Canad. Math. Bull. Vol. 65 (1), 2022, pp. 234–252
http://dx.doi.org/10.4153/S0008439521000242
© Canadian Mathematical Society 2021

Cancellation of two classes of dirichlet coefficients over Beatty sequences

Qiang Ma

Abstract. Let π be an automorphic irreducible cuspidal representation of GL_m over $\mathbb{Q}$. Denoted by $\lambda_\pi(n)$ the nth coefficient in the Dirichlet series expansion of $L(s,\pi)$ associated with π. Let π_1 be an automorphic irreducible cuspidal representation of $\mathrm{SL}(2,\mathbb{Z})$. Denoted by $\lambda_{\pi_1\times\pi_1}(n)$ the nth coefficient in the Dirichlet series expansion of $L(s,\pi_1\times\pi_1)$ associated with $\pi_1\times\pi_1$. In this paper, we study the cancellations of $\lambda_\pi(n)$ and $\lambda_{\pi_1\times\pi_1}(n)$ over Beatty sequences.

1 Introduction

It is an important problem in number theory to study the mean value of Dirichlet coefficients. For instance, in 1899, de la Vallée Poussin [15] proved that

$$\psi(x)=\sum_{n\le x}\Lambda(n)=x+O(xe^{-c\sqrt{\log x}}).$$

Moreover, it is a central problem in the study of L-function to study the mean value of its Dirichlet coefficients. According to the conjectures of Langlands, the most general L-functions arise from automorphic representations of GL_n over a number field, and these can be decomposed into products of general L-functions, which are attached to cuspidal automorphic representations of GL_m over the field of rational numbers and we call primitive L-functions of degree m. Let $m\ge 2$ be an integer and π be an automorphic irreducible cuspidal representation of GL_m over $\mathbb{Q}$ with unitary central character. Throughout the paper, p denotes a prime number. Then π is a restricted tensor product,

$$\pi=\otimes_p\pi_p,$$

when p runs over all prime numbers and the archimedean places. To π one associates an Euler product

$$L(s,\pi)=\prod_p L(s,\pi_p) \tag{1.1}$$

given by a product of local factors. Outside of a finite set of primes, π_p is unramified. At every finite place p where π_p is unramified we associate a semi-simple conjugacy

Received by the editors December 30, 2020; revised April 10, 2021, accepted April 10, 2021.
Published online on Cambridge Core April 20, 2021.
AMS subject classification: **11M32, 11F70, 11B83.**
Keywords: Automorphic representation, Dirichlet coefficients, Beatty sequences.

https://doi.org/10.4153/S0008439521000242 Published online by Cambridge University Press

class

$$A_\pi(p) = \begin{pmatrix} \alpha_{\pi,p}(1) & & \\ & \ddots & \\ & & \alpha_{\pi,p}(m) \end{pmatrix}$$

and define the local L-function for the finite place p as

$$L\left(s, \pi_p\right) = \det\left(I - p^{-s} A_\pi(p)\right)^{-1} = \prod_{j=1}^{m} \left(1 - \alpha_{\pi,p}(j) p^{-s}\right)^{-1},$$

where $\left\{\alpha_{\pi,j}(p)\right\}_{j=1}^{m}$ are local parameters associated with π_p. The generalized Ramanujan conjecture states that for unramified places

$$\left|\alpha_{\pi,j}(p)\right| = 1.$$

The best known record is due to Kim and Sarnak [20] $(2 \le m \le 4)$ and Luo et al. [29] $(m \ge 5)$:

$$\left|\alpha_{\pi,j}(p)\right| \le p^{\theta_m}$$

for all primes p and $1 \le j \le m$, where

$$\theta_2 = \frac{7}{64}, \quad \theta_3 = \frac{5}{14}, \quad \theta_4 = \frac{9}{22}, \quad \theta_m = \frac{1}{2} - \frac{1}{m^2+1} (m \ge 5). \tag{1.2}$$

By (1.1), the product over primes gives a Dirichlet series representation,

$$L(s, \pi) = \sum_{n=1}^{\infty} \frac{\lambda_\pi(n)}{n^s},$$

which converges absolutely for $\Re s > 1$.

In the case of $m = 2, \lambda_\pi(n)$ is the nth normalized Fourier coefficient of a holomorphic cusp form. In 1927, Hecke [12] proved that

$$\sum_{n \le x} \lambda_\pi(n) \ll x^{\frac{1}{2}}.$$

Many scholars have made further improvements to this result, such as Kloosterman [23], Davenport [6], Weil [36], and Deligne [7]. As for $m \ge 2, \lambda_\pi(n)$ is the nth normalized Fourier coefficient of a holomorphic or Maass cusp form form. Lü [26] has proved that for any $\varepsilon > 0$,

$$\sum_{n \le x} \lambda_\pi(n) \ll_{\varepsilon,\pi} \begin{cases} x^{\frac{71}{192}+\varepsilon} & \text{if} \quad m = 2 \\ x^{\frac{m^2-m}{m^2+1}+\varepsilon} & \text{if} \quad m \ge 3. \end{cases}$$

