In particular, given any theorem φ, the immediate table for $\Box\varphi$ will always present a partial valuation v such that $v(\Box\varphi)$ is not a designated value, thus invalidating the rule of necessitation. Grätz noticed that the problem lies in the definition of *partial valuation*: such partial valuations are precisely the ones that cannot be extended to full level valuations; thus, they should be ruled out by some appropriate definition.

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

	$\neg^{S4}$
2	{0}
1	{0}
0	{1,2}

	$\Box^{S4}$
2	{2}
1	{0}
0	{0}

$\rightarrow^{S4}$	2	1	0
2	{1,2}	{1}	{0}
1	{1,2}	{1,2}	{0}
0	{1,2}	{1,2}	{1,2}

FIGURE 3. Tables for $\mathcal{M}_{S4}$.

Grätz's Nmatrix for *S4*. Instead of 4 truth-values, Grätz uses a set with three values, 0, 1, and 2, where each one has the following intended meaning:

$v(\varphi)$=0: means that φ is false;
$v(\varphi)$=1: means that φ is contingently true, i.e., φ is true but possibly false;
$v(\varphi)$=2: means that φ is necessarily true.

From this interpretation, defining the subset $\mathcal{D} = \{2, 1\}$ as the set of designated values is natural. As primitive operators, it is used for negation ($\neg$), implication ($\rightarrow$), and box ($\Box$). The Nmatrix is defined as follows.

DEFINITION 5.1 ($\mathcal{M}_{S4}$). Let $\Sigma_b = (\{\neg, \Box\}, \{\rightarrow\})$ be the signature for modal logic *S4*, and let $For(\Sigma_b)$ be the corresponding set of formulas. Grätz's Nmatrix over Σ_b is the tuple $\mathcal{M}_{S4} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V} = \{0, 1, 2\}$, $\mathcal{D} = \{1, 2\}$, and $\mathcal{O}(\#) := \#^{S4}$, for a connective # in Σ_b, is defined as in Figure 3.

To understand the Nmatrix for the modal operator $\Box$, we must recall the axioms that define *S4*. In particular, the axioms $\Box\varphi \rightarrow \varphi$ (reflexivity) and $\Box\varphi \rightarrow \Box\Box\varphi$ (transitivity). Consider the contrapositive of the reflexivity axiom: $\neg\varphi \rightarrow \neg\Box\varphi$. This means that if φ is false, then $\Box\varphi$ is also false. The Nmatrix for $\Box$ reflects this behavior when $v(\varphi) = 0$. Next, when $v(\varphi) = 2$ (indicating that φ is necessary, or in other words, $\Box\varphi$ is true), we have $v(\Box\varphi) = \{2\}$. This signifies that $\Box\varphi$ is also necessary, which implies that $\Box\Box\varphi$ is true as well. Finally, when $v(\varphi) = 1$, it indicates that φ is contingent, meaning it is possibly false, hence $\Box\varphi$ is false.

DEFINITION 5.2. (Grätz's level valuations for *S4*)
Let $Val^{nd}(\mathcal{M}_{S4})$ be the set of valuations over $\mathcal{M}_{S4}$. We define the set of nth-level valuations $\mathcal{L}_k^{S4}$, where $n \in \omega$, as follows:

i) $\mathcal{L}_0^{S4} = Val^{nd}(\mathcal{M}_{S4})$;
ii) $\mathcal{L}_{n+1}^{S4} = \{v \in \mathcal{L}_n^{S4} \mid \forall \alpha \in For(\Sigma_b), \mathcal{L}_n^{S4}(\alpha) \subseteq \{1, 2\} \Rightarrow v(\alpha) = 2\}$,

where $\mathcal{L}_n^{S4}(\alpha) = \{v(\alpha) \mid v \in \mathcal{L}_n^{S4}\}$.
The set of level valuations in $\mathcal{M}_{S4}$ is the intersection of the sets $\mathcal{L}_n^{S4}$, for $n \geq 0$:

$$\mathcal{L}_{S4} = \bigcap_{n \geq 0}^{\infty} \mathcal{L}_n^{S4}. \tag{2}$$

DEFINITION 5.3 (Grätz's RNmatrix for *S4*). The RNmatrix for *S*4 defined by Grätz is the pair $\mathcal{R}(\mathcal{M}_{S4}) = \langle \mathcal{M}_{S4}, \mathcal{L}_{S4} \rangle$.

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

Let $\vdash^b_{S4}$ be the consequence relation of the standard Hilbert calculus $\mathcal{H}^b_{S4}$ for *S4* over Σ_b. As usual, $\Gamma \vdash^b_{S4} \varphi$ iff either φ is a theorem of $\mathcal{H}^b_{S4}$ or there exists a non-empty set $\{\beta_1, \ldots, \beta_n\} \subseteq \Gamma$ such that $\beta_1 \to (\beta_2 \to (\ldots \to (\beta_n \to \varphi)\ldots))$ is a theorem of $\mathcal{H}^b_{S4}$.

THEOREM 5.4 [12, Theorems 3.5 and 3.10]. *For every* Γ *and* φ *over* Σ_b: $\Gamma \vdash^b_{S4} \varphi$ *iff* $\Gamma \models_{\mathcal{R}(\mathcal{M}_{S4})} \varphi$.

In order to obtain a decision procedure for *S4* based on the 3-valued RNmatrix $\mathcal{R}(\mathcal{M}_{S4})$, Grätz proposes the notion of partial level valuation. The idea is as follows: given a formula φ to be checked as a tautology in *S4*, construct the (finite) table generated by the Nmatrix $\mathcal{M}_{S4}$ for φ and all of its subformulas. Then, by means of a decision procedure, it is possible to determine which rows of the table are allowed (since they correspond to level valuations); the non-allowed ones correspond to valuations over $\mathcal{M}_{S4}$ which *do not* correspond to level valuations, hence they are deleted. If φ only receives a designated value in the allowed rows, then it is declared to be valid in *S4*, otherwise it is declared to be non-valid. Formally:

DEFINITION 5.5 (Partial valuation in $\mathcal{M}_{S4}$). A partial valuation in $\mathcal{M}_{S4}$ is a function $\tilde{v}_p : \Lambda \to \mathcal{V}$ such that $\Lambda \subseteq For(\Sigma_b)$ is closed under subformulas[1] and, for every $\alpha, \beta \in \Lambda$:

– if $\# \in \{\neg, \Box\}$ and $\#\alpha \in \Lambda$ then $\tilde{v}_p(\#\alpha) \in \#^{S4}(\tilde{v}_p(\alpha))$;
– if $\alpha \to \beta \in \Lambda$ then $\tilde{v}_p(\alpha \to \beta) \in \to^{S4}(\tilde{v}_p(\alpha), \tilde{v}_p(\beta))$.

DEFINITION 5.6 (Grätz's partial level valuation). A partial valuation $v_p : \Lambda \to \{0, 1, 2\}$ in $\mathcal{M}_{S4}$ is a *partial level valuation* in $\mathcal{R}(\mathcal{M}_{S4})$ ($v_p \in PLV$) iff $\forall \alpha \in \Lambda$ such that $v_p(\alpha) = 1$, there exists $w_p \in PLV$ such that $w_p(\alpha) = 0$ and, $\forall \beta \in \Lambda$, $w_p(\beta) = 2$ whenever $v_p(\beta) = 2$.

REMARK 5.7. The set PLV plays a fundamental role in the proof of soundness and completeness of the truth-tables decision procedure introduced in [12]. Some comments on this notion are in order at this point.

The existence (and uniqueness) of the set PLV is the crucial fact used in his proof. In this respect, [12, p. 15] presents an algorithm (Algorithm 3.4) for computing the set PLV of partial level valuations as a (constructive) truth-table. He argues that this process is well-defined since the set of all partial valuations with domain Λ is finite, so is its powerset. Uniqueness is tacitly assumed, given the nature of the algorithm. Moreover, he proves that this notion is analytic in $\mathcal{M}_{S4}$ (i.e., every partial level valuation in $\mathcal{M}_{S4}$ can be extended to a level valuation in $\mathcal{M}_{S4}$) and co-analytic (i.e., for any level valuation v in $\mathcal{M}_{S4}$, the restriction of v to a partial level valuation v_p defined over a finite set Λ closed under subformulas is a partial level valuation). This argument, justified by the uniqueness of the set PLV, allows us to conclude that the truth-table procedure corresponds exactly to the level valuation semantics. A similar and fully detailed argument will be given for *IPL* in Section 8.

[1] That is: if $\alpha \in \Lambda$ and β is a subformula of α then $\beta \in \Lambda$.

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

From the observations above, one can easily prove that the truth-table method for *S4* is sound and complete, that is:

THEOREM 5.8 [12, Theorem 3.15]. *Let φ be a formula over Σ_b. Then, $\models_{\mathcal{R}(\mathcal{M}_{S4})} \varphi$ if, and only if, for every partial valuation v_p in $\mathcal{M}_{S4}$ defined over the set Λ of subformulas of φ: if $v_p \in PLV$, then $v_p(\varphi) \in \mathcal{D}$.*

COROLLARY 5.8.1. *For every φ over Σ_b: $\vdash^b_{S4} \varphi$ if, and only if, for every partial valuation v_p in $\mathcal{M}_{S4}$ defined over the set Λ of subformulas of φ, if $v_p \in PLV$ then $v_p(\alpha) \in \mathcal{D}$.*

PROOF. It is an immediate consequence of Theorems 5.8 and 5.4. ⊣

Therefore, to construct a truth-table, compute every partial valuation and remove any non-PLV in a finite number of steps. This algorithmic process is a sound and complete decision procedure for *S4*. Clearly, the soundness and completeness Theorem 5.8 (and hence the decision procedure) can be extended to inferences in *S4* from a finite set of premises, since $\vdash^b_{S4}$ (and thus $\models_{\mathcal{R}(\mathcal{M}_{S4})}$) satisfies the deduction metatheorem (DM).

§6. First steps towards an RNmatrix for *IPL*. As mentioned in Section 1, Kearns's level valuations fail to provide an effective decision procedure for *S4*, while Grätz method produces an algorithm to decide that logic. Clearly, the composition of Gödel's translation from *IPL* to *S4* with Grätz's decision procedure for *S4* naturally produces a decision procedure for *IPL*.

Let $\Sigma = (\{\neg, \Box\}, \{\wedge, \vee, \rightarrow\})$ be the full signature of *S4*, and let $\Omega = (\{\neg\}, \{\wedge, \vee, \rightarrow\})$ be the signature for *IPL*.

DEFINITION 6.1. The *complexity* $\mathsf{c}(\alpha)$ of a formula $\alpha \in For(\Omega)$ is defined as follows: $\mathsf{c}(p) = 0$ if p is a propositional variable; $\mathsf{c}(\neg\alpha) = \mathsf{c}(\alpha) + 1$; and $\mathsf{c}(\alpha \# \beta) = \mathsf{c}(\alpha) + \mathsf{c}(\beta) + 1$, for $\# \in \{\rightarrow, \vee, \wedge\}$.

Recall the Gödel–Tarski–McKinsey Box translation $t : For(\Omega) \rightarrow For(\Sigma)$ defined as follows:

i) $t(p) = \Box p$, if p is a propositional variable;
ii) $t(\neg\alpha) = \Box\neg t(\alpha)$;
iii) $t(\alpha \# \beta) = \Box(t(\alpha) \# t(\beta))$, where $\# \in \{\rightarrow, \vee, \wedge\}$.

It is widely known that this translation is faithful (see [26]; for a more recent proof see [27]), that is:

THEOREM 6.2 [26]. *Let φ be a formula in $For(\Omega)$. Then: φ is valid in IPL if and only if $t(\varphi)$ is valid in S4.*

As seen in Section 5, the signature considered in [12] contains only the connectives $\rightarrow$, $\neg$, and $\Box$. Using the standard definition of classical conjunction and disjunction, $\alpha \wedge \beta := \neg(\alpha \rightarrow \neg\beta)$ and $\alpha \vee \beta := \neg\alpha \rightarrow \beta$, we extend the signature of $\mathcal{M}_{S4}$ by computing the truth-table for $\neg(p \rightarrow \neg q)$ and $\neg p \rightarrow q$.

Note that the Nmatrices in Figure 4 above were obtained by computing truth-tables, and therefore filtered sets of partial valuations. Because of this, some rows were excluded. In particular, if $v(\beta) = 2$ and $v(\alpha \rightarrow \beta) = 1$, we cannot find a

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

	$\neg'^{S4}$
2	{0}
1	{0}
0	{1,2}

	$\Box'^{S4}$
2	{2}
1	{0}
0	{0}

$\rightarrow'^{S4}$	2	1	0
2	{2}	{1}	{0}
1	{2}	{1,2}	{0}
0	{2}	{1,2}	{1,2}

$\vee'^{S4}$	2	1	0
2	{2}	{2}	{2}
1	{2}	{1,2}	{1,2}
0	{2}	{1,2}	{0}

$\wedge'^{S4}$	2	1	0
2	{2}	{1}	{0}
1	{1}	{1}	{0}
0	{0}	{0}	{0}

FIGURE 4. Tables for $\mathcal{M}'_{S4}$.

partial valuation w such that $w(\alpha \rightarrow \beta) = 0$ and $w(\beta) = 2$. Therefore, such a partial valuation cannot be a partial level valuation and we can safely remove it from the definition of implication. This was also discussed in [12, p. 17].

Grätz's results and algorithm can be extended to *S4* defined over Σ, now based on $\mathcal{M}'_{S4}$ (see [22] for details[2]). Then, by Theorem 6.2 and by Corollary 5.8.1 extended to Σ, a formula α in $For(\Omega)$ is valid in *IPL* iff the reduced truth-table obtained by taking the partial valuations in $\mathcal{M}'_{S4}$ for the set of subformulas of the formula $t(\alpha)$ in $For(\Sigma)$ shows that $t(\alpha)$ is a valid formula in *S4*. This produces obviously a decision procedure for *IPL*, based on an RNmatrix for *S4* and a translation between *IPL* and *S4*. It is a challenging question how to define a direct decision procedure for *IPL* based on a finite-valued RNmatrix for *IPL*, with level valuations (and with an associate truth-tables decision procedure) specifically designed for this logic. Moreover, the soundness and completeness of the RNmatrix (and the decision procedure obtained from it) should not rely on the corresponding soundness and completeness results of Gödel's [11] and Grätz's constructions. A solution to this question will be given in the next two sections.

§7. An RNmatrix for *IPL*. This section describes a new semantics for *IPL* based on a 3-valued RNmatrix with a very intuitive interpretation. We obtain this semantics by abstracting the composed procedure for *IPL* mentioned at the end of the previous section but defined in a direct way. That is, we describe a 'pure' RNmatrix for *IPL* (i.e., one that does not depend on any other construction), which turns out to be a suitable semantics for *IPL*. Moreover, Section 8 presents an algorithm for deciding validity in *IPL* by removing spurious rows from the truth-tables generated by the procedure. Hence, the soundness and completeness of the semantics and the associated decision procedure are proven independently of Gödel's 1933 results (proved in [26]) and those in [12].

[2]In [22], a technical report companion to the present article, we provide all the definitions for $\mathcal{M}'_{S4}$, including detailed proofs of Grätz's results.

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

7.1. Inducing the Nmatrix $\mathcal{M}_{IPL}$ for IPL. The truth-tables for intuitionistic logic can be defined based on the Box translation t described above, combined with a semi-translation t_0 to be defined now.

DEFINITION 7.1. The *semi-translation function* $t_0 : For(\Omega) \to For(\Sigma)$ is defined as follows:

i) $t_0(p) = p$, if p is a propositional variable;
ii) $t_0(\neg\alpha) = \neg t(\alpha)$;
iii) $t_0(\alpha \# \beta) = t(\alpha) \# t(\beta)$, where $\# \in \{\to, \vee, \wedge\}$.

LEMMA 7.2. *For every* $\alpha \in For(\Omega)$, $t(\alpha) = \Box(t_0(\alpha))$.

PROOF. Straightforward, by induction on the complexity of α. ⊣

REMARK 7.3. Observe that the interpretation of $\Box$ in $\mathcal{M}'_{S4}$ is deterministic (see Figure 4). Let us denote by $\Box^4(a)$ the only element of $\Box'^{S4}(a)$, for $a \in \mathcal{V}$. Hence, if v is a valuation in $\mathcal{M}'_{S4}$ then $v(\Box\alpha) = \Box^4(v(\alpha))$, for every $\alpha \in For(\Sigma)$. By Lemma 7.2, $v(t(\alpha)) = \Box^4(v(t_0(\alpha)))$, for every $\alpha \in For(\Omega)$.

To check the validity in IPL of $\alpha \in For(\Omega)$, one must construct a truth-table in $\mathcal{M}'_{S4}$ for $t(\alpha) \in For(\Sigma)$. Assume that the set of subformulas of α is arranged as a sequence $\alpha_1, \ldots, \alpha_n = \alpha$ in non-decreasing order of complexity (the complexity of formulas in $For(\Omega)$ is defined as usual, see Definition 6.1 above).

