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The parameter space of cubic laminations with a fixed critical leaf

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Abstract. Thurston parameterized quadratic invariant laminations with a non-invariant lamination, the quotient of which yields a combinatorial model for the Mandelbrot set. As a step toward generalizing this construction to cubic polynomials, we consider slices of the family of cubic invariant laminations defined by a fixed critical leaf with non-periodic endpoints. We parameterize each slice by a lamination just as in the quadratic case, relying on the techniques of smart criticality previously developed by the authors.

1. *Introduction*

One of the major goals of complex dynamics is to describe the structure of the space of complex polynomials of degree d up to Moebius conjugation. In the quadratic case, this space of polynomials can be identified with the family $P_c(z) = z^2 + c$ with $c \in \mathbb{C}$ parameterizing the family. Since c is the critical value of $P_c(z)$, it shows that dynamically motivated points in the plane (that is, the critical value of each polynomial) can be used to parameterize the space of all polynomials.

A major achievement by Thurston [**Thu85**] was his combinatorial interpretation of the above space. To achieve this, he introduced several important dynamical and combinatorial notions. One of them was that of a *lamination*. Laminations are equivalence relations on the unit circle in the plane with finite (equivalence) classes whose convex hulls in the unit disk are pairwise disjoint. A lamination is σ_2*-invariant* if its equivalence classes are

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preserved under σ_2. Using a Riemann map, one can see that every quadratic polynomial with a locally connected Julia set corresponds to a unique σ_2-invariant lamination.

Below, we give a modified (but equivalent) description of Thurston's results [**Thu85**]. Every σ_2-invariant lamination $\sim$ has a unique *minor* set $m_\sim$ which is the convex hull of the image of a $\sim$-class of maximal diameter. A striking result by Thurston is that the minor sets are pairwise disjoint and form a lamination of the unit circle! The quotient space of the unit circle under this lamination then parameterizes the set of all σ_2-invariant laminations and serves as a model of the Mandelbrot set.

In this paper, we will consider a slice of the set of all σ_3-laminations. Fix a chord $D = \overline{ab}$ such that $\sigma_3(a) = \sigma_3(b)$, where both a and b are not periodic. We consider the set LAM_D of all σ_3-invariant laminations $\sim$ such that $a \sim b$, except for laminations of *Siegel capture type* defined below. For each lamination $\sim$ in LAM_D, we define its minor set $m_\sim$ similarly to the above. Then we apply machinery from [**BOPT14**] to show that, in a way that is similar to the case of the Mandelbrot set, the collection of all such minor sets is a lamination itself so that the quotient space of the unit circle by this lamination parameterizes LAM_D.

We now describe how the paper is organized. While the precise definitions are given later, here we give heuristic versions of some of them. Also, even though all notions can be introduced for any degree d, our main focus here is cubic, and hence some definitions are given only in the cubic case.

In §2, we provide the motivating background from complex dynamics, define laminations, geolaminations (which will always be denoted by $\mathcal{L}$), leaves, (infinite, all-critical) gaps and related notions. In particular (cf. [**BMOV13**]), we define *proper geolaminations* as geolaminations such that no leaf can have one periodic and one non-periodic endpoint. We also review the relevant combinatorial machinery developed in [**BOPT14**]. Then we define less standard notions, some of which were introduced in [**BOPT14**]. We give an overview of these notions here. In the rest of the Introduction we will mostly consider the cubic case.

A set is a *generalized (critical) quadrilateral* if it is a critical leaf, an all-critical triangle (which collapses to a point), a *collapsing quadrilateral* (that is, a quadrilateral which maps to a chord) or an all-critical quadrilateral (in the cubic case the last case is impossible). If $\mathcal{L}$ is cubic, a *quadratically critical portrait (qc-portrait) of* $\mathcal{L}$ is an ordered pair of *distinct* generalized quadrilaterals that are leaves or gaps of $\mathcal{L}$. In fact, cubic qc-portraits can be defined without geolaminations: an *admissible (cubic) qc-portrait* is an ordered pair of generalized quadrilaterals such that (1) a generalized quadrilateral with a periodic vertex either has a degenerate image or, otherwise, has a periodic edge and (2) they and all their images intersect at most over a common edge or vertex. By a standard pullback construction, an admissible cubic qc-portrait can be included in a cubic geolamination. Here we study only the family $\mathcal{QCP}_3^{\mathrm{np}}$ of admissible qc-portraits QCP such that the second set in QCP is a critical leaf D with non-periodic endpoints.

In general, a geolamination may have no qc-portrait; it has a qc-portrait if and only if all its critical sets are collapsing quadrilaterals, critical leaves or all-critical gaps [**BOPT14**]. However, invariant geolaminations can be *tuned* to acquire a qc-portrait without compromising the dynamics. To this end, generalized quadrilaterals (which form

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a qc-portrait) are inserted into critical gaps of a given proper cubic geolamination $\mathcal{L}$. Using a well-known procedure, one can complete the inserted leaves and their images with their pullbacks and, in this way, define a new tuned invariant geolamination. Observe that the new tuned geolamination is not necessarily proper as we may need to insert new leaves which connect periodic and non-periodic points.

Note that, here, we consider proper geolaminations $\mathcal{L}$ which have a critical leaf D with non-periodic endpoints. For such geolaminations there are a few possibilities concerning their critical sets (see §2.3). First, $\mathcal{L}$ can have a finite first critical set C. By properties of invariant geolaminations, C is a gap or a leaf on which σ_3 acts two-to-one (unless D is an edge of C and so the point $\sigma_3(D)$ has all three of its preimages in C). Thus, if C is finite, there are two cases. First, C can be a $(2n+1)$-gon with D being one of its edges such that one can break down all its edges into pairs of 'sibling edges', that is, leaves with the same image except for D (one can say that the 'sibling edge' of D is the vertex of C not belonging to D and with the same image as D). Second, C can be a $2m$-gon such that D is not an edge of C, in which case $\sigma_3|_C$ is two-to-one.

Also, C could be an infinite gap. Then there are two cases. First, C may be a periodic Fatou gap of some period k and degree two (observe that, in this case, D may well be an edge of C). Second, there may exist a periodic Siegel gap U with D being one of its edges and an infinite gap V such that $\sigma_3|_{\mathrm{Bd}(V)}$ is two-to-one and V eventually maps to U. We call such (geo)laminations *Siegel (geo)laminations of capture type*.

Now, let $\mathbb{LP}_3^{\mathrm{np}}$ be the family of all cubic proper geolaminations that have a critical leaf with non-periodic endpoints *except for Siegel geolaminations of capture type*. If $\mathcal{L} \in \mathbb{LP}_3^{\mathrm{np}}$ has a critical leaf D with non-periodic endpoints, then a qc-portrait $\mathrm{QCP} = (Q, D)$ is called *privileged for* $\mathcal{L}$ if and only if $Q \subset C$, where $C \neq D$ is a critical set of $\mathcal{L}$ and either C is finite or C is a periodic Fatou gap of degree two and period k and Q is a collapsing quadrilateral which is the convex hull of a (possibly degenerate) edge ℓ of C of period k and another edge $\hat{\ell}$ of C such that $\sigma_3(\ell) = \sigma_3(\hat{\ell})$.

In §3, we state our main results. We show that, for each $\mathcal{L} \in \mathbb{LP}_3^{\mathrm{np}}$, there are only a finite number of privileged qc-portraits and, for every admissible qc-portrait $\mathrm{QCP} \in \mathcal{QCP}_3^{\mathrm{np}}$, there exists $\mathcal{L}_{\mathrm{QCP}} \in \mathbb{LP}_3^{\mathrm{np}}$ such that QCP is privileged for $\mathcal{L}_{\mathrm{QCP}}$ and the lamination induced by $\mathcal{L}_{\mathrm{QCP}}$ is unique. For a critical leaf D with non-periodic endpoints, $\mathcal{S}_D$ denotes the collection of all admissible qc-portraits (Q, D) of $\mathcal{QCP}_3^{\mathrm{np}}$ with D as the second element. To each such qc-portrait (Q, D) we associate the set (a chord or a point called *minor*) $\sigma_3(Q) \subset \overline{\mathbb{D}}$. For each such chord we identify its endpoints, extend this identification by transitivity and define the corresponding equivalence relation $\sim_D$ on $\mathbb{S}$. Our main result is that $\sim_D$ is itself a lamination whose quotient is a parameterization of LAM_D.

Section 4 describes how we will tag σ_3-invariant geolaminations with privileged qc-portraits and shows that the tagging generalizes Thurston's tagging of σ_2-invariant geolaminations by minors. We show that any qc-portrait which contains a critical leaf without periodic endpoints is privileged for some geolamination. This implies that when we fix a critical leaf with non-periodic endpoints we obtain a closed set of qc-portraits.

In §5, we study possible intersections of minors of our qc-portraits. Finally, in §6 we construct the parameter lamination $\sim_D$.

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2. *Preliminaries*

We assume basic knowledge of complex dynamics and use standard notation. Let $P: \mathbb{C} \to \mathbb{C}$ be a polynomial of degree $d \geq 2$, A_∞ be the basin of attraction of infinity and $J_P = \mathrm{Bd}(A_\infty)$ be the Julia set of P. When J_P is connected, A_∞ is simply connected and conformally isomorphic to $\mathbb{C}\backslash\mathbb{D}$ by a unique isomorphism $\phi: \mathbb{C}\backslash\overline{\mathbb{D}} \to A_\infty$, asymptotic to the identity at ∞. By a theorem of Böttcher (see, for example, [**Mil00**, Theorem 9.1]), ϕ conjugates $P|_{A_\infty}$ with $z^d|_{\mathbb{C}\backslash\overline{\mathbb{D}}}$. If J_P is locally connected, then ϕ extends continuously to a semiconjugacy $\overline{\phi}$ between $\sigma_d = z \mapsto z^d|_{\mathbb{S}}$ and $P|_{J_P}$

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{\sigma_d|_{\mathbb{S}}} & \mathbb{S} \\ {\scriptstyle\overline{\phi}}\downarrow & & \downarrow{\scriptstyle\overline{\phi}} \\ J_P & \xrightarrow{P|_{J_P}} & J_P \end{array} \tag{2.1}$$

The *lamination generated by* P is the equivalence relation $\sim_P$ on $\mathbb{S}$ whose classes are $\overline{\phi}$*-fibers*, that is, point-preimages under $\overline{\phi}$. We call $\widehat{J}_P = \mathbb{S}/\sim_P$ a *topological Julia set* and $\widehat{P}$, the corresponding map induced on $\widehat{J}_P$ by σ_d, a *topological polynomial*. Evidently $P|_{J_P}$ and $\widehat{P}|_{\widehat{J}_P}$ are conjugate. The collection $\mathcal{L}_P$ of chords of $\overline{\mathbb{D}}$ which are edges of convex hulls of $\sim_P$ classes is called the *geolamination generated by* P.

2.1. *Laminations, geolaminations, and their properties.*

Laminations and geolaminations can be defined independently of polynomials. This approach is due to Thurston [**Thu85**] who constructed a combinatorial model of the Mandelbrot set by parameterizing quadratic geolaminations. We define laminations to be equivalence relations without underlying dynamics.

Definition 2.1. (Laminations) An equivalence relation $\sim$ on the unit circle $\mathbb{S}$ is called a *lamination* if either $\mathbb{S}$ is one $\sim$-class (such laminations are called *degenerate*) or the following hold:

(E1) the graph of $\sim$ is a closed subset of $\mathbb{S} \times \mathbb{S}$;
(E2) the convex hulls of distinct equivalence classes are disjoint; and
(E3) each equivalence class of $\sim$ is finite.

These properties are satisfied by laminations generated by polynomials. Property (E1) is necessary for $\mathbb{S}/\sim$ to be a metric space, (E2) follows from the semiconjugacy $\overline{\phi}$ (a proof can be found in [**Sch09**, Proposition II.3.3]) and (E3) follows from [**Mil00**, Lemma 18.12] for (pre)periodic points and from Kiwi [**Kiw02**] for points with infinite orbit. Definition 2.2 enforces dynamics on laminations consistent with those generated by polynomials.

Definition 2.2. (Laminations and dynamics) An equivalence relation $\sim$ is called (σ_d-)*invariant* if:

(D1) $\sim$ is *forward invariant*: for a $\sim$-class $\mathfrak{g}$, the set $\sigma_d(\mathfrak{g})$ is a $\sim$-class; and
(D2) for any $\sim$-class $\mathfrak{g}$, the map $\sigma_d: \mathfrak{g} \to \sigma_d(\mathfrak{g})$ extends to $\mathbb{S}$ as an orientation preserving covering map such that $\mathfrak{g}$ is the full preimage of $\sigma_d(\mathfrak{g})$ under this covering map.

There is a useful geometric object associated with each lamination.

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Definition 2.3. For a lamination $\sim$, take convex hulls of all its classes and consider the family of chords which are their edges (together with all single points of the unit circle). This family of (possibly degenerate) chords is called a *geolamination generated by* $\sim$ and is denoted by $\mathcal{L}_\sim$.

Definition 2.3 makes sense for non-invariant and invariant laminations; for invariant laminations we use notation described in Definition 2.4.

Definition 2.4. The family of all invariant geolaminations generated by laminations is denoted by $\mathbb{L}\mathcal{Q}$; if we consider only σ_d-invariant laminations, the corresponding family of geolaminations is denoted by $\mathbb{L}\mathcal{Q}_d$. Geolaminations from $\mathbb{L}\mathcal{Q}$ ($\mathbb{L}\mathcal{Q}_d$) are called $\mathbb{L}\mathcal{Q}$-*geolaminations* ($\mathbb{L}\mathcal{Q}_d$-*geolaminations*).

Similar notions were defined by Thurston in [**Thu85**] with no equivalence relations involved. For a collection $\mathcal{R}$ of chords of $\mathbb{D}$, set $\mathcal{R}^+ = \bigcup \mathcal{R}$.

Definition 2.5. (Geolaminations) Two chords of $\overline{\mathbb{D}}$ are called *unlinked* if they are disjoint in $\mathbb{D}$. A *geolamination* is a collection $\mathcal{L}$ of (perhaps degenerate) pairwise unlinked chords of $\overline{\mathbb{D}}$ called *leaves* such that $\mathcal{L}^+$ is closed, and all points of $\mathbb{S}$ are elements of $\mathcal{L}$. A *gap* of $\mathcal{L}$ is the closure of a component of $\mathbb{D}\backslash\mathcal{L}^+$. If $\mathcal{L} = \mathcal{L}_\sim$ is a geolamination generated by a lamination $\sim$, gaps of $\mathcal{L}_\sim$ will also be called *gaps of* $\sim$. A gap G is *finite (infinite)* depending on whether $G \cap \mathbb{S}$ is finite (infinite). Gaps G such that $G \cap \mathbb{S}$ is uncountable are called *Fatou gaps*. For Fatou gaps G, we define the monotone map $\psi_G : \mathrm{Bd}(G) \to \mathbb{S}$ which collapses all edges of G to points.

If $x, y \in \mathbb{S}$ are the endpoints of a chord ℓ, set $\ell = \overline{xy}$. Given a closed set $A \subset \mathbb{S}$, let $\sigma_d(\mathrm{CH}(A)) = \mathrm{CH}(\sigma_d(A))$ (in particular, $\sigma_d(\overline{xy}) = \overline{\sigma_d(x)\sigma_d(y)}$). Any non-degenerate chord ℓ such that $\sigma_d(\ell)$ is a point is called *critical*. Given a geolamination $\mathcal{L}$, we linearly extend σ_d over leaves of $\mathcal{L}$; clearly, this extension is continuous and well defined. With this extension, we can define σ_d-*invariant geolaminations*. Here we rely on [**BMOV13**], where the definition is a bit different from Thurston's [**Thu85**].

Definition 2.6. (Invariant sibling geolaminations [**BMOV13**]) A geolamination $\mathcal{L}$ is (σ_d-)*invariant (sibling) geolamination* provided:

(1) for each $\ell \in \mathcal{L}$, we have $\sigma_d(\ell) \in \mathcal{L}$;
(2) for each $\ell \in \mathcal{L}$, there exists $\ell^* \in \mathcal{L}$ so that $\sigma_d(\ell^*) = \ell$; and
(3) for each $\ell \in \mathcal{L}$ such that $\sigma_d(\ell)$ is non-degenerate, there exist d **pairwise disjoint** leaves $\ell_1, \ldots, \ell_d$ in $\mathcal{L}$ so that $\ell = \ell_1$ and $\sigma_d(\ell_i) = \sigma_d(\ell)$ for all $i = 1, \ldots, d$.

If a geolamination satisfies only condition (1), we call it a *forward* (σ_d-)*invariant sibling geolamination*.

Any leaf ℓ^* with $\sigma_d(\ell^*) = \ell$ is called a *pullback* of ℓ. Also, two leaves with the same σ_d-image are called *siblings*, and the leaves $\ell_1 = \ell, \ldots, \ell_d$ in (3) are said to form a *disjoint sibling collection of* ℓ [**BMOV13**]. The space of all σ_d-invariant sibling geolaminations is denoted by $\mathbb{L}_d$.