In this paper, we shall consider the cancellation of the nth coefficient $\lambda_\pi(n)$ over a Beatty sequence, i.e.,

$$S(\alpha, \beta, X) = \sum_{n \leqslant X} \lambda_\pi\left(\left[\alpha n + \beta\right]\right),$$

where $\alpha \ge 1, \beta \ge 0$, $[x]$ denotes the greatest integer not exceeding x. The so-called Beatty sequence is the sequence of integers defined by $\mathcal{B}_{\alpha,\beta} = \{[\alpha n + \beta]\}_{n=1}^{\infty}$. Under

https://doi.org/10.4153/S0008439521000242 Published online by Cambridge University Press

some additional hypotheses, Güloğlu and Nevans [10] have obtained a general result. Lao [25] and Sun [33] studied the cancellation behavior of Fourier coefficients of cusp forms over Beatty sequences. Let $e(z) = e^{2\pi iz}$. For any $\varepsilon > 0$, any $\vartheta \in \mathbb{R}$ and $\theta < 1$, Sun [33] has proved

$$S(\alpha, \beta, X) = \sum_{n \leqslant X} A(n_1, \ldots, n_{m-2}, [\alpha n + \beta]) \ll_{f,\alpha,\varepsilon,n_1,\ldots,n_{m-2}} X^{\theta+\varepsilon},$$

provided that

$$\sum_{n_{m-1} \leqslant X} A(n_1, \ldots, n_{m-1})\, e(n_{m-1}\vartheta) \ll_{f,\varepsilon} X^{\theta+\varepsilon},$$

where α is of type 1 and $A(n_1, n_2, \ldots, n_{m-1})$ are the normalized Fourier coefficients of a Maass cusp form on GL_m. Type τ is explained later in the next section. The aim of this paper is to study the mean value of Dirichlet coefficients over Beatty sequences of the L-function without assumptions. Firstly, we consider the sum,

$$S_{\lambda_\pi}(x) = \sum_{n \leq x} \lambda_\pi(n) e(n\alpha).$$

By the Rankin–Selberg theory, there is the trivial bound

$$S_{\lambda_\pi}(x) \ll_\pi x.$$

We expect that nontrivial and uniform cancellation exists in the sum $S_{\lambda_\pi}(x)$. In the case of $m = 2$, when $\lambda_\pi(n)$ are either the normalized Fourier coefficients of a modular form, or a Maass form on the upper half plane, one has the classical estimate on linear polynomials (see, for example, [14, Theorem 8.1])

$$\sum_{n \leq x} \lambda_\pi(n) e(n\alpha) \ll_{\pi,\varepsilon} x^{\frac{1}{2}+\varepsilon}$$

for any $\alpha \in \mathbb{R}$. For the $m = 3$ case, $\lambda_\pi(n)$ being coefficients of the L-function of automorphic cusp form π on GL_m over $\mathbb{Q}$, Miller [30] proved the first result and showed that

$$\sum_{n \leq x} \lambda_\pi(n) e(n\alpha) \ll_{\pi,\varepsilon} x^{\frac{3}{4}+\varepsilon}.$$

For $m \geq 3$, Jiang and Lü [18] used a technical hypothesis concerning the coefficients of $L(s, \pi)$, which is the well known Hypothesis H of Rudnick and Sarnak [32], to give that

$$S_{\lambda_\pi}(x) = \sum_{n \leq x} \lambda_\pi(n) e(n\alpha) \ll_\pi x \frac{\log\log x}{\log x}.$$

This is a nontrivial, uniform estimate for $S_{\lambda_\pi}(x)$.

Hypothesis H *For any fixed $k \geq 2$,*

$$\sum_p \frac{|\lambda_\pi(p^k)|^2 (\log p)^2}{p^k} < \infty.$$

https://doi.org/10.4153/S0008439521000242 Published online by Cambridge University Press

Applying the result of Jiang and Lü [18], we can have a result without assumptions. So by the methods in [2, 24, 33], we can get

Theorem 1.1 *Let $\alpha > 1$ be of finite type τ, $\tau \geq 1$, and π be an automorphic irreducible cuspidal representation of GL_m over $\mathbb{Q}$ with unitary central character. Under Hypothesis H, we have*

$$S(\alpha, \beta, X) = \sum_{n \leqslant X} \lambda_\pi \left([\alpha n + \beta] \right) \ll_{\pi,\alpha} X \frac{(\log\log X)^2}{\log X}$$

uniformly in $\beta \geq 0$.

Remark 1.2 In the cases of $m = 2, 3$, Hypothesis H follows from the Rankin–Selberg theory [32]. The GL_4 case and the symmetric fourth power $sym^4\pi$ of a cuspidal representation π of GL_2 were proved by Kim [21].

By Theorem 1.1 and Remark 1.1, we have

Corollary 1.3 *Let $\alpha > 1$ and $2 \leq m \leq 4$ or $\pi = sym^4\pi_1$ with an automorphic irreducible cuspidal representation π_1 of GL_2. We have*

$$S(\alpha, \beta, X) = \sum_{n \leqslant X} \lambda_\pi \left([\alpha n + \beta] \right) \ll_{\pi,\alpha} X \frac{(\log\log X)^2}{\log X}$$

uniformly in $\beta \geq 0$.

Similarly, we are also interested in the cancellation of Rankin–Selberg coefficients over Beatty sequences. The Rankin–Selberg L-function $L(s, \pi \times \hat{\pi})$ associated to π and its contragredient $\hat{\pi}$ is defined as a product of local factors

$$L(s, \pi \times \hat{\pi}) = \prod_{p < \infty} L\left(s, \pi_p \times \hat{\pi}_p\right).$$

By a result of Jacquet and Shalika [16], the product $\prod_p L\left(s, \pi_p \times \hat{\pi}_p\right)$ converges absolutely in $\Re s > 1$. We write this product as a Dirichlet series

$$L(s, \pi \times \hat{\pi}) = \prod_p \sum_{k=0}^{\infty} \frac{\lambda_{\pi\times\hat{\pi}}\left(p^k\right)}{p^{ks}} = \sum_{n=1}^{\infty} \frac{\lambda_{\pi\times\hat{\pi}}(n)}{n^s}.$$

In order to obtain the mean value of Rankin–Selberg coefficients over Beatty sequences, we need