But then, the columns of the truth-table for $t(\alpha)$ in $\mathcal{M}'_{S4}$ can be organized in pairs of formulas of the form $\langle t_0(\alpha_1); t(\alpha_1)\rangle, \ldots, \langle t_0(\alpha_n); t(\alpha_n)\rangle = \langle t_0(\alpha); t(\alpha)\rangle$. Indeed, the sequence of formulas $t_0(\alpha_1), t(\alpha_1), t_0(\alpha_2), \ldots, t_0(\alpha_n), t(\alpha_n) = t(\alpha)$ in $For(\Sigma)$ induced by the sequence of pairs of formulas is the sequence of formulas that can be considered to construct a truth-table for $t(\alpha)$.[3] For instance, in order to check $\alpha = p \vee \neg p$ it is considered the sequence of pairs $\langle p; \Box p\rangle$, $\langle \neg\Box p; \Box\neg\Box p\rangle$, $\langle \Box p \vee \Box\neg\Box p; \Box(\Box p \vee \Box\neg\Box p)\rangle$ based on the sequence $p, \neg p, p \vee \neg p$ for α. Observe that $t(\alpha) = \Box(\Box p \vee \Box\neg\Box p)$.

This suggests considering for IPL an Nmatrix (derived from $\mathcal{M}'_{S4}$) with domain $\{\langle c; \Box^4(c)\rangle \mid c \in \mathcal{V}\} = \{\langle 2;2\rangle, \langle 1;0\rangle, \langle 0;0\rangle\}$ and multioperations defined as follows:

1. $\neg^{IPL}(\langle a;b\rangle) = \{\langle c; \Box^4(c)\rangle \mid c \in \neg'^{S4}(b)\}$
2. $\#^{IPL}(\langle a;b\rangle, \langle c;d\rangle) = \{\langle e; \Box^4(e)\rangle \mid e \in \#'^{S4}(b,d)\}$,

where $\# \in \{\to, \vee, \wedge\}$.

Let us now consider a label for the truth-values above as follows: $\mathbf{T} := \langle 2;2\rangle$, $\mathbf{U} := \langle 1;0\rangle$, and $\mathbf{F} := \langle 0;0\rangle$. The observations above lead us to the following.

DEFINITION 7.4 (Intuitionistic Nmatrix $\mathcal{M}_{IPL}$). The intuitionistic Nmatrix over signature Ω is given by $\mathcal{M}_{IPL} = \langle V_{IPL}, \mathcal{D}_{IPL}, \mathcal{O}_{IPL}\rangle$, where $V_{IPL} = \{\mathbf{T}, \mathbf{U}, \mathbf{F}\}$, $\mathcal{D}_{IPL} = \{\mathbf{T}\}$, and $\mathcal{O}_{IPL}(\#) = \#^{IPL}$ for each connective $\#$ in Ω is defined as in Figure 5.

[3] Rigorously speaking, if α depends on the propositional variables $p_1, \ldots, p_k$ then usually it is considered $\alpha_i = p_i$ (for $1 \le i \le k$), and then $t(\alpha_i) = \Box p_i$ is placed at a latter stage, say α_{i+k}. Hence, the first $2k$ columns of a 'standard' table for $t(\alpha)$ start with formulas $p_1, \ldots, p_k, \Box p_1, \ldots \Box p_k$ instead of $p_1, \Box p_1, p_2, \Box p_2, \ldots p_k, \Box p_k$. After this, the pairs produce the final columns of a table for $t(\alpha)$.

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

	$\neg^{IPL}$
T	{**F**}
U	{**U**, **T**}
F	{**U**, **T**}

$\rightarrow^{IPL}$	**T**	**U**	**F**
T	{**T**}	{**F**}	{**F**}
U	{**T**}	{**U**, **T**}	{**U**, **T**}
F	{**T**}	{**U**, **T**}	{**U**, **T**}

$\vee^{IPL}$	**T**	**U**	**F**
T	{**T**}	{**T**}	{**T**}
U	{**T**}	{**F**}	{**F**}
F	{**T**}	{**F**}	{**F**}

$\wedge^{IPL}$	**T**	**U**	**F**
T	{**T**}	{**F**}	{**F**}
U	{**F**}	{**F**}	{**F**}
F	{**F**}	{**F**}	{**F**}

FIGURE 5. Tables for $\mathcal{M}_{IPL}$.

Observe that, in a given valuation over $\mathcal{M}_{IPL}$, a formula is false when it receives a non-designated value, **U** or **F**. By looking at the tables of the Nmatrix, apparently both values have the same behavior. But it is not the case.

The first thing to note in Figure 5 is that the truth-value **U**, which intuitively means that the epistemic status of α is *unknown*, only appears in the definitions of $\neg^{IPL}$ and $\rightarrow^{IPL}$. As it is going to be more detailed in the completeness proof, the intended meaning is the following (see also Remark 7.15 below):

$v(\neg\alpha)=(\mathbf{U})$: only happens when both $\neg\alpha$ and α are false; that is, when there is no proof of (or knowledge or information about) neither α nor $\neg\alpha$, hence excluded middle fails for α (intuitionistic, non-classical scenario);

$v(\neg\alpha)=(\mathbf{F})$: means that $\neg\alpha$ is false, but α is true; that is, although $\neg\alpha$ is unknown, α is known, therefore excluded middle holds for α (classical scenario);

$v(\alpha \rightarrow \beta)=(\mathbf{U})$: only happens when both α and $\alpha \rightarrow \beta$ are false; that is, there is no proof of (or knowledge or information about) neither α nor $\alpha \rightarrow \beta$, hence Peirce's law $\alpha \vee (\alpha \rightarrow \beta)$ fails for α and $\alpha \rightarrow \beta$ (intuitionistic, non-classical scenario);

$v(\alpha \rightarrow \beta)=(\mathbf{F})$: means that $\alpha \rightarrow \beta$ is false, but α is true; that is, although $\alpha \rightarrow \beta$ is unknown, α is known, therefore Peirce's law holds for α and $\alpha \rightarrow \beta$ (classical scenario).

In turn, the behavior of conjunction and disjunction is 'classical' (in terms of designated/non-designated values), and that is to be expected: the inference rules for conjunction and disjunction are identical in classical and intuitionistic logic. Because of this, **U** never appears as a value assigned to a conjunction or a disjunction.

The third point to note is that, if we disregard the truth-value **U**, the remaining Nmatrix corresponds exactly to classical logic. In fact, it is straightforward to see how classical logic can be fully recovered from the intuitionistic Nmatrix. This recovery can be achieved by refining $\mathcal{M}_{IPL}$ (in the sense of [3, 4]) to the truth-values

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

T, **F**. The resulting is an Nmatrix with zero degree of non-determinism, equivalent to the standard (deterministic) matrix for classical logic.

7.2. Level valuations for $\mathcal{M}_{IPL}$. The following step is the definition of a suitable set of valuations (level valuations) over the Nmatrix for *IPL* just defined. The construction of the sets of level valuations given by Kearns and Grätz is based on the idea of *super designated truth-values*, that is: designated values a such that any element of $\Box(a)$ is still designated, where $\Box$ is the multioperator associated with $\Box$. Thus, if a formula gets a designated value for every nth-level valuation, it must receive a super-designated value in every $(n+1)$th-level valuation. This guarantees the validity of the *Necessitation* inference rule, level by level. In this way, if a formula is provable in *S4* with n applications of the necessitation rule, then it is valid in $\mathcal{L}_n^{S4}$. From this, soundness of syntactical proofs in *S4* w.r.t. the RNmatrix semantics follows easily. *Necessitation* is usually called a *global* inference rule: in order to be sound, the hypothesis of the rule must be provable, not merely assumed. However, *IPL* has no global inference rules in its usual Hilbert-style presentation (being *Modus Ponens* the only inference rule, which is *local* in opposition to global).

Taking into account that we are abstracting the composition of the Box translation with Grätz's truth-tables algorithm for *S4*, the set of nth-level valuations will be constructed based on partial level valuations (Definition 5.6) instead of the Kearns-like Definition 5.2 of level valuations. Thus, the requirement that every valuation (row) v_p such $v_p(\alpha) = 1$ must be supported by another valuation (row) w_p with $w_p(\alpha) = 0$ while preserving the values 2 can be easily adapted to the present framework, by changing 1, 0, and 2 by **U**, **F**, and **T**, respectively (see Definition 7.6 below).

However, as observed above, there is no counterpart to *Necessitation* rule in the usual Hilbert-style calculus for *IPL*, therefore the level valuations seem, in principle, unrelated to syntactical proofs in *IPL*. The key is to consider a standard sequent calculus presentation for *IPL*, and so it can be established, in a similar way to the case of *S4*, a relationship between syntactical proofs in *IPL* and *nth*-level valuations: if a sequent is proved in *IPL* with a derivation of height n then it is valid w.r.t. *nth*-level valuations for *IPL* (see Proposition 7.11). Using this, the soundness of the sequent calculus for *IPL* w.r.t. its RNmatrix semantics can be easily obtained.

In addition, intuitionistic level valuations have a very interesting interpretation. Indeed, a suitable and very natural 3-valued characteristic function associated with φ-saturated sets in *IPL* will be introduced in Definition 7.14. Hence, in Proposition 7.17 it will be proven that these characteristic functions correspond to level valuations.

The relationship between level valuations and 3-valued characteristic functions of φ-saturated sets also has an interesting consequence in terms of complexity: at level 0 (and hence at any level) the valuations can only assign the classical truth-values **T** and **F** to propositional variables. Indeed, atomic formulas contain neither negation nor implication connectives, so it makes no sense to assign them the value **U**, which, as discussed above, is exclusively related to the failure of the law of excluded middle and Peirce's law. This feature will reduce the complexity of the algorithm, as we will discuss in Section 9.

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

DEFINITION 7.5. Let $Val^{nd}(\mathcal{M}_{IPL})$ be the set of valuations over $\mathcal{M}_{IPL}$, and let $Val^{nd}_{+}(\mathcal{M}_{IPL})$ be the set of valuations v in $Val^{nd}(\mathcal{M}_{IPL})$ such that $v(p) \in \{\mathbf{T}, \mathbf{F}\}$ for every propositional variable p.

DEFINITION 7.6. (Level valuations for IPL) The set of nth-level valuations $\mathcal{L}^{IPL}_n$ for IPL, where $n \in \omega$, is defined as follows:

i) $\mathcal{L}^{IPL}_0 = Val^{nd}_{+}(\mathcal{M}_{IPL})$;
ii) $\mathcal{L}^{IPL}_{n+1} = \{v \in \mathcal{L}^{IPL}_n \mid \forall \alpha \in For(\Omega)$, if $v(\alpha) = \mathbf{U}$ then there exists $w \in \mathcal{L}^{IPL}_n$ such that $w(\alpha) = \mathbf{F}$ and, for every $\beta \in For(\Omega)$: $v(\beta) = \mathbf{T}$ implies $w(\beta) = \mathbf{T}\}$.

The set of level valuations in $\mathcal{M}_{IPL}$ is defined as the intersection of the sets of nth-level valuations, for $n \geq 0$:

$$\mathcal{L}_{IPL} = \bigcap_{n \geq 0}^{\infty} \mathcal{L}^{IPL}_n. \tag{3}$$

DEFINITION 7.7 (RNmatrix for IPL). Let $\mathcal{M}_{IPL}$ be the Nmatrix defined in Definition 7.4. The RNmatrix for IPL is the pair $\mathcal{R}(\mathcal{M}_{IPL}) = \langle \mathcal{M}_{IPL}, \mathcal{L}_{IPL} \rangle$.

7.3. Soundness and completeness of IPL w.r.t. $\mathcal{R}(\mathcal{M}_{IPL})$. From now on, **LI** will stand for a standard sequent calculus for IPL over Ω. In fact, the sequent calculus **LI** presented here is obtained from Gentzen's system **LJ** (see, for instance, [31, Chapter 1]) with minor modifications. Concretely, we consider sequents formed by pairs of finite multisets of formulas instead of pairs of sequences (which allows to eliminate the *exchange* rule), and by adding contexts to the axiom (which allows to eliminate the *left weakening* rule). Certain rules can also be simplified in this setting. Clearly, both calculi are equivalent.

By a *sequent* we mean an expression $\Gamma \Rightarrow \Delta$ such that Γ and Δ are (non simultaneously empty) finite multisets, and Δ contains at most one formula. The calculus **LI** is given by the following axiom and inference rules:

Axiom

$$(Ax)\ \alpha, \Gamma \Rightarrow \alpha.$$

Structural rules

$$(contr \Rightarrow) \frac{\alpha, \alpha, \Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow w) \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}$$

$$(cut) \frac{\Gamma \Rightarrow \alpha \quad \alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}.$$

Logical rules

$$(\vee \Rightarrow) \frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \vee_1) \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} \qquad (\Rightarrow \vee_2) \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}$$

$$(\wedge \Rightarrow) \frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow \wedge) \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}$$

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

$$(\rightarrow\Rightarrow)\ \frac{\Gamma \Rightarrow \alpha \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \rightarrow \beta, \Gamma \Rightarrow \Delta} \qquad (\Rightarrow\rightarrow)\ \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \rightarrow \beta}$$

$$(\neg\Rightarrow)\ \frac{\Gamma \Rightarrow \alpha}{\neg\alpha, \Gamma \Rightarrow} \qquad (\Rightarrow\neg)\ \frac{\alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg\alpha}.$$

The consequence relation of **LI** will be denoted by $\vdash_{IPL}$. Hence, for any set $\Gamma \cup \{\alpha\} \subseteq For(\Omega)$, $\Gamma \vdash_{IPL} \alpha$ iff there exists a finite set $\Gamma_0 \subseteq \Gamma$ such that the sequent $\Gamma_0 \Rightarrow \alpha$ is derivable in **LI**. The logic generated by **LI**, with consequence relation $\vdash_{IPL}$, is precisely IPL.

REMARK 7.8. Being a presentation of IPL, the DM holds for the consequence relation $\vdash_{IPL}$ associated with **LI**:

$$(DM)\ \ \Gamma, \alpha \vdash_{IPL} \beta \ \text{ iff } \ \Gamma \vdash_{IPL} \alpha \rightarrow \beta$$

for every set of formulas $\Gamma \cup \{\alpha, \beta\}$. The validity of (DM) can be checked directly by using (Ax) and the rules for implication. Indeed, the 'only if' part follows by $(\Rightarrow\rightarrow)$. For the 'if' part, it can be observed that $\Gamma_0, \alpha, \alpha \rightarrow \beta \Rightarrow \beta$ is provable in **LI** for every finite multiset Γ_0 and formulas α and β: this is a consequence of applying $(\rightarrow\Rightarrow)$ to the axioms $\Gamma_0, \alpha \Rightarrow \alpha$ and $\Gamma_0, \alpha, \beta \Rightarrow \beta$. From this, the result follows from the hypothesis $\Gamma_0 \Rightarrow \alpha \rightarrow \beta$ (for a finite set $\Gamma_0 \subseteq \Gamma$) and (cut).

DEFINITION 7.9. Let Π be a derivation of a sequent $\Gamma \Rightarrow \Delta$ in **LI** (that is, Π is a tree of sequents with root $\Gamma \Rightarrow \Delta$ and axioms as leaves). The *depth* of Π is its height (as a tree). If a sequent $\Gamma \Rightarrow \Delta$ admits a derivation in **LI** of depth k (for $k \geq 0$), we write $\Gamma \vdash^{k}_{IPL} \Delta$.

The proof of soundness of **LI** w.r.t. the RNmatrix $\mathcal{R}(\mathcal{M}_{IPL})$ requires to establish, in a precise way, soundness of derivations of depth k w.r.t. valuations of level k.

DEFINITION 7.10. Let $k \geq 0$ and $v \in \mathcal{L}^{IPL}_k$. We say that v *satisfies a formula* α, written as $v \models \alpha$, if $v(\alpha) = \mathbf{T}$. If Γ is a non-empty (multi)set of formulas, we say that v *satisfies* Γ, denoted by $v \models \Gamma$, if $v \models \gamma$ for every $\gamma \in \Gamma$. We say that v satisfies a sequent $\Gamma \Rightarrow \Delta$, denoted by $v \models \Gamma \Rightarrow \Delta$, if the following holds:

(1) $\Delta = \emptyset$. In this case, $v \not\models \Gamma$, that is, $v(\beta) \in \{\mathbf{U}, \mathbf{F}\}$ for some $\beta \in \Gamma$; or
(2) $\Delta = \{\gamma\}$, for some formula γ. In this case, $v \models \gamma$ whenever $v \models \Gamma$.

A sequent $\Gamma \Rightarrow \Delta$ is said to be *valid* in $\mathcal{L}^{IPL}_k$, denoted by $\Gamma \models_k \Delta$, if it is satisfied by any $v \in \mathcal{L}^{IPL}_k$.

PROPOSITION 7.11. *For every* $k \geq 0$ *and every sequent* $\Gamma \Rightarrow \Delta$: *if* $\Gamma \vdash^{k}_{IPL} \Delta$, *then* $\Gamma \models_k \Delta$.

PROOF. By induction on k.