Thurston defines σ_d-invariant geolaminations differently. He requires that geolaminations satisfy (1) above, be such that each leaf ℓ has d *disjoint* pullbacks and be *gap invariant*.

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Definition 2.7. (Gap invariance [**Thu85**]) Suppose that, for any gap G of a geolamination $\mathcal{L}$, the set $H = \sigma_d(G)$ is either a point of $\mathbb{S}$, a leaf of $\mathcal{L}$ or a gap of $\mathcal{L}$. Moreover, if H is a gap, then $\sigma_d : \mathrm{Bd}(G) \to \mathrm{Bd}(H)$ is the positively oriented composition of a monotone map and a covering map. Then $\mathcal{L}$ is said to be *gap invariant*.

Let us now define the degree of σ_d on a gap or leaf of $\mathcal{L}$.

Definition 2.8. (Degree of σ_d on a gap or leaf) Let G be a gap or leaf of a gap invariant geolamination and let $H = \sigma_d(G)$. Then the *degree of* $\sigma_d|_G$ is either the cardinality of $G \cap \mathbb{S}$ (if H is a point) or, otherwise, the number of components of the set $\sigma_d^{-1}(x) \cap \mathrm{Bd}(G)$, where x is any point of $\mathrm{Bd}(H)$.

It is easy to see that the degree of $\sigma_d|_G$ does not depend on the choice of the point $x \in \mathrm{Bd}(H)$. In the case when $\sigma_d(G)$ is a gap, the degree of $\sigma_d|_G$ can be defined as the degree of a covering map involved in the representation of $\sigma_d|_{\mathrm{Bd}(G)}$ as the composition of a monotone map and a covering map.

Definition 2.9. A gap G, all of whose edges are critical, is said to be *all critical*. A gap G is called *critical* if the degree of $\sigma_d|_G$ is greater than one.

Definition 2.6 conveniently deals only with leaves. This leads to useful results on properties of σ_d-invariant sibling geolaminations [**BMOV13**]. We will agree to endow any family of compact subsets of $\overline{\mathbb{D}}$ with the Hausdorff metric and the induced topology. In particular, we define a natural topology on $\mathbb{L}_d$ induced by the Hausdorff metric on subcontinua $\mathcal{L}^+$ of $\overline{\mathbb{D}}$, where $\mathcal{L} \in \mathbb{L}_d$ (from now on talking about $\mathbb{L}_d$ we mean the just defined topological space rather than a family of geolaminations).

THEOREM 2.10. [**BMOV13**] *The following results hold.*

(1) $\mathbb{L}\mathcal{Q}_d \subset \mathbb{L}_d$ *(that is, geolaminations generated by* σ_d*-invariant laminations are* σ_d*-invariant sibling geolaminations);*
(2) $\mathbb{L}_d$ *is compact (thus,* $\mathbb{L}_d$ *contains all limits of geolaminations from* $\mathbb{L}\mathcal{Q}_d$*); and*
(3) σ_d*-invariant sibling geolaminations are* σ_d*-invariant in the sense of Thurston.*

We will also need a few technical results of [**BMOV13**]. Denote by $<$ the *positive (circular) order on* $\mathbb{S}$. That is, given points $x, y, z \in \mathbb{S}$, $x < y < z$, if the counterclockwise arc of $\mathbb{S}$ from x to y contains y in its interior. A consequence of gap invariance is that if an arc in the boundary of a gap of a σ_d-invariant geolamination is mapped injectively by σ_d, then σ_d preserves the (circular) order of points of that arc. In [**BMOV13**] this property was explored for leaves of σ_d-invariant sibling geolaminations emanating from the same point of $\mathbb{S}$, which may or may not be edges of a common gap.

LEMMA 2.11. [**BMOV13**, Lemma 3.7] *Let* $\mathcal{L}$ *be a* σ_d*-invariant sibling geolamination and* $T \subset \mathcal{L}^+$ *be an arc consisting of two leaves with a common endpoint or a triod consisting of three leaves with a common endpoint. Suppose that* $Y \subset \mathcal{L}^+$ *is an arc (triod) such that* $\sigma_d(Y) = T$ *and* $\sigma_d|_Y$ *is one-to-one. Then* $\sigma_d|_{Y \cap \mathbb{S}}$ *preserves circular order.*

Corollary 2.12 easily follows from Lemma 2.11.

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COROLLARY 2.12. *Let $\mathcal{L}$ be a σ_d-invariant sibling geolamination. Let $x < y < z < x$ be the endpoints of two non-critical leaves $\overline{xy}$ and $\overline{yz}$ of $\mathcal{L}$ such that $\sigma_d(x) = \sigma_d(z)$. If now $z < t < x$ and $\overline{yt}$ is a leaf of $\mathcal{L}$, then either $\sigma_d(t) = \sigma_d(y)$ or $\sigma_d(t) = \sigma_d(x)$; in particular, there are at most $2d - 3$ such leaves $\overline{yt}$.*

Proof. Suppose otherwise. Then, if we apply Lemma 2.11 to $\overline{xy} \cup \overline{yt}$, we see that $\sigma_d(x) < \sigma_d(y) < \sigma_d(t) < \sigma_d(x)$. On the other hand, if we apply Lemma 2.11 to $\overline{yz} \cup \overline{yt}$, we see that $\sigma_d(y) < \sigma_d(z) < \sigma_d(t) < \sigma_d(y)$. Since $\sigma_d(x) = \sigma_d(z)$, this is a contradiction. □

Corollary 2.13 follows from Corollary 2.12. By definition, if a non-critical leaf ℓ of a σ_d-invariant sibling geolamination has overall $d - 1$ other leaves of $\mathcal{L}$ which map to $\sigma_d(\ell)$, then all these d leaves are pairwise disjoint and form a unique disjoint sibling collection of leaves with image $\sigma_d(\ell)$.

COROLLARY 2.13. *There are a finite number of leaves ℓ such that more than d leaves have the image $\sigma_d(\ell)$.*

Proof. We may assume that ℓ is non-critical. If ℓ has more than $d - 1$ other leaves with the image $\sigma_d(\ell)$, then at least two of them will share an endpoint (say, y) with the two other endpoints (say, x and z) mapped to the same point. Set $\mathrm{CH}(x, y, z) = T_\ell$. Call $\overline{xz}$ the *base* of T_ℓ and call y the *top vertex* of T_ℓ. It is easy to see that there are at most a finite number of (in fact, no more than $2d - 3$) triangles whose bases are pairwise unlinked; let $T^1, \dots, T^k$ be a maximal collection of such triangles with pairwise unlinked bases. Any remaining triangle T_ℓ is such that its base crosses the base of some triangle T^j, $1 \le j \le k$. Since sides of all these triangles which are not their bases are *unlinked* leaves, then T_ℓ and T^j must share a vertex. Clearly, the number of vertices of triangles $T^1, \dots, T^k$ is finite. By Corollary 2.12, each of them can be the top vertex of a finite number of triangles T_ℓ. Hence the overall number of such triangles is finite as desired. □

This helps in studying disjoint sibling collections.

LEMMA 2.14. *If ℓ is a non-critical non-isolated leaf, then there exists a disjoint sibling collection which contains ℓ and consists of non-isolated leaves: that is, if $\hat{\ell}$ is a non-degenerate non-isolated from one (two) sides leaf, then there is a disjoint sibling collection of leaves mapped to $\hat{\ell}$ such that leaves in this collection are non-isolated from the appropriate side(s).*

Proof. Let $\ell_i^1 \to \ell^1 = \ell$ be a sequence of leaves which converge to ℓ from one side. By Corollary 2.13, we may assume that, for each i, the leaf ℓ_i is a member of a disjoint sibling collection $L_i = \{\ell_i^1, \ell_i^2, \dots, \ell_i^d\}$, which includes *all* leaves mapped to $\sigma_d(\ell_i^1)$. By compactness, we may assume that leaves in these collections are ordered so that, for each $1 \le j \le d$, $\lim_i \ell_i^j = \ell^j$. Clearly, $\sigma_d(\ell^j) = \sigma_d(\ell)$ and all leaves ℓ^j are non-isolated.

To show that all these leaves are pairwise disjoint, observe that since the leaves ℓ_i^1 approach ℓ^1 from one side, for any j, the leaves ℓ_i^j approach ℓ^j from one side. Thus, the leaves $\sigma_d(\ell_i^j)$ approach $\sigma_d(\ell^j)$ from one side too. If now, say, $\ell^2 = \ell^3$, then it would follow that leaves ℓ_i^2, ℓ_i^3 are disjoint, close to each other and have the same image, which is impossible. Suppose that $\ell^2 \ne \ell^3$ come out of the same point but are otherwise disjoint.

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Then the leaves ℓ^2 and ℓ^3 form a wedge, and the leaves ℓ_i^2, ℓ_i^3 are located outside the wedge close to the appropriate sides of the wedge (recall that all these leaves come from the same geolamination and are therefore pairwise unlinked). Then their images approach $\sigma_d(\ell^1)$ from opposite sides, which is a contradiction. □

Lemma 2.14 allows us to apply a certain 'cleaning' procedure to construct other σ_d-invariant sibling geolaminations out of a given geolamination.

THEOREM 2.15. *Let $\mathcal{L}$ be a σ_d-invariant sibling geolamination and $\mathcal{A}$ be a family of grand orbits of its leaves. Construct a collection of leaves $\mathcal{L}'$ by removing from $\mathcal{L}$ all isolated leaves belonging to $\mathcal{A}$. Then $\mathcal{L}'$ is a σ_d-invariant sibling geolamination.*

Proof. Since non-isolated leaves map to non-isolated leaves, $\mathcal{L}'$ is forward invariant. Let $\ell' \in \mathcal{L}'$ be non-degenerate. Then either $\ell' \notin \mathcal{A}$ or $\ell' \in \mathcal{A}$ is non-isolated. In the former case by definition, there is a leaf $\ell'' \in \mathcal{L}$ such that $\sigma_d(\ell'') = \ell'$ and, since $\ell' \notin \mathcal{A}$, it follows that $\ell'' \notin \mathcal{A}$ and hence $\ell'' \in \mathcal{L}'$. In the latter case, the claim follows from Lemma 2.14. Thus, any non-degenerate leaf from $\mathcal{L}'$ has a preimage in $\mathcal{L}'$. Finally, let $\ell' \in \mathcal{L}'$ be non-critical. Then, again, either $\ell' \notin \mathcal{A}$ or $\ell' \in \mathcal{A}$ is non-isolated. In the former case, the claim is obvious as all disjoint sibling collections of ℓ' are disjoint from $\mathcal{A}$. In the latter, case the claim follows from Lemma 2.14. □

Arguments, similar to those in the proof of Corollary 2.12 and based upon Lemma 2.11, prove Corollary 2.16; we leave the proof to the reader.

COROLLARY 2.16. *If $\mathcal{L}$ is a σ_d-invariant sibling geolamination and $\overline{ab}$ is a leaf of $\mathcal{L}$ with periodic endpoints, then a and b are periodic of the same period.*

The paper [**BMOV13**] also explored how disjoint sibling collections must be located in $\overline{\mathbb{D}}$. To state the corresponding lemma, we need a notational agreement. If X is a collection of pullbacks of a leaf $\overline{ab}$, we denote the endpoints of chords of X by the same letters as for the endpoints of their images, but with a hat and distinct subscripts, and call them correspondingly (a-points, b-points and so on). Lemma 2.17 is typically used in the situation of Figure 1, when no leaves cross the critical chord joining two sibling leaves. We say that two distinct chords *cross* each other, or are *linked*, if they intersect inside the open unit disk $\mathbb{D}$.

LEMMA 2.17. [**BMOV13**, Lemma 3.8] *Let X be a disjoint sibling collection of leaves mapped to a leaf $\overline{ab}$ and $\overline{\hat{a}_1\hat{b}_1}$, $\overline{\hat{a}_2\hat{b}_2}$ be two leaves from X. Then the number of leaves from X crossing the chord $\overline{\hat{a}_1\hat{a}_2}$ inside $\mathbb{D}$ is even if and only if either $\hat{a}_1 < \hat{b}_1 < \hat{a}_2 < \hat{b}_2$ or $\hat{a}_1 < \hat{b}_2 < \hat{a}_2 < \hat{b}_1$. In particular, if there exists a concatenation Q of chords connecting $\hat{a}_1$ and $\hat{a}_2$, disjoint with leaves of X except the points $\hat{a}_1$, $\hat{a}_2$, then either $\hat{a}_1 < \hat{b}_1 < \hat{a}_2 < \hat{b}_2$ or $\hat{a}_1 < \hat{b}_2 < \hat{a}_2 < \hat{b}_1$.*

We will also need the following easy corollary of Lemma 2.17.

COROLLARY 2.18. *Let X be a disjoint sibling collection of leaves mapped to a leaf $\overline{ab}$ and $\overline{\hat{a}_1\hat{b}_1}$, $\overline{\hat{a}_2\hat{b}_2}$ be two leaves from X. Suppose that there exists a concatenation Q of pairwise non-crossing critical chords connecting $\hat{a}_1$ and $\hat{a}_2$, with endpoints separated by $\overline{\hat{a}_1\hat{a}_2}$ from*

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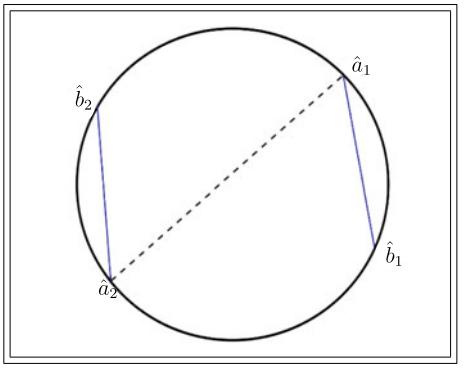


FIGURE 1. Siblings and critical leaves. Siblings must be on opposite sides of the chord $\overline{\hat{a}_1\hat{a}_2}$, which is not crossed by leaves of the disjoint sibling collection.

$\hat{b}_1$ inside $\mathbb{D}$, which are disjoint from the leaves of X except for their endpoints. Then either $\hat{a}_1 < \hat{b}_1 < \hat{a}_2 < \hat{b}_2$ or $\hat{a}_1 < \hat{b}_2 < \hat{a}_2 < \hat{b}_1$.

Proof. Without loss of generality, assume that $\hat{a}_1 < \hat{a}_2 < \hat{b}_1$. By the assumptions of the lemma, there is a chain Y of pairwise non-crossing critical chords $\overline{\hat{a}_1 x_1}$, $\overline{x_1 x_2}$, $\ldots$, $\overline{x_n \hat{a}_2}$ with $\hat{a}_1 < x_1 < x_2 < \cdots < x_n < \hat{a}_2$, each of which does not cross leaves from X. By definition, X must contain n disjoint leaves with endpoints $x_1, \ldots, x_n$, respectively. Moreover, applying Lemma 2.17 step by step and using the fact that chords in Y do not cross chords of X, we see that X contains leaves $\overline{y_1 x_1}, \ldots, \overline{y_n x_n}$ such that

$$\hat{a}_1 < y_1 < x_1 < y_2 < x_2 < \cdots < y_n < x_n < \hat{b}_2 < \hat{a}_2,$$

which implies that $\hat{a}_1 < \hat{b}_2 < \hat{a}_2 < \hat{b}_1$, as desired. □

One benefit of working with geolaminations is the ability to construct σ_d-invariant geolaminations from a small generating set. This is due to Thurston [**Thu85**] for his definition of σ_d-invariance, but is easily modified for σ_d-invariant sibling geolaminations. The geolamination of Theorem 2.19 is constructed by iteratively adding preimages of leaves of $\widehat{\mathcal{L}}$ and taking the closure after a countable number of steps.

THEOREM 2.19. (Cf. [**Thu85**]) *Let $\widehat{\mathcal{L}}$ be a forward σ_d-invariant sibling geolamination. Then there exists a σ_d-invariant geolamination $\mathcal{L} \supset \widehat{\mathcal{L}}$ for which grand orbits of leaves of $\widehat{\mathcal{L}}$ are dense.*

In what follows, we call σ_2-invariant and σ_3-invariant sibling geolaminations *quadratic* and *cubic* (sibling) geolaminations, respectively.

2.2. *Accordions and linked geolaminations.* By [**Thu85**], a quadratic geolamination $\mathcal{L}$ has one or two longest leaves called *majors* of $\mathcal{L}$. If the major is unique, then it must be a diameter, otherwise the two majors are disjoint. In any case, the majors are denoted by $M_{\mathcal{L}}$ and $M'_{\mathcal{L}}$. The *minor* $m_{\mathcal{L}}$ of $\mathcal{L}$ is defined as $m_{\mathcal{L}} = \sigma_2(M_{\mathcal{L}}) = \sigma_2(M'_{\mathcal{L}})$. Thurston proves that no two minors intersect in $\mathbb{D}$ and the collection of all quadratic minors is a geolamination itself called the *quadratic minor lamination* (QML).

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In [**BOPT14**], we partially generalize this result to cubic geolaminations. Using minors alone for such geolaminations is impossible. While, in the quadratic case, $m_{\mathcal{L}}$ is uniquely pulled back under σ_2 to a pair of majors, there are multiple ways to pull leaves back under σ_3, which result in very different geolaminations. To avoid this ambiguity, we associate cubic geolaminations with ordered pairs of *generalized critical quadrilaterals*; such pairs are called *qc-portraits*. In the case of σ_2, this is equivalent to associating to a quadratic geolamination $\mathcal{L}$ the set $\mathrm{CH}(\sigma_2^{-1}(m_{\mathcal{L}}))$.