Theorem 1.4 *Let π be a unitary automorphic cuspidal representation of $\mathrm{SL}(2, \mathbb{Z})$ and $\pi \simeq \hat{\pi}$ where $\hat{\pi}$ is the representation contragredient to π. Suppose that $|\alpha - a/q| \leq 1/q^2$, $(a, q) = 1$ and $2 \leq \log^3 x \leq q \leq x^{(1-\delta)}/(\log^3 x)$ with some $0 < \delta < 1/10$. Then we deduce*

$$S_{\lambda_{\pi\times\pi}}(x) = \sum_{n \leq x} \lambda_{\pi\times\pi}(n) e\left(n\alpha\right) \ll_\pi x \frac{\log\log x}{(\log x)^{\frac{1}{2}}}.$$

https://doi.org/10.4153/S0008439521000242 Published online by Cambridge University Press

Remark 1.5 We know when $m = 2$, $\lambda_\pi(n)$ essentially is the nth normalized Fourier coefficient of a holomorphic or Maass cusp form. When π is an $\mathrm{SL}(2,\mathbb{Z})$ holomorphic cusp form, under the conditions of Theorem 1.4, Lü [28] has proved

$$\sum_{n\le x}\lambda_{\pi\times\pi}(n)e(n\alpha) \ll_{\pi,\varepsilon} x^{\frac{4}{5}+\varepsilon},$$

for every $\varepsilon > 0$. However, as π is a holomorphic cusp form or even a Maass cusp form, we obtain a uniform result in Theorem 1.4.

It follows from $\pi \simeq \hat{\pi}$ that $\lambda_{\pi\times\pi} > 0$. Hence, due to Landau's lemma (see [3]), we have

$$\sum_{n\le X}\lambda_{\pi\times\pi}(n) = c_\pi X + O_\pi\left(X^{\frac{3}{5}}\right),$$

where c_π is a positive constant. Further, we can prove an asymptotic formula,

Theorem 1.6 *Let $\alpha > 1, \beta \ge 0$ and suppose that α is irrational and of finite type τ. Let π be a unitary and irreducible automorphic cuspidal representation of $\mathrm{SL}(2,\mathbb{Z})$ and $\pi \simeq \hat{\pi}$ where $\hat{\pi}$ is the representation contragredient to π. Then we have*

$$S(\alpha,\beta,X) = \sum_{n\le X}\lambda_{\pi\times\pi}([\alpha n+\beta]) = c_{\alpha,\pi}X + O_{\pi,\alpha}\left(X\frac{(\log\log X)^2}{(\log X)^{\frac{1}{2}}}\right),$$

where $c_{\alpha,\pi}$ is a positive constant.

Let

$$M(f,x) = 1 + \max_{1\le m\le x}|f(m)|$$

$$S_\alpha(f,x) = \sum_{\substack{1\le m\le x\\ m\in\mathcal{B}(\alpha)}} f(m), \quad S(f,x) = \sum_{1\le m\le x} f(m),$$

where $f:\mathbb{N}\to\mathbb{C}$ is an arbitrary arithmetic function. For any $\varepsilon > 0, \alpha > 1$ in a set depending on ε and f, of full Lebesgue measure, Abercrombie et al. [1] have proved

$$S_\alpha(f,x) = \alpha^{-1}S(f,x) + O_{f,\alpha,\varepsilon}\left(x^{2/3+\varepsilon}M(f,x)\right).$$

Assume that f does not grow too quickly in the sense that

$$\limsup_{r\to\infty}\frac{\|f|_{2^{r+1}}\|_2}{\|f|_{2^r}\|_2} < \infty,$$

where

$$\|f|_x\|_2^2 = \sum_{1\le m\le x}|f(m)|^2.$$

For all $\alpha > 1$ in a set of full Lebesgue measure (depending on f), Technau and Zafeiropoulos [34] have proved

$$S_\alpha(f,x) = \alpha^{-1}S(f,x) + O_{f,\alpha,\varepsilon}\left(x^{1/2+\varepsilon}M(f,x)\right)$$

https://doi.org/10.4153/S0008439521000242 Published online by Cambridge University Press

uniformly for $x \geq 8$. By (1.2), in the case that f is an automorphic irreducible cuspidal representation of GL_2 over $\mathbb{Q}$, the result of Abercrombie et al. improves error terms Theorem 1.1 and Corollary 1.2. Moreover, by (1.2), the result of Technau and Zafeiropoulos can improve error terms in Theorem 1.1, Corollary 1.3 and Theorem 1.6. However, the results of [1] and [34] yield no further information on the set of admissible $\alpha > 1$ other than having full Lebesgue measure and are, therefore, rather less explicit than our results which apply for all α of finite type (see Section 2).

2 Preliminaries

Following Kuipers and Niederreiter [24], we give the definition of an irrational number of type τ.

Definition 2.1 Let τ be a positive real number or infinity. The irrational number α is said to be of type τ if τ is the supremum of all t for which

$$\tau = \sup\left\{t \in \mathbb{R} : \liminf_{n\to\infty} n^t \|\alpha n\| = 0\right\},$$

where q runs through the positive integers. Here $\|\cdot\|$ denotes the distance to a nearest integer.

Note that the result of Khintchine [19] asserts that almost all real numbers are of finite type 1. By the Jarník–Besicovitch theorem [4, 17], we know that the Hausdorff dimension of all real numbers of type $> \tau$ is

$$\dim_{\mathrm{H}}\{x \in \mathbb{R} \text{ of type } > \tau\} = \frac{2}{\tau+1}.$$

Roth [31] has shown that all real algebraic irrational numbers are of type 1.

We recall that the *discrepancy* $D(M)$ of a sequence of (not necessarily distinct) real numbers $a_1, \ldots, a_M \in [0,1)$ is defined by

$$D(M) = \sup_{\mathcal{I}\subseteq[0,1)} \left|\frac{V(\mathcal{I},M)}{M} - |\mathcal{I}|\right|, \tag{2.1}$$

where the supremum is taken over all subintervals $\mathcal{I} = (c,d)$ of the interval $[0,1)$, $V(\mathcal{I},M)$ is the number of positive integers $m \leq M$ such that $a_m \in \mathcal{I}$, and $|\mathcal{I}| = d - c$ is the length of $\mathcal{I}$.