Base $k = 0$: Suppose that $\Gamma \vdash^{0}_{IPL} \Delta$. Then $\Gamma \Rightarrow \Delta$ is an instance of the axiom (Ax), that is: $\Gamma \Rightarrow \Delta = \alpha, \Gamma' \Rightarrow \alpha$ for some multiset Γ'. Let $v \in \mathcal{L}^{IPL}_0$ such that $v \models \Gamma' \cup \{\alpha\}$. In particular, $v \models \alpha$, hence $v \models \Gamma \Rightarrow \Delta$. From this, $\Gamma \models_0 \Delta$.

Inductive step: Suppose that, for any sequent $\Gamma \Rightarrow \Delta$, if $\Gamma \vdash^{j}_{IPL} \Delta$ then $\Gamma \models_j \Delta$, for every $j \leq k$, for a given $k \geq 0$ (induction hypothesis (IH)). Let $\Gamma \Rightarrow \Delta$ be a sequent such that $\Gamma \vdash^{k+1}_{IPL} \Delta$. If $\Gamma \Rightarrow \Delta$ is an instance of the axiom, it can be proven as

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

above that $\Gamma \models_{k+1} \Delta$. Suppose now that $\Gamma \Rightarrow \Delta$ was obtained from other sequents by applying a rule of **LI** in a derivation of depth $k+1$. We have the following cases:

(*contr* $\Rightarrow$): Suppose that $\Gamma \Rightarrow \Delta = \alpha, \Gamma' \Rightarrow \Delta$ is obtained from $\alpha, \alpha, \Gamma' \Rightarrow \Delta$ by applying rule (*contr* $\Rightarrow$). Then $\alpha, \alpha, \Gamma' \vdash^{k}_{IPL} \Delta$ and so $\alpha, \alpha, \Gamma' \models_k \Delta$, by (IH). But the latter clearly implies that $\alpha, \Gamma' \models_k \Delta$ and so $\alpha, \Gamma' \models_{k+1} \Delta$, given that $\mathcal{L}^{IPL}_{k+1} \subseteq \mathcal{L}^{IPL}_{k}$.

($\Rightarrow w$): Suppose that $\Gamma \Rightarrow \Delta = \Gamma \Rightarrow \alpha$ is obtained from $\Gamma \Rightarrow$ by ($\Rightarrow w$). Then $\Gamma \vdash^{k}_{IPL}$ and so, by (IH), $\Gamma \models_k$. This means that, for every $v \in \mathcal{L}^{IPL}_{k}$, $v \not\models \Gamma$ and so $v \models \Gamma \Rightarrow \alpha$. From this it follows that $\Gamma \models_{k+1} \Delta$, given that $\mathcal{L}^{IPL}_{k+1} \subseteq \mathcal{L}^{IPL}_{k}$.

(*cut*): Assume that $\Gamma \Rightarrow \Delta$ follows from $\Gamma \Rightarrow \alpha$ and $\alpha, \Gamma \Rightarrow \Delta$ by (*cut*) rule. Then $\Gamma \vdash^{k}_{IPL} \alpha$ and $\alpha, \Gamma \vdash^{k}_{IPL} \Delta$ and so, by (IH), $\Gamma \models_k \alpha$ and $\alpha, \Gamma \models_k \Delta$. Suppose first that $\Delta = \emptyset$. Let $v \in \mathcal{L}^{IPL}_{k+1}$ such that $v \models \Gamma$; in particular, $v \in \mathcal{L}^{IPL}_{k}$ and so $v \models \Gamma \Rightarrow \alpha$. From this it follows that $v \models \alpha$. But this is impossible, given that $v \models \Gamma \cup \{\alpha\} \Rightarrow$, by (IH). From this, $v \not\models \Gamma$ for every $v \in \mathcal{L}^{IPL}_{k+1}$. That is, $v \models \Gamma \Rightarrow$, for every $v \in \mathcal{L}^{IPL}_{k+1}$. Now, suppose that $\Delta = \{\gamma\}$ for some formula γ, and let $v \in \mathcal{L}^{IPL}_{k+1}$ such that $v \models \Gamma$. Once again, since $v \models \Gamma \Rightarrow \alpha$ we infer that $v \models \alpha$. But then $v \models \Gamma \cup \{\alpha\}$ and so $v \models \gamma$, given that $v \models \Gamma \cup \{\alpha\} \Rightarrow \gamma$. This means that $v \models \Gamma \Rightarrow \Delta$, for every $v \in \mathcal{L}^{IPL}_{k+1}$. That is, $\Gamma \models_{k+1} \Delta$.

($\vee \Rightarrow$): Assume that $\Gamma \Rightarrow \Delta = \alpha \vee \beta, \Gamma' \Rightarrow \Delta$ follows from $\alpha, \Gamma' \Rightarrow \Delta$ and $\beta, \Gamma' \Rightarrow \Delta$ by ($\vee \Rightarrow$). By (IH), $\alpha, \Gamma' \models_k \Delta$ and $\beta, \Gamma' \models_k \Delta$. Suppose first that $\Delta = \emptyset$. By hypothesis, if $v \in \mathcal{L}^{IPL}_{k+1}$ is such that $v \models \Gamma'$, then $v \not\models \alpha$ and $v \not\models \beta$, and so $v \not\models \alpha \vee \beta$. This means that $v \models \Gamma' \cup \{\alpha \vee \beta\} \Rightarrow$, for every $v \in \mathcal{L}^{IPL}_{k+1}$. Now, if $\Delta = \{\gamma\}$ for some formula γ, let $v \in \mathcal{L}^{IPL}_{k+1}$. By hypothesis, if $v \models \Gamma'$ and $v \models \alpha$, then $v \models \gamma$; and if $v \models \Gamma'$ and $v \models \beta$, then $v \models \gamma$. From this, if $v \models \Gamma'$ and $v \models \alpha \vee \beta$ then, by definition of $\mathcal{M}_{IPL}$, either $v \models \alpha$ or $v \models \beta$ and so, by hypothesis, $v \models \gamma$. That is, $v \models \Gamma \Rightarrow \Delta$ for every $v \in \mathcal{L}^{IPL}_{k+1}$ and so $\Gamma \models_{k+1} \Delta$.

($\Rightarrow \vee_i$): It is enough to observe that, if either $v \models \alpha$ or $v \models \beta$ then $v \models \alpha \vee \beta$, by definition of $\mathcal{M}_{IPL}$.

($\wedge \Rightarrow$): Suppose that $\Gamma \Rightarrow \Delta = \alpha \wedge \beta, \Gamma' \Rightarrow \Delta$ follows from $\alpha, \beta, \Gamma' \Rightarrow \Delta$ by ($\wedge \Rightarrow$). Assume first that $\Delta = \emptyset$. Then, if $v \in \mathcal{L}^{IPL}_{k+1}$ such that $v \models \Gamma'$ it follows that $v \not\models \{\alpha, \beta\}$, since $\mathcal{L}^{IPL}_{k+1} \subseteq \mathcal{L}^{IPL}_{k}$ and $\alpha, \beta, \Gamma' \models_k \Delta$ by (IH). From this, either $v \not\models \alpha$ or $v \not\models \beta$. By definition of $\mathcal{M}_{IPL}$, $v \not\models \alpha \wedge \beta$. That is, $v \not\models \Gamma' \cup \{\alpha \wedge \beta\}$ for every $v \in \mathcal{L}^{IPL}_{k+1}$, i.e., $\alpha \wedge \beta, \Gamma' \models_{k+1}$. In turn, if $\Delta = \{\gamma\}$ for some formula γ, let $v \in \mathcal{L}^{IPL}_{k+1}$ such that $v \models \Gamma' \cup \{\alpha \wedge \beta\}$. By definition of $\mathcal{M}_{IPL}$, $v \models \Gamma' \cup \{\alpha, \beta\}$ and so $v \models \gamma$, by (IH) and the fact that $v \in \mathcal{L}^{IPL}_{k}$. That is, $v \models \Gamma' \cup \{\alpha \wedge \beta\} \Rightarrow \Delta$ for every $v \in \mathcal{L}^{IPL}_{k+1}$.

($\Rightarrow \wedge$): It is enough to observe that, if both $v \models \alpha$ and $v \models \beta$ then $v \models \alpha \wedge \beta$, by definition of $\mathcal{M}_{IPL}$.

($\rightarrow\Rightarrow$) Suppose that $\Gamma \Rightarrow \Delta = \alpha \rightarrow \beta, \Gamma' \Rightarrow \Delta$ is obtained from $\Gamma' \Rightarrow \alpha$ and $\beta, \Gamma' \Rightarrow \Delta$ by ($\rightarrow\Rightarrow$). By (IH), $\Gamma' \models_k \alpha$ and $\beta, \Gamma' \models_k \Delta$. Assume first that $\Delta = \emptyset$. Let $v \in \mathcal{L}^{IPL}_{k+1}$ such that $v \models \Gamma' \cup \{\alpha \rightarrow \beta\}$. Then, $v \in \mathcal{L}^{IPL}_{k}$ such that $v \models \Gamma'$ and so $v \models \alpha$, by hypothesis. Since $v \models \alpha \rightarrow \beta$ then $v \models \beta$, by definition of $\mathcal{M}_{IPL}$. That is, $v \in \mathcal{L}^{IPL}_{k}$ is such that $v \models \Gamma' \cup \{\beta\}$, which contradicts the fact that $\beta, \Gamma' \models_k$. From this, $v \not\models \Gamma' \cup \{\alpha \rightarrow \beta\}$ for every $v \in \mathcal{L}^{IPL}_{k+1}$, which means that $\alpha \rightarrow \beta, \Gamma' \models_{k+1}$. Now, suppose that $\Delta = \{\gamma\}$ for some formula γ. Let $v \in \mathcal{L}^{IPL}_{k+1}$ such that $v \models \Gamma' \cup \{\alpha \rightarrow \beta\}$. By reasoning as above, $v \in \mathcal{L}^{IPL}_{k}$ is such that $v \models \Gamma' \cup \{\beta\}$.

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

But then $v \models \gamma$, by hypothesis. This implies that $v \models \Gamma' \cup \{\alpha \to \beta\} \Rightarrow \gamma$ for every $v \in \mathcal{L}^{IPL}_{k+1}$.

$(\Rightarrow\to)$: Assume that $\Gamma \Rightarrow \Delta = \Gamma \Rightarrow \alpha \to \beta$ is obtained from $\alpha, \Gamma \Rightarrow \beta$ by $(\Rightarrow\to)$. By (IH), $\alpha, \Gamma \models_k \beta$. Let $v \in \mathcal{L}^{IPL}_{k+1}$ such that $v \models \Gamma$. Suppose that $v(\alpha \to \beta) = \mathbf{F}$. By definition of $\mathcal{M}_{IPL}$, $v(\alpha) = \mathbf{T}$ and $v(\beta) \in \{\mathbf{U}, \mathbf{F}\}$. That is, $v \models \Gamma \cup \{\alpha\}$ but $v \not\models \beta$, which contradicts the fact that $\alpha, \Gamma \models_k \beta$, given that $v \in \mathcal{L}^{IPL}_k$. Hence, $v(\alpha \to \beta) \neq \mathbf{F}$. Suppose now that $v(\alpha \to \beta) = \mathbf{U}$. By Definition 7.6 of level valuations, there exists $w \in \mathcal{L}^{IPL}_k$ such that $w(\alpha \to \beta) = \mathbf{F}$ and $w \models \Gamma$. By reasoning as above, $w(\alpha) = \mathbf{T}$ and $w(\beta) \in \{\mathbf{U}, \mathbf{F}\}$. But then $w \models \Gamma \cup \{\alpha\}$ and $w \not\models \beta$, which contradicts the fact that $\alpha, \Gamma \models_k \beta$. From this, it follows that $v(\alpha \to \beta) = \mathbf{T}$. That is, $v \models \Gamma \Rightarrow \alpha \to \beta$ for every $v \in \mathcal{L}^{IPL}_{k+1}$.

$(\neg\Rightarrow)$: Assume that $\Gamma \Rightarrow \Delta = \neg\alpha, \Gamma' \Rightarrow$ is obtained from $\Gamma' \Rightarrow \alpha$ by $(\neg\Rightarrow)$. By (IH), $\Gamma' \models_k \alpha$. Let $v \in \mathcal{L}^{IPL}_{k+1}$ such that $v \models \Gamma' \cup \{\neg\alpha\}$. Then, $v \in \mathcal{L}^{IPL}_k$ such that $v \models \Gamma'$ and so $v \models \alpha$, by hypothesis, which contradicts the fact that $v \models \neg\alpha$. From this, $v \not\models \Gamma' \cup \{\neg\alpha\}$ for every $v \in \mathcal{L}^{IPL}_{k+1}$, which means that $\neg\alpha, \Gamma' \models_{k+1}$.

$(\Rightarrow\neg)$: Assume that $\Gamma \Rightarrow \Delta = \Gamma \Rightarrow \neg\alpha$ is obtained from $\alpha, \Gamma \Rightarrow$ by $(\Rightarrow\neg)$. By (IH), $\alpha, \Gamma \models_k$. Let $v \in \mathcal{L}^{IPL}_{k+1}$ such that $v \models \Gamma$. Suppose that $v(\neg\alpha) = \mathbf{F}$. By definition of $\mathcal{M}_{IPL}$, $v(\alpha) = \mathbf{T}$, hence $v \models \Gamma \cup \{\alpha\}$, which contradicts the fact that $\alpha, \Gamma \models_k$, since $v \in \mathcal{L}_k$. Now, suppose that $v(\neg\alpha) = \mathbf{U}$. By Definition 7.6 of level valuations, there exists $w \in \mathcal{L}^{IPL}_k$ such that $w(\neg\alpha) = \mathbf{F}$ and $w \models \Gamma$. As proven above, $w(\alpha) = \mathbf{T}$ and so $w \models \Gamma \cup \{\alpha\}$, contradicting the fact that $\alpha, \Gamma \models_k$. From this, it follows that $v(\neg\alpha) = \mathbf{T}$. That is, $v \models \Gamma \Rightarrow \neg\alpha$ for every $v \in \mathcal{L}^{IPL}_{k+1}$.

This concludes the proof. ⊣

THEOREM 7.12 (Soundness of **LI** w.r.t. $\mathcal{R}(\mathcal{M}_{IPL})$). *For every set* $\Gamma \cup \{\varphi\}$ *of formulas over* Ω: $\Gamma \vdash_{IPL} \varphi$ *implies that* $\Gamma \models_{\mathcal{R}(\mathcal{M}_{IPL})} \varphi$.

PROOF. Suppose that $\Gamma \vdash_{IPL} \varphi$. By definition, there exists a finite set $\Gamma_0 \subseteq \Gamma$ such that the sequent $\Gamma_0 \Rightarrow \varphi$ is derivable in **LI**. But then, there exists $k \geq 0$ such that $\Gamma_0 \vdash^k_{IPL} \varphi$. By Proposition 7.11, $\Gamma_0 \models_k \varphi$. This implies that $\Gamma \models_{\mathcal{R}(\mathcal{M}_{IPL})} \varphi$, given that $\mathcal{L}_{IPL} \subseteq \mathcal{L}^{IPL}_k$. ⊣

We will concentrate now on the completeness proof of **LI** w.r.t. $\mathcal{R}(\mathcal{M}_{IPL})$.

REMARK 7.13. Recall that, given a Tarskian and finitary logic **L** and a formula φ, a set Δ of formulas is said to be *φ-saturated in* ***L*** if $\Delta \nvdash_{\mathbf{L}} \varphi$, but $\Delta, \alpha \vdash_{\mathbf{L}} \varphi$, whenever $\alpha \notin \Delta$. It is a well-known result that, if $\Gamma \nvdash_{\mathbf{L}} \varphi$, there exists a φ-saturated set Δ in **L** such that $\Gamma \subseteq \Delta$. If Δ is φ-saturated in **L** then it is closed, that is: $\Delta \vdash_{\mathbf{L}} \alpha$ iff $\alpha \in \Delta$. In particular, this holds in the logic generated by **LI** (that is, *IPL*). Moreover, it is easy to prove that, if Δ is a φ-saturated set in *IPL* then: $\alpha \in \Delta$ implies that $\neg\alpha \notin \Delta$; $\alpha \vee \beta \in \Delta$ iff $\alpha \in \Delta$ or $\beta \in \Delta$; $\alpha \wedge \beta \in \Delta$ iff $\alpha \in \Delta$ and $\beta \in \Delta$; $\alpha \to \beta \in \Delta$ and $\alpha \in \Delta$ implies that $\beta \in \Delta$; and $\beta \in \Delta$ implies that $\alpha \to \beta \in \Delta$.