Definition 2.20. (Generalized critical quadrilaterals) A *generalized quadrilateral* Q is the circularly ordered 4-tuple $[a_0, a_1, a_2, a_3]$ of marked points $a_0 \le a_1 \le a_2 \le a_3 \le a_0$ in $\mathbb{S}$. Such Q is said to be *(σ_d-)critical* if $\overline{a_0a_2}$ and $\overline{a_1a_3}$ are non-degenerate σ_d-critical chords (called *spikes*). Generalized σ_d-critical quadrilaterals $[a_0, a_1, a_2, a_3]$, $[a_1, a_2, a_3, a_0]$ and the other circular permutations of the vertices are viewed as equal. The chords $\overline{a_0a_1}$, $\overline{a_1a_2}$, $\overline{a_2a_3}$, $\overline{a_3a_0}$ are called *edges* of Q.

Note that a generalized critical quadrilateral Q is either a critical leaf, an all-critical triangle, an all-critical quadrilateral (impossible if the degree is three) or a quadrilateral which collapses to a leaf (a *collapsing quadrilateral*). Although the definition of a qc-portrait is suitable for σ_d-invariant sibling geolaminations, we simplify it here for σ_3. When saying that a geolamination $\mathcal{L}$ *has (or is with) a generalized critical quadrilateral* Q, we mean that Q is a leaf or a gap of $\mathcal{L}$.

Definition 2.21. (Quadratic criticality) Let $(\mathcal{L}, \mathrm{QCP})$ be a cubic geolamination with an ordered pair QCP of *distinct* generalized critical quadrilaterals which do not intersect in $\mathbb{D}$ other than over a common edge/vertex. Then QCP is called a *quadratically critical portrait* (*qc-portrait*) *for* $\mathcal{L}$, while the pair $(\mathcal{L}, \mathrm{QCP})$ is called a *geolamination with qc-portrait*.

Note that we do *not* require that the two generalized critical quadrilaterals be disjoint. In fact, one may even be a subset of the other. However, we do require that they be distinct. Thus, a critical leaf repeated twice is *not* a qc-portrait.

In the quadratic case, the notion of a qc-portrait reduces to that of a critical leaf or collapsing quadrilateral (recall, that a collapsing quadrilateral is a critical quadrilateral whose image is a non-degenerate chord). In this case, the image of a collapsing quadrilateral, or a critical leaf of a quadratic geolamination $\mathcal{L}$, is the minor $m_{\mathcal{L}}$ of $\mathcal{L}$.

Now we discuss *strong linkage* between generalized critical quadrilaterals and then *linkage* between cubic geolaminations with qc-portraits. For a generalized quadrilateral Q, call a component of $\mathbb{S}\setminus Q$ a *hole* of Q.

Definition 2.22. (Strongly linked quadrilaterals) Let A and B be generalized quadrilaterals. Say that A and B are *strongly linked* if the vertices of A and B can be numbered so that

$$a_0 \le b_0 \le a_1 \le b_1 \le a_2 \le b_2 \le a_3 \le b_3 \le a_0,$$

where a_i, $0 \le i \le 3$ are vertices of A and b_i, $0 \le i \le 3$ are vertices of B. Note that the vertices of the two quadrilaterals here need not strictly alternate on the circle.

Definition 2.23 is a cubic version of the corresponding one from [**BOPT14**].

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Definition 2.23. (Linked qc-portraits) Let qc-portraits $\mathrm{QCP}_1 = (C_1^1, C_1^2)$ and $\mathrm{QCP}_2 = (C_2^1, C_2^2)$ be such that for every $1 \le i \le 2$ the sets C_1^i and C_2^i are either strongly linked generalized critical quadrilaterals or share a spike. Then the qc-portraits are said to be *linked* while the critical sets C_1^i and C_2^i, $i = 1, 2$ are called *associated*. If $(\mathcal{L}_1, \mathrm{QCP}_1)$ and $(\mathcal{L}_2, \mathrm{QCP}_2)$ are geolaminations with qc-portraits then they are called *linked* if either QCP_1 and QCP_2 are linked, or $\mathcal{L}_1$ and $\mathcal{L}_2$ share an all-critical triangle.

Observe that, in [**BOPT14**], geolaminations $\mathcal{L}_1$, $\mathcal{L}_2$, from Definition 2.23, for which either both pairs of associated generalized critical quadrilaterals in qc-portraits share a spike or $\mathcal{L}_1$, $\mathcal{L}_2$ themselves share an all-critical triangle, are said to be *essentially equal* rather than linked. However, for our purposes, this fine distinction does not matter much, and thus, in this paper, we use the slightly less precise definition above.

Thurston's result that no quadratic geolaminations have linked minors is equivalent to the statement that there are no quadratic geolaminations with linked qc-portraits. Also, in [**BOPT14**], the notion of linked geolaminations was introduced and various results were obtained in the degree d case; the results stated in the rest of this subsection are restatements of the results of [**BOPT14**] for the cubic case. The main tool used in [**BOPT14**] to study linked σ_d-invariant sibling geolaminations with qc-portraits is an *accordion*, designed to track linked leaves from linked geolaminations.

Definition 2.24. Let ℓ_1, ℓ_2 be leaves of σ_d-invariant sibling geolaminations $\mathcal{L}_1, \mathcal{L}_2$. Denote by $A_{\ell_2}(\ell_1)$ the collection of leaves from the forward orbit of ℓ_2 linked with ℓ_1 together with ℓ_1. We call $A_{\ell_2}(\ell_1)$ the *accordion of* ℓ_1 *with respect to* ℓ_2. Abusing notation, we often denote $A^+_{\ell_2}(\ell_1)$ by $A_{\ell_2}(\ell_1)$.

An important property of accordions is introduced in Definition 2.25.

Definition 2.25. Say that ℓ_1 has *order-preserving accordions* with respect to ℓ_2 if $A_{\ell_2}(\ell_1) \neq \{\ell_1\}$ and, for all $k \ge 0$, the restriction of σ_d to $A_{\ell_2}(\sigma_d^k(\ell_1)) \cap \mathbb{S}$ is order preserving. If ℓ_1 has order-preserving accordions with respect to ℓ_2, and *vice versa*, then these accordions are said to be *mutually order preserving*, while ℓ_1, ℓ_2 are said to have *mutually order-preserving accordions*.

Mutually order-preserving accordions are not gaps of a single sibling invariant geolamination as convex hulls of their forward images may have intersecting interiors. Nevertheless, their dynamics resemble that of gaps of sibling invariant geolaminations, so leaves from mutually order-preserving accordions have a rigid structure. This is supported by the findings made in [**BOPT14**, §3]; some of the main results from there are stated here.

LEMMA 2.26. [**BOPT14**, Lemma 3.7] *If $\ell_a = \overline{ab}$ and $\ell_x = \overline{xy}$, $a < x < b < y$, are linked leaves with mutually order-preserving accordions and if a, b are of period k, then x, y are also of period k.*

To state the main result of [**BOPT14**, §3], we need Definition 2.27.

Definition 2.27. Let $\ell_a = \overline{ab}$, $\ell_x = \overline{xy}$ be linked chords. Set $B = \mathrm{CH}(\ell_a, \ell_x)$. Say that the *$\sigma_d$-dynamics of B is orderly* if the following hold.

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(1) The order of vertices of B is preserved under iterations of σ_d.
(2) One of the following holds.
   (a) All images of B are pairwise disjoint.
   (b) There exist the minimal r, m and n with $\sigma_d^r(B) \cap \sigma_d^{r+m}(B) \neq \emptyset$ and $\sigma_d^{r+mn}(B) = B$ and, if, for any $i \geq 0$, we set $\sigma_d^{r+im}(a) = a_i$, $\sigma_d^{r+im}(b) = b_i$, $\sigma_d^{r+im}(x) = x_i$, $\sigma_d^{i+rm}(y) = y_i$, then either

$$a_0 < x_0 < b_0 \leq a_1 < y_0 \leq x_1 < b_1 \leq a_2 < y_1 \leq x_2 < b_2 \leq \cdots$$

   or

$$x_0 < a_0 < y_0 \leq x_1 < b_0 \leq a_1 < y_1 \leq x_2 < b_1 \leq a_2 < y_2 \leq \cdots$$

   and the set $\bigcup_{i=0}^{n-1} \sigma_d^{r+im} B$ is a component of the orbit of B.
   (c) The leaves ℓ_a, ℓ_x are (pre)periodic of the same eventual period of endpoints.

Despite its appearance, the order among vertices of B and some of its images given in Definition 2.27 can be easily described.

Definition 2.28. If there are chords $\overline{a_0b_0}, \overline{a_1b_1}, \ldots, \overline{a_nb_n}$ with $a_0 \leq a_1 \leq \cdots \leq a_n \leq a_0$ and $b_0 \leq b_1 \leq \cdots \leq b_n \leq b_0$, we say that these chords are *positively ordered* and denote this by $\overline{a_0b_0} \leq \overline{a_1b_1} \cdots \leq \overline{a_nb_n} \leq \overline{a_0b_0}$.

Here is how one can restate the long inequalities from Definition 2.27. Set $\sigma_d^{r+im}(\ell_a) = \ell_a^i$, $\sigma_d^{r+im}(\ell_x) = \ell_x^i$. Then

$$\ell_a^0 \leq \ell_x^0 \leq \ell_a^1 \leq \ell_x^1 \leq \cdots \leq \ell_a^{n-1} \leq \ell_x^{n-1} \leq \ell_a^n = \ell_a^0 \leq \ell_x^n = \ell_x^0$$

or

$$\ell_x^0 \leq \ell_a^0 \leq \ell_x^1 \leq \ell_a^1 \leq \cdots \leq \ell_x^{n-1} \leq \ell_a^{n-1} \leq \ell_x^n = \ell_x^0 \leq \ell_a^n = \ell_a^0.$$

In addition, only consecutive images of ℓ_a (ℓ_x) in this collection can intersect and, at most, at their endpoints.

The following is the main result of [**BOPT14**, §3].

THEOREM 2.29. [**BOPT14**, Theorem 3.12] *Consider linked chords ℓ_a, ℓ_x with mutually order-preserving accordions and set $B = \mathrm{CH}(\ell_a, \ell_x)$. Then the σ_d-dynamics of B is orderly.*

In the rest of this subsection, we assume that $\mathcal{L}_1$, $\mathcal{L}_2$ *are linked cubic geolaminations*. Linked leaves of $\mathcal{L}_1$, $\mathcal{L}_2$ will often have images which are linked and have mutually order-preserving accordions. The only alternative is that they eventually map to leaves which are not linked but share an endpoint. We call this behavior *collapsing (around a chain of spikes)*.

Definition 2.30. Non-disjoint leaves $\ell_1 \neq \ell_2$ are said to *collapse around chains of spikes* if there are two chains of spikes (one chain in each of our geolaminations) connecting the same two adjacent endpoints of ℓ_1, ℓ_2.

We now list a few results of [**BOPT14**] on accordions of leaves of $\mathcal{L}_1(\mathcal{L}_2)$. Observe that the results obtained in [**BOPT14**] hold in the degree d case (that is, for σ_d); however, in this paper we confine ourselves to the case $d = 3$ (in particular, we define the linkage

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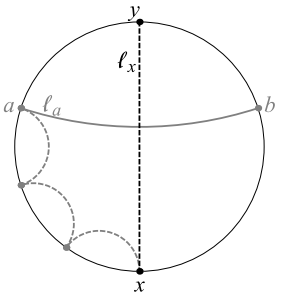


FIGURE 2. This figure illustrates Definition 2.30. Here the leaves ℓ_a, ℓ_x collapse around a chain of spikes, shown as dashed grey geodesics.

between geolaminations only for cubic geolaminations) which is why we state results from [**BOPT14**] in the cubic case only. If $X \subset \mathbb{S}$ is a set of points such that $\sigma_3|_X$ preserves orientation except that some points may be mapped to one, we say that $\sigma_3|_X$ *weakly preserves orientation*.

LEMMA 2.31. [**BOPT14**, Corollary 4.2] *Let $\ell_1 \in \mathcal{L}_1$, $\ell_2 \in \mathcal{L}_2$; then, for all $k \geq 0$, the restriction of σ_d to $A_{\ell_2}(\ell_1) \cap \mathbb{S}$ is (non-strictly) order preserving. Moreover, if ℓ_1 and ℓ_2 are linked leaves or share an endpoint, then the order among their endpoints is weakly preserved under σ_3 unless the common endpoint of ℓ_1, ℓ_2 is a common vertex of associated critical quadrilaterals of our geolaminations.*

Corollary 4.2 in [**BOPT14**] contains much more detail than Lemma 2.31. It serves as a major tool as it shows that, even though geolaminations $\mathcal{L}_1$, $\mathcal{L}_2$ are distinct, crossing leaves of those geolaminations form a set which behaves more or less like a gap of one geolamination.

LEMMA 2.32. [**BOPT14**, Lemma 4.3] *Suppose that:*

(1) *$\ell_1 = \overline{a_1 b_1} \in \mathcal{L}_1$, $\ell_2 = \overline{a_1 b_2} \in \mathcal{L}_2$ are linked leaves or share an endpoint (in the latter case, the shared endpoint is not a common vertex of two associated generalized quadrilaterals);*
(2) *at least of these leaves is not critical; and*
(3) *the order among the endpoints of these leaves is weakly preserved so that, say, $a_1 \neq a_2$, while $\sigma_3(a_1) = \sigma_3(a_2)$.*

Then there are chains of spikes of $\mathcal{L}_1$ and of $\mathcal{L}_2$ connecting a_1 and a_2 with endpoints in the arc with endpoints a_1, a_2 not containing points b_1, b_2.

The main problem with applications of Lemma 2.31 is that the order among the endpoints of linked leaves of geolaminations $\mathcal{L}_1$, $\mathcal{L}_2$ is only weakly preserved. In other words, endpoints of these leaves may collide. Lemma 2.32 describes how this can happen and shows that if two linked leaves ℓ_1, ℓ_2 of linked geolaminations have colliding endpoints, then these endpoints are connected with specifically located chains of spikes. This allows us to apply Corollary 2.18 and prove Proposition 2.33, which shows that the colliding endpoints are, in fact, endpoints of specifically located sibling leaves of ℓ_1, ℓ_2, respectively.

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PROPOSITION 2.33. *Suppose that $\mathcal{L}_1$, $\mathcal{L}_2$ are linked geolaminations and that the leaves $\overline{ab} \in \mathcal{L}_1$ and $\overline{xy} \in \mathcal{L}_2$ are non-critical, linked, and such that $a < x < b < y < a$ and $\sigma_3(a) = \sigma_3(y)$. Then there exists a leaf $\overline{ax'}$ which is a sibling of $\overline{xy}$ such that $x' < a < x < y < x'$.*

Proof. Consider a disjoint sibling collection $\mathcal{A}$ of the leaf $\overline{xy}$ (formed by leaves of $\mathcal{L}_2$). By Lemma 2.32, there is a chain of spikes of $\mathcal{L}_2$ from a to y with endpoints contained in $[y, a]$. Choose the leaf $\overline{ax'} \in \mathcal{A}$; then, by Corollary 2.18, $x' < a < x < y < x'$. □

Finally, we state here [**BOPT14**, Lemma 4.5].

LEMMA 2.34. [**BOPT14**, Lemma 4.5] *If $\ell_1 \in \mathcal{L}_1$, $\ell_2 \in \mathcal{L}_2$ are linked leaves or share an endpoint and $\sigma_d^t(\ell^1)$ and $\sigma_d^t(\ell^2)$ do not collapse around chains of critical chords for any t, then there exists an N such that the σ_d^N-images of ℓ^1, ℓ^2 are linked and have mutually order-preserving accordions. Thus, if $\overline{m}_1 \in \mathcal{L}_1$, $\overline{m}_2 \in \mathcal{L}_2$ are non-disjoint leaves, then, for any n, the leaves $\sigma_3^n(\overline{m}_1)$, $\sigma_d^n(\overline{m}_2)$ are non-disjoint.*

2.3. *Proper geolaminations.* Given a lamination $\sim$, the generated geolamination $\mathcal{L}_\sim$ is formed by the edges of $\sim$-classes (see Definition 2.3); if $\sim$ is invariant, the family of all such geolaminations is denoted by $\mathbb{L}\mathcal{Q}$. While $\mathbb{L}\mathcal{Q}$-geolaminations are natural objects, they are often difficult to work with. To overcome these difficulties, we define a wider and often more convenient class of geolaminations. This will also be helpful in dealing with geolaminations in the parameter space.

2.3.1. *Basic definitions and properties.* First, we introduce an inverse process to that from Definition 2.3.

Definition 2.35. Suppose that a family $\mathcal{F}$ of chords in $\overline{\mathbb{D}}$ is given. Define an equivalence relation $\sim_{\mathcal{F}}$ as follows: $x \sim_{\mathcal{F}} y$ if and only if there exists a finite concatenation of chords of $\mathcal{F}$ joining x and y. We say that $\mathcal{F}$ *generates (or gives rise to)* the equivalence relation $\sim_{\mathcal{F}}$.