The following result is taken from [24].

Lemma 2.2 *Let α be a fixed irrational number of type τ. For all $\beta \in \mathbb{R}$, we have*

$$D_{\alpha,\beta}(M) \ll_\alpha M^{-1/\tau+o(1)} \quad (M \to \infty),$$

where the function implied by $o(1)$ depends only on α.

Also, it is important to have the following result (see [2, Lemma 3.4]).

https://doi.org/10.4153/S0008439521000242 Published online by Cambridge University Press

Lemma 2.3 *Let $\alpha > 1$. An integer m has the form $m = \alpha n + \beta$ for some integer n if and only if*

$$0 < \{\alpha^{-1}(m - \beta + 1)\} \leq \alpha^{-1}.$$

The value of n is determined uniquely by m.

Let $\psi(x)$ be a periodic function of period 1 for which

$$\psi(x) = \begin{cases} 1 & \text{if } 0 < x \leqslant \gamma \\ 0 & \text{if } \gamma < x \leqslant 1. \end{cases}$$

Then by a classical result of Vinogradov (see [35, Chapter 2 and Lemma 2]), for any $\triangle \in \mathbb{R}$ such that $0 < \triangle < 1/8$ and $\triangle \leqslant 1/2 \min\{\gamma, 1-\gamma\}$, there exists a real-valued function $\psi_\triangle(x)$ of satisfying the following properties:

(1) $\psi_\triangle(x)$ is periodic with period one;
(2) $0 \leqslant \psi_\triangle(x) \leqslant 1$ for all $x \in \mathbb{R}$;
(3) $\psi_\triangle(x) = \psi(x)$ if $\triangle \leqslant x \leqslant \gamma - \triangle$ or $\gamma + \triangle \leqslant x \leqslant 1 - \triangle$; and
(4) $\psi_\triangle(x)$ can be represented as a Fourier series

$$\psi_\triangle(x) = \gamma + \sum_{j=1}^{\infty} (g_j e(jx) + h_j e(-jx)), \tag{2.2}$$

where the coefficients g_j and h_j satisfy the uniform bound

$$\max\{|g_j|, |h_j|\} \ll \min\{j^{-1}, j^{-2}\triangle^{-1}\}, \quad (j \geqslant 1). \tag{2.3}$$

Finally, we need the following result (see Jiang and Lü [18]).

Lemma 2.4 *Under Hypothesis H, π is an automorphic irreducible cuspidal representation of GL_m over $\mathbb{Q}$. For any real number α, we have*

$$S_{\lambda_\pi, k}(x) = \sum_{n \leqslant x} \lambda_\pi(n) e\left(n^k \alpha\right) \ll_{\pi,k} x \frac{\log\log x}{\log x},$$

where $k \geq 1$.

3 Proof of Theorem 1.1

Let $\alpha > 1$ be of type τ and $\gamma = \alpha^{-1}$. Put $\delta = \alpha^{-1}(1-\beta)$, $x_0 = [\alpha + \beta - 1]$ and $x = [\alpha X + \beta]$. Then by Lemma 2.3 and the definition of $\psi(x)$, we have

$$S(\alpha, \beta, X) = \sum_{n \leqslant X} \lambda_\pi([\alpha n + \beta]) = \sum_{\substack{x_0 \leqslant n \leqslant x \\ 0 < \{\gamma n + \delta\} \leqslant \gamma}} \lambda_\pi(n) = \sum_{x_0 \leqslant n \leqslant x} \lambda_\pi(n) \psi(\gamma n + \delta).$$

Applying the properties of $\psi_\triangle(x)$, we obtain

$$S(\alpha, \beta, X) = \sum_{x_0 \leqslant n \leqslant x} \lambda_\pi(n) \psi_\triangle(\gamma n + \delta) + O\Bigg(\sum_{\substack{x_0 \leqslant n \leqslant x \\ \{\gamma n + \delta\} \in \mathcal{J}}} |\lambda_\pi(n)| \Bigg), \tag{3.1}$$

https://doi.org/10.4153/S0008439521000242 Published online by Cambridge University Press

where $\mathcal{I} = [0, \triangle) \cup (\gamma - \triangle, \gamma + \triangle) \cup (1 - \triangle, 1)$ with $|\mathcal{I}| \ll \triangle$. Let $\mathcal{V}(\mathcal{I}, x_0, x) = \#\{x_0 \leqslant n \leqslant x \mid \{\gamma n + \delta\} \in \mathcal{I}\}$. Since the length of each interval above is at most $2\triangle$, by (2.1) and Lemma 2.2, we have

$$\mathcal{V}(\mathcal{I}, x_0, x) \ll_\alpha X^{1-1/(2\tau)} + \triangle X.$$

By virtue of the Rankin–Selberg theory, it is easy to see that

$$\sum_{n \leq x} |\lambda_\pi(n)|^2 \leq \sum_{n \leq x} \lambda_{\pi \times \hat{\pi}}(n) \sim c_\pi x,$$

where $\lambda_{\pi \times \hat{\pi}}(n)$ is the Dirichlet series coefficient of the Rankin–Selberg L-function $L(s, \pi \times \hat{\pi})$ associated to π and its contragredient $\hat{\pi}$. Thus, by Cauchy's inequality, we see that the O-term in (3.1) is bounded by

$$\ll \Bigg(\sum_{x_0 \leqslant n \leqslant x} |\lambda_\pi(n)|^2 \Bigg)^{1/2} \Bigg(\sum_{\substack{x_0 \leqslant n \leqslant x \\ \{\gamma n + \delta\} \in \mathcal{I}}} 1 \Bigg)^{1/2} \ll_{\pi, \alpha} X^{1/2} (X^{1-1/(2\tau)} + \triangle X)^{1/2}.$$