DEFINITION 7.14 (Valuation associated with a φ-saturated set in *IPL*). Let $\Delta \subseteq For(\Omega)$ be a φ-saturated set in the logic generated by **LI** (that is, IPL). The *valuation associated with* Δ *in* IPL is the function $v_\Delta : For(\Omega) \to V_{IPL}$ defined inductively as follows (here, p is a propositional variable and $\alpha, \beta \in For(\Omega)$):

$$v_\Delta(p) = \begin{cases} \mathbf{T} & if\ p \in \Delta, \\ \mathbf{F} & if\ p \notin \Delta; \end{cases}$$

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

$$v_\Delta(\neg\alpha) = \begin{cases} \mathbf{T} & if\ \neg\alpha \in \Delta, \\ \mathbf{U} & if\ \neg\alpha \notin \Delta\ and\ \alpha \notin \Delta, \\ \mathbf{F} & if\ \neg\alpha \notin \Delta\ and\ \alpha \in \Delta; \end{cases}$$

$$v_\Delta(\alpha \to \beta) = \begin{cases} \mathbf{T} & if\ \alpha \to \beta \in \Delta, \\ \mathbf{U} & if\ \alpha \to \beta \notin \Delta\ and\ \alpha \notin \Delta, \\ \mathbf{F} & if\ \alpha \to \beta \notin \Delta\ and\ \alpha \in \Delta; \end{cases}$$

$$v_\Delta(\alpha \wedge \beta) = \begin{cases} \mathbf{T} & if\ \alpha \in \Delta\ and\ \beta \in \Delta, \\ \mathbf{F} & if\ \alpha \notin \Delta\ or\ \beta \notin \Delta; \end{cases}$$

$$v_\Delta(\alpha \vee \beta) = \begin{cases} \mathbf{T} & if\ \alpha \in \Delta\ or\ \beta \in \Delta, \\ \mathbf{F} & if\ \alpha \notin \Delta\ and\ \beta \notin \Delta. \end{cases}$$

REMARK 7.15. The observations about the meaning of **U** made after Definition 7.4 can be recast now. Observe that $v_\Delta(\neg\alpha) = \mathbf{U}$ whenever α is unknown in Δ, that is: $\neg\alpha \notin \Delta$ and $\alpha \notin \Delta$. Hence, the failure of excluded middle $\alpha \vee \neg\alpha$ (a principle valid in classical logic but not in *IPL*) occurs only for formulas α with such undetermined status in Δ. Analogously, $v_\Delta(\alpha \to \beta) = \mathbf{U}$ whenever α and $\alpha \to \beta$ are jointly unknown in Δ, that is: neither α nor $\alpha \to \beta$ belong to Δ. Hence, the failure of Peirce's law $\alpha \vee (\alpha \to \beta)$ (a principle valid in classical logic but not in *IPL*) occurs only for formulas α and $\alpha \to \beta$ with such joint undetermined status in Δ. The function v_Δ can be seen as a 3-valued characteristic map associated with Δ, in which the third value **U** reflects the undetermined status of a sentence that does not belong to Δ. Only sentences of the form $\neg\alpha$ or $\alpha \to \beta$ can be unknown or undetermined (in this sense) w.r.t. a φ-saturated set in *IPL*.

PROPOSITION 7.16. *Let $\Delta \subseteq For(\Omega)$ be a φ-saturated set in the logic generated by **LI**. Then, the valuation v_Δ associated with Δ in IPL is a valuation in the Nmatrix $\mathcal{M}_{IPL}$ which belongs to $Val^{nd}_{+}(\mathcal{M}_{IPL})$ (recall Definition 7.5) such that, for every α: $v_\Delta(\alpha) = \mathbf{T}$ iff $\alpha \in \Delta$.*

PROOF. Observe that, by definition and by Remark 7.13, $v_\Delta(\alpha) = \mathbf{T}$ iff $\alpha \in \Delta$, for every formula α. Moreover, $v_\Delta(p) \in \{\mathbf{T}, \mathbf{F}\}$ for every propositional variable p.

Negation: Let us prove first that $v_\Delta(\neg\alpha) \in \neg^{IPL}(v_\Delta(\alpha))$ for every α. If $v_\Delta(\neg\alpha) = \mathbf{T}$ then $\neg\alpha \in \Delta$, by definition. Hence, $\alpha \notin \Delta$ and so $v_\Delta(\alpha) \in \{\mathbf{U}, \mathbf{F}\}$. From this, $v_\Delta(\neg\alpha) = \mathbf{T} \in \{\mathbf{U}, \mathbf{T}\} = \neg^{IPL}(v_\Delta(\alpha))$. Now, if $v_\Delta(\neg\alpha) = \mathbf{U}$ then $\alpha \notin \Delta$, by definition. Hence $v_\Delta(\alpha) \in \{\mathbf{U}, \mathbf{F}\}$ and so $v_\Delta(\neg\alpha) = \mathbf{U} \in \{\mathbf{U}, \mathbf{T}\} = \neg^{IPL}(v_\Delta(\alpha))$. Finally, if $v_\Delta(\neg\alpha) = \mathbf{F}$ then $\alpha \in \Delta$, by definition. Hence $v_\Delta(\alpha) = \mathbf{T}$ and so $v_\Delta(\neg\alpha) = \mathbf{F} \in \{\mathbf{F}\} = \neg^{IPL}(v_\Delta(\alpha))$.

Implication: There are three cases:

(I1) Suppose that $v_\Delta(\alpha \to \beta) = \mathbf{T}$. Then $\alpha \to \beta \in \Delta$, by definition.

– If $\alpha \notin \Delta$ then $v_\Delta(\alpha) \in \{\mathbf{U}, \mathbf{F}\}$. Hence, $v_\Delta(\alpha \to \beta) = \mathbf{T} \in \to^{IPL}(v_\Delta(\alpha), v_\Delta(\beta))$.

– If $\alpha \in \Delta$ then $\beta \in \Delta$, by (DM). Hence $v_\Delta(\alpha) = v_\Delta(\beta) = \mathbf{T}$ and so $v_\Delta(\alpha \to \beta) = \mathbf{T} \in \{\mathbf{T}\} = \to^{IPL}(v_\Delta(\alpha), v_\Delta(\beta))$.

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

(I2) Suppose that $v_\Delta(\alpha \to \beta) = \mathbf{U}$. Then $\alpha \to \beta \notin \Delta$ and $\alpha \notin \Delta$, by definition. Moreover, $\beta \notin \Delta$, by Remark 7.13. Then $v_\Delta(\alpha) \in \{\mathbf{U}, \mathbf{F}\}$ and $v_\Delta(\beta) \in \{\mathbf{U}, \mathbf{F}\}$. Hence, $v_\Delta(\alpha \to \beta) = \mathbf{U} \in \{\mathbf{U}, \mathbf{T}\} = \to^{IPL}(v_\Delta(\alpha), v_\Delta(\beta))$.

(I3) Suppose that $v_\Delta(\alpha \to \beta) = \mathbf{F}$. Then $\alpha \to \beta \notin \Delta$ and $\alpha \in \Delta$, by definition. Moreover, $\beta \notin \Delta$, by Remark 7.13. Then $v_\Delta(\alpha) = \mathbf{T}$ and $v_\Delta(\beta) \in \{\mathbf{U}, \mathbf{F}\}$. Hence, $v_\Delta(\alpha \to \beta) = \mathbf{F} \in \{\mathbf{F}\} = \to^{IPL}(v_\Delta(\alpha), v_\Delta(\beta))$.

Conjunction: There are two cases:

(C1) Suppose that $v_\Delta(\alpha \wedge \beta) = \mathbf{T}$. Then $\alpha \in \Delta$ and $\beta \in \Delta$, by definition. From this, $v_\Delta(\alpha) = v_\Delta(\beta) = \mathbf{T}$. Hence, $v_\Delta(\alpha \wedge \beta) = \mathbf{T} \in \{\mathbf{T}\} = \wedge^{IPL}(v_\Delta(\alpha), v_\Delta(\beta))$.

(C2) Suppose that $v_\Delta(\alpha \wedge \beta) = \mathbf{F}$. Then, either $\alpha \notin \Delta$ or $\beta \notin \Delta$, by definition. From this, $v_\Delta(\alpha) \in \{\mathbf{U}, \mathbf{F}\}$ or $v_\Delta(\beta) \in \{\mathbf{U}, \mathbf{F}\}$. Hence, $v_\Delta(\alpha \wedge \beta) = \mathbf{F} \in \{\mathbf{F}\} = \wedge^{IPL}(v_\Delta(\alpha), v_\Delta(\beta))$.

Disjunction: There are two cases:

(D1) Suppose that $v_\Delta(\alpha \vee \beta) = \mathbf{T}$. Then, either $\alpha \in \Delta$ or $\beta \in \Delta$, by definition. From this, $v_\Delta(\alpha) = \mathbf{T}$ or $v_\Delta(\beta) = \mathbf{T}$. Hence, $v_\Delta(\alpha \vee \beta) = \mathbf{T} \in \{\mathbf{T}\} = \vee^{IPL}(v_\Delta(\alpha), v_\Delta(\beta))$.

(D2) Suppose that $v_\Delta(\alpha \vee \beta) = \mathbf{F}$. Then $\alpha \notin \Delta$ and $\beta \notin \Delta$, by definition. From this, $v_\Delta(\alpha) \in \{\mathbf{U}, \mathbf{F}\}$ and $v_\Delta(\beta) \in \{\mathbf{U}, \mathbf{F}\}$. Hence, $v_\Delta(\alpha \wedge \beta) = \mathbf{F} \in \{\mathbf{F}\} = \vee^{IPL}(v_\Delta(\alpha), v_\Delta(\beta))$.

This concludes the proof. ⊣

PROPOSITION 7.17. *Let $\Delta \subseteq For(\Omega)$ be a φ-saturated set in the logic generated by **LI**. Then, the valuation v_Δ associated with Δ in IPL is a level valuation in $\mathcal{M}_{IPL}$ such that, for every α: $v_\Delta(\alpha) = \mathbf{T}$ iff $\alpha \in \Delta$.*

PROOF. By Proposition 7.16, $v_\Delta(\alpha) = \mathbf{T}$ iff $\alpha \in \Delta$ for every α. It will be shown, by induction on n, that $v_\Delta \in \mathcal{L}_n^{IPL}$ for every $n \geq 0$.

Base $n = 0$: $v_\Delta \in \mathcal{L}_0^{IPL} = Val_+^{nd}(\mathcal{M}_{IPL})$, by Proposition 7.16.

Inductive step: Assume that, for every φ' and for every φ'-saturated set Δ', $v_{\Delta'} \in \mathcal{L}_k^{IPL}$ for every $k \leq n$, for a given $n \geq 0$ (IH). Let us prove that $v_\Delta \in \mathcal{L}_{n+1}^{IPL}$. By IH, $v_\Delta \in \mathcal{L}_n^{IPL}$. Let $\alpha \in For(\Omega)$ such that $v_\Delta(\alpha) = \mathbf{U}$. By Definition 7.14, $\alpha \neq p$ for every propositional variable p; and $\alpha \neq \beta \# \gamma$ for $\# \in \{\wedge, \vee\}$ and every formulas β, γ. Then, there are two cases to analyze:

(1) $\alpha = \neg\beta$ for some β. By Definition 7.14, $\neg\beta \notin \Delta$ and $\beta \notin \Delta$. Suppose that $\Delta, \beta \vdash_{IPL} \neg\beta$. By the DM, $\Delta \vdash_{IPL} \beta \to \neg\beta$, and so the sequent $\Delta_0 \Rightarrow \beta \to \neg\beta$ is provable in **LI**, for some finite set $\Delta_0 \subseteq \Delta$. But, for every finite multiset Γ, the sequent $\Gamma, \beta \to \neg\beta \Rightarrow \neg\beta$ is provable in **LI**. Indeed, from axiom $\Gamma, \beta \Rightarrow \beta$ it follows $\Gamma, \beta, \neg\beta \Rightarrow$, by $(\neg \Rightarrow)$. From this, and by using axiom $\Gamma, \beta \Rightarrow \beta$ once again, by $(\to\Rightarrow)$ we infer $\Gamma, \beta, \beta \to \neg\beta \Rightarrow$. Finally, $\Gamma, \beta \to \neg\beta \Rightarrow \neg\beta$ follows by $(\Rightarrow \neg)$. In particular, $\Delta_0, \beta \to \neg\beta \Rightarrow \neg\beta$ is provable in **LI** and so, by (cut) with $\Delta_0 \Rightarrow \beta \to \neg\beta$, it follows that $\Delta_0 \Rightarrow \neg\beta$ is provable in **LI**. That is, $\Delta \vdash_{IPL} \neg\beta$, hence $\neg\beta \in \Delta$, a contradiction. From this, $\Delta, \beta \nvdash_{IPL} \neg\beta$. Then, there exists a $\neg\beta$-saturated set Δ' in *IPL* such that $\Delta \cup \{\beta\} \subseteq \Delta'$. By IH, $v_{\Delta'} \in \mathcal{L}_n^{IPL}$ such that, by Definition 7.14, $v_{\Delta'}(\neg\beta) = \mathbf{F}$. Moreover, for every $\delta \in For(\Omega)$, if $v_\Delta(\delta) = \mathbf{T}$ then $v_{\Delta'}(\delta) = \mathbf{T}$, since $\Delta \subseteq \Delta'$.

(2) $\alpha = \beta \to \gamma$ for some β and γ. By Definition 7.14, $\beta \to \gamma \notin \Delta$ and $\beta \notin \Delta$. Suppose that $\Delta, \beta \vdash_{IPL} \beta \to \gamma$. By (DM), $\Delta, \beta \vdash_{IPL} \gamma$ and so, by (DM) again, $\Delta \vdash_{IPL} \beta \to \gamma$, a contradiction. From this, $\Delta, \beta \nvdash_{IPL} \beta \to \gamma$. Then, there exists a $(\beta \to \gamma)$-

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

saturated set Δ'' in *IPL* such that $\Delta \cup \{\beta\} \subseteq \Delta''$. By IH, $v_{\Delta''} \in \mathcal{L}_n^{IPL}$ such that, by Definition 7.14, $v_{\Delta''}(\beta \to \gamma) = \mathbf{F}$. Moreover, for every $\delta \in For(\Omega)$, if $v_\Delta(\delta) = \mathbf{T}$ then $v_{\Delta''}(\delta) = \mathbf{T}$, since $\Delta \subseteq \Delta''$.

From the previous analysis we conclude that $v_\Delta \in \mathcal{L}_{n+1}^{IPL}$. This shows that $v_\Delta \in \mathcal{L}_n^{IPL}$ for every $n \geq 0$ and so $v_\Delta \in \mathcal{L}_{IPL}$. ⊣

THEOREM 7.18 (Completeness of **LI** w.r.t. $\mathcal{R}(\mathcal{M}_{IPL})$). *For every* Γ *and* φ: $\Gamma \models_{\mathcal{R}(\mathcal{M}_{IPL})} \varphi$ *implies that* $\Gamma \vdash_{IPL} \varphi$.

PROOF. Suppose that $\Gamma \nvdash_{IPL} \varphi$. Then, there exists a φ-saturated set Δ in **LI** such that $\Gamma \subseteq \Delta$. By Proposition 7.17, v_Δ is a level valuation in $\mathcal{M}_{IPL}$ such that $v_\Delta(\alpha) = \mathbf{T}$ for every $\alpha \in \Gamma$, but $v_\Delta(\varphi) \neq \mathbf{T}$. This shows that $\Gamma \not\models_{\mathcal{R}(\mathcal{M}_{IPL})} \varphi$. ⊣

REMARK 7.19. Observe that the semantical characterization of *IPL* w.r.t. level valuation semantics is quite interesting from a theoretical point of view. However, it has little interest from a practical point of view, given that level valuations are infinitary objects, and it is far from obvious how to check the validity of a formula in *IPL* by using $\mathcal{R}(\mathcal{M}_{IPL})$. The next step is the definition of an (algorithmic) way to construct *intuitionistic truth-tables*, based precisely on level valuations.

From soundness and completeness of *IPL* w.r.t. $\mathcal{R}(\mathcal{M}_{IPL})$, several important consequences can be obtained. Some of these consequences will be extremely useful to prove the soundness and completeness of the (non-deterministic) truth-tables decision method for *IPL* to be defined in the next section.

COROLLARY 7.19.1 (Finitariness for $\mathcal{R}(\mathcal{M}_{IPL})$). *Let* $\Gamma \cup \{\varphi\} \subseteq For(\Omega)$. *If* $\Gamma \models_{\mathcal{R}(\mathcal{M}_{IPL})} \varphi$ *then* $\Gamma_0 \models_{\mathcal{R}(\mathcal{M}_{IPL})} \varphi$ *for some finite subset* Γ_0 *of* Γ.

PROOF. It follows from Theorems 7.18 and 7.12 and the fact that **LI** is finitary. ⊣

COROLLARY 7.19.2 (DM for $\mathcal{R}(\mathcal{M}_{IPL})$). *Let* $\Gamma \cup \{\alpha, \beta\} \subseteq For(\Omega)$. *Then,* $\Gamma, \alpha \models_{\mathcal{R}(\mathcal{M}_{IPL})} \beta$ *if, and only if,* $\Gamma \models_{\mathcal{R}(\mathcal{M}_{IPL})} \alpha \to \beta$.

PROOF. It follows from Theorems 7.18 and 7.12 and the fact that **LI** satisfies the DM (recall Remark 7.8). ⊣

PROPOSITION 7.20. *Let* $v \in \mathcal{L}_{IPL}$ *and let* $\Gamma = \{\alpha \in For(\Omega) \mid v(\alpha) = \mathbf{T}\}$. *Then,* Γ *is a closed theory in IPL, i.e.: if* $\Gamma \vdash_{IPL} \alpha$ *then* $\alpha \in \Gamma$.