Clearly, $\sim_{\mathcal{F}}$ *is* an equivalence relation for *any* collection of chords $\mathcal{F}$ ($\mathcal{F}$ does not have to be invariant or even closed). We are interested in families $\mathcal{F}$ such that the equivalence relation $\sim_{\mathcal{F}}$ is a lamination with the specific properties from Definition 2.1 (in the non-invariant case) and, in addition, from Definition 2.2 (in the invariant case). Observe that there are invariant geolaminations not from $\mathbb{L}\mathcal{Q}$ which generate invariant laminations. For example, inserting a leaf into the interior of a finite gap G of an $\mathbb{L}\mathcal{Q}$-geolamination and then inserting leaves from its grand orbit into appropriate sets from the grand orbit of G does not change the underlying equivalence relation. Whether this can be done depends on the dynamics of G (for example, it can always be done if G is critical or wandering), but the resulting geolamination is no longer in $\mathbb{L}\mathcal{Q}$.

Definition 2.36. A family of chords $\mathcal{F}$ such that the generated equivalence relation $\sim_{\mathcal{L}}$ is a lamination is said to be *proper*. Denote by $\mathbb{LP}$ the family of all *invariant* proper geolaminations and by $\mathbb{LP}_d$ the set of σ_d-invariant geolaminations from $\mathbb{LP}$.

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The following lemma can be found in [**BMOV13**].

LEMMA 2.37. *Suppose that a σ_d-invariant geolamination $\mathcal{L}$ is proper (that is, generates a lamination $\sim_{\mathcal{L}}$). Then $\sim_{\mathcal{L}}$ is a σ_d-invariant lamination.*

There are useful criteria for an invariant geolamination to be proper.

Definition 2.38. Two non-degenerate leaves with common endpoint v and the same non-degenerate image are said to form a *critical wedge* (the point v is said to be its *vertex*). Call a non-degenerate leaf *improper* if it has exactly one periodic endpoint.

The dynamics of improper leaves is described in Lemma 2.39.

LEMMA 2.39. *Let $\ell = \overline{px}$ be an improper leaf with p periodic of period k and x non-periodic. Then it has an eventual image-leaf which is either a critical leaf with a periodic endpoint or a leaf with exactly one periodic endpoint which maps to a periodic leaf (in the latter case, this image-leaf and the appropriate image of it form a critical wedge).*

Proof. Consider the leaves $\sigma_d^{kn}(\ell)$. Suppose that all points $p, x, \sigma_d^k(x), \sigma_d^{2k}(x), \ldots$ are distinct. Then, by Lemma 2.11 applied to σ_d^k, we may assume that $p < x < \sigma_d^k(x) < \sigma_d^{2k}(x) < \cdots$. Since σ_d is expanding, this is impossible. Hence there exists a minimal N such that either $\sigma_d^{Nk}(x) = p$ or (again, by Lemma 2.11) $\sigma_d^{Nk}(x)$ is a σ_d^k-fixed point distinct from p. In the first case, there exists $0 \le i < k$ such that $\sigma_d^{k(N-1)+i}(\ell)$ is a critical leaf with a periodic endpoint and, in the second case, there exists $0 \le i < k$ such that $\sigma_d^{k(N-1)+i}(\ell) \cup \sigma_d^{k(N-1)+i+1}$ is a critical wedge with a periodic edge. □

Lemma 2.40 presents the above mentioned criteria for an invariant geolamination to be proper.

LEMMA 2.40. (Cf. [**BMOV13**]) *The following properties are equivalent.*

(1) *The σ_d-invariant geolamination $\mathcal{L}$ is proper (that is, $\sim_{\mathcal{L}}$ is a lamination).*
(2) *The σ_d-invariant geolamination $\mathcal{L}$ has no critical leaves (wedges) with a periodic endpoint (vertex).*
(3) *The σ_d-invariant geolamination $\mathcal{L}$ has no improper leaves.*
(4) *The σ_d-invariant geolamination has neither a critical leaf with a periodic endpoint nor a critical wedge with a periodic leaf.*

In particular, concatenations of leaves of $\mathcal{L}$ are finite.

Proof. Parts (1)–(3) of the lemma are equivalent, by [**BMOV13**]. Clearly, (3) implies (4). The opposite direction follows from Lemma 2.39. □

Consider the difference between a proper geolamination $\mathcal{L}$ and the $\mathbb{L}\mathcal{Q}$-geolamination $\mathcal{L}_{\sim_{\mathcal{L}}}$ generated by the lamination $\sim_{\mathcal{L}}$.

Definition 2.41. Let $\mathcal{L}$ be a proper geolamination. For a $\sim_{\mathcal{L}}$-class $\mathfrak{g}$, denote by $\mathfrak{g}_{\mathcal{L}}$ the collection of all leaves of $\mathcal{L}$ connecting points of $\mathfrak{g}$.

Lemma 2.42 easily follows from the definitions.

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LEMMA 2.42. *Let $\mathcal{L}$ be a proper geolamination and let $\mathfrak{g}_{\mathcal{L}}$ be a $\sim_{\mathcal{L}}$-class. Then all leaves of $\mathcal{L}$ intersecting $\mathfrak{g}_{\mathcal{L}}$ are disjoint from all other leaves of $\mathcal{L}$; for any points $a, b \in \mathfrak{g}_{\mathcal{L}}$ there is a finite chain of leaves from $\mathfrak{g}_{\mathcal{L}}$ connecting a and b.*

For a σ_d-invariant geolamination, by Theorem 2.10 and Definition 2.7, for any gap G, the map $\sigma_d|_{\mathrm{Bd}(G)}$ is the composition of a monotone map and a covering map of some degree. If G is of period k, then the same holds for $\sigma_d^k : \mathrm{Bd}(G) \to \mathrm{Bd}(G)$; the degree of the corresponding map is called the *degree of* G. A concatenation of chords is *trivial* if it consists of one chord.

Consider a lamination $\sim$. By [**BL02**], for any infinite gap G of $\sim$, the set $G \cap \mathbb{S}$ is a Cantor set (and thus, by Definition 2.5, G is a Fatou gap). Recall that $\psi_G : \mathrm{Bd}(G) \to \mathbb{S}$ is the monotone map which collapses all edges of G to points (since $G \cap \mathbb{S}$ is a Cantor set, there are no non-trivial concatenations of edges of G). By [**BL02**], a periodic infinite gap G of $\sim$ of period k can be of one of two types. If G is of degree one, we associate to it its irrational *rotation number* ρ and $\sigma_d^k|_{\mathrm{Bd}(G)}$ is semiconjugate by ψ_G to the circle rotation by the angle ρ. Such gaps are called *Siegel* gaps. Now, $\sigma_d^k|_{\mathrm{Bd}(G)}$ can be of degree $r > 1$. Then by [**BL02**], the map ψ_G semiconjugates $\sigma_d^k|_{\mathrm{Bd}(G)}$ and $\sigma_r|_{\mathbb{S}}$. Such gaps G are called *Fatou gaps of degree* r. Observe, finally, that, by [**BL02**], any infinite gap G of $\sim$ is (pre)periodic.

Now, if $\mathcal{L}$ is a proper geolamination, we can construct the lamination $\sim_{\mathcal{L}}$, the geolamination $\mathcal{L}_{\sim_{\mathcal{L}}}$ and apply to $\sim_{\mathcal{L}}$ and $\mathcal{L}_{\sim_{\mathcal{L}}}$ the results of [**BL02**] quoted in the previous paragraph, together with Lemma 2.42. This gives the following description of infinite gaps of proper geolaminations. A periodic gap of period k is said to be *identity return (under* σ_d^k*)* if σ_d^k fixes its vertices.

LEMMA 2.43. *Any infinite gap G of a proper geolamination $\mathcal{L}$ is a (pre)periodic Fatou gap. If G is a k-periodic gap of $\mathcal{L}$ of degree r, then the following hold.*

(1) *If $r = 1$, then the map ψ_G semiconjugates $\sigma_d^k|_{\mathrm{Bd}(G)}$ and the rotation of $\mathbb{S}$ by an irrational number ρ_G.*
(2) *If $r > 1$, then the map ψ_G semiconjugates $\sigma_d^k|_{\mathrm{Bd}(G)}$ and σ_r. If a maximal concatenation L of periodic leaves on the boundary of G is non-trivial and $\sigma_d^{kn}(L) \cap L \neq \emptyset$, then its convex hull H is a periodic identity return gap under σ^{kn}.*

Proof. We only need to prove the last claim from part (2). Suppose that $L = \ell_1 \cup \cdots \cup \ell_m$ is a maximal concatenation (in this order) of periodic leaves on the boundary of G and $m > 1$. Then the convex hull H of L can be obtained by connecting the appropriate endpoints of ℓ_1 and ℓ_m with a chord, which we denote by $\overline{n}$. If kn is the period of all the leaves in L, then $\sigma_d^{kn}(H) = H$ and σ_d^k fixes all the vertices of H (observe that, by Lemma 2.26, all leaves in L have the same period). Suppose that there exists a number $s < k$ such that $\sigma_d^s(H) = H$. Then leaves in L are not fixed under σ_d^s and one of them will have to be mapped onto $\overline{n}$, which is a contradiction. □

Definition 2.44. Periodic gaps of degree r of proper geolaminations are said to be either *Siegel gaps* (if $r = 1$) or *Fatou gaps of degree* r (if $r > 1$). For any gap G of period k, a k-periodic edge of G is said to be *refixed*; $\sigma_d^k|_{\mathrm{Bd}(G)}$ is called a *remap*.

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Let $\mathcal{NP}$ be the set of all critical leaves with non-periodic endpoints.

LEMMA 2.45. *Let $\mathcal{L}$ be a proper geolamination with a critical leaf $D \in \mathcal{NP}$. Let A be the convex hull of the $\sim_{\mathcal{L}}$-class containing the endpoints of D. Then A is a leaf or a finite gap and the following cases are possible.*

(1) *The set A is critical such that $\sigma_3|_A$ is three-to-one. Moreover, either A is an all-critical triangle, or there is a critical gap or leaf $C \subset A$, $C \neq D$ of $\mathcal{L}$ such that $\sigma_3|_C$ is of degree two.*

(2) *Otherwise, there is a critical set C that is either an infinite gap or the convex hull of a $\sim_{\mathcal{L}}$-class such that $\sigma_3|_C$ is of degree two and one of the following holds.*
   (a) *$C \cap \mathbb{S}$ is finite, C is disjoint from A and $\sigma_3|_C$ is two-to-one.*
   (b) *C is a periodic Fatou gap.*
   (c) *C is a preperiodic infinite gap which eventually maps to a periodic Siegel gap U, with D being an edge of U.*

Proof. Since $\sim_{\mathcal{L}}$ is a lamination, then A is either a leaf (and then it coincides with D) or a finite critical gap. If A is of degree three, then it is the unique critical set of $\sim_{\mathcal{L}}$ because we are considering the cubic case. The rest of case (1) is left to the reader (recall that the degree of a gap is defined in Definition 2.8).

Now, suppose that A is of degree two and denote the second critical set of $\sim_{\mathcal{L}}$ by C. If $C \cap \mathbb{S}$ is finite, then (a) holds. Thus we may assume that C is an infinite gap. If C is periodic, then, by Lemma 2.43, (b) holds. If C is preperiodic, then an eventual image of C is a periodic gap U of degree one (it cannot be of degree two because we work with the cubic case and there is not enough criticality). By Lemma 2.43, U is a Siegel gap. It is well known that, in any cycle of Siegel gaps, there must be a gap with at least one critical edge (see, for example, [**BOPT14**, Lemma 2.13]). Hence we may assume that U has D as its critical edge (under the assumption that D is the unique critical leaf of $\mathcal{L}$ as, otherwise, C would have been another critical edge of U, while we assume that C is an infinite gap). □

Proper geolaminations, described in case (2)(c) of Lemma 2.45, are said to be *Siegel geolaminations of capture type*.

2.3.2. *Finding proper sub-geolaminations.* By Theorem 2.10, the space $\mathbb{L}_d$ of all σ_d-invariant sibling geolaminations is compact. This allows one to assign geolamination(s) to every polynomial approximated by polynomials with locally connected Julia sets: if $P_i \to P$ and P_i are polynomials with locally connected Julia sets and geolaminations $\mathcal{L}_{P_i}$, then any limit geolamination $\lim_{i\to\infty} \mathcal{L}_{P_i}$ can be associated to P. However, while $\mathcal{L}_{P_i}$ are proper, their limit geolaminations might be not proper because the set of proper geolaminations $\mathbb{LP}_d \subset \mathbb{L}_d$ is not closed. To overcome this difficulty, we will develop techniques to associate a proper geolamination to many geolaminations of $\mathbb{L}_3$. A fact which plays a central role in this procedure is that improper leaves are isolated. Say that a leaf ℓ of a geolamination $\mathcal{L}$ is a *limit* leaf if it is the limit of a sequence of leaves of $\mathcal{L}$ distinct from ℓ itself.

Recall (Definition 2.38) that a non-degenerate leaf is *improper* if it has exactly one periodic endpoint. Also, given an arc $I \subset \mathbb{S}$, set $|I|$ to be its length.

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PROPOSITION 2.46. *Let $\mathcal{L} \in \mathbb{L}_d$ be a σ_d-invariant sibling geolamination. Then every improper leaf of $\mathcal{L}$ is isolated in $\mathcal{L}$.*

Proof. Let ℓ be an improper leaf of $\mathcal{L}$. By Lemma 2.39, an eventual image $\sigma_d^m(\ell) = \overline{px}$ of ℓ is either (1) a critical leaf with one periodic endpoint p of period k or (2) a leaf with exactly one periodic endpoint of period k whose σ_d^k-image $\sigma^k(\sigma_d^m(\ell))$ is a σ_d^k-periodic leaf ($\sigma_d^m(\ell)$ and $\sigma_d^{m+k}(\ell)$ form a critical wedge for σ_d^k). We show, by way of contradiction, that $\sigma_d^m(\ell) = \overline{px}$ is isolated in either case.

(1) In this case, $\sigma_d^k(x) = p$. Since σ_d is locally order preserving, for any chord $\overline{p'x'}$ close to, but disjoint from $\overline{px}$, its σ_d^k-image crosses $\overline{px}$. Hence $\overline{px}$ is a limit of leaves which meet $\overline{px}$. Since σ_d is expanding, the σ_d^k-image of a chord $\overline{xp'}$ close to $\overline{px}$ crosses itself. On the other hand, let $\overline{px'}$ be a leaf of $\mathcal{L}$, and assume (without loss of generality) that $p < x' < x$ and that x' is close to x. Set $\sigma_d^k(\overline{px'}) = \overline{py}$; clearly, $x < y < p$. Then, by Lemma 2.17, there exists a sibling leaf $\overline{xz}$ of $\overline{py}$ emanating from x and such that $p < z < x$. Since x' is close to x, we may assume that $[y, p] = \sigma_d^k[x', x]$. Since σ_d^k is expanding, $[y, p] = d \cdot [x', x] > |[x', x]|$. On the other hand, the fact that $\overline{py}$ and $\overline{xz}$ are siblings and the fact that $[y, p]$ is a small arc (because x' and x are close) imply that the $|[z, x]| \geq |[y, p]|$. Thus, $|[z, x]| > |[x', x]|$. Hence, $\overline{zx}$ crosses $\overline{px'}$, which is a contradiction.

(2) Let $\sigma_d^m(\ell) = \overline{px}$, $\sigma_d^{m+k}(\ell) = \overline{px'}$, $\sigma_d^k(x) = x' < p < x$ (and thus $\overline{px} \cup \overline{px'}$ is a critical wedge). Then, by Corollary 2.12, there are at most a finite number of leaves separating $\overline{px}$ and $\overline{px'}$. This implies that if the leaf $\overline{px}$ is a limit leaf, then there exists a sequence of leaves $\overline{qt}$ which converge to $\overline{px}$ and are such that $p \leq q < t \leq x$. Since σ_d is expanding, if $p < q$, then the σ_d^k-image of $\overline{qt}$ crosses $\overline{px}$. Thus $p = q$. Then, by continuity, a infinite number of leaves $\sigma_d^k(\overline{qt})$ separate $\overline{px}$ and $\overline{px'}$, which contradicts the above. □

Call a disjoint sibling collection *improper* if it includes an improper leaf.

LEMMA 2.47. *If a leaf ℓ is periodic or has non-preperiodic endpoints, then no disjoint sibling collection of ℓ is improper.*

Proof. The case when the endpoints of ℓ are non-preperiodic is obvious. If $\ell = \overline{xy}$ is periodic and ℓ' is a disjoint sibling of ℓ, then both endpoints of ℓ' are non-periodic, as desired. □

We need the following definition.

Definition 2.48. Let $\mathcal{L} \in \mathbb{L}_d \backslash \mathbb{LP}_d$. Let $\mathcal{L}^p \subset \mathcal{L}$ be the set of proper leaves $\ell \in \mathcal{L}$ such that they and all their (eventual) non-critical images have disjoint sibling collections consisting of proper leaves.