Therefore, by (2.2), we deduce that

$$\begin{aligned}
S(\alpha, \beta, X) &= \sum_{x_0 \leqslant n \leqslant x} \lambda_\pi(n) \psi_\triangle(\gamma n + \delta) + O_{\pi, \alpha}\left(X^{1/2} (X^{1-1/(2\tau)} + \triangle X)^{1/2} \right) \\
&= \gamma \sum_{x_0 \leqslant n \leqslant x} \lambda_\pi(n) + \sum_{j=1}^{\infty} g_j e(\delta j) \sum_{x_0 \leqslant n \leqslant x} \lambda_\pi(n) e(\gamma j n) \\
&\quad + \sum_{j=1}^{\infty} h_j e(-\delta j) \sum_{x_0 \leqslant n \leqslant x} \lambda_\pi(n) e(-\gamma j n) + O_{\pi, \alpha}\left(X^{1/2} (X^{1-1/(2\tau)} + \triangle X)^{1/2} \right) \\
&\ll_{\pi, \alpha} x \frac{\log\log x}{\log x} + x \frac{\log\log x}{\log x} \Bigg\{ \sum_{j \leqslant \triangle^{-1}} j^{-1} + \sum_{j > \triangle^{-1}} \triangle^{-1} j^{-2} \Bigg\} \\
&\quad + X^{1/2} (X^{1-1/(2\tau)} + \triangle X)^{1/2} \\
&\ll_{\pi, \alpha} x \frac{\log\log x}{\log x} + x \frac{\log\log x}{\log x} \log \triangle^{-1} + X^{1/2} (X^{1-1/(2\tau)} + \triangle X)^{1/2}.
\end{aligned}$$

On taking $\triangle = (\log x)^{-2}$, we conclude that

$$S(\alpha, \beta, X) = \sum_{n \leqslant X} \lambda_\pi([\alpha n + \beta]) \ll_{\pi, \alpha} X \frac{(\log\log X)^2}{\log X}.$$

This finishes the proof of Theorem 1.1.

4 Four Lemmas for Theorem 1.4

Theorem 1.4 depends on the following lemmas.

Firstly, we recall some background from Bump [5] and Iwaniec [13]. The symmetric square L-function associated to π is

$$L(s, sym^2 \pi) = \sum_{n=1}^{\infty} \frac{\lambda_{sym^2 \pi}(n)}{n^s}.$$

https://doi.org/10.4153/S0008439521000242 Published online by Cambridge University Press

When π is an automorphic cuspidal representation of $\mathrm{SL}(2,\mathbb{Z})$ and $\Re s > 1$, it is well known that

$$L(s,\pi\times\pi) = \zeta(s)L(s,sym^2\pi).$$

Then we have

$$\lambda_{\pi\times\pi}(n) = \sum_{d|n}\lambda_{sym^2\pi}(d).$$

Although we lack the automorphy of $\pi\times\pi$, it is known from the work of Kim and Shahidi [22] that $sym^2\pi$ is an automorphic irreducible cuspidal representation.

Lemma 4.1 *For any $\varepsilon > 0$, let π be a unitary automorphic cuspidal representation of $\mathrm{SL}(2,\mathbb{Z})$. Then we have*

$$\sum_{n\le x}\lambda_{\pi\times\pi}(n)^2 \ll_\pi x\log x.$$

Proof Let

$$L(s) = \sum_{n\le x}\frac{\lambda_{\pi\times\pi}(n)^2}{n^s}.$$

By the same method as in [27, Lemma 2.2], we have

$$L(s) = \zeta(s)^2L(s,sym^2\pi)^3L(s,sym^4\pi)H(s), \tag{4.1}$$

where the function $H(s)$ is a Dirichlet series absolutely convergent in $\Re s > 1/2$ and $H(s)\neq 0$ for $\Re s = 1$. Suppose that for $\Re s > 1$,

$$\zeta(s)^2L(s,sym^2\pi)^3L(s,sym^4\pi) = \sum_{n=1}^{\infty}b(n)n^{-s}. \tag{4.2}$$

Then by Landau's lemma (see [3]), we have

$$\sum_{n\le x}b(n) = xP(\log x) + O_{\pi,\varepsilon}(x^{\frac{15}{16}+\varepsilon}), \tag{4.3}$$

where $P(t)$ is a polynomial depending on π of degree 1.

By (4.1), we have

$$\lambda_{\pi\times\pi}(n)^2 = \sum_{n=uv}b(u)c(v)$$

and

$$\sum_{v\ge1}|c(v)|v^{-\sigma} \ll_\sigma 1 \quad \left(\sigma > \frac{1}{2}\right). \tag{4.4}$$

Hence, by (4.2)-(4.4) and partial summation, we obtain

$$\sum_{n\le x}\lambda_{\pi\times\pi}(n)^2 = \sum_{n\le x}\sum_{n=uv}b(u)c(v) = \sum_{u\le x}b(u)\sum_{v=1}^{[\frac{x}{u}]}c(v) \ll_\pi x\log x.$$

■

https://doi.org/10.4153/S0008439521000242 Published online by Cambridge University Press

The second lemma is given by Hall and Tenenbaum [11, Theorem 1]. We can get an upper bound for a class of general multiplicative functions.

Lemma 4.2 *Let f be a multiplicative function such that $f(n) \geq 0$ for all n. Assume that there exist positive constants A and B such that for $x > 1$ both inequalities*

$$\sum_{p \leq x} f(p) \log p \leq Ax \quad \text{and} \quad \sum_{p} \sum_{k \geq 2} \frac{f\left(p^k\right) \log p^k}{p^k} \leq B$$

hold. Then

$$\sum_{n \leq x} f(n) \leq (A + B + 1) \frac{x}{\log x} \sum_{n \leq x} \frac{f(n)}{n}.$$

The next result is due to Rudnick and Sarnak [32]. It can be viewed as Mertens' theorem or the Selberg orthogonality conjecture for automorphic L-functions.