PROOF. Let $\Gamma = \{\alpha \in For(\Omega) \mid v(\alpha) = \mathbf{T}\}$ and suppose that $\Gamma \vdash_{IPL} \alpha$. By the soundness of *IPL* w.r.t. level valuation semantics (Theorem 7.12), $\Gamma \models_{\mathcal{R}(\mathcal{M}_{IPL})} \alpha$. Given that $v \in \mathcal{L}_{IPL}$ such that $v \models \Gamma$, it follows that $v \models \alpha$. That is, $v(\alpha) = \mathbf{T}$ and so $\alpha \in \Gamma$. ⊣

The following result will be crucial for proving the soundness and completeness of intuitionistic truth-tables.

PROPOSITION 7.21. *Let* $v \in \mathcal{L}_{IPL}$, *and* $\alpha \in For(\Omega)$ *such that* $v(\alpha) = \mathbf{U}$. *Then, there exists* $w \in \mathcal{L}_{IPL}$ *such that* $w(\alpha) = \mathbf{F}$ *and* $w(\gamma) = \mathbf{T}$ *for every formula* γ *such that* $v(\gamma) = \mathbf{T}$.

PROOF. $v \in \mathcal{L}_{IPL}$, and $\alpha \in For(\Omega)$ such that $v(\alpha) = \mathbf{U}$. Since, in particular, $v \in Val_+^{nd}(\mathcal{M}_{IPL})$ (recall Definitions 7.5 and 7.6), $v(\alpha) = \mathbf{U}$ implies that α cannot be a propositional variable. By definition of the conjunction and disjunction tables

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

in $\mathcal{M}_{IPL}$ (Figure 5), and since $v(\alpha) = \mathbf{U}$, it follows that α cannot be either a conjunction or a disjunction. Let $\Gamma = \{\gamma \in For(\Omega) \mid v(\gamma) = \mathbf{T}\}$. Then, $\alpha \notin \Gamma$. By the observations above, there are only two cases to consider:

(1) $\alpha = \neg\beta$ for some β. As noted above, $\alpha \notin \Gamma$, i.e., $\neg\beta \notin \Gamma$. Hence $\Gamma \nvdash_{IPL} \neg\beta$, given that Γ is closed by Proposition 7.20. If $\beta \in \Gamma$ then $v(\beta) = \mathbf{T}$ and so, by definition of $\mathcal{M}_{IPL}$, $v(\alpha) = v(\neg\beta) = \mathbf{F}$, a contradiction. Then, $\beta \notin \Gamma$. Suppose that $\Gamma, \beta \vdash_{IPL} \neg\beta$. As in the proof of Proposition 7.17 we infer that $\Gamma \vdash_{IPL} \neg\beta$. But this leads us to a contradiction since, as pointed out before, $\Gamma \nvdash_{IPL} \neg\beta$. From this, $\Gamma, \beta \nvdash_{IPL} \neg\beta$. Thus, there exists a $\neg\beta$-saturated set Δ in IPL such that $\Gamma \cup \{\beta\} \subseteq \Delta$. By Proposition 7.17, the function v_Δ is a level valuation such that $v_\Delta(\alpha) = \mathbf{F}$ (since $v_\Delta(\beta) = \mathbf{T}$) and $v_\Delta(\gamma) = \mathbf{T}$ for every formula γ such that $v(\gamma) = \mathbf{T}$, as required.

(2) $\alpha = \beta \to \delta$ for some β and δ. Since $\Gamma, \delta \vdash_{IPL} \beta \to \delta$ and $\Gamma \nvdash_{IPL} \beta \to \delta$ then $\Gamma \nvdash_{IPL} \delta$. Suppose that $\Gamma, \beta \vdash_{IPL} \beta \to \delta$. By (DM), $\Gamma, \beta \vdash_{IPL} \delta$ and so, by (DM) again, $\Gamma \vdash_{IPL} \beta \to \delta$, a contradiction. From this, $\Gamma, \beta \nvdash_{IPL} \beta \to \delta$. Thus, there exists a $(\beta \to \delta)$-saturated set Δ in IPL such that $\Gamma \cup \{\beta\} \subseteq \Delta$. By reasoning as above, we conclude that $\delta \notin \Delta$. By Proposition 7.17, $v_\Delta \in \mathcal{L}_{IPL}$ is such that $v_\Delta(\alpha) = \mathbf{F}$ (since $v_\Delta(\beta) = \mathbf{T}$ and $v(\delta) \neq \mathbf{T}$) and $v_\Delta(\gamma) = \mathbf{T}$ for every formula γ such that $v(\gamma) = \mathbf{T}$, as required. ⊣

§8. A truth-tables procedure for *IPL*. After obtaining a suitable RNmatrix for *IPL*, the next step is the definition of a decision procedure from this. As in the case of *S4*, this procedure is given by means of branching truth-tables with an algorithm for deleting inadequate rows.

8.1. Intuitionistic partial (level) valuations and intuitionistic truth-tables.

DEFINITION 8.1. Consider the sets

$$CS(\Omega) = \{\Lambda \subseteq For(\Omega) \mid \Lambda \text{ is non-empty and closed under subformulas}\},$$

$$FCS(\Omega) = \{\Lambda \subseteq For(\Omega) \mid \Lambda \text{ is finite, non-empty, and closed under subformulas}\}.$$

DEFINITION 8.2 (Partial valuation in $\mathcal{M}_{IPL}$). Let $\Lambda \in CS(\Omega)$. A *partial valuation in* $\mathcal{M}_{IPL}$ is a function $\tilde{v}_p : \Lambda \to V_{IPL}$ such that, for every $\alpha, \beta \in \Lambda$:

– if $\alpha \in \mathcal{P} \cap \Lambda$ then $\tilde{v}_p(\alpha) \in \{\mathbf{T}, \mathbf{F}\}$;
– if $\neg\alpha \in \Lambda$ then $\tilde{v}_p(\neg\alpha) \in \neg^{IPL}(\tilde{v}_p(\alpha))$;
– if $\# \in \{\to, \vee, \wedge\}$ and $\alpha\#\beta \in \Lambda$ then $\tilde{v}_p(\alpha\#\beta) \in \#^{IPL}(\tilde{v}_p(\alpha), \tilde{v}_p(\beta))$.

Let $iPV(\Lambda)$ be the set of partial valuations in $\mathcal{M}_{IPL}$ with domain Λ.

DEFINITION 8.3 (Intuitionistic partial level valuation over Λ). Let $\Lambda \in FCS(\Omega)$. An *intuitionistic partial level valuation* in $\mathcal{M}_{IPL}$ over Λ is a partial valuation $\tilde{v}_p \in iPV(\Lambda)$ satisfying the following:

$\forall\alpha \in \Lambda$ such that $\tilde{v}_p(\alpha) = \mathbf{U}$, there exists an intuitionistic partial level valuation $\tilde{w}_p$ in $\mathcal{M}_{IPL}$ over Λ such that $\tilde{w}_p(\alpha) = \mathbf{F}$ and, $\forall\beta \in \Lambda$, $\tilde{w}_p(\beta) = \mathbf{T}$ whenever $\tilde{v}_p(\beta) = \mathbf{T}$.

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

The set of intuitionistic partial level valuations in $\mathcal{M}_{IPL}$ over Λ will be denoted by $iPLV(\Lambda)$.

REMARK 8.4. Given v, w, and α let $\mathsf{P}_\Lambda(v, w, \alpha)$ iff $w(\alpha) = \mathbf{F}$ and, $\forall \beta \in \Lambda, w(\beta) = \mathbf{T}$ whenever $v(\beta) = \mathbf{T}$. If $\Lambda \in FCS(\Omega)$ then

$$iPLV(\Lambda) = \{\tilde{v}_p \in iPV(\Lambda) \mid \forall \alpha \in \Lambda\big(\tilde{v}_p(\alpha) = \mathbf{U} \text{ implies that } \mathsf{P}_\Lambda(\tilde{v}_p, \tilde{w}_p, \alpha) \text{ for some } \tilde{w}_p \in iPLV(\Lambda)\big)\}.$$

REMARK 8.5. As in the case of **S4** discussed in Remark 5.7, the impredicative definition of $iPLV(\Lambda)$ does not lead to circularity. Indeed, by using an algorithm (see Algorithm 1 below) similar to Algorithm 3.4 given in [12, p. 15] for $\mathcal{M}_{S4}$, Definition 8.3 can be transformed into a predicative one (see Definition 8.11 and Proposition 8.19). This is the consequence of the fact that $iPLV(\Lambda)$ is obtained as the output of Algorithm 1, being the unique subset of $iPV(\Lambda)$ satisfying the condition of Remark 8.4, as it will be shown in Proposition 8.7. The set $iPLV(\Lambda)$ is precisely a (intuitionistic) truth-table, in the wider sense proposed here, based on $\mathcal{R}(\mathcal{M}_{IPL})$. It is obtained by means of a mechanical procedure, where its elements correspond to the rows of the table, and its columns correspond to the values assigned in each row to the elements of Λ.

NOTATION 8.6. *Given $\Lambda \in FCS(\Omega)$ with $n \geq 1$ elements, consider an arrangement $\alpha_1, \ldots, \alpha_n$ of this set such that the complexity of α_i is less than or equal to the complexity of α_j if $i < j$ (so the atomic formulas occurring in Λ appear as the first elements of the sequence, in arbitrary order). In this way, any subformula of α_j appears as α_i for some unique $i \leq j$. Suppose that the set $iPV(\Lambda)$ has $m \geq 1$ elements. Consider now an arrangement $v_p^1, \ldots, v_p^m$ of its elements, ordered by lexicographical order, by stipulating that $\mathbf{F} < \mathbf{U} < \mathbf{T}$. In this way, $iPV(\Lambda)$ can be represented as a two-dimensional array (i.e., an array of arrays, or a matrix) V such that each row $V[i]$, for $1 \leq i \leq m$, corresponds to the ith element v_p^i of $iPV(\Lambda)$. In turn, $V[i][j]$ is the truth-value assigned to the formula $\alpha_j \in \Lambda$ by the partial valuation $V[i] = v_p^i$. That is, $V[i][j] = v_p^i(\alpha_j)$. This notation will be used in Algorithm 1, to be described in the sequel, also for subsets of $iPV(\Lambda)$.*

Algorithm 1 (in Figure 6) for obtaining $iPLV(\Lambda)$. *Let $\Lambda \in FCS(\Omega)$, and consider the notation stipulated in Notation 8.6. Hence, $\Lambda = \{\alpha_1, \ldots, \alpha_n\}$ and $iPV(\Lambda) = \{v_p^1, \ldots, v_p^m\}$ (which can also be seen as sequences). Let $V := iPLV(\Lambda)$. For each element v_p^i of V, if $v_p^i(\alpha_j) = \mathbf{U}$, determine if there is some $v_p^{i'}$ such that $\mathsf{P}_\Lambda(v_p^i, v_p^{i'}, \alpha_j)$. This is represented by the function* `isSupported`(V, i, j) *in Figure 6. If, for some j,* `isSupported`(V, i, j) *is false (meaning no valuation in V supports v_p^i w.r.t. α_j) then v_p^i is removed from the candidate set for $iPLV(\Lambda)$ and will not be considered in subsequent algorithm iterations. After checking all the elements of V using the elements of V itself, let IPV be the set of remaining valuations. If no valuation was removed, i.e., if $IPV = V$, the algorithm stops at this point and returns IPV as being the set $iPLV(\Lambda)$ of partial level valuations. The set IPV is the result of a* refinement *function applied to V, represented by* `refine`(V) *in Figure 6. In case $IPV \neq V$, it means that IPV is a proper subset of V (i.e., some valuations were effectively removed in this round of the process). The set IPV needs to be checked again with the function*

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

Algorithm 1: Refinement algorithm.

```
Input  : A matrix of values in {F, U, T} (see Notation 8.6)
Output: A matrix of values in {F, U, T} (of cardinality less than or equal to that of the input)
Function refine(V):
    iPLV ← V
    m ← |V|                                          /* The number of arrays in V.  */
    for i ← 1 to m do
        v ← V[i]                                     /* A partial valuation with domain in Λ.  */
        n ← |v|                                      /* The size of Λ.  */
        for j ← 1 to n do
            if v[j] = U then
                if isSupported(V, i, j) then
                    continue
                else
                    iPLV ← iPLV \ {v}
            j ← j + 1
        i ← i + 1
    return iPLV
V ← iPV(Λ)
iPLV ←refine(V)
while iPLV ≠ V do
    V ← iPLV
    iPLV ←refine(V)
```

FIGURE 6. Algorithm for obtaining the set $iPLV(\Lambda)$.

refine, to guarantee that every row is still supported. Then, the set V is updated to be equal to IPV (and so V is, again, the universe of partial valuations to be checked), it is renumbered as a sequence $V[1], \ldots, V[m']$ (where $m' < m$ is the cardinal or IPV) and then we repeat the process with the updated V until no changes occur. The set $iPLV(\Lambda)$ is then defined to be the fixpoint $V = IPV$ (see Figure 6).

The next result shows that, thanks to Algorithm 1, the set $iPLV(\Lambda)$ is well-defined, it is unique, and it is non-empty.

PROPOSITION 8.7 (Existence and uniqueness of intuitionistic partial level valuations). *Let $\Lambda \in FCS(\Omega)$. Then, there exists a unique set $iPLV(\Lambda) \subseteq iPV(\Lambda)$ satisfying the condition of Remark 8.4:*

$$iPLV(\Lambda) = \{\tilde{v}_p \in iPV(\Lambda) \mid \forall \alpha \in \Lambda\big(\tilde{v}_p(\alpha) = \mathbf{U} \text{ implies that } \mathsf{P}_\Lambda(\tilde{v}_p, \tilde{w}_p, \alpha) \text{ for some } \tilde{w}_p \in iPLV(\Lambda)\big)\}.$$

Moreover, the set $iPLV(\Lambda)$ is non-empty, having at least 2^k elements, where k is the number of different propositional variables occurring in Λ.

PROOF. Given $\Lambda \in FCS(\Omega)$, define the sequences $(\alpha_1, \ldots, \alpha_n)$ and $(v_p^1, \ldots, v_p^m)$ representing Λ and $iPV(\Lambda)$ as in Notation 8.6, and then execute Algorithm 1 as described. Notice that $iPV(\Lambda)$ is a finite set, and the algorithm operates on a subset $V \subseteq iPV(\Lambda)$ in each round. If the algorithm does not stop in a given round, it deletes some elements from V and restarts the process with a new subset that is strictly smaller than the previous one. This shows that the process will eventually terminate. There are two possible termination conditions: either every element of V is eliminated, or a set $IPV = V$ is produced. In the second case, every element of IPV is

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

supported (as no removal is possible at this point); hence, $iPLV(\Lambda) := IPV$ satisfies the condition of Remark 8.4. In turn, the first possibility (arriving to a set $IPV = \emptyset$) is impossible by the following reason: every partial valuation $v : \Lambda \to V_{IPL}$ such that $v(\alpha) \in \{\mathbf{T}, \mathbf{F}\}$ for every $\alpha \in \Lambda$ (i.e., every classical partial valuation) is never removed from $iPV(\Lambda)$ by means of Algorithm 1. Indeed, any classical valuation v never satisfies that $v(\alpha_j) = \mathbf{U}$ for some j, hence it will never be removed. Since there are exactly 2^k classical valuations in $iPV(\Lambda)$ (where k is the number of different propositional variables occurring in Λ), the set IPV produced by Algorithm 1 is non-empty, having at least 2^k elements. We have proven, therefore, the existence of a (non-empty) subset IPV of $iPV(\Lambda)$ satisfying the condition of Remark 8.4.

To prove the uniqueness of such a set of partial valuations, observe that the order of the valuations $V[i]$ in the array V is irrelevant for Algorithm 1. The algorithm's execution does not depend on this initial ordering. In each round, every valuation $V[i]$ is tested against every formula α_j in Λ. If $V[i][j] = \mathbf{U}$, the function `isSupported`(V, i, j) checks all other valuations $V[i']$ in V to determine whether $\mathsf{P}_\Lambda(V[i], V[i'], \alpha_j)$ holds. If this condition fails for every i', $V[i]$ is removed in the next step. This procedure is clearly independent of the lexicographical order used in the formal presentation of Algorithm 1: that order was chosen solely to provide a precise description of the algorithm's steps. In other words, regardless of the initial arrangement of $iPV(\Lambda)$, the first round will eliminate the same valuations and produce a unique set V_1 for the second stage. The same argument applies: any arrangement of V_1 yields the same subset V_2 after the second round. By induction, the algorithm's output—the fixpoint $V_s = IPV$ reached in the final step s—must be unique.

This completes the proof. ⊣

REMARK 8.8. Observe that it is possible to have a set $iPLV(\Lambda)$ depending on k different propositional variables $p_1, \ldots, p_k$ having *exactly* 2^k elements. It is enough considering a formula $\alpha(p_1, \ldots, p_k)$ constructed by using exclusively conjunction and disjunction. Let Λ be the set of subformulas of α. Clearly, the truth-table $iPLV(\Lambda)$ for α only contains classical values $\mathbf{T}$ and $\mathbf{F}$, coinciding so with $iPV(\Lambda)$. It has, therefore, exactly 2^k elements. Moreover, by expanding this table with a column for $\alpha \to \alpha$, i.e., by considering $iPV(\Lambda \cup \{\alpha \to \alpha\})$, the last column only contains values $\mathbf{U}$ or $\mathbf{T}$. But then the rows assigning $\mathbf{U}$ to $\alpha \to \alpha$ are not supported by any other row, hence they are removed in the first round of Algorithm 1, arriving so to a set $iPLV(\Lambda \cup \{\alpha \to \alpha\})$ with 2^k elements.