Arguments similar to those used in the proof of Theorem 2.15 allow us to find proper sub-geolaminations.

LEMMA 2.49. *Let $\mathcal{L} \in \mathbb{L}_d \backslash \mathbb{LP}_d$. Then $\mathcal{L}^p$ is a proper σ_d-invariant sibling geolamination containing all periodic leaves of $\mathcal{L}$, all leaves of $\mathcal{L}$ with non-preperiodic endpoints and all critical leaves of ℓ with non-periodic endpoints.*

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Proof. Let $\ell \in \mathcal{L}^p$. Then ℓ is proper. By definition, $\sigma_d(\ell) \in \mathcal{L}^p$. Moreover, by definition, ℓ has a disjoint sibling collection consisting of proper leaves all of which also belong to $\mathcal{L}^p$. Now consider pullbacks of ℓ and show that among them we can choose a disjoint sibling collection consisting of proper leaves. Indeed, the only way ℓ can have an improper pullback ℓ' is when ℓ is periodic while ℓ' has exactly one periodic endpoint. To handle this case, notice that ℓ must also have a purely periodic pullback (for example, if ℓ is of period s, we can always choose $\sigma_d^{s-1}(\ell)$ as such a leaf). Choosing this pullback and its disjoint sibling collection, we see that this entire collection consists of proper leaves, as desired.

By definition, it remains to show that $\mathcal{L}^p$ is closed. To this end, we need to check whether all removed leaves are isolated. Indeed, a leaf ℓ is removed if, for some $i \geq 0$, all disjoint sibling collections of $\sigma_d^i(\ell)$ include an improper leaf (which is isolated, by Proposition 2.46). By Lemma 2.14, this implies that $\sigma_d^i(\ell)$ is isolated. Indeed, if $\sigma_d^i(\ell)$ is non-isolated, then, by Lemma 2.14, it has a disjoint sibling collection consisting of non-isolated leaves which (by Proposition 2.46) are all proper, which is a contradiction. Hence $\sigma_d^i(\ell)$ is isolated, as desired. The last claim of the lemma follows from Lemma 2.47 and the definition of $\mathcal{L}^p$. □

Proposition 2.50 provides conditions whereby (the grand orbit of) a periodic leaf can be added to a proper geolamination $\mathcal{L}$ to create another proper geolamination.

PROPOSITION 2.50. *Let $\mathcal{L}$ be a proper geolamination and L be a cycle of periodic leaves which do not cross leaves of $\mathcal{L}$. Then there exists a proper geolamination $\widehat{\mathcal{L}}$ containing $L \cup \mathcal{L}$ such that $\widehat{\mathcal{L}} \setminus \mathcal{L}$ consists of leaves which either eventually map to L or are limits of such leaves.*

Proof. The proof is rather straightforward, so we only sketch it. To construct $\widehat{\mathcal{L}}$, we need to pull back L in a step-by-step fashion so that, on each step, disjoint sibling collections are formed. This is immediate if a leaf which is being pulled back is not contained in a gap which itself is the image of a critical set. Otherwise, it suffices to choose the pullbacks of the leaf in question inside of the appropriate critical sets so that, again, we will be getting one or several disjoint sibling collections. Repeating this a countable number of times we will make the first step in the construction. It is easy to see that, when we take the closure, the resulting geolamination is a σ_d-invariant sibling geolamination [**BMOV13**]. It follows, from Proposition 2.46, that $\widehat{\mathcal{L}}$ is a proper geolamination. □

3. *Main results*

Consider cubic (geo)laminations. We need the following definition.

Definition 3.1. Two generalized critical quadrilaterals form an *admissible (cubic) qc-portrait* if (1) a generalized critical quadrilateral has a periodic vertex and a non-degenerate image, and thus must have a periodic edge, and (2) the elements of the qc-portrait and all their images intersect at most over a common edge or vertex. By Theorem 2.19, the sets of the orbits of elements of an admissible qc-portrait QCP can be pulled back to form a σ_3-invariant geolamination; any such geolamination is denoted by $\mathcal{L}_{\mathrm{QCP}}$.

Recall that $\mathcal{NP}$ is the set of critical leaves with non-periodic endpoints.

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Definition 3.2. Let $\mathbb{LP}_3^{\rm np}$ be the set of all proper geolaminations with a critical leaf from $\mathcal{NP}$, except for Siegel geolaminations of capture type. Let $\mathcal{QCP}_3^{\rm np}$ be the collection of all admissible qc-portraits with a critical leaf from $\mathcal{NP}$ as the second element. For $D \in \mathcal{NP}$, $\mathcal{S}_D$ is the collection of admissible qc-portraits with D as the second element.

Recall that the sets $\mathbb{LP}_3^{\rm np}$, $\mathcal{QCP}_3^{\rm np}$ and $\mathcal{S}_D$ are endowed with the Hausdorff metric and the induced topology. From now on, whenever we talk about a privileged qc-portrait of a geolamination $\mathcal{L} \in \mathbb{LP}_3^{\rm np}$ (the notion of a privileged qc-portrait can be found in the Introduction and is formally given in Definition 4.1) we always assume that its second element is a critical leaf with non-periodic endpoints.

THEOREM A. *Each $\mathcal{L} \in \mathbb{LP}_3^{\rm np}$ has at least one and no more than a finite number of privileged qc-portraits. For every* $\mathrm{QCP} \in \mathcal{QCP}_3^{\rm np}$*, there exists $\mathcal{L} \in \mathbb{LP}_3^{\rm np}$ such that* QCP *is privileged for $\mathcal{L}$. Moreover, $\mathcal{S}_D$ is compact.*

It turns out that privileged portraits properly capture dynamics.

THEOREM B. *Suppose that* $\mathrm{QCP}_1 = (Q_1, D), \mathrm{QCP}_2 = (Q_2, D)$ *are privileged qc-portraits for geolaminations $\mathcal{L}_1, \mathcal{L}_2 \in \mathbb{LP}_3^{\rm np}$ such that $\sigma_3(Q_1) \cap \sigma_3(Q_2) \neq \emptyset$. Then* $\sim_{\mathcal{L}_1} = \sim_{\mathcal{L}_2}$.

Observe that if $\mathrm{QCP}_1, \mathrm{QCP}_2$ are linked privileged qc-portraits for geolaminations $\mathcal{L}_1, \mathcal{L}_2 \in \mathbb{LP}_3^{\rm np}$, then there exists a critical leaf $D \in \mathcal{NP}$ such that $\mathrm{QCP}_1, \mathrm{QCP}_2 \in \mathcal{S}_D$.

In contrast to the quadratic case, where there are no linked minors, there may be linked qc-portraits in $\mathcal{QCP}_3^{\rm np}$, but the laminations specified by these linked qc-portraits are the same. We use this result to study $\mathcal{S}_D$, which is naturally identified with the set of $\mathbb{LP}_3$-geolaminations containing D. Each $(Q, D) \in \mathcal{S}_D$ is tagged by the chord or point (the *minor*) $\sigma_3(Q)$. Denote by CML_D the family of all such leaves $\sigma_3(Q)$. We will show that CML_D is an appropriate cubic analog of Thurston's QML.

THEOREM C. *The family* CML_D *is proper, so that the generated equivalence relation $\sim_{\mathrm{CML}_D} = \sim_D$ is a lamination. Each $\sim_D$-class corresponds to a unique cubic lamination $\mathcal{L}$. Conversely, every cubic lamination which is not of capture Siegel type and is such that endpoints of D are equivalent corresponds to a $\sim_D$-equivalence class.*

If $D = \overline{ab}$, then Theorem C equips the set LAM_D of cubic laminations $\approx$ such that $a \approx b$ and which are not of Siegel capture type with the quotient topology of the unit circle. Also, suppose that a lamination $\approx$ with $a \approx b$ is of Siegel capture type. Then there is only one way that this can happen. Namely, $\approx$ must then have a periodic Siegel gap U, with D being an edge of U, and a non-periodic pullback V of U, which maps forward in a two-to-one fashion.

Finally we study the $\sim_D$-classes. First, we modify the classical notion of the *minor* of a geolamination in the quadratic case.

Definition 3.3. (Minor sets in the quadratic case [**Thu85**]) For a quadratic lamination $\sim$, let C be the critical set of $\sim$. If C is finite, let $m_\sim = \sigma_2(C)$. If C is a Fatou gap of degree two, let $m_\sim$ be the σ_2-image of the refixed edge of C. The set $m_\sim$ is called the *minor set* of $\sim$.

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Note that, in the quadratic case, minor sets of quadratic laminations coincide with the convex hulls of $\sim_{\mathrm{QML}}$-classes. We want to extend these ideas to the cubic case. In the cubic case, similarly to the quadratic case, minor sets can be introduced for all laminations $\sim$ from LAM_D as images of the first critical set of $\sim$ (if it is finite) or the image of the refixed edge of the periodic Fatou gap U of degree two of $\sim$ (if it exists). However, in the cubic case, there is a new phenomenon which causes these collections of minor sets taken for laminations $\sim$ from LAM_D to be insufficient. We overcome this difficulty by modifying Definition 3.3, below.

Definition 3.4. (Minor sets in the cubic case) If $\sim \in \mathrm{LAM}_D$, let $D_\sim$ be the $\sim$-class containing the endpoints of D. Also, if $\sim$ has a k-periodic critical Fatou gap U of degree two, let $M_\sim$ be the $\sim$-class of the unique edge of U of period k. Let $C_\sim$ be either the first critical set of $\sim$ (if it is different from $D_\sim$ and finite), $M_\sim$ (if the first critical set of $\sim$ is a periodic Fatou gap U of degree two) or $\mathrm{CH}(D_\sim)$ (if $\sim$ has a unique critical class $D_\sim$). Set $\sigma_3(C_\sim) = m_\sim$ and call $m_\sim$ the *minor set* of $\sim$.

Now we are ready to state Theorem D.

THEOREM D. *Classes of $\sim_D$ coincide with the minor sets $m_\sim$, where $\sim \in \mathrm{LAM}_D$.*

4. *Privileged QC-portraits for proper geolaminations*

Let us recall the definition of a privileged portrait.

Definition 4.1. If $\mathcal{L} \in \mathbb{LP}_3^{\mathrm{np}}$ has a critical leaf D with non-periodic endpoints, then a qc-portrait $\mathrm{QCP} = (Q, D)$ is called *privileged for* $\mathcal{L}$ if and only if $Q \subset C$, where $C \neq D$ is a critical set of $\mathcal{L}$ and either C is finite or C is a periodic Fatou gap of degree two and period k and Q is a collapsing quadrilateral which is a convex hull of a (possibly degenerate) edge ℓ of C of period k and another edge $\hat{\ell}$ of C such that $\sigma_3(\ell) = \sigma_3(\hat{\ell})$.

Observe that edges of the set Q, from Definition 4.1, are not necessarily leaves of $\mathcal{L}$. For example, if Q is contained in a periodic Fatou gap of degree two, then improper edges of Q are not leaves of the proper geolamination $\mathcal{L}$.

Lemma 4.2 immediately follows from the definitions and Lemma 2.45 and is stated here without proof.

LEMMA 4.2. *Each $\mathcal{L} \in \mathbb{LP}_3^{\mathrm{np}}$ has at least one and no more than a finite number of privileged qc-portraits.*

In the rest of this section we show that every $\mathrm{QCP} \in \mathcal{QCP}_3^{\mathrm{np}}$ is a privileged qc-portrait of some $\mathcal{L} \in \mathbb{LP}_3^{\mathrm{np}}$. Recall Definition 3.1.

Definition 4.3. An *admissible (cubic) qc-portrait* is an ordered pair of generalized critical quadrilaterals (A, B) such that $A \neq B$ and, moreover, A, B and all their images intersect at most over a common edge or vertex and, if A or B has a periodic vertex, then it either has a degenerate image or has a periodic edge.

It turns out that if $B \in \mathcal{NP}$, then it suffices to make sure that the first part of the definition of an admissible qc-portrait holds, as the second one will automatically follow.

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To prove this, given a qc-portrait QCP, one needs to use Theorem 2.19 and construct corresponding (sibling) invariant geolaminations, each of which is denoted by $\mathcal{L}_{\mathrm{QCP}}$.

PROPOSITION 4.4. *Let (Q, D) be a cubic qc-portrait with $D \in \mathcal{NP}$ and such that $Q \neq D$ and all their images intersect at most over a common edge or vertex. Moreover, if Q is an all-critical triangle viewed as a generalized quadrilateral, then we assume that if Q has a periodic vertex, then this vertex is considered as an edge of Q. Then (Q, D) is admissible. Moreover, if Q has no periodic vertices, then any geolamination containing $\{Q, D\}$ is proper and* $\mathrm{QCP} = (Q, D)$ *is a privileged qc-portrait for it.*

Proof. Let $\mathcal{L}_{\mathrm{QCP}}$ denote a geolamination which contains (Q, D). If $\mathcal{L}_{\mathrm{QCP}}$ is not proper, then there is a periodic point p and a critical leaf/wedge L with vertex at p. Since the endpoints of D are non-periodic, $p \notin D$. Thus, if vertices of Q are non-periodic, then $p \notin Q$, which implies that such a set L does not exist, $\mathcal{L}_{\mathrm{QCP}}$ is proper and, by definition, QCP is a privileged qc-portrait for $\mathcal{L}_{\mathrm{QCP}}$. This proves the second claim of the lemma.

To prove the first claim, assume that Q is a collapsing quadrilateral with a periodic vertex p. Clearly, two edges of Q with periodic vertex, say, p, form a critical wedge L. Moreover, it follows that $\mathcal{L}$ has no critical leaves with a periodic endpoint and the unique critical wedge of $\mathcal{L}$ is L. Since, by Lemma 2.39, $\mathcal{L}$ has at least one critical wedge with a periodic edge, the second claim of the lemma follows. □

Corollary 4.5 immediately follows.

COROLLARY 4.5. *Let* $\mathrm{QCP} = (Q, D) \in \mathcal{QCP}_3^{\mathrm{np}}$. *Then $\sigma_3^k(Q)$ is not an improper leaf for all $k > 0$.*

Proof. If Q has a periodic vertex, then, by Proposition 4.4, it has a periodic edge and the result is immediate. Otherwise, by Proposition 4.4, any pullback geolamination $\mathcal{L}_{\mathrm{QCP}}$ is proper. Since all σ_3-images of Q are in $\mathcal{L}_{\mathrm{QCP}}$, none of them are improper. □

To prove that if $\mathrm{QCP} = (Q, D) \in \mathcal{QCP}_3^{\mathrm{np}}$, then QCP is a privileged qc-portrait of a proper cubic geolamination, we use the following strategy. First, we use Theorem 2.19 and construct a pullback geolamination $\mathcal{L}_{\mathrm{QCP}}$. Then we clean $\mathcal{L}_{\mathrm{QCP}}$ using Lemma 2.49 and obtain a proper geolamination $\mathcal{L}^{\mathrm{pr}}$. Now consider two cases. The case when sets from QCP have no periodic vertices is easier to handle. In this case, by Proposition 4.4, any pullback geolamination $\mathcal{L}_{\mathrm{QCP}}$ is proper. By definition, $\mathcal{L}_{\mathrm{QCP}}$ is not of Siegel capture type. Since Q and D, by construction, remain critical sets of $\mathcal{L}_{\mathrm{QCP}}$, QCP is privileged for $\mathcal{L}_{\mathrm{QCP}}$, by definition.

Now consider the case when Q has a periodic vertex. By Proposition 4.4, Q is either a critical leaf, an all-critical triangle or a collapsing quadrilateral with a periodic edge. To show that (Q, D) is a privileged qc-portrait for some proper cubic geolamination, we need the following fact.

PROPOSITION 4.6. [**Sch09**, **Thu85**] *Let $\overline{c}$ be a σ_2-critical leaf with a periodic endpoint p of period $k > 1$. Then there exists a unique leaf $M_{\overline{c}} = \overline{px}$ and a Fatou gap $V_{\overline{c}}$ such that $V_{\overline{c}}$ is of period k and $M_{\overline{c}}$ is its refixed leaf.*

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Now, construct a proper geolamination $\mathcal{L}^{\rm pr}$ out of $\mathcal{L}_{\rm QCP}$, using Definition 2.48 and Lemma 2.49. If QCP is a privileged qc-portrait for $\mathcal{L}^{\rm pr}$, there is nothing to prove. Suppose that QCP is not a privileged qc-portrait for $\mathcal{L}^{\rm pr}$. By definition, the only way it can happen is when the set Q is contained in a quadratic Fatou gap U of $\mathcal{L}^{\rm pr}$ and, moreover, the period of the periodic edge of Q is greater than the period of the gap U. In this case, we apply the map ψ_U, which sends Q to a σ_2-critical leaf $\overline{c}$ with a periodic endpoint, say, p of period $k > 1$ ($k > 1$ *exactly because* QCP is not privileged for $\mathcal{L}_{\rm QCP}$). Then we use Proposition 4.6, find a σ_2-periodic Fatou gap V for which p is an endpoint of a refixed edge and pull V back to U using the projection map ψ_U. This, finally, produces a proper geolamination for which QCP is a privileged qc-portrait.