Lemma 4.3 *Under Hypothesis H, we have,*

$$\sum_{p \leq x} \frac{|\lambda_\pi(p)|^2}{p} = \log \log x + O(1).$$

Let $\mathcal{P}$ be defined as

$$\mathcal{P} = \{\text{primes} \quad p : z < p \leq w\},$$

where z and w are integers that are to be chosen later in Section 5. We denote by P the product of primes which belong to the set $\mathcal{P}$, that is $P = \prod_{p \in \mathcal{P}} p$.

Lemma 4.4 *Let $z < w < x$. Then we have,*

$$\sum_{\substack{n \leq x \\ (n,P)=1}} |\lambda_{\pi \times \pi}(n)|^2 \ll x \prod_{p \mid P} \left(1 - \frac{1}{p}\right),$$

where the implied constant depends on π.

Proof We shall apply Lemma 4.2 with the arithmetic function

$$f(n) = \begin{cases} |\lambda_{\pi \times \pi}(n)|^2 & \text{if } (n, P) = 1 \\ 0 & \text{if } (n, P) > 1. \end{cases}$$

Clearly, $f(n)$ is multiplicative and non-negative. Next we verify the conditions in Lemma 4.2. By Cauchy's inequality and Lemma 4.3, we have

https://doi.org/10.4153/S0008439521000242 Published online by Cambridge University Press

$$\sum_{p\le x}\frac{|\lambda_{sym^2\pi}(p)|}{p}=\sum_{p\le x}\left(\frac{|\lambda_{sym^2\pi}(p)|}{\sqrt{p}}\right)\left(\frac{1}{\sqrt{p}}\right)$$
$$=\left(\sum_{p\le x}\left(\frac{|\lambda_{sym^2\pi}(p)|}{\sqrt{p}}\right)^2\right)^{\frac{1}{2}}\left(\sum_{p\le x}\left(\frac{1}{\sqrt{p}}\right)^2\right)^{\frac{1}{2}}$$
$$=\log\log x+O(1),$$

and we can get a result similar to Lemma 4.3,

$$\sum_{p\le x}\frac{|\lambda_{\pi\times\pi}(p)|^2}{p}=\sum_{p\le x}\frac{\left|\sum_{d|p}\lambda_{sym^2\pi}(d)\right|^2}{p}$$
$$=\sum_{p\le x}\frac{|1+\lambda_{sym^2\pi}(p)|^2}{p}$$
$$\ll\log\log x. \tag{4.5}$$

We obtain by using summation by parts,

$$\sum_{p\le x}|\lambda_{\pi\times\pi}(p)|^2\log p\ll x.$$

By the work of Gelbart and Jacquet [9], we know that the $sym^2\pi$ is an automorphic cuspidal representation of GL_3 over $\mathbb{Q}$. So by Remark 1.1, Hypothesis H implies that

$$\sum_p\sum_{k\ge2}\frac{\left|\lambda_{sym^2\pi}\left(p^k\right)\right|^2\log p}{p^k}\ll1.$$

Furthermore, we can get

$$\sum_p\sum_{k\ge2}\frac{|\lambda_{\pi\times\pi}(p^k)|^2\log p}{p^k}=\sum_p\sum_{k\ge2}\frac{\left|\sum_{d|p^k}\lambda_{sym^2\pi}(d)\right|^2\log p}{p^k}\ll1. \tag{4.6}$$

Hence, by Lemma 4.2, we get

$$\sum_{\substack{n\le x\\(n,P)=1}}|\lambda_{\pi\times\pi}(n)|^2\ll\frac{x}{\log x}\sum_{\substack{n\le x\\(n,P)=1}}\frac{|\lambda_{\pi\times\pi}(n)|^2}{n}. \tag{4.7}$$

Using the multiplicative property of $|\lambda_{\pi\times\pi}(n)|^2$ and the estimate (4.6), the logarithmic average of $|\lambda_{\pi\times\pi}(n)|^2$ can be estimated as follows:

https://doi.org/10.4153/S0008439521000242 Published online by Cambridge University Press

$$\begin{aligned}\sum_{\substack{n\le x\\(n,P)=1}}\frac{|\lambda_{\pi\times\pi}(n)|^2}{n} &\ll \prod_{\substack{p\le x\\p\nmid P}}\left(1+\frac{|\lambda_{\pi\times\pi}(p)|^2}{p}+\sum_{k\ge 2}\frac{|\lambda_{\pi\times\pi}(p^k)|^2\log p^k}{p^k}\right)\\ &\ll \exp\left(\sum_{\substack{p\le x\\p\nmid P}}\left(\frac{|\lambda_{\pi\times\pi}(p)|^2}{p}+\sum_{k\ge 2}\frac{|\lambda_{\pi\times\pi}(p^k)|^2\log p^k}{p^k}\right)\right)\\ &\ll \exp\left(\sum_{\substack{p\le x\\p\nmid P}}\frac{|\lambda_{\pi\times\pi}(p)|^2}{p}+O(1)\right)\\ &\ll \exp\left(\sum_{\substack{p\le x\\p\nmid P}}\frac{|\lambda_{\pi\times\pi}(p)|^2}{p}\right).\end{aligned} \tag{4.8}$$

By (4.5), we derive

$$\begin{aligned}\sum_{\substack{p\le x\\p\nmid P}}\frac{|\lambda_{\pi\times\pi}(p)|^2}{p} &= \sum_{p\le x}\frac{|\lambda_{\pi\times\pi}(p)|^2}{p}-\sum_{p\le w}\frac{|\lambda_{\pi\times\pi}(p)|^2}{p}+\sum_{p\le z}\frac{|\lambda_{\pi\times\pi}(p)|^2}{p}\\ &= \log\log x-\log\log w+\log\log z+O(1).\end{aligned} \tag{4.9}$$