DEFINITION 8.9. An *intuitionistic truth-table in* $\mathcal{M}_{IPL}$ is an exhaustive list of every and only intuitionistic partial level valuations for a given finite domain closed under subformulas.

REMARK 8.10. Each of these partial valuations corresponds to a row of the table.

EXAMPLE. In what follows, we present an example of a truth-table produced by Algorithm 1 which keeps track of the dependencies of each partial valuation[4]. The

[4]This example was automatically generated by a Rocq module for RNmatrices written by R. Leme (see https://github.com/renatoleme/Forest).

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

Row (ID)	p	$\neg p$	$p \vee \neg p$	$\neg(p \vee \neg p)$	$\neg\neg(p \vee \neg p)$
(1)	**F**	**U**	**F**	**U**	**U**
(2)	**F**	**U**	**F**	**U**	**T**
(3)	**F**	**U**	**F**	**T**	**F**
(4)	**F**	**T**	**T**	**F**	**U**
(5)	**F**	**T**	**T**	**F**	**T**
(6)	**T**	**F**	**T**	**F**	**U**
(7)	**T**	**F**	**T**	**F**	**T**

(A) Initial table.

Row (ID)	p	$\neg p$	$p \vee \neg p$	$\neg(p \vee \neg p)$	$\neg\neg(p \vee \neg p)$
(1)	×	6, 7	×	4, 5, 6, 7	3
(2)	×	7	×	5, 7	×
(3)	×	∅	×	×	×
(4)	×	×	×	×	∅
(5)	×	×	×	×	×
(6)	×	×	×	×	∅
(7)	×	×	×	×	×

(B) First cycle of dependencies.

Row (ID)	p	$\neg p$	$p \vee \neg p$	$\neg(p \vee \neg p)$	$\neg\neg(p \vee \neg p)$
(1)	**F**	**U**	**F**	**U**	**U**
(2)	**F**	**U**	**F**	**U**	**T**
(5)	**F**	**T**	**T**	**F**	**T**
(7)	**T**	**F**	**T**	**F**	**T**

(C) Intermediate table.

Row (ID)	p	$\neg p$	$p \vee \neg p$	$\neg(p \vee \neg p)$	$\neg\neg(p \vee \neg p)$
(1)	×	7	×	5, 7	∅
(2)	×	7	×	5, 7	×
(5)	×	×	×	×	×
(7)	×	×	×	×	×

(D) Second cycle of dependencies.

Row (ID)	p	$\neg p$	$p \vee \neg p$	$\neg(p \vee \neg p)$	$\neg\neg(p \vee \neg p)$
(2)	**F**	**U**	**F**	**U**	**T**
(5)	**F**	**T**	**T**	**F**	**T**
(7)	**T**	**F**	**T**	**F**	**T**

(E) Final table.

Row (ID)	p	$\neg p$	$p \vee \neg p$	$\neg(p \vee \neg p)$	$\neg\neg(p \vee \neg p)$
(2)	×	7	×	5, 7	×
(5)	×	×	×	×	×
(7)	×	×	×	×	×

(F) Final dependency table.

FIGURE 7. Example of executions of the refinement algorithm. Note that each partial valuation is identified by its ID, which is an index $n \in \mathbb{N}$. The dependency tables are obtained in two steps: first, find every v such that there is some α such that $v(\alpha) = \mathbf{U}$. Then, for each of these partial valuations, find every valuation w such that $w(\alpha) = \mathbf{F}$ and, for every β, $w(\beta) = \mathbf{T}$ whenever $v(\beta) = \mathbf{T}$.

first table (Figure 7a) is the full table for $\neg\neg(p \vee \neg p)$, a valid intuitionistic formula, and was obtained by applying the Nmatrix for *IPL* in the usual way.

In Figure 7, the tables on the left correspond to an execution of Algorithm 1 (see Figure 6), while the tables on the right represent the dependency tables. Whenever $v(\alpha) = \mathbf{U}$ occurs in a table on the left, the corresponding table on the right contains a set of supports (i.e., partial valuations w such that $w(\alpha) = \mathbf{F}$ and $w(\beta) = \mathbf{T}$ whenever $v(\beta) = \mathbf{T}$) in the correspondent cell. If no such support exists for that case, then the corresponding partial valuation is removed in the subsequent execution of Algorithm 1, which can be observed in the transition between successive tables on the left.

By removing rows 3, 4, and 6 from Figure 7a, we obtain Figure 7b. In this new table, the row 1, which was previously supported by the row 3, no longer has that support available. Running the algorithm again with the updated table, we find that the row 1 must now also be removed. This results in Figure 7e. At this point, Figure 7f contains no occurrences of $\emptyset$, which means that every instance of **U** is now adequately supported. Hence, there are no more rows to remove and the algorithm is terminated. Finally, by inspecting the resulting Figure 7e, we can confirm that the formula is indeed valid.

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

Algorithm 1 is fundamentally the same as the one presented by Grätz [12, Algorithm 3.4], but is applied to the new RNmatrix for *IPL*. As a result, it remains within the same complexity class using exponential space. However, it has a lower upper bound because the atomic valuation considers only two truth-values: **F** and **T**. Section 9 provides a further discussion on this topic.

8.2. Soundness and completeness of the intuitionistic truth-tables. Now, it will be shown that the generalized notion of truth-tables for *IPL* proposed in Section 8.1 is sound and complete. This will be established by first proving analyticity and co-analyticity with respect to level valuations in $\mathcal{M}_{IPL}$. To this end, and adapting [12, Definition 3.12], we introduce the intermediary notion of intuitionistic partial level valuation in $\mathcal{R}(\mathcal{M}_{IPL})$ over Λ.

DEFINITION 8.11 (Intuitionistic partial' level valuation over Λ). Let $\Lambda \in CS(\Omega)$ and $v_p \in iPV(\Lambda)$. Then, $\tilde{v}_p$ is an *intuitionistic partial' level valuation* in $\mathcal{R}(\mathcal{M}_{IPL})$ over Λ iff

$\forall \alpha \in \Lambda$ such that $\tilde{v}_p(\alpha) = \mathbf{U}$, there exists a *level valuation* w in $\mathcal{L}_{IPL}$ such that $w(\alpha) = \mathbf{F}$ and, $\forall \beta \in \Lambda$, $w(\beta) = \mathbf{T}$ whenever $\tilde{v}_p(\beta) = \mathbf{T}$.

The set of intuitionistic partial' level valuations in $\mathcal{R}(\mathcal{M}_{IPL})$ over Λ will be denoted by $iPLV'(\Lambda)$.

Definition 8.11 above is similar to Definition 8.3, except for two important aspects: (1) the domain Λ of the partial valuation $\tilde{v}_p$ no longer needs to be finite and (2) whenever there is a formula $\alpha \in \Lambda$ such that $\tilde{v}_p(\alpha) = \mathbf{U}$, the support w must be a level valuation.

This difference is central for proving both co-analyticity and analyticity. Aspect (1) is crucial for analyticity (Lemma 8.18), since we aim to prove that any domain closed under subformulas (possibly infinite) can be extended to a domain over the entire set of formulas. Aspect (2), in turn, is essential for co-analyticity (Lemma 8.13), as it relies on the previous result obtained in Proposition 7.21, which establishes the existence of a certain level valuation under appropriate circumstances.

The notion of $iPLV'$ is well-suited for proof-theoretic purposes, but it is not sufficient for obtaining a decision procedure, since it depends on the set of level valuations. For this reason, we later prove that, for finite domains (which correspond to truth-tables), Definitions 8.11 and 8.3 are equivalent (see Proposition 8.19). This result, a direct consequence of Proposition 8.7, allows us to replace the notion of $iPLV'$ with its finitary counterpart $iPLV$, showing that, in order to decide the validity of a formula (or argument), it is sufficient to restrict the analysis to the finite universe of its subformulas (see Theorem 8.22).

REMARK 8.12. Recall the predicate $\mathsf{P}_\Lambda(v, w, \alpha)$ introduced in Remark 8.4. Clearly,

$$iPLV'(\Lambda) = \{\tilde{v}_p \in iPV(\Lambda) \mid \forall \alpha \in \Lambda\big(\tilde{v}_p(\alpha) = \mathbf{U} \text{ implies that } \mathsf{P}_\Lambda(\tilde{v}_p, w, \alpha) \text{ for some } w \in \mathcal{L}_{IPL}\big)\}.$$

LEMMA 8.13 (Co-analyticity lemma for $\mathcal{R}(\mathcal{M}_{IPL})$). *Let $v \in \mathcal{L}_{IPL}$ and $\Lambda \in CS(\Omega)$. Then, the restriction $\tilde{v}_p := v_{|\Lambda}$ of v to the domain Λ belongs to $iPLV'(\Lambda)$.*

PROOF. Let $v \in \mathcal{L}_{IPL}$ and $\tilde{v}_p := v_{|\Lambda}$. Clearly, $\tilde{v}_p \in iPV(\Lambda)$. Let $\alpha \in \Lambda$ such that $\tilde{v}_p(\alpha) = \mathbf{U}$. Hence, $v(\alpha) = \mathbf{U}$. By Proposition 7.21, there exists $w \in \mathcal{L}_{IPL}$ such that

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

$w(\alpha) = \mathbf{F}$ and $w(\beta) = \mathbf{T}$ for every β such that $v(\beta) = \mathbf{T}$. In particular, $\mathsf{P}_\Lambda(\tilde{v}_p, w, \alpha)$ holds. From this, $\tilde{v}_p \in iPLV'(\Lambda)$ by Remark 8.12. ⊣

A key concern regarding truth-tables is whether a given partial valuation can always be extended to a full valuation, i.e., whether a domain closed under subformulas can be extended to a complete domain. This property, called *analyticity*, is central for proving the completeness of the method (see Theorem 8.22). To establish analyticity, we start with a domain over a set closed under subformulas and conservatively add new formulas to this domain. At the limit, we obtain the function given by the union of all such extensions, which corresponds to a function defined over the set of all formulas. However, a formula can be evaluated only if all of its subformulas are already evaluated. To ensure that this condition always holds, we consider a suitable ordering on the set of formulas.

DEFINITION 8.14. Given $n \geq 0$, the set of formulas over Ω with complexity less than or equal to n will be denoted by $For(\Omega)_n$. That is: $For(\Omega)_n = \{\alpha \in For(\Omega) \mid \mathsf{c}(\alpha) \leq n\}$.

REMARK 8.15. Given the ordinal ω, let $\omega^2 := \omega \cdot \omega$ be its ordinal multiplication. Observe that it is possible to define an enumeration $\alpha_1, \alpha_2, \ldots \alpha_m, \ldots$ (for $m \in \omega^2$) of $For(\Omega)$ such that $\mathsf{c}(\alpha_i) \leq \mathsf{c}(\alpha_j)$ if $i \leq j$ and, for every i such that $\mathsf{c}(\alpha_i) > 0$, if β is a strict subformula of α_i then $\beta = \alpha_j$ for some $j < i$. Such an enumeration of $For(\Omega)$ can be defined as follows: every formula has an index in ω^2, which is a denumerable ordinal, such that all the formulas with complexity 0 (which form a denumerable set) are placed first, with indexes in $I_0 := \omega$; after this, all the formulas with complexity 1 (which form a denumerable set) are placed with an index in $I_1 := \omega \cdot 2 \setminus \omega = \{\omega, \omega + 1, \ldots\}$; in general, the formulas with complexity n (which form a denumerable set) have an index in $I_n := \omega \cdot (n+1) \setminus \omega \cdot n = \{\omega \cdot n, \omega \cdot n + 1, \ldots\}$.

Based on the enumeration of $For(\Omega)$ given in the previous remark, consider the following sets.

DEFINITION 8.16. Let $\alpha_1, \alpha_2, \ldots \alpha_m, \ldots$ (for $m \in \omega^2$) be an enumeration of $For(\Omega)$ as in Remark 8.15. For every $n, m \in \omega$ let $\bar{\Lambda}_n := \Lambda \cup For(\Omega)_n$; $\bar{\Lambda}_n^0 := \bar{\Lambda}_n$; and $\bar{\Lambda}_n^{m+1} = \bar{\Lambda}_n^m \cup \{\alpha_{\omega\cdot(n+1)+m}\}$.

REMARK 8.17. By definition, $\bar{\Lambda}_n^{m+1} = \bar{\Lambda}_n \cup \{\alpha_{\omega\cdot(n+1)}, \alpha_{\omega\cdot(n+1)+1}, \ldots, \alpha_{\omega\cdot(n+1)+m}\}$, and $\alpha_{\omega\cdot(n+1)+m} \in \Lambda$ if and only if $\bar{\Lambda}_n^{m+1} = \bar{\Lambda}_n^m$. Hence, $\bar{\Lambda}_n^{m+1}$ is obtained from $\bar{\Lambda}_n$ by adding the first $m+1$ formulas (of the given enumeration) with complexity $n+1$, and so $\bar{\Lambda}_n^m$ adds at most m formulas to $\bar{\Lambda}_n$, for $m \in \omega$. It is worth noting that $\bar{\Lambda}_n^m \in CS(\Omega)$ for every $n, m \in \omega$; in particular, $\bar{\Lambda}_n \in CS(\Omega)$ for every $n \in \omega$. Clearly $\bar{\Lambda}_n^m \subseteq \bar{\Lambda}_n^{m+1}$, $\bar{\Lambda}_{n+1} = \bigcup_{m\in\omega} \bar{\Lambda}_n^m$, and $For(\Omega) = \bigcup_{n\in\omega} \bar{\Lambda}_n$.

LEMMA 8.18 (Analyticity lemma for $\mathcal{R}(\mathcal{M}_{IPL})$). *Let $\Lambda \in CS(\Omega)$ and $\tilde{v}_p \in iPLV'(\Lambda)$. Then, there exists $v \in \mathcal{L}_{IPL}$ such that the restriction $v_{|\Lambda}$ of v to the domain Λ coincides with $\tilde{v}_p$.*

PROOF. It is an adaptation of the proof of [12, Lemma 3.14] to $\mathcal{R}(\mathcal{M}_{IPL})$.

Suppose first that $\Lambda = For(\Omega)$. By induction on $n \in \omega$, it will be shown that $\tilde{v}_p \in \mathcal{L}_n^{IPL}$ for every n. By Definition 8.2 of partial valuation in $\mathcal{M}_{IPL}$, $\tilde{v}_p \in Val_+^{nd}(\mathcal{M}_{IPL}) = \mathcal{L}_0^{IPL}$. Suppose that $\tilde{v}_p \in \mathcal{L}_n^{IPL}$ for a given $n \geq 0$, and let $\alpha \in For(\Omega)$

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

such that $\tilde{v}_p(\alpha) = \mathbf{U}$. By Definition 8.11 of intuitionistic partial' level valuation, there exists a level valuation w in $\mathcal{L}_{IPL}$ (and so, $w \in \mathcal{L}_n^{IPL}$) such that $\mathsf{P}_\Lambda(\tilde{v}_p, w, \alpha)$. Since $\Lambda = For(\Omega)$, this means that $\tilde{v}_p \in \mathcal{L}_{n+1}^{IPL}$. From this, $\tilde{v}_p \in \mathcal{L}_{IPL}$ and the result clearly holds by taking $v := \tilde{v}_p$.

Suppose now that $\Lambda \neq \emptyset$ is a proper subset of $For(\Omega)$. Consider an enumeration of $For(\Omega)$ as in Remark 8.15. The valuation v will be defined by induction on the complexity n of the formulas. More precisely, with terminology as in Definition 8.16 and from the observations in Remark 8.17, an extension v_n^m of $\tilde{v}_p$ (i.e., $\tilde{v}_p \subseteq v_n^m$) will be defined for every $(n, w) \in \omega \times \omega$ such that $v_n^m \in iPLV'(\bar{\Lambda}_n^m)$ and $v_i^j \subseteq v_n^m$ if $(i, j) \leq (n, m)$, where: $(i, j) \leq (n, m)$ iff $i \leq n$ or $i = n$ and $j \leq m$ (and where $\omega \times \omega$ denotes, as usual, the Cartesian product of ω with itself).

Base $n = m = 0$: Observe that $\bar{\Lambda}_0 = \bar{\Lambda}_0^0 = \Lambda \cup \mathcal{P}$. Define $v_0^0(\alpha) = \tilde{v}_p(\alpha)$ if $\alpha \in \Lambda$, and $v_0^0(p) = \mathbf{F}$ if p is a propositional variable which does not belong to Λ. Clearly, $\tilde{v}_p \subseteq v_0^0$ and $v_0^0 \in iPLV'(\bar{\Lambda}_0) = iPLV'(\bar{\Lambda}_0^0)$. Let $v_0 := v_0^0$.