THEOREM 4.7. *Every* $\mathrm{QCP} = (Q, D) \in \mathcal{QCP}_3^{\rm np}$ *is a privileged qc-portrait of a geolamination from* $\mathbb{LP}_3^{\rm np}$.

Proof. We may assume that Q has a periodic (possibly degenerate) edge ℓ_Q of vertex period N and its sibling edge $\hat{\ell}_Q$. Consider the family $\mathcal{A}_{\rm QCP}$ of *all* cubic geolaminations containing Q and D (by Theorem 2.19, $\mathcal{A}_{\rm QCP} \neq \emptyset$). Among geolaminations from $\mathcal{A}_{\rm QCP}$, choose a geolamination $\mathcal{L}$ with a maximal family of non-degenerate periodic leaves of periods at most N. Clearly, such a geolamination $\mathcal{L}$ exists. By Lemma 2.49, we can find a proper geolamination $\mathcal{L}^{\rm pr} \subset \mathcal{L}$ with $\ell_Q \in \mathcal{L}^{\rm pr}$.

By way of contradiction, we may assume that QCP is not privileged for $\mathcal{L}^{\rm pr}$. Then there must exist a periodic Fatou gap U_Q which contains Q. On the other hand, by Lemma 2.49, D is a leaf of $\mathcal{L}^{\rm pr}$. It follows that U_Q is a Fatou gap such that $\sigma_3|_{\mathrm{Bd}(U_Q)}$ is of degree two. Consider the case when ℓ_Q is non-degenerate. Since, by Lemma 2.49, $\ell_Q \in \mathcal{L}^{\rm pr}$, ℓ_Q is an edge of U_Q. Let us show that $\hat{\ell}_Q \in \mathcal{L}^{\rm pr}$. Indeed, by Lemma 2.49, we include in $\mathcal{L}^{\rm pr}$ those leaves of $\mathcal{L}$ which have disjoint sibling collections consisting of proper leaves, all images of which have the same property. Now, take $\hat{\ell}_Q$ and its disjoint sibling collection (which must exist by definition of a sibling invariant geolamination). It follows that such a collection *must* include ℓ_Q. Hence $\hat{\ell}_Q \in \mathcal{L}^{\rm pr}$, as desired. By construction, ℓ_Q and $\hat{\ell}_Q$ are edges of U_Q. Now, if ℓ_Q is degenerate, then Q can be a critical leaf or an all-critical triangle (viewed as a generalized quadrilateral). In either case, ℓ_Q is a vertex of (an edge of) U_Q and $\hat{\ell}_Q$ is either a vertex of U_Q or an edge of U_Q (the latter holds if Q is an all-critical triangle).

Let us show that, in that case, $\mathcal{L}^{\rm pr}$ cannot be a Siegel geolamination of capture type. Indeed, suppose otherwise. Then U_Q eventually maps onto a periodic Siegel gap, so the original set Q has edges with non-preperiodic leaves, which a contradiction. Hence $\mathcal{L}^{\rm pr}$ is not a Siegel geolamination of capture type.

The map $\phi_{U_Q} : \mathrm{Bd}(U_Q) \to \mathbb{S}$ collapses concatenations of edges in $\mathrm{Bd}(U_Q)$ to points and semiconjugates the remap of $\mathrm{Bd}(U_Q)$ to σ_2. Clearly, $\phi_{U_Q}(Q) = \overline{ab}$ is a σ_2-critical leaf with a periodic endpoint, say, z. If $z = 0$, then ℓ_Q is a refixed edge of U_Q and the proof is complete. Assume that $z \neq 0$ and bring this to a contradiction. Set $M = M_{\overline{ab}}$, as in Proposition 4.6, and use $\phi_{U_Q}^{-1}$ to pull the Fatou gap $V = V_{\overline{ab}}$ from Proposition 4.6 back to a gap $V_Q \subset U_Q$, by taking full ψ_{U_Q}-preimages of points from $V \cap \mathbb{S}$, and then taking the convex hull of such preimages. Then ℓ_Q is an edge of V_Q and a periodic edge M_Q of V_Q of vertex period N maps to M under ψ_U. By construction, $Q \subset V_Q$, while M_Q is a periodic

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leaf whose orbit consists of leaves which do not cross Q. This contradicts the choice of the geolamination $\mathcal{L}$ as a geolamination with the maximal number of non-degenerate periodic leaves of period at most N. □

Proof of Theorem A. By Lemma 4.2, there is at least one and at most a finite number of qc-portraits privileged for a given geolamination from $\mathbb{LP}_3^{\mathrm{np}}$. On the other hand, by Theorem 4.7, every qc-portrait $\mathrm{QCP} \in \mathcal{QCP}_3^{\mathrm{np}}$ is a privileged qc-portrait of a geolamination from $\mathbb{LP}_3^{\mathrm{np}}$.

Recall that, for $D \in \mathcal{NP}$, $\mathcal{S}_D$ is the collection of admissible qc-portraits with D as the second element. To show that $\mathcal{S}_D$ is compact, let $(Q_n, D) \to (Q, D)$. By definition, the fact that Q_n are generalized critical quadrilaterals implies that Q is a generalized critical quadrilateral. Moreover, if (Q, D) is not admissible, then either there exist numbers i, j with $\sigma_3^i(Q)$ crossing $\sigma_3^j(Q)$ or there exists a number k such that $\sigma_3^k(Q)$ crosses D. In either case, the same crossing will have to take place for the corresponding images of Q_n and D, which contradicts the fact that (Q_n, D) is admissible. By Proposition 4.4, (Q, D) is admissible. □

Notice that if QCP is a privileged qc-portrait of a proper geolamination $\mathcal{L}$ and all vertices of sets from QCP are non-periodic, then QCP remains a privileged qc-portrait of the $\mathbb{LQ}$-geolamination $\mathcal{L}_{\sim_{\mathcal{L}}}$ generated by $\mathcal{L}$.

5. *Intersecting minors of* $\mathcal{QCP}_3^{\mathrm{np}}$

By definition, a proper family of chords $\mathcal{F}$ generates a lamination $\sim_{\mathcal{F}}$ and, if $\mathcal{F} = \mathcal{L}$ is also a (σ_d-)invariant geolamination, then $\sim_{\mathcal{L}}$ is a (σ_d-)invariant lamination. Throughout the rest of this section, the following holds.

Standing assumption. We fix a critical leaf $D = \overline{ab} \in \mathcal{NP}$ with a positive arc $(a, b) \subset \mathbb{S}$ of length $\frac{1}{3}$, set $\sigma_3(D) = d$ and denote by Δ the all-critical triangle $\mathrm{CH}(a, b, v)$. Fix admissible qc-portraits $(Q_1, D) = \mathrm{QCP}_1$ and $(Q_2, D) = \mathrm{QCP}_2$ such that their *minors* $\sigma_3(Q_1)$ and $\sigma_3(Q_2)$ are non-disjoint. Assume that QCP_i is a privileged qc-portrait for a proper geolamination $\mathcal{L}_i$ for $i = 1, 2$ such that neither $\mathcal{L}_1$ nor $\mathcal{L}_2$ is of Siegel capture type. Set $\sim_{\mathcal{L}_1} = \sim_1$, $\sim_{\mathcal{L}_2} = \sim_2$. Let $\widehat{\mathcal{L}}_1 = \mathcal{L}_{\sim_1}$ and $\widehat{\mathcal{L}}_2 = \mathcal{L}_{\sim_2}$.

By definition, $Q_1 \neq D$, $Q_2 \neq D$. On the other hand, it is possible that $Q_1 = \Delta$ or $Q_2 = \Delta$.

Our aim is to prove Theorem B, that is, to prove that $\sim_1 = \sim_2$. If Q_1 and Q_2 are strongly linked or share a spike, tools from [**BOPT14**] apply and eventually imply the desired result. However, we need to prove Theorem B under the weaker assumptions that $\sigma_3(Q_1)$ and $\sigma_3(Q_2)$ are non-disjoint. Thus we need to study the case when $\sigma_3(Q_1)$ and $\sigma_3(Q_2)$ are non-disjoint but Q_1 and Q_2 are neither strongly linked nor share a spike. Lemma 5.1 shows that this is an exceptional case.

LEMMA 5.1. *Qc-portraits* QCP_1 *and* QCP_2 *are linked unless one of the sets* Q_1, Q_2 *coincides with* (a, x, v, x') *while the other one coincides with* (b, y, v, y').

Proof. Let us show that if $\sigma_3(Q_1) \cap \sigma_3(Q_2) \neq \{d\}$, then QCP_1 and QCP_2 are linked. Observe that $\sigma_3|_{[b,a]}$ is two-to-one, except that d has three preimages $a, b, v \in [b, a]$.

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Consider some cases. First, assume that $\sigma_3(Q_1) \cap \sigma_3(Q_2) \cap \mathbb{D} \neq 0$. Since both minors have points inside $\mathbb{D}$, they are non-degenerate and the generalized quadrilaterals Q_1, Q_2 are true quadrilaterals. If the minors do not coincide, it immediately follows from properties of $\sigma_3|_{[b,a]}$ and the assumptions that Q_1 and Q_2 are strongly linked. If minors coincide and do not have d as an endpoint, it follows that $Q_1 = Q_2$ (and hence, by definition, Q_1 and Q_2 are strongly linked). Assume that $\sigma_3(Q_1) = \sigma_3(Q_2) = \overline{dx}$ and denote by $x', x'' \in (b, a)$ two points with $\sigma_3(x') = \sigma_3(x'') = x$. Then either $Q_1 = Q_2$ or, otherwise, we may assume that $Q_1 = (a, x', v, x'')$ and $Q_2 = \mathrm{CH}(b, x', v, x'')$. Clearly, in this case, again, Q_1 and Q_2 are strongly linked. Assume now that $\sigma_3(Q_1) \cap \sigma_3(Q_2) = \{y\}$, where $y \neq d$. Let $y', y'' \in (b, a)$ so that $\sigma_3(y') = \sigma_3('') = y$. Then both Q_1 and Q_2 share the spike $\overline{y'y''}$. By definition, in all these cases, QCP_1 and QCP_2 are linked. If now $\sigma_3(Q_1) \cap \sigma_3(Q_2) = \{d\}$, then both Q_1 and Q_2 have either $\overline{av}$ or $\overline{bv}$ as a spike. If either Q_1 or Q_2 is Δ, $\mathcal{L}_1$ and $\mathcal{L}_2$ are linked. Hence one of the sets Q_1, Q_2 coincides with (a, x, v, x'), while the other one coincides with (b, y, v, y'). □

For brevity, if QCP_1 and QCP_2 are not linked, we will say that *Case V* holds; without loss of generality, we will then always assume that $Q_1 = (a, x, v, x')$ and $Q_2 = (b, y, v, y')$. We use Lemma 5.1 to handle Case V. It turns out that then either Theorem B follows from known results, or specific restructuring allows us to find geolaminations which *are* linked and generate the same laminations as the given geolaminations. This shows that it is enough to prove Theorem B in the case when QCP_1 and QCP_2 *are* linked.

LEMMA 5.2. *Suppose that Case V holds and v is not periodic. Then $\sim_1 = \sim_2$.*

Proof. Since a, b and v are not periodic, Q_1 and Q_2 cannot be critical quadrilaterals generated by periodic Fatou gaps. Hence $\sim_1$ ($\sim_2$) has a unique finite critical class $\mathfrak{g}_1$ ($\mathfrak{g}_2$) on which σ_3 is three-to-one. Clearly, $\{a, b, v\}$ is contained in both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ so that $d = \sigma_3(a) \in \sigma_3(\mathfrak{g}_1) \cap \sigma_3(\mathfrak{g}_2)$. By **[Sch04]**, $\mathfrak{g}_1 = \mathfrak{g}_2$. A standard pullback argument then shows that $\sim_1 = \sim_2$. □

To complete considering Case V, we now assume that v *is* periodic. We need a few lemmas from **[CHMMO14]**, where they are proven in the degree d case (we restate them for the cubic case).

LEMMA 5.3. **[CHMMO14**, Lemma 1.5] *Suppose that $\mathcal{L}$ is a cubic geolamination. Then, for any leaf $\ell = \overline{ab}$ of $\mathcal{L}$, there exists $i \geq 0$ such that both circle arcs with endpoints $\sigma_3^i(a)$, $\sigma_3^i(b)$ are of length at least $\frac{1}{4}$.*

Given a circle chord $\ell = \overline{xy}$ which is not a diameter, let $\|\ell\|$ be the length of the smallest of the two arcs in $\mathbb{S}$ with endpoints x, y. Let $\ell_0 = \overline{ab}$ be a chord with $0 < \|\ell_0\| = b - a < \frac{1}{3}$. Let $I_0 = [b, a + \frac{1}{3}]$; let I_1 be I_0 rotated by $\frac{1}{3}$ and I_2 be I_0 rotated by $\frac{2}{3}$. Define the *central strip* $\mathrm{CS}(\ell_0)$ of ℓ_0 as the convex hull of $I_0 \cup I_1 \cup I_3$.

LEMMA 5.4. **[CHMMO14**, Theorem 2.10] *Suppose that $\mathcal{L}$ is a cubic geolamination and ℓ is a leaf of $\mathcal{L}$ such that $\frac{1}{4} \leq \|\ell\| < \frac{1}{3}$. Let $j > 0$ be the least number such that $\sigma_3^j(\ell)$ has an endpoint in $\mathrm{CS}(\ell)$. Then $\sigma_3^j(\ell)$ has endpoints in two distinct circle arcs on the boundary of $\mathrm{CS}(\ell)$.*

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We use these tools to prove the next lemma.

LEMMA 5.5. *Let a proper geolamination $\mathcal{L}$ have a periodic critical Fatou gap U of degree two and a refixed edge ℓ. Let ℓ' be the sibling of ℓ contained in* $\mathrm{Bd}(U)$. *If ℓ' is contained in a finite critical gap G' of $\sim_{\mathcal{L}}$, then ℓ is the unique refixed edge of U and $\frac{1}{4} \le \|\ell\| < \frac{1}{3}$. Moreover, the entire orbit of U except for U itself is disjoint from* $\mathrm{CS}(\ell)$.

A weaker version of the last claim of the lemma, which may be easier to apply, is that the images of ℓ do not intersect the two all-critical triangles, one of whose vertices is an endpoint of ℓ.

Proof. Suppose that $\ell = \overline{pq}$. Then $p + \frac{1}{3}$, $p + \frac{2}{3} \in G'$. Since ℓ does not cross edges or diagonals of G', $\|\ell\| \le \frac{1}{3}$. Moreover, since ℓ is refixed, $\|\ell\| < \frac{1}{3}$. Let T be the all-critical triangle with vertex p. Clearly, images of ℓ or any other periodic edge of U do not cross edges of T. Now consider any refixed edge $\hat{\ell}$ of U and prove that $\|\hat{\ell}\| \ge \frac{1}{4}$.

Indeed, if $\|\hat{\ell}\| < \frac{1}{4}$, choose i with $\|\sigma_3^i(\hat{\ell})\|$ maximal. Clearly, $T \subset \mathrm{CS}(\sigma_3^i(\hat{\ell}))$. By Lemma 5.3, $\frac{1}{4} \le \|\sigma_3^i(\hat{\ell})\| < \frac{1}{3}$, the set $\mathrm{CS}(\sigma_3^i(\hat{\ell}))$ is well defined and $\sigma_3^i(\hat{\ell}) \ne \hat{\ell}$. Since $\hat{\ell}$ is itself an eventual image of $\sigma_3^i(\hat{\ell})$, $\sigma_3^i(\hat{\ell})$ at some later moment enters $\mathrm{CS}(\sigma_3^i(\hat{\ell}))$. However, by Lemma 5.4, the *first time* $\sigma_3^i(\hat{\ell})$ enters $\mathrm{CS}(\sigma_3^i(\hat{\ell}))$, the corresponding image of $\sigma_3^i(\hat{\ell})$ will have its endpoints in distinct arcs of the boundary of $\mathrm{CS}(\sigma_3^i(\hat{\ell}))$, which contradicts the maximality of $\|\sigma_3^i(\hat{\ell})\|$. Hence $\|\hat{\ell}\| \ge \frac{1}{4}$, and the circle arcs on the boundary of $\mathrm{CS}(\hat{\ell})$ are of length at most $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$. Hence ℓ is the unique refixed edge of U. The last claim of the lemma follows from Lemma 5.4. □

Now we can go back to the case of two proper geolaminations for which Case V holds.

LEMMA 5.6. *Suppose that Case V holds and v is periodic. Then we can replace the quadrilateral Q_2 by a quadrilateral Q_2' so that the new qc-portrait (Q_2', D) is still privileged for $\mathcal{L}_2$ and is linked with* QCP_1.

Proof. Since v is periodic and QCP_2 is privileged for $\mathcal{L}_2$, either $\overline{yv}$ or $\overline{vy'}$ is a refixed edge of a periodic Fatou gap U of degree two. Assume that $\overline{vy'}$ is a refixed edge of U. Then the leaves $\overline{vy}$, $\overline{by'}$ are not leaves of $\mathcal{L}_2$.