By inserting (4.9) into (4.8) and Mertens' theorem, we have

$$\sum_{\substack{n\le x\\(n,P)=1}}\frac{|\lambda_{\pi\times\pi}(n)|^2}{n}\ll(\log x)\prod_{p\mid P}\left(1-\frac{1}{p}\right). \tag{4.10}$$

Finally, our lemma immediately follows from the above estimate together with (4.7). ∎

5 Proof of Theorem 1.4

Now it remains to prove Theorem 1.4. We mainly use the multiplicity of $\lambda_{\pi\times\pi}(n)$ and the method of sifting. In Section 5.1, we begin by splitting the range of integration $[1,x]$ into two disjoint sets $\mathcal{I}$ and $\mathcal{J}$. Obviously, we have

$$S_{\lambda_{\pi\times\pi}}(x)=S_{\mathcal{I}}(x)+S_{\mathcal{J}}(x).$$

Hence it remains to estimate for $S_{\mathcal{I}}(x)$ and $S_{\mathcal{J}}(x)$. In Section 5.2, we can estimate for $S_{\mathcal{I}}(x)$ using the bound for exponential sums and the multiplicity of the involved functions. In Section 5.3, we need to apply Lemmas 4.1 and 4.4 to estimate the contribution of $S_{\mathcal{J}}(x)$.

We always assume that x is sufficiently large.

5.1 Set up

We assume that $H=H(x)$ and $K=K(x)$ be two parameters satisfying

$$\log^{\delta}x<H<K<x^{\delta} \tag{5.1}$$

with some $0<\delta<1/10$. Moreover, we assume that z and w are integers satisfying

$$H^2/2\le z<w\le 2K^2.$$

https://doi.org/10.4153/S0008439521000242 Published online by Cambridge University Press

Let $v \in [H, K]$,

$$\mathcal{I}_v = ((v-1)^2, v^2], \quad P_v = \prod_{(H-1)^2 < p \le v^2} p$$

$$\mathcal{P}_v = \{\text{primes} \quad p : p \in \mathcal{I}_v\}, \quad \mathcal{M}_v = \left\{ m \in \left[1, \frac{x}{v^2}\right] : (m, P_v) = 1 \right\} \tag{5.2}$$

$$\mathcal{P}_v \mathcal{M}_v = \{pm : p \in \mathcal{P}_v, m \in \mathcal{M}_v\},$$

$$\mathcal{I} = \bigcup_{H \le v \le K} \mathcal{P}_v \mathcal{M}_v \text{ and } \mathcal{J} = [1, x] \backslash \mathcal{I}.$$

Each $n \in \mathcal{P}_v \mathcal{M}_v$ can be written as $n = pm$ with $p \in \mathcal{P}_v$ and $m \in \mathcal{M}_v$, hence $|\mathcal{P}_v \mathcal{M}_v| = |\mathcal{P}_v \| \mathcal{M}_v|$, and that $\mathcal{P}_v \mathcal{M}_v \subset [1, x]$. For $H \le v \le K$, the sets $\mathcal{P}_v \mathcal{M}_v$ are pairwise disjoint.

Later on we will need certain bounds related to the sets $\mathcal{P}_v$, for $H \le v \le K$. Clearly, in view of the definition of $\mathcal{P}_v$, the Brun–Titchmarsh theorem [8] immediately implies that

$$|\mathcal{P}_v| \ll \frac{v}{\log v}. \tag{5.3}$$

Moreover, a standard sieve estimate gives

$$\left| \left\{ n \in [1, x] : n \text{ has no prime factors in } \bigcup_{H \le v \le K} \mathcal{P}_v \right\} \right| = \sum_{\substack{n \le x \\ (n, P_K) = 1}} 1 \ll x \prod_{p | P_K} \left(1 - \frac{1}{p}\right). \tag{5.4}$$

Note that $K^2 \le x^{2\delta}$ due to (5.1). We can estimate the product over primes in (5.4) using Mertens' theorem,

$$\prod_{p | P_K} \left(1 - \frac{1}{p}\right) \ll \frac{\log H}{\log K}. \tag{5.5}$$

By the definition of intervals $\mathcal{I}$ and $\mathcal{J}$, this leads to the basic decomposition

$$\begin{aligned} \sum_{n \le x} \lambda_{\pi \times \pi}(n) e(n\alpha) &= \sum_{n \in \mathcal{I}} \lambda_{\pi \times \pi}(n) e(n\alpha) + \sum_{n \in \mathcal{J}} \lambda_{\pi \times \pi}(n) e(n\alpha) \\ &= S_{\mathcal{I}}(x) + S_{\mathcal{J}}(x). \end{aligned} \tag{5.6}$$

5.2 Estimating $S_{\mathcal{I}}(x)$

Since the sets $\mathcal{P}_v \mathcal{M}_v$ are pairwise disjoint for $H \le v \le K$, we obtain that

$$S_{\mathcal{I}}(x) \ll \sum_{H \le v \le K} \left| \sum_{pm \in \mathcal{P}_v \mathcal{M}_v} \lambda_{\pi \times \pi}(pm) e(pm\alpha) \right|. \tag{5.7}$$