Inductive step: Assume that $v_i^j(\alpha)$ was defined for every $\alpha \in \bar{\Lambda}_i^j$ such that $v_i^j(\alpha) = \tilde{v}_p(\alpha)$ if $\alpha \in \Lambda$, $v_i^j \in iPLV'(\bar{\Lambda}_i^j)$, $v_i^j \subseteq v_k^r$ if $(i, j) \leq (k, r) \leq (n, m)$, for given $n, m \geq 0$ (IH). Now it will be shown how to extend v_n^m to a function v_n^{m+1} with domain $\bar{\Lambda}_n^{m+1} = \bar{\Lambda}_n^m \cup \{\alpha_{\omega\cdot(n+1)+m}\}$. That is, it will be shown how to define a value for $\alpha_{\omega\cdot(n+1)+m}$ while preserving the values assigned by v_n^m, in such a manner that $v_n^{m+1} \in iPLV'(\bar{\Lambda}_n^{m+1})$. To start with, define $v_n^{m+1}(\beta) = v_n^m(\beta)$ for every $\beta \in \bar{\Lambda}_n^m$.

Let $\alpha = \alpha_{\omega\cdot(n+1)+m}$. If $\alpha \in \Lambda$ then $\bar{\Lambda}_n^{m+1} = \bar{\Lambda}_n^m$. Then $v_n^{m+1} := v_n^m$, and the required hypotheses are obviously satisfied by v_n^{m+1}. Suppose now that $\alpha \notin \Lambda$. Observe that $v_n^{m+1}(\beta) = v_n^m(\beta)$ was already defined, for every strict subformula β of α. Since $v_n^m \in iPLV'(\bar{\Lambda}_n^m)$ then, for every $\delta \in \bar{\Lambda}_n^m$ such that $v_n^m(\delta) = \mathbf{U}$ there exists $w \in \mathcal{L}_{IPL}$ such that $\mathsf{P}_{\bar{\Lambda}_n^m}(v_n^m, w, \delta)$. Such a w will be denoted by w^δ (observe that it is possible to have more than one w^δ for each δ, and it is possible to have $w^\delta = w^\gamma$ for $\delta \neq \gamma$). Recall that $\bar{\Lambda}_n^m$ is closed under subformulas. There are three cases to analyze:

(1) $\alpha = \neg\beta$ such that $\neg^{IPL}(v_n^m(\beta)) = \{\mathbf{U}, \mathbf{T}\}$, or $\alpha = \beta \to \gamma$ such that $\to^{IPL}(v_n^m(\beta), v_n^m(\gamma)) = \{\mathbf{U}, \mathbf{T}\}$. There are three subcases to analyze:

(1.1) There is $\delta \in \bar{\Lambda}_n^m$ with $v_n^m(\delta) = \mathbf{U}$ and there exists some w^δ such that $w^\delta(\alpha) = \mathbf{F}$. Define $v_n^{m+1}(\alpha) = \mathbf{U}$. Hence $v_n^{m+1} \in iPLV'(\bar{\Lambda}_n^{m+1})$ such that $w^\alpha = w^\delta$.

(1.2) There is $\delta \in \bar{\Lambda}_n^m$ with $v_n^m(\delta) = \mathbf{U}$ and there exists some w^δ such that $w^\delta(\alpha) = \mathbf{U}$. By Proposition 7.21, there exists $w'' \in \mathcal{L}_{IPL}$ such that $\mathsf{P}_{For(\Omega)}(w^\delta, w'', \alpha)$. Define $v_n^{m+1}(\alpha) = \mathbf{U}$. Hence, $v_n^{m+1} \in iPLV'(\bar{\Lambda}_n^{m+1})$ such that $w^\alpha = w''$.

(1.3) For every $\delta \in \bar{\Lambda}_n^m$ such that $v_n^m(\delta) = \mathbf{U}$, $w^\delta(\alpha) = \mathbf{T}$ for every w^δ. Define $v_n^{m+1}(\alpha) = \mathbf{T}$. Clearly, $v_n^{m+1} \in iPLV'(\bar{\Lambda}_n^{m+1})$.

(2) $\alpha = \neg\beta$ such that $\neg^{IPL}(v_n^m(\beta)) = \{\mathbf{F}\}$, or $\alpha = \beta\#\gamma$ such that $\#^{IPL}(v_n^m(\beta), v_n^m(\gamma)) = \{\mathbf{F}\}$ (for some $\# \in \{\vee, \wedge, \to\}$). Define $v_n^{m+1}(\alpha) = \mathbf{F}$. It is clear that $v_n^{m+1} \in iPLV'(\bar{\Lambda}_n^{m+1})$.

(3) $\alpha = \beta\#\gamma$ such that $\#^{IPL}(v_n^m(\beta), v_n^m(\gamma)) = \{\mathbf{T}\}$ (for some $\# \in \{\vee, \wedge, \to\}$). Define $v_n^{m+1}(\alpha) = \mathbf{T}$. It is clear that $v_n^{m+1} \in iPLV'(\bar{\Lambda}_n^{m+1})$.

We have shown in cases (1)–(3) how to extend the domain of v_n^m to the additional formula $\alpha_{\omega\cdot(n+1)+m} \in \bar{\Lambda}_n^{m+1}$, showing that the resulting function v_n^{m+1} is in $iPLV'(\bar{\Lambda}_n^{m+1})$, and $\tilde{v}_p \subseteq v_i^j \subseteq v_k^r$, for every $(i, j) \leq (k, r) \leq (n, m+1)$. That

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

is, we show how to increase the superscript m. In order to increase the subscript n, let $v_{n+1} = v^0_{n+1} := \bigcup_{m\in\omega} v^m_n$. By the procedure described above, it is immediate to see that v^0_{n+1} is in $iPLV'(\bigcup_{m\in\omega} \bar{\Lambda}^m_n) = iPLV'(\bar{\Lambda}_{n+1}) = iPLV'(\bar{\Lambda}^0_{n+1})$, and $\tilde{v}_p \subseteq v^j_i \subseteq v^r_k$, for every $(i,j) \leq (k,r) \leq (n+1,0)$. By repeating the process, it is possible to define v^m_n for every $(n,m) \in \omega \times \omega$ with the required properties. Finally, let $v := \bigcup_{n\in\omega} v_n$. From the manner in which the construction was made, it follows that v is in $iPLV'(\bigcup_{n\in\omega} \bar{\Lambda}_n) = iPLV'(For(\Omega))$, and $\tilde{v}_p \subseteq v$. By the first part of the proof, we conclude that $v \in \mathcal{L}_{IPL}$. ⊣

PROPOSITION 8.19. *Let* $\Lambda \in FCS(\Omega)$. *Then* $iPLV(\Lambda) = iPLV'(\Lambda)$.

PROOF. Let $\tilde{v}_p \in iPLV'(\Lambda)$. Then, $\tilde{v}_p \in iPV(\Lambda)$. Suppose that $\alpha \in \Lambda$ such that $\tilde{v}_p(\alpha) = \mathbf{U}$. Then, there exists $w \in \mathcal{L}_{IPL}$ such that $\mathsf{P}_\Lambda(\tilde{v}_p, w, \alpha)$. Let $\tilde{w}_p = w_{|\Lambda}$. By Lemma 8.13, $\tilde{w}_p \in iPLV'(\Lambda)$ such that $\mathsf{P}_\Lambda(\tilde{v}_p, \tilde{w}_p, \alpha)$. Now, suppose that there exists $\tilde{w}'_p \in iPLV'(\Lambda)$ such that $\mathsf{P}_\Lambda(\tilde{v}_p, \tilde{w}'_p, \alpha)$. By Lemma 8.18, there exists $v' \in \mathcal{L}_{IPL}$ such that the restriction of v' to Λ coincides with $\tilde{w}'_p$. Hence, $\mathsf{P}_\Lambda(\tilde{v}_p, v', \alpha)$. This shows that

$$iPLV'(\Lambda) = \{\tilde{v}_p \in iPV(\Lambda) \mid \forall \alpha \in \Lambda\big(\tilde{v}_p(\alpha) = \mathbf{U} \text{ implies that } \mathsf{P}_\Lambda(\tilde{v}_p, \tilde{w}_p, \alpha) \text{ for some } \tilde{w}_p \in iPLV'(\Lambda)\big)\}.$$

From this, it follows that $iPLV(\Lambda) = iPLV'(\Lambda)$, by Proposition 8.7. ⊣

REMARK 8.20. In Proposition 8.7 the set $iPLV(\Lambda)$ was characterized constructively, as the output of an algorithm (Algorithm 1). In this section, we define an alternative subset $iPLV'(\Lambda) \subseteq iPV(\Lambda)$ in a predicative way. This subset connects the RNmatrix semantics for *IPL* (based on level valuations) to the truth-table procedure given by $iPLV(\Lambda)$. Analyticity and co-analyticity of $iPLV'(\Lambda)$ for level valuations allow us to show that, for a finite Λ, $iPLV'(\Lambda)$ satisfies the condition of Remark 8.4. By uniqueness (Proposition 8.7), $iPLV'(\Lambda)$ must coincide with $iPLV(\Lambda)$, i.e., with the truth-table generated by Λ. But then we obtain analyticity and co-analyticity of $iPLV(\Lambda)$ for level valuations, which implies the soundness and completeness of truth-tables for *IPL* (see Theorem 8.22).

DEFINITION 8.21. Let $\Gamma \cup \{\varphi\}$ be a finite subset of $For(\Omega)$. Let Λ be the (finite) set of subformulas of $\Gamma \cup \{\varphi\}$. We say that φ is a consequence of Γ w.r.t. intuitionistic truth-tables, denoted by $\Gamma \models_{iPLV} \varphi$, if, for every intuitionistic partial level valuation $\tilde{v}_p \in iPLV(\Lambda)$, if $\tilde{v}_p(\beta) = \mathbf{T}$ for every $\beta \in \Gamma$ then $\tilde{v}_p(\varphi) = \mathbf{T}$. In particular, $\emptyset \models_{iPLV} \varphi$, denoted simply by $\models_{iPLV} \varphi$, if $\tilde{v}_p(\varphi) = \mathbf{T}$ for every $\tilde{v}_p \in iPLV(\Lambda)$, where Λ is the set of subformulas of φ.

Finally, we establish the correspondence between the RNmatrix for *IPL* and the intuitionistic truth-tables.

THEOREM 8.22 (Soundness and completeness of intuitionistic truth-tables w.r.t. $\mathcal{R}(\mathcal{M}_{IPL})$). *Let* $\Gamma \cup \{\varphi\}$ *be a finite subset of* $For(\Omega)$. *Then*: $\Gamma \models_{\mathcal{R}(\mathcal{M}_{IPL})} \varphi$ *if, and only if,* $\Gamma \models_{iPLV} \varphi$.

PROOF. Let Λ be the set of subformulas of $\Gamma \cup \{\varphi\}$. Observe that $\Lambda \in FCS(\Omega)$. Suppose that $\Gamma \models_{\mathcal{R}(\mathcal{M}_{IPL})} \varphi$. Let $\tilde{v}_p \in iPLV(\Lambda)$ such that $\tilde{v}_p(\beta) = \mathbf{T}$ for every $\beta \in \Gamma$. By Proposition 8.19, $\tilde{v}_p \in iPLV'(\Lambda)$. By Lemma 8.18, there exists $v \in \mathcal{L}_{IPL}$

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

such that the restriction $v_{|\Lambda}$ of v to Λ coincides with $\tilde{v}_p$. Then, $v(\beta) = \mathbf{T}$ for every $\beta \in \Gamma$ and so, by hypothesis, $v(\varphi) = \mathbf{T}$. That is, $\tilde{v}_p(\varphi) = \mathbf{T}$. This shows that $\Gamma \models_{iPLV} \varphi$.

Conversely, suppose that $\Gamma \models_{iPLV} \varphi$. Let $v \in \mathcal{L}_{IPL}$ such that $v(\beta) = \mathbf{T}$ for every $\beta \in \Gamma$. By Lemma 8.13, the restriction $\tilde{v}_p := v_{|\Lambda}$ of v to Λ belongs to $iPLV'(\Lambda)$. By Proposition 8.19, $\tilde{v}_p \in iPLV(\Lambda)$ such that $\tilde{v}_p(\beta) = \mathbf{T}$ for every $\beta \in \Gamma$. By hypothesis, $\tilde{v}_p(\varphi) = \mathbf{T}$. Then, $v(\varphi) = \mathbf{T}$ and so $\Gamma \models_{\mathcal{R}(\mathcal{M}_{IPL})} \varphi$. ⊣

COROLLARY 8.22.1 (DM for intuitionistic truth-tables). *Let $\Gamma \cup \{\alpha, \beta\}$ be a finite subset of $For(\Omega)$. Then, $\Gamma, \alpha \models_{iPLV} \beta$ if, and only if, $\Gamma \models_{iPLV} \alpha \to \beta$.*

PROOF. It follows from Theorem 8.22 and Corollary 7.19.2. ⊣

It is worth noting that the latter result is useful to reduce the complexity of the truth-tables: instead of generating the (reduced) truth-table for a formula $\delta = \beta_1 \to (\beta_2 \to (\ldots \to (\beta_n \to \varphi)\ldots))$ to be tested as a tautology, it suffices generating the (reduced) truth-table for $\{\beta_1, \ldots, \beta_n, \varphi\}$ and then determine if, in every row such that $\beta_1, \ldots, \beta_n$ receive simultaneously the value $\mathbf{T}$, φ also receives the value $\mathbf{T}$. This reduces the number of rows required to test δ.

Finally, we arrive to the main result of the article, recalling that **LI** is a standard sequent calculus for *IPL*.

THEOREM 8.23 (Soundness and completeness of **LI** w.r.t. intuitionistic truth-tables). *Let $\Gamma \cup \{\varphi\}$ be a finite subset of $For(\Omega)$. Then: $\Gamma \vdash_{IPL} \varphi$ if, and only if, $\Gamma \models_{iPLV} \varphi$.*

PROOF. It follows from Theorems 7.12, 7.18, and 8.22. ⊣

The latter result shows that the procedure of constructing (non-deterministic) truth-tables in Definition 8.9 determines a decision method for propositional intuitionistic logic *IPL*.

§9. Some considerations on complexity. Unlike deterministic truth-tables, the size of non-deterministic truth-tables cannot be determined solely by the number of atoms. This is because, in principle, each subformula may (or may not) create new rows. Thus, we must also consider the number and shape of every evaluated subformula.

A workaround for this problem is to establish size limits using boundaries. More formally, we can define a general upper bound for Nmatrices by identifying their branching factor as follows.

DEFINITION 9.1 (Branching factor). Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be some Nmatrix over a signature Θ. The branching factor of $\mathcal{M}$ is the lowest κ such that, for every n-ary $\# \in \Theta_n$ and every $a_1, \ldots, a_n \in \mathcal{V}$, $|\mathcal{O}(\#)(a_1, \ldots, a_n)| \leq \kappa$.

DEFINITION 9.2 (Upper bound). Let $sub(\alpha)$ be the set of all subformulas of α, $atoms(\alpha)$ the set of subformulas with complexity equal to 0, $\kappa \geq 1$ the branching factor, and $\theta \leq |\mathcal{V}|$ the number of truth-values used for atomic valuations. Then, the maximum number of rows of the initial matrix for α in $\mathcal{R}(\mathcal{M})$ is given by

$$ub_{\mathcal{M}}(\alpha) = \frac{\kappa^{|sub(\alpha)|}}{\kappa^{|atoms(\alpha)|}} \times \theta^{|atoms(\alpha)|}.$$

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

EXAMPLE 9.3. In $\mathcal{R}(\mathcal{M}_{IPL})$, $\kappa = 2$ and $\theta = 2$. Hence,

$$ub_{\mathcal{M}_{IPL}} = 2^{|sub(\alpha)|}. \tag{4}$$

A fortiori, we can infer that the upper bound for the Nmatrix is also an upper bound for the RNmatrix. Therefore, we can expect no more than $2^{|sub(\alpha)|}$ rows in a truth-table created for α with $\mathcal{R}(\mathcal{M}_{IPL})$. The truth-table may be shorter than $2^{|sub(\alpha)|}$, but not less than $2^{|atoms(\alpha)|}$, given that there are at least $2^{|atoms(\alpha)|}$ (trivially) valid partial valuations in the truth-table.

This exponential growth in the number of subformulas may be problematic; a formula with only one variable can be arbitrarily large and produce immense tables. However, truth-tables are meaningful. By constructing a truth-table, one generates significant information about a given proposition or argument. A future work direction would be improving the average-case runtime by refining the algorithm with state-of-the-art strategies.

§10. Related works. The idea of using finite non-deterministic semantics for characterizing intuitionistic logic and modal logic is not new. This section will mention some approaches to this topic proposed in the literature. We do not intend to present a detailed description and comparison between all these approaches here since they are beyond the scope of the present article.

(1) Andrea Loparic already obtained in 1977, announced in [24] and finally published in [25], a semantical characterization of Johannson–Kolmogorov minimal logic and *IPL* in terms of a 2-valued, non-truth-functional valuation semantics which induces a decision procedure for these logics. Similar to Grätz's method (and to the adaptation to *IPL* proposed here), Loparic's approach is based on a criterion for distinguishing between what she called *semivaluations* and *valuations*, the latter being the useful ones. This would correspond to level valuations, since such valuations are associated with φ-saturated sets, and they are the ones that characterize the logic. The associated algorithm generates a non-deterministic truth-table for a given formula, as in the cases of *S4* and *IPL* presented in this article. Then, using formal criteria based on implicative formulas, it deletes certain rows. Thus, a formula is a tautology of the given logic if and only if it receives the value 1 in all remaining rows.