By Lemma 5.5, $\overline{vy'}$ is actually the unique refixed edge of U. Let us show that, in this case, no edge of U crosses the critical chord $\overline{va}$. Indeed, suppose otherwise. Clearly, the only way an edge ℓ of U can cross $\overline{va}$ is when it has b as an endpoint. By [**BMOV13**], this implies that $\mathcal{L}$ has a leaf $\overline{zv}$ with $b < z < v$ which is a sibling leaf of ℓ. Since $\mathcal{L}_2$ is proper, the leaf $\overline{vz}$ must be periodic of the same period as ℓ. Since ℓ is the unique refixed edge of U, the leaf $\overline{zv}$ cannot be an edge of U. This implies that, for geometric reasons, there still must exist an edge of U coming out of v and different from $\overline{vy}$. However, as above, this edge of U will then have to be refixed, which, again, contradicts Lemma 5.5.

Thus, we can remove $\overline{vy}$ and $\overline{y'b}$ (that is, two sides of Q_2 which are not edges of U) and replace them by the critical leaf $\overline{va}$. By definition, the qc-portrait $(\overline{va}, D)$ is privileged for $\mathcal{L}_2$. On the other hand, by definition, $(\overline{va}, D)$ is linked with QCP_1. This completes the proof. □

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Thus, to prove Theorem B, it suffices to consider the case when Q_1, Q_2 are linked as by Lemma 5.6 Case V can be reduced to this case. So, from now on, we may assume that Q_1, Q_2 are linked and, therefore, $(\mathcal{L}_1, Q_1)$ and $(\mathcal{L}_2, Q_2)$ are linked.

We will need [**BMOV13**, Lemma 4.8].

LEMMA 5.7. [**BMOV13**, Lemma 4.8] *If $\mathcal{L}$ is a σ_d-invariant geolamination and $\ell \in \mathcal{L}$ is a leaf such that $\sigma_d^n(\ell)$ is concatenated with ℓ for some n, then the endpoints of ℓ are (pre)periodic.*

Proposition 5.8 studies possible intersections between various leaves and edges of Δ.

PROPOSITION 5.8. *If* QCP $= (Q, D) \in \mathcal{S}_D$ *with $d \in \sigma_3(Q)$, then no eventual σ_3-image of Q crosses an edge of Δ. If, moreover, $\mathcal{L}$ is a proper geolamination for which* QCP *is privileged, then no periodic leaf of $\mathcal{L}$ crosses an edge of Δ.*

Proof. Let $\overline{av}$ be a spike of Q. To prove the first claim of the lemma, assume that $Q = (a, x, v, x')$ is a true quadrilateral, and show that no eventual σ_3-image of Q crosses $\overline{bv}$. Indeed, suppose that $\sigma_3^k(Q)$ crosses $\overline{bv}$. Then, since images of Q cannot cross D or Q, it follows that the leaf $\sigma_3^k(Q)$ equals $\overline{ay}$ and is concatenated with the leaf $\overline{xa}$. By Lemma 5.7, points a and y are (pre)periodic. By the assumptions, a is not periodic. By Corollary 4.5, $\overline{ay}$ and all its images are proper leaves. However if we now follow the concatenation of leaves $\overline{ay}$, $\sigma_3^k(\overline{ay})$ and so on, we will, at some point, encounter the situation when exactly one endpoint of some image of $\overline{ay}$ is periodic, which is a contradiction.

Consider the above situation for the sake of definiteness. Assume that QCP is a privileged qc-portrait of a cubic proper geolamination $\mathcal{L}$; then leaves of $\mathcal{L}$ cannot cross $\overline{av}$ or D. Hence, if a leaf ℓ of $\mathcal{L}$ crosses an edge of Δ, then $\ell = \overline{az}$, which cannot be periodic because a is not periodic. □

For a point $z \in \mathbb{S}$, let $F(z)$ be the set of endpoints of leaves from both $\mathcal{L}_1$ and $\mathcal{L}_2$ containing z. Note that $z \in F(z)$. Recall that, by Lemma 2.40(3), if an endpoint of a leaf ℓ of a proper geolamination is periodic, then the other endpoint of ℓ is also periodic of the same period. Thus, the notion of vertex period is well defined for periodic leaves.

LEMMA 5.9. *If $z \in \mathbb{S}$ is periodic under σ_3, then $\sigma_3|_{F(z)}$ is order preserving.*

Proof. By Corollary 2.16 and since $\mathcal{L}_1$, $\mathcal{L}_2$ are proper, all points in $F(z)$ are periodic of the same period. It suffices to show that, for any distinct points $x, y \neq z$ of $F(z)$, the points x, y, z are kept in order by σ_3. By Lemma 2.11, we may assume that $\overline{xz} \in \mathcal{L}_1$ and $\overline{yz} \in \mathcal{L}_2$. We show that *there is only one point such that when z coincides with this point it becomes possible for the order among the points x, y, z to be reversed under σ_3.*

First, let Q_1, Q_2 be neither strongly linked nor share a spike. By Lemma 5.1, we may assume that Q_1 has a spike $\overline{av}$ and Q_2 has a spike $\overline{bv}$. If $z \neq v$, then $z \notin \Delta$. By Proposition 5.8, $\overline{xz}$, $\overline{yz}$ do not cross Δ. Hence x, y, z belong either to $[a, b]$, to $[b, v]$ or to $[v, a]$. Also, $a, b \notin \{x, y, z\}$ because a, b are non-periodic. It follows that $\sigma_3|_{\{x,y,z\}}$ preserves the order. Hence, in this case, the unique location for z, for which the order among a, y, z can be reversed, is when $z = v$.

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It remains to consider the case when Q_1 and Q_2 are strongly linked or share a spike. In this case, the qc-portraits QCP_1, QCP_2 are linked and, by Lemma 2.31, the order among the points x, y, z can be reversed only if z is a common vertex of associated quadrilaterals. If there is such a unique periodic vertex, then the claim is proved. Now, assume that Q_1 and Q_2 have more than one common *periodic* vertex. If so, then both Q_1 and Q_2 must have at least three vertices (because for any critical chord only one of its endpoints can be periodic). If, say, Q_1 is a triangle, then it must coincide with Δ, and hence, in that case, the unique location of z, which allows for the reversal of orientation, is $z = v$, and the claim is once again verified.

Finally, consider the case when both Q_1 and Q_2 are quadrilaterals sharing an edge $\overline{ut}$, with both endpoints periodic. Let $\mathfrak{g}_i$ be the class of $\sim_i$ which contains u and t. Since the class $\mathfrak{g}_i$ is periodic and finite, it is disjoint from $\overline{ab}$. Hence the siblings of u and t in (b, a) are unique and $Q_1 = Q_2 = Q$ is the same quadrilateral. Let us show that, it this case, $\sigma_3|_{\{x,y,z\}}$ is order preserving. We may assume that $z = u$. Then there are two cases. First, the points x and y may be separated by the spike of Q with an endpoint u. Then the remaining spike of Q and D cut $\overline{\mathbb{D}}$ in pieces, one of which contains x, y and z, which implies that $\sigma_3|_{\{x,y,z\}}$ is order preserving. Otherwise, the points x and y are not separated by the spike of Q with an endpoint u. Then this very spike and D again cut $\overline{\mathbb{D}}$ in pieces, one of which contains x, y and z, which implies that $\sigma_3|_{\{x,y,z\}}$ is order preserving. By induction, $\sigma_3^k|_{\{x,y,z\}}$ is order preserving for all k.

Thus, there is a unique location of z for which the orientation of the points x, y, z can be reversed. However, the periods of point x, y, z are the same, and hence the power of σ_3 which maps z back to itself sends both x and y back to themselves too. Since, by the above claim, the reversal of orientation can happen along the way only once, it follows that it does not take place at all. We conclude that the claim holds for all periodic z. □

Suppose that, say, $\mathrm{CH}(Q_1) = \Delta$. By definition, either $Q_1 = (a, b, b, v)$, or $Q_1 = (a, b, v, v)$, or $Q_1 = (a, a, b, v)$. Then either Δ is a gap of $\mathcal{L}_2$ as well, Q_1 and Q_2 are strongly linked or Q_1 and Q_2 share a spike. Let us show that either $\overline{va}$ or $\overline{bv}$ is a spike of Q_2. This is obvious if Δ is a gap of $\mathcal{L}_2$. Suppose that Δ is not a gap of $\mathcal{L}_2$ and neither $\overline{va}$ nor $\overline{bv}$ is a spike of Q_2. Since, by definition, $\mathrm{CH}(Q_2) \neq D$, it is easy to see that no vertex of Q_2 may coincide with a, b or v. However, it follows that Q_2 and Q_1 cannot be strongly linked. Thus, if $\mathrm{CH}(Q_1) = \Delta$, then we can always assume that either $\overline{va}$ or $\overline{bv}$ is a spike of Q_2.

Proposition 5.10 studies vertex periods of linked leaves of $\mathcal{L}_1$, $\mathcal{L}_2$.

PROPOSITION 5.10. *Suppose that a leaf $\ell_1 \in \mathcal{L}_1$ is periodic of vertex period n and a leaf $\ell_2 \in \mathcal{L}_2$ meets ℓ_1. Then the order among the endpoints of ℓ_1 and ℓ_2 is preserved under σ_3 and ℓ_2 is periodic with vertex period n.*

Proof. If ℓ_1 and ℓ_2 meet at an endpoint, the claim follows from Lemma 5.9. Otherwise, we may assume that $\ell_1 = \overline{p_1q_1}$, $\ell_2 = \overline{p_2q_2}$ and $p_1 < p_2 < q_1 < q_2$. Observe that ℓ_2 cannot be critical. Indeed, suppose otherwise. Since ℓ_2 and ℓ_1 cross, $\ell_2 \neq D$. Then there are the following options for Q_2: $\mathrm{CH}(Q_2) = \Delta$, $\mathrm{CH}(Q_2)$ is the edge of Δ not equal to D or ℓ_2 and, finally $\mathrm{CH}(Q_2) = \ell_2$. Let us first assume that $\mathrm{CH}(Q_2) = \ell_2 = \overline{st}$. Then

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$Q_2 = (s, s, t, t)$. Since ℓ_1 crosses ℓ_2, Q_1 must be located on one side of ℓ_1. However, this would imply that Q_1 and Q_2 can neither be strongly linked nor share a spike, which is a contradiction.

Thus, either $\mathrm{CH}(Q_2) = \Delta$, $\mathrm{CH}(Q_2)$ is the edge of Δ not equal to D or ℓ_2, and ℓ_2 is either $\overline{bv}$ or $\overline{va}$. Observe that, in either case, a, a, a, b or b, b are consecutive vertices of Q_2. On the other hand, the leaf ℓ_1 crosses ℓ_2 and, therefore, is linked with both $\overline{bv}$ and $\overline{va}$. As before, it means that Q_1 is located on one side of ℓ_1 and hence, by definition, Q_1 cannot be linked with Q_2, which is a contradiction. So we may assume that ℓ_2 is not critical.

Let us show that the order among the endpoints of ℓ_1, ℓ_2 is preserved. Indeed, suppose otherwise. By Lemma 2.31, without loss of generality, we may assume that $\sigma_3(p_1) = \sigma_3(q_2)$. By Lemma 2.33, there is a sibling $\overline{p_1 p_2'}$ of $\overline{p_2 q_2}$ with $q_2 < p_2' < p_1$. Since $\mathcal{L}_2$ is proper, p_2' is periodic. Then, by Lemma 5.9, $\sigma_3(p_2') = \sigma_3(p_2) < \sigma_3(p_1) < \sigma_3(q_1)$. On the other hand, since $(\mathcal{L}_1, \mathrm{QCP}_1)$, $(\mathcal{L}_2, \mathrm{QCP}_2)$ are linked, by Lemma 2.31, the order among the points p_1, p_2, q_1, q_2 is weakly preserved under σ_3 so that $\sigma_3(p_1) \le \sigma_3(p_2) \le \sigma_3(q_1)$, which is a contradiction. Thus the order among the endpoints of ℓ_1, ℓ_2 is preserved. In particular, ℓ_2 is not (pre)critical.

Let us show that ℓ_2 is (pre)periodic. Consider $A_{\ell_2}(\ell_1) = A$. By the above, the order of endpoints of ℓ_1, ℓ_2 is preserved under σ_3^{nk} for every $k \ge 0$. Hence the sequence of points of intersection of leaves $\sigma_3^{nk}(\ell_2)$ with ℓ_1 is monotone on ℓ_1 while all leaves $\sigma_3^{nk}(\ell_2)$ are pairwise unlinked. Then there exists a leaf ℓ_∞ equal to the limit of leaves $\sigma_3^{nk}(\ell_2)$. Since σ_3 is locally expanding, ℓ_2 is (pre)periodic. Moreover, by Lemma 2.26, the vertex period of the eventual periodic image of ℓ_2 is n.

Now, let us show that, in fact, ℓ_2 is periodic itself. Suppose otherwise. Then for some i, $\sigma_3^i(\ell_2)$ and $\sigma_3^{n+i}(\ell_2)$ are disjoint siblings which are both linked with $\sigma_3^i(\ell_1)$. Clearly, D does not separate $\sigma_3^i(\ell_2)$ and $\sigma_3^{n+i}(\ell_2)$ as, otherwise, D and ℓ_1 would cross. Thus $\sigma_3^i(\ell_2)$ and $\sigma_3^{n+i}(\ell_2)$ are located in the closure $\overline{W}$ of the component W of $\overline{\mathbb{D}}\setminus D$ with boundary circle arc of length $\frac{2}{3}$. Moreover, since $\sigma_3^i(\ell_2)$ and $\sigma_3^{n+i}(\ell_2)$ are disjoint siblings, both spikes of Q_2 must separate $\sigma_3^i(\ell_2)$ and $\sigma_3^{n+i}(\ell_2)$. Thus both spikes of Q_2 cross ℓ_1. Since Q_1 and Q_2 are strongly linked or share a spike, it follows that Q_1 has a spike linked with ℓ_1, which is a contradiction. Thus ℓ_2 is periodic. □

Observe that if $\mathcal{L}$ is proper, then all $\sim_{\mathcal{L}}$-classes are finite. Hence, for any point $x \in \mathbb{S}$, there are at most a finite number of leaves of $\mathcal{L}$ containing x.

PROPOSITION 5.11. *No leaf $\ell_1 \in \mathcal{L}_1$ (respectively, $\ell_2 \in \mathcal{L}_2$) can intersect an infinite number of leaves of $\mathcal{L}_2$ (respectively, $\mathcal{L}_1$). In particular, no leaf $\ell_1 \in \mathcal{L}_1$ (respectively, $\ell_2 \in \mathcal{L}_2$) is linked with a limit leaf of $\mathcal{L}_2$ (respectively, $\mathcal{L}_1$).*

Proof. If $\ell_1 \in \mathcal{L}_1$ intersects an infinite number of leaves $\{\ell_2^j\}_1^\infty$ of $\mathcal{L}_2$, then, by Lemma 2.31, $\sigma_3^i(\ell_1)$ intersects σ_3^i-images of all leaves $\{\ell_2^j\}_1^\infty$ for any i. Since, for any point $x \in \mathbb{S}$, there are at most a finite number of leaves of $\mathcal{L}$ containing x, $\sigma_3^i(\ell_1)$ is linked with an infinite number of leaves of $\mathcal{L}_2$. If $\sigma_3^i(\ell_1)$ is periodic for some i, this would contradict Proposition 5.10. Thus we may assume that the orbits of the endpoints of ℓ_1 are infinite. As there are only a finite number of chains of spikes of $\mathcal{L}_1$ or $\mathcal{L}_2$, for some i_0 and any $i \ge i_0$, the leaf $\sigma_3^i(\ell_1)$ cannot collapse around any chain of spikes. Since $\sigma_3^{i_0}(\ell_1)$ is linked with an

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infinite number of leaves of $\mathcal{L}_2$, Lemma 2.31 implies that $\sigma_3^{i_0}(\ell_1)$ and those leaves of $\mathcal{L}_2$ have mutually order-preserving accordions. Since true quadrilaterals cannot be wandering under σ_3 [**Kiw02**], by Theorem 2.29, ℓ_1 is (pre)periodic, which is a contradiction. □

Let us study Fatou gaps of our geolaminations. For a Fatou gap U of $\mathcal{L}_1$, let $\phi_U : \partial U \to \mathbb{S}$ be the map collapsing all leaves in $\mathrm{Bd}(U)$ to points. For a leaf $\ell_2 \in \mathcal{L}_2$ with $\ell_2 \cap \mathrm{Bd}(U) \neq 0$, let ℓ', ℓ'' be edges of $\mathrm{Bd}(U)$ which intersect ℓ_2 such that $\phi_U(\ell') \neq \phi_U(\ell'')$, if possible. If so, define $\phi_U(\ell_2) = \overline{\phi_U(\ell')\phi_U(\ell'')}$ and define $\phi_U(\ell_2) = \phi_U(\ell_2 \cap \mathrm{Bd}(U))$, otherwise. Observe that the map ϕ_U is defined not only for periodic but also for non-periodic Fatou gaps (recall that all Fatou gaps are (pre)periodic).