(2) J. Kearns proposed in [18] a semantics for (first-order) intuitionistic logic based on the notion of *justifications*. A 3-valued Nmatrix, whose domain is not formed by truth-values but by values of justification (+ for 'justified', o for 'weakly unjustified', and – for 'strongly unjustifiable'), is considered for interpreting the connectives of *IPL*. A sentence is weakly unjustified if it is not justified by the currently justified sentences but might become justified later. In turn, strongly unjustifiable sentences are those already ruled out by the currently justified sentences. Given two valuations v and v' over the Nmatrix, $v \preceq v'$ (v *is contained in* v') if v' preserves the values + (and so, the values –) assigned by v to each formula. Hence, if $v'(\alpha) = o$ then $v(\alpha) = o$, for any α. The intuitive idea is that v' is a 'possible future' of v (hence v' has more 'knowledge' than v, provided that the weak epistemic status o may be corrected to the stronger + or – in the future). Kearns constructed a hierarchy of level valuations (as he would later do for modal logics) starting with level 0,

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

which is formed by all valuations on the Nmatrix. The $(n+1)$th-level valuations are the nth-level valuations satisfying certain restrictions defined in terms of the sets $X_n^v = \{v' \mid v' \text{ is an } n\text{th-level valuation such that } v \preceq v'\}$, for any nth-level valuation v. A basic feature is that, if v is an $(n+1)$th-level valuation and $v'(\alpha) \neq +$ for every $v' \in X_n^v$ then $v(\alpha) = -$. That is, if α is not justified in any possible future of v, then α must be strongly unjustifiable at present v. In a second paper [20] on the subject, published two years after [18] and seven months after [19], Kearns improved the interpretation of the justification semantics and simplified its formal presentation by using trees instead of levels. The *contained* relation $\preceq$ is crucial to define the trees whose nodes are valuations. It should be stressed that Kearns's proposal constitutes a 3-valued RNmatrix semantics for *IPL*, although it differs significantly from our approach. In particular, Kearns's RNmatrix does not seem to induce a decision procedure for *IPL*.

(3) Concerning modal logics, in [21], right after the publication of [12], O. Lahav and Y. Zohar proposed a decision procedure for systems **K** and **KT** based on 4- and 3-valued RNmatrices, respectively, by using a decision procedure different (but closely related) to Grätz's approach. Indeed, for $\mathbf{L} \in \{\mathbf{K}, \mathbf{KT}\}$ they considered a sequent calculus $\mathcal{G}_{\mathbf{L}}$ and sets $\mathbb{V}_{\mathbf{L}}^{\mathcal{F},n}$ of nth-level valuations relative to a set $\mathcal{F}$ of formulas (the domain of such valuations) and n is the maximum number of applications of the necessitation rule validated by such valuations. Given a sequent $\Gamma \Rightarrow \varphi$, the expression $\vdash_{\mathcal{G}_{\mathbf{L}}}^{\mathcal{F},n} \Gamma \Rightarrow \varphi$ denotes that there exists a derivation in $\mathcal{G}_{\mathbf{L}}$ of $\Gamma \Rightarrow \varphi$ with a derivation containing exclusively formulas in $\mathcal{F}$ and such that the number of applications of the inference rule of necessitation $\Delta \Rightarrow \psi / \Box\Delta \Rightarrow \Box\psi$ in any of the branches of the derivation is limited by n. They proved that $\vdash_{\mathcal{G}_{\mathbf{L}}}^{\mathcal{F},n} \Gamma \Rightarrow \varphi$ if and only if φ is a consequence of Γ w.r.t. nth-level valuations in $\mathbb{V}_{\mathbf{L}}^{\mathcal{F},n}$. Moreover, they obtained a decision procedure for **K** by considering, for each sequent $\Gamma \Rightarrow \varphi$ to be tested, the set of level valuations $\mathbb{V}_{\mathbf{L}}^{\mathcal{F},n}$, where $\mathcal{F}$ is the set of subformulas of $\Gamma \cup \{\varphi\}$ and $n = 4^{|\mathcal{F}|}$. For **KT**, it is enough considering $n = 3^{|\mathcal{F}|}$, since the RNmatrix for **KT** is 3-valued. Moreover, that RNmatrix coincides with the one proposed in [12] for **KT**. It is worth noting that [21] considers the reduced version of the underlying Nmatrix, expanded to the full signature. This produces the Nmatrix $\mathcal{M}'_{S4}$ for *S4* of Figure 4, but where $\Box'(2) = \{1, 2\}$. Despite these similarities, the decision procedure in [21] (as mentioned earlier) differs from that in [12]. It consists of generating the set of nth-level valuations over $\mathcal{F}$ for appropriate n and $\mathcal{F}$.

(4) More recently, A. Solares-Rojas proposed a 3-valued Nmatrix semantics for *IPL* based on intuitionistic Kripke models and the novel notion of *depth-bounded approximations* (see [30, Chapter 5]). Within this semantic approach to *IPL*, a 3-valued Nmatrix—whose truth-values are intuitionistic truth (1), falsity (0), and indeterminacy (u)—gives the meaning of the connectives in terms of information that a given agent holds. The valuations over the Nmatrix aim to model actual information states, ordered by a certain *refinement relation* $\sqsubseteq_3$. This relation models how the agent's knowledge evolves when new information from reliable external sources is acquired. The semantics for *IPL* is defined via intuitionistic Kripke frames expanded with a family of valuations indexed by the worlds, where the refinement order corresponds to the frame's accessibility relation. Some similarities between this approach and Kearns's (mentioned in point **(2)**) may be observed. Conceptually,

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

both approaches aim to model the evolution of states of knowledge: the information state of an agent in [30] and the knowledge of an idealized community in [18, 20]. Regarding technical aspects, by identifying 1, 0, and u with $+$, $-$, and o, respectively, the Nmatrix in [30, Chapter 5] coincides with that in [18], with one exception: while $o \vee o = \{o, +\}$ in Kearns's work, $u \vee u = \{u\}$ (i.e., $o \vee o = \{o\}$) in Solares-Rojas's approach. Additionally, the refinement relation $\sqsubseteq_3$ between valuations coincides with the *contained* relation $\preceq$ introduced in [18]. The relationship between both approaches (in particular, the connections with the improved version [20], which is closer to Kripke and Beth's semantics) deserves to be investigated. Regarding the connections with our approach, although the 3-valued Nmatrices proposed in [30, Chapter 5] and the one we gave in Definition 7.4 are quite different, it is not immediate whether the semantics proposed by Solares-Rojas could be seen as given by an RNmatrix. This is the subject of future investigation, as well as the relationship between our approach and the semantics of justification introduced by Kearns.

§11. Concluding remarks. In this article, we proposed a new decision procedure for *IPL* based on a 3-valued RNmatrix semantics. This procedure was obtained by abstracting the composition of Grätz's decision procedure for *S4* [12] and Gödel's faithful translation from *IPL* to *S4*. A 3-valued Nmatrix with an intuitive interpretation in epistemic terms was proposed for *IPL*. By adapting Grätz's algorithm for *S4*, a suitable notion of level valuations was obtained, which allowed us to define a straightforward algorithm for deleting spurious rows in the non-deterministic truth-tables generated by the Nmatrix for any formula. The soundness and completeness proofs for the RNmatrix and the resulting truth-table algorithm for *IPL* were presented in full technical detail.

It is worth noting that the Nmatrix for *IPL* introduced in Definition 7.4 has a clear and intuitive interpretation in terms of knowledge (see comments after Definition 7.4) and constructibility. Indeed, recall that the valuation associated with a φ-saturated set Δ in *IPL* (given in Definition 7.14) acts as a 3-valued characteristic function of Δ, and note the observations in Remark 7.15.

By considering finite and decidable RNmatrix semantics as a generalization of standard finite-valued deterministic truth-tables, the results presented here provide a way to overcome the limitations obtained by Gödel and Dugundji for *IPL* and *S4*, respectively, as mentioned in Section 1.

For future research, we intend to explore the meaning of the partial valuations corresponding to the rows of the non-deterministic truth-tables in *IPL* and *S4*. In particular, we are studying the possibility of generating finite Kripke models for *IPL*, *KT*, and *S4* from the resulting truth-tables. This could be connected to *finite approximability* [5, p. 49 and Chapter 11] or to *mini-canonical models* [13, Chapter 8]. Finally, the extension of this methodology to other modal logics, as well as to Prawitz's ecumenical systems [28], will also be investigated. Some steps in this direction were taken in [23].

Acknowledgments. We thank the anonymous referee for the insightful and precise comments on an earlier version of this article. Their feedback was essential in improving the overall quality of the manuscript.

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

Funding. The first and second authors thank the support of the São Paulo Research Foundation (FAPESP, Brazil) through the Ph.D. scholarship grant no. 21/01025-3 and the Thematic Project RatioLog no. 20/16353-3. The second author also acknowledges support from the National Council for Scientific and Technological Development (CNPq, Brazil), through the individual research grant no. 309830/2023-0. The third author thanks Fundação Carlos Chagas de Amparo à Pesquisa do Estado do Rio de Janeiro (FAPERJ, Brazil), through the research grant no. E-26.210.971/2019.

REFERENCES

[1] A. Avron, *Non-deterministic semantics for logics with a consistency operator*. ***International Journal of Approximate Reasoning***, vol. 45 (2007), no. 2, pp. 271–287.

[2] A. Avron and I. Lev, *Canonical propositional Gentzen-type systems*, ***Automated Reasoning. IJCAR 2001*** (R. Goré, A. Leitsch, and T. Nipkow, editors), volume 2083 of Lecture Notes in Computer Science, Berlin, Heidelberg, Springer, 2001, pp. 529–544.

[3] A. Avron and Y. Zohar, *Rexpansions of nondeterministic matrices and their applications in nonclassical logics*. ***The Review of Symbolic Logic***, vol. 12 (2019), no. 1, pp. 173–200.

[4] C. Caleiro and S. Marcelino, *On axioms and rexpansions*, ***Arnon Avron on Semantics and Proof Theory of Non-classical Logics*** (O. Arieli and A. Zamansky, editors), volume 21 of Outstanding Contributions to Logic Series, Springer, Cham, 2021, pp. 39–69.

[5] A. Chagrov and M. Zakharyaschev, ***Modal Logic***, Oxford University Press, New York, 1997.

[6] M. E . Coniglio, L F. del Cerro, N. M . Peron, and N. M. Peron, *Errata and addenda to 'finite non-deterministic semantics for some modal systems'*. ***Journal of Applied Non-Classical Logics***, vol. 26 (2016), no. 4, pp. 336–345.

[7] M. E. Coniglio and G. V. Toledo, *Two decision procedures for da costa's C_n logics based on restricted Nmatrix semantics*. ***Studia Logica***, vol. 110 (2022), no. 3, pp. 601–642.

[8] J. Dugundji, *Note on a property of matrices for Lewis and Langford's calculi of propositions*. ***The Journal of Symbolic Logic***, vol. 5 (1940), no. 4, pp. 150–151.

[9] S. Feferman, J. W. Dawson Jr., S. C. Kleene, G. H. Moore, R. M. Solovay, and J. van Heijenoort, ***Kurt Gödel Collected Works, Volume I, Publications 1929–1936***, Oxford University Press, New York, 1986.

[10] K. Gödel, *Zum intuitionistischen aussagenkalkül*. ***Anzeiger der Akademie der Wissenschaften in Wien***, vol. 69 (1932), pp. 65–66. English translation as 'On the intuitionistic propositional calculus'. In [9], pp. 223–225.

[11] ———, *Eine interpretation des intuitionischen aussagenkalküls*, ***Ergebnisse eines Mathematischen Kolloquiums***, vol. 4 (1933), pp. 39–40. English translation as 'An interpretation of the intuitionistic propositional calculus'. In [9], page 301.

[12] L. Grätz, *Truth tables for modal logics T and S4, by using three-valued non-deterministic level semantics*. ***Journal of Logic and Computation***, vol. 32 (2022), no. 1, pp. 129–157.

[13] G. E . Hughes and M. J. Cresswell, ***A New Introduction to Modal Logic***, Routledge, London, 1996.

[14] Y. V. Ivlev, *Tablitznoe postrojenie propozicionalnoj modalnoj logiki (Truth-tables for systems of propositional modal logic, in Russian)*. ***Vestnik Moskovskogo Universiteta. Seriya 7. Filosofiya***, vol. 6 (1973), pp. 51–61.

[15] ———, ***Soderzhatel'naya semantika modal'noy logiki (Content semantics for modal logic, in Russian)***, Ph.D. thesis, Moscow State University M. V. Lomonossov, 1986.

[16] ———, *A semantics for modal calculi*. ***Bulletin of the Section of Logic***, vol. 17 (1988), nos. 3–4, pp. 114–121.

[17] ———, *Quasi-matrix (quasi-functional) logic*, ***Many-Valued Semantics and Modal Logics: Essays in Honor to Yu. V. Ivlev*** (M. E. Coniglio, E. Kubishkina, and D. Zaitsev, editors), Synthese Library—Studies in Epistemology, Logic, Methodology, and Philosophy of Science Series, 485 Springer Nature, Cham, 2024, pp. 3–51.

[18] J. T. Kearns, *Intuitionist logic, a logic of justification*. ***Studia Logica***, vol. 37 (1978), no. 3, pp. 243–260.

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press

[19] ———, *Modal semantics without possible worlds*. ***The Journal of Symbolic Logic***, vol. 46 (1981), no. 1, pp. 77–86.

[20] ———, *A more satisfactory description of the semantics of justification*. ***Notre Dame Journal of Formal Logic***, vol. 22 (1981), no. 2, pp. 109–119.

[21] O. Lahav and Y. Zohar, *Effective semantics for the modal logics K and KT via non-deterministic matrices*, ***Proceedings of the International Joint Conference on Automated Reasoning—IJCAR 2022*** (J. Blanchette, L. Kovács, and D. Pattinson, editors), Lecture Notes in Computer Science (LNAI), 13385, Springer, Cham, 2022, pp. 468–485.

[22] R. Leme, M. E. Coniglio, and B. Lopes, *A new decision method for intuitionistic logic by 3-valued non-deterministic truth-tables (pre-print version)*, preprint, 2023, arXiv:2308.13664. https://arxiv.org/abs/2308.13664

[23] R. Leme, C. Olarte, E. Pimentel, and M. E. Coniglio, *The modal cube revisited: Semantics without worlds*, ***Automated Reasoning with Analytic Tableaux and Related Methods. Proceedings of the 34th International Conference, TABLEAUX 2025*** (G. L. Pozzato and T. Uustalu, editors), Lecture Notes in Computer Science (LNAI), 15980, pages 181–200 Springer, 2025.

[24] A. Loparic, *Une étude sémantique de quelques calculs propositionnels*. ***Comptes Rendus de l'Académie de Sciences de Paris (A)***, vol. 284 (1977), pp. 835–838.

[25] ———, *Valuation semantics for intuitionistic propositional calculus and some of its subcalculi*. ***Principia***, vol. 14 (2010), no. 1, pp. 125–133.

[26] J. C. C. McKinsey and A. Tarski, *Some theorems about the sentential calculi of Lewis and Heyting*. ***The Journal of Symbolic Logic***, vol. 13 (1948), no. 1, pp. 1–15.

[27] G. Mints, *The Gödel-Tarski translations of intuitionistic propositional formulas*. ***Correct Reasoning: Essays on Logic-Based AI in Honour of Vladimir Lifschitz***, (2012), pp. 487–491.

[28] D. Prawitz, *Classical versus intuitionistic logic*, ***Why Is this a Proof? Festschrift for Luiz Carlos Pereira*** (E. H. Haeusler, W. Sanz, and B. Lopes, editors), Tributes, 27, College Publications, London, 2015, pp. 15–32.

[29] N. Rescher, *Quasi-truth-functional systems of propositional logic*. ***The Journal of Symbolic Logic***, vol. 27 (1962), no. 1, pp. 1–10.

[30] A. Solares-Rojas, ***Tractable depth-bounded approximations to some propositional logics: Towards more realistic models of logical agents***, Ph.D. thesis, Università degli Studi di Milano, 2022.

[31] G. Takeuti, ***Proof Theory***, North-Holland, Amsterdam, 1987.

INSTITUTE OF PHILOSOPHY AND HUMAN SCIENCES (IFCH) AND
CENTRE FOR LOGIC, EPISTEMOLOGY AND THE HISTORY OF SCIENCE (CLE)
STATE UNIVERSITY OF CAMPINAS (UNICAMP)
BRAZIL

E-mail: rntreisleme@gmail.com
E-mail: coniglio@unicamp.br

COMPUTING INSTITUTE (IC)
FLUMINENSE FEDERAL UNIVERSITY (UFF)
BRAZIL

E-mail: bruno@ic.uff.br

https://doi.org/10.1017/jsl.2025.10174 Published online by Cambridge University Press