LEMMA 5.12. *If U is a Fatou gap of $\mathcal{L}_1$, then, for any leaf $\ell_2 \in \mathcal{L}_2$ such that $\ell_2 \cap \partial U \neq \emptyset$, the set $\phi_U(\ell_2)$ is degenerate.*

Proof. Assume that $\phi_U(\ell_2)$ is non-degenerate. Let us show that we may assume that U is periodic. Indeed, any Fatou gap eventually maps to a periodic Fatou gap. Moreover, since $\mathcal{L}_1$ is not of capture Siegel type, then there are no critical preperiodic Fatou gaps (in our case, that is, with D being a part of $\mathcal{L}_1$: a critical preperiodic Fatou gap which maps to a periodic Fatou gap of degree greater than one is impossible). Thus, if W is a non-periodic Fatou gap of $\mathcal{L}_1$ and $\overline{h}$ and $\overline{k}$ are its edges which do not belong to the same concatenation of edges of W, then their images are distinct. Hence, if $\phi_U(\ell_2)$ is not degenerate, we can keep mapping U forward until U maps to a periodic gap W and consider intersections of images ℓ_2 with appropriate images of U. By Lemma 2.31, the intersections of ℓ_2 with edges of U are preserved under iteration of σ_3. Since, along the way to W, all preperiodic images of U are non-critical, we conclude that the image of ℓ_2 which intersects W gives rise to a non-degenerate ϕ_W-image. Thus we can assume, without loss of generality, that U is periodic from the very beginning.

Now, for geolaminations of $\mathbb{LP}_3^{\mathrm{np}}$, all periodic Fatou gaps are of degree one or two. Assume that U is of degree two. Then Q_1 has an edge M which is a refixed edge of U. Denote the sibling edge of M in U by M'. Thus $\phi_U(Q_1) = \overline{0\frac{1}{2}}$. Since σ_3-images of ℓ_2 do not cross one another, σ_2-images of $\phi_U(\ell_2)$ do not cross one another either. Then there exists k such that either (1) $\sigma_2^k(\phi_U(\ell_2))$ crosses $\overline{0\frac{1}{2}}$ or (2) $\sigma_2^k(\phi_U(\ell_2)) = \overline{0\frac{1}{2}}$. If (1) holds, then $\sigma_3^k(\ell_2)$ crosses two improper leaves of Q_1, which contradicts the fact that Q_2 is strongly linked with Q_1 or shares a spike with Q_1. If (2) holds, $\sigma_3^k(\ell_2)$ meets one refixed edge of U and one edge of U which is not periodic, which contradicts Proposition 5.10 (indeed, if $\sigma_3^k(\ell_2)$ is not periodic, it cannot meet any refixed edge of U while, if $\sigma_3^k(\ell_2)$ is periodic, it cannot meet a non-periodic edge of U). Finally, if U is of degree one, then $\phi_U(\ell_2)$ will eventually cross itself under irrational rotation induced on $\phi_U(\mathrm{Bd}(U)) = \mathbb{S}$, which is a contradiction. □

Recall that $\widehat{\mathcal{L}}_1$ and $\widehat{\mathcal{L}}_2$ are canonical geolaminations constructed for laminations $\sim_1$, $\sim_2$, in turn generated by the given geolaminations $\mathcal{L}_1$, $\mathcal{L}_2$.

Proof of Theorem B. First, we prove that Fatou gaps of $\widehat{\mathcal{L}}_1$ and $\widehat{\mathcal{L}}_2$ are the same. A Fatou gap of either geolamination $\widehat{\mathcal{L}}_i$ cannot have a non-trivial (consisting of more than one) concatenation of its edges as it will have to be completed with an edge separating it

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from the rest of the Fatou gap in question. Let $\widehat{U}_1$ be a periodic Fatou gap of $\widehat{\mathcal{L}}_1$. The corresponding Fatou gap U_1 of $\mathcal{L}_1$ is of the same degree as $\widehat{U}_1$. If, for a leaf $\hat{\ell}_2 \in \widehat{\mathcal{L}}_2$, the chord $\phi_{\widehat{U}_1}(\hat{\ell}_2)$ is non-degenerate, then there must exist a leaf $\ell_2 \in \mathcal{L}_2$ with non-degenerate image $\phi_{U_1}(\ell_2)$, which contradicts Lemma 5.12. Hence there exists an infinite gap $\widehat{U}_2$ of $\widehat{\mathcal{L}}_2$ containing $\widehat{U}_1$. Similarly, there exists an infinite gap of $\widehat{\mathcal{L}}_1$ containing $\widehat{U}_2$. Thus $\widehat{U}_1 = \widehat{U}_2$.

We claim now that leaves of $\widehat{\mathcal{L}}_1$ and $\widehat{\mathcal{L}}_2$ coincide. Call a leaf of $\widehat{\mathcal{L}}_i$ a *limit* leaf if it is the limit of leaves of $\widehat{\mathcal{L}}_i$. We claim the limit leaves of $\widehat{\mathcal{L}}_1$ and the limit leaves of $\widehat{\mathcal{L}}_2$ form the same family of leaves. Indeed, we let $\hat{\ell}_1 \in \widehat{\mathcal{L}}_1$ be a limit leaf, and prove that, in this case, $\hat{\ell}_1$ is a leaf of $\widehat{\mathcal{L}}_2$. Observe that $\hat{\ell}_1$ must be a leaf of $\mathcal{L}_1$ too. By Proposition 5.11, $\hat{\ell}_1$ is not linked with any leaf of $\mathcal{L}_2$, as any such leaf of $\mathcal{L}_2$ will be crossed by an infinite number of leaves of $\mathcal{L}_1$. This easily implies that $\hat{\ell}_1$ is, in fact, not linked with leaves of $\widehat{\mathcal{L}}_2$ either. Moreover, the same arguments show that no leaf of $\widehat{\mathcal{L}}_2$ can share an endpoint with $\hat{\ell}_1$ and be otherwise located on the side of $\hat{\ell}_1$ from which $\hat{\ell}_1$ is approached by an infinite number of leaves of $\widehat{\mathcal{L}}_1$. Suppose that $\hat{\ell}_1$ is not a leaf of $\widehat{\mathcal{L}}_2$. Then, by the above, $\hat{\ell}_1$ is contained (except the endpoints) in a Fatou gap V of $\widehat{\mathcal{L}}_2$, which contradicts the above.

On the other hand, suppose that $\hat{\ell}_1 \in \widehat{\mathcal{L}}_1$ is not a limit leaf of $\widehat{\mathcal{L}}_1$. Then, on at least one side, a Fatou gap is attached to $\hat{\ell}_1$, which implies that $\hat{\ell}_1$ is a leaf of $\widehat{\mathcal{L}}_2$ too. Thus $\widehat{\mathcal{L}}_1 = \widehat{\mathcal{L}}_2$, as desired. □

6. *Unlinkage of QC-portraits of $\mathcal{QCP}_3^{\mathrm{np}}$*

In what follows, we always assume that $\mathrm{QCP}_1 = (Q_1, D), \mathrm{QCP}_2 = (Q_2, D) \in \mathcal{QCP}_3^{\mathrm{np}}$ are *linked* and *distinct* privileged portraits of proper geolaminations $\mathcal{L}_1, \mathcal{L}_2 \in \mathcal{S}_D$. Recall that $\Delta = \mathrm{CH}(a, b, v)$ is the all-critical triangle with edge $D = \overline{ab}$, where (a, b) is a circle arc of length $\frac{1}{3}$. Observe that (Δ, D) is an admissible qc-portrait.

LEMMA 6.1. *If $Q_1 = \Delta$, then Q_2 is an edge of Δ, distinct from D, or Q_2 is a collapsing quadrilateral whose one spike is an edge of Δ not equal to D and whose edges do not cross D.*

Proof. Suppose that Δ is represented as a generalized quadrilateral by assigning to it the following vertices in the positive direction: (a, b, b, v). Denoting the vertices of Q_2 by $x_1 \le x_2 \le x_3 \le x_4$ we may assume that

$$a \le x_1 \le b \le x_2 \le b \le x_3 \le v \le x_4 \le a,$$

which implies that $x_2 = b$. Since $\overline{x_2x_4}$ is a critical chord, either $a = x_4$ or $v = x_4$. Clearly, the argument can be repeated in other cases too; it shows that Q_2 has a spike which is an edge of Δ. In the degenerate case Q_2 coincides with an edge of Δ; recall here, that, by definition, $Q_2 \neq D$. If Q_2 is a triangle, then $Q_2 = \Delta$, which contradicts the assumption that $Q_1 \neq Q_2$. Now, suppose that Q_2 is a true quadrilateral. Then it cannot have D as a spike. Hence its spike, which is an edge of Δ, must be either $\overline{av}$ or $\overline{bv}$. Clearly, the edges of Q_2 cannot cross D. □

It will be convenient to separate the following easy fact into a lemma.

LEMMA 6.2. *The sets Q_1 and Q_2 share at most one periodic vertex.*

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Proof. Indeed, otherwise, the two shared vertices are not endpoints of D and are not the endpoints of a critical chord. Hence, if they are shared by Q_1 and Q_2, then $Q_1 = Q_2$, which contradicts the assumptions. □

6.1. *Slices of* $\mathbb{LP}_3^{\mathrm{np}}$. For a family of chords $\mathcal{F}$, the equivalence relation $\sim_{\mathcal{F}}$ is defined as an equivalence relation on $\mathbb{S}$ under which two points of $\mathbb{S}$ are declared to be *equivalent* if and only if a finite chain of leaves of $\mathcal{F}$ connects them; we call the equivalence relation $\sim_{\mathcal{F}}$ *generated* by $\mathcal{F}$. We claim that each admissible qc-portrait $\mathrm{QCP} \in \mathcal{S}_D$ is associated with a unique lamination denoted by $\sim_{\mathrm{QCP}}$. Indeed, by Theorem A, there is a proper geolamination $\mathcal{L}_{\mathrm{QCP}}$ for which QCP is privileged; by definition, $\mathcal{L}_{\mathrm{QCP}}$ gives rise to an equivalence relation $\sim_{\mathcal{L}_{\mathrm{QCP}}} = \sim_{\mathrm{QCP}}$. By Theorem B, $\sim_{\mathrm{QCP}}$ is unique for a given admissible qc-portrait QCP from $\mathcal{S}_D$. Indeed, if there are two distinct geolaminations $\mathcal{L}_{\mathrm{QCP}}$ and QCP is privileged for both of them, these two equal qc-portraits can be viewed as linked, which implies that the laminations generated by the two proper geolaminations in question are the same (so that our notation $\sim_{\mathrm{QCP}}$ is appropriate). Set $\widehat{\mathcal{L}}_{\mathrm{QCP}} = \mathcal{L}_{\sim_{\mathrm{QCP}}}$. Denote by $\mathbb{LP}_{3,D}^{\mathrm{np}}$ the family of all proper geolaminations with a critical leaf D, except for geolaminations of Siegel capture type.

Since $\mathrm{QCP} \in \mathcal{S}_D$, the endpoints a and b of D are equivalent under $\sim_{\mathrm{QCP}}$. Now, if $\sim$ is a cubic lamination not of Siegel capture type and if a and b are $\sim$-equivalent, we can find a qc-portrait $\mathrm{QCP} = (Q, D)$ privileged for $\mathcal{L}_{\sim}$. Thus, the family LAM_D of laminations $\sim_{\mathrm{QCP}}$, obtained as described above for admissible qc-portraits $\mathrm{QCP} \in \mathcal{S}_D$, is in fact the family of all cubic laminations $\sim$ not of Siegel capture type with $a \sim b$. Observe that if $\sim$ has a periodic Fatou gap U of degree two, then the non-degenerate refixed edge of U is unique (if it exists) because the remap is of degree two and canonical geolaminations associated with laminations cannot have concatenations of edges on their boundaries. Also, if $\sim$ has a unique critical set, then it has to coincide with the $\sim$-class of a, b and is therefore finite.

Recall that every σ_2-invariant lamination $\sim$ has a unique *minor* set $m_{\sim}$ which is the convex hull of the image of a $\sim$-class of maximal diameter. By Thurston [**Thu85**], the minor sets of quadratic invariant laminations are pairwise disjoint and form a lamination of the unit circle. That is, minor sets of quadratic laminations form classes of QML. Domains of $\overline{\mathbb{D}}$/QML (that is, components of $\overline{\mathbb{D}}$/QML$\setminus\mathbb{S}$/QML) come from critical sets of quadratic laminations $\sim$ which are periodic Fatou gaps U of degree two; each such domain will be called the *main cardioid of* U. In particular, U can coincide with the entire closed unit disk, in which case, we have the main cardioid of $\overline{\mathbb{D}}$ (or the main cardioid of the Mandelbrot set, as it is usually called). In general, U can be of period n, in which case, its main cardioid can be described as follows. Take all *rotational* sets G of period n inside U. Each such set has an unique longest edge denoted by M_G. This includes Siegel disks S of period n inside U, in which case, the critical edge of S is its longest edge. Then the boundary of the main cardioid of U is formed by sets $\sigma_2(M)$ taken over all rotational sets $G \subset U$ of period n.

Let us go back to the cubic case. Recall that LAM_D is the set of all σ_3-invariant laminations $\sim$ such that $a \sim b$, except for laminations of *Siegel capture type*, and that $\mathcal{S}_D$ is the family of all admissible qc-portraits (Q, D) with D as the second element.

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Each $(Q, D) \in \mathcal{S}_D$ is tagged by the chord or point (the *minor*) $\sigma_3(Q)$, and CML_D is the family of all such chords $\sigma_3(Q)$.

By Theorem A, every admissible qc-portrait is a privileged qc-portrait of a *proper* geolamination. Typically, but not always, an admissible qc-portrait is privileged for a geolamination canonically associated with a cubic lamination from LAM_D (to each class of a lamination one associates its convex hull and then considers a geolamination formed by the edges of these convex hulls). Indeed, suppose that $\sim$ is a lamination from LAM_D which has a periodic critical Fatou gap U of degree two and period k with a refixed edge M which is also an edge of an *identity return* triangle T (that is, of a periodic triangle T of period k). Denote the other two edges of T by $\overline{m}$ and $\overline{n}$. If we remove the grand orbit of M from $\widehat{\mathcal{L}}_\sim$, we will get a new proper geolamination $\mathcal{L}$ for which the gap U becomes a new gap V with refixed edges $\overline{m}$ and $\overline{n}$. The convex hulls of $\overline{m}$ ($\overline{n}$) with their siblings are two admissible qc-portraits privileged for $\mathcal{L}$.

This shows that all edges of the triangle $\sigma_3(T)$ belong to CML_D. However, $\sigma_3(\overline{m})$ or $\sigma_3(\overline{n})$ are *not* minors of any geolaminations canonically associated to a lamination from LAM_D. Call the triangle T the *major set* of $\sim$; if such T does not exist we call M the *major set* of U. In other words, the major set of $\sim$ is the $\sim$-class of the refixed edge of U.

Definition 6.3. (Minor sets in the cubic case) For $\sim \in \mathrm{LAM}_D$ let $\mathfrak{g}_D$ be the $\sim$-class of a; moreover, if $\sim$ has a periodic Fatou gap U of degree two, let $M_\sim$ be the major set of U. Let $C_\sim$ be either the first critical set of $\mathcal{L}_\sim$ (if it is different from $\mathfrak{g}_D$ and finite), the major set of U (if the first critical set of $\sim$ is a periodic Fatou gap U of degree two) or $\mathrm{CH}(\mathfrak{g}_D)$ (if $\sim$ has a unique critical class $\mathfrak{g}_D$). Set $\sigma_3(C_\sim) = m_\sim$ and call $m_\sim$ the *minor set* of $\sim$.

We are ready to prove Theorems C and D.

THEOREM 6.4. *The family of chords* CML_D *is proper and gives rise to the lamination* $\sim_D$. *Convex hulls of* $\sim_D$*-classes are minor sets* $m_\sim$, *where* $\sim$ *belongs to* LAM_D.

Proof. Since, by Theorem A, the set $\mathcal{S}_D$ is compact, CML_D is a closed family of chords (minors). Consider the equivalence relation $\sim_D = \sim_{\mathrm{CML}_D}$. Clearly, $\sim_D$ is closed (because CML_D is closed). Assume that a lamination $\approx$ from LAM_D is a lamination such that its first critical set $C_\approx$ is a finite gap disjoint from D. Then we can insert collapsing quadrilaterals in $C_\approx$ in several ways and associate to it the corresponding proper geolaminations. Images of these collapsing quadrilaterals are minors of the corresponding proper geolaminations. Since these images are edges (and possibly some diagonals) of $m_\approx$, in this case $m_\approx$ is a class of equivalence of $\sim_D$.

Similar arguments show that the same holds if $\approx$ has a unique critical set coinciding with the convex hull $\mathrm{CH}(\mathfrak{g}_\approx)$ of the $\approx$-class $\mathfrak{g}_\approx$ of a. On the other hand, the analysis before Definition 6.3 shows that if the first critical set $C_\approx$ is a periodic Fatou gap of degree two, then the corresponding class of equivalence of $\sim_D$ is the image $m_\approx$ of the major set $M_\approx$ of $\approx$. This covers all possibilities for the lamination $\approx$. Since each minor from CML_D can be associated with a proper geolamination, and then with a lamination from LAM_D, these cases cover the entire CML_D and complete the proof. □

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