For $pm \in \mathcal{P}_v \mathcal{M}_v$, we have $p \in \mathcal{P}_v$ and $m \in \mathcal{M}_v$. The $\lambda_{\pi \times \pi}(pm)$ in (5.7) can be factored as $\lambda_{\pi \times \pi}(p) \lambda_{\pi \times \pi}(m)$ by its multiplicativity. Thus, we use the Cauchy–Schwarz inequal-

https://doi.org/10.4153/S0008439521000242 Published online by Cambridge University Press

ity and (5.2) to estimate the inner sum

$$\begin{aligned} S_{\mathcal{I}_v}(x) &\ll \Bigg(\sum_{m\in\mathcal{M}_v}|\lambda_{\pi\times\pi}(m)|^2\Bigg)^{1/2}\Bigg(\sum_{m\in\mathcal{M}_v}\Bigg|\sum_{p\in\mathcal{P}_v}\lambda_{\pi\times\pi}(p)e(pm\alpha)\Bigg|^2\Bigg)^{1/2}\\ &\ll \Bigg(\sum_{m\in\mathcal{M}_v}|\lambda_{\pi\times\pi}(m)|^2\Bigg)^{1/2}\Bigg(\sum_{m\in\mathcal{M}_v}\sum_{p\in\mathcal{P}_v}|\lambda_{\pi\times\pi}(p)|^2\Bigg|\sum_{p,q\in\mathcal{P}_v}e((p-q)m\alpha)\Bigg|\Bigg)^{1/2}\\ &\ll \Bigg(\sum_{m\in\mathcal{M}_v}|\lambda_{\pi\times\pi}(m)|^2\Bigg)^{1/2}\Bigg(\sum_{m\in\mathcal{M}_v}\sum_{p\in\mathcal{P}_v}|\lambda_{\pi\times\pi}(p)|^2\Bigg|\sum_{|h|\le 2v}e(hm\alpha)\Bigg|\Bigg)^{1/2}\\ &\ll \Bigg(\sum_{pm\in\mathcal{P}_v\mathcal{M}_v}|\lambda_{\pi\times\pi}(pm)|^2\Bigg)^{1/2}\Bigg(\sum_{|h|\le 2v}\Bigg|\sum_{m\le x/v^2}e(hm\alpha)\Bigg|\Bigg)^{1/2}, \end{aligned}$$

where $h = p - q$. We use the following estimate [15, Lemma 13.7] about exponential sums, for any $M, N \ge 1$

$$\sum_{|n|\le N}\Bigg|\sum_{m\le M}e(\alpha mn)\Bigg| \ll \Bigg(M+N+\frac{MN}{q}+q\Bigg)\log q, \tag{5.8}$$

to have

$$\ll \Bigg(\sum_{pm\in\mathcal{P}_v\mathcal{M}_v}|\lambda_{\pi\times\pi}(pm)|^2\Bigg)^{1/2}\Bigg(\frac{x}{v^2}+\frac{x}{vq}+v\log q+q\log q\Bigg)^{1/2}. \tag{5.9}$$

By summing over v, and applying Lemma 4.1 and the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} S_{\mathcal{J}}(x) &\ll \Bigg(\sum_{n\le x}|\lambda_{\pi\times\pi}(n)|^2\Bigg)^{1/2}\Bigg(\sum_{H\le v\le K}\Bigg(\frac{x}{v^2}+\frac{x}{vq}+v\log q+q\log q\Bigg)\Bigg)^{1/2}\\ &\ll_\pi x(\log x)^{\frac{1}{2}}\Bigg(\frac{1}{H}+\frac{\log K}{q}+\frac{K^2\log q}{x}+\frac{qK\log q}{x}\Bigg)^{1/2}. \end{aligned} \tag{5.10}$$

5.3 Estimating $S_{\mathcal{J}}(x)$

We first introduce the following subsets of $[1, x]$:

$$\begin{aligned} \mathcal{J}_1^{(v)} &= \Bigg\{n\in[1,x] : n \text{ has exactly one divisor in } \mathcal{P}_v \text{ and none in } \bigcup_{H\le h<v}\mathcal{P}_h\Bigg\},\\ \mathcal{J}_1 &= \bigcup_{H\le v\le K}\mathcal{J}_1^{(v)},\\ \mathcal{J}_2 &= \Bigg\{n\in[1,x] : n \text{ has at least one prime factor in } \bigcup_{H\le v\le K}\mathcal{P}_v\Bigg\},\\ \mathcal{J}_3 &= \Bigg\{n\in[1,x] : n \text{ has no prime factors in } \bigcup_{H\le v\le K}\mathcal{P}_v\Bigg\}. \end{aligned} \tag{5.11}$$

https://doi.org/10.4153/S0008439521000242 Published online by Cambridge University Press

By the definitions of these subsets of $[1,x]$, it is clear that $\mathcal{P}_v\mathcal{M}_v \subset \mathcal{J}_1^{(v)}$. Thus, we have $\mathcal{I} \subset \mathcal{J}_1$. Moreover, $\mathcal{J}_2 \cup \mathcal{J}_3 = [1,x]$ and $\mathcal{J}_2 \cap \mathcal{J}_3 = \varnothing$. Thus, we get

$$\mathcal{J} \subset (\mathcal{J}_1 \backslash \mathcal{I}) \cup (\mathcal{J}_2 \backslash \mathcal{J}_1) \cup \mathcal{J}_3.$$

As a consequence, by Cauchy's inequality and Lemma 4.1, we obtain that

$$\begin{aligned} |S_{\mathcal{J}}(x)| &\ll \sum_{n\in\mathcal{J}_1\backslash\mathcal{I}} |\lambda_{\pi\times\pi}(n)| + \sum_{n\in\mathcal{J}_2\backslash\mathcal{J}_1} |\lambda_{\pi\times\pi}(n)| + \sum_{n\in\mathcal{J}_3} |\lambda_{\pi\times\pi}(n)| \\ &\ll_\pi x^{\frac{1}{2}}(\log x)^{\frac{1}{2}}\left(|\mathcal{J}_1\backslash\mathcal{I}|^{\frac{1}{2}} + |\mathcal{J}_2\backslash\mathcal{J}_1|^{\frac{1}{2}}\right) + \left(\sum_{n\in\mathcal{J}_3} |\lambda_{\pi\times\pi}(n)|^2\right)^{\frac{1}{2}} |\mathcal{J}_3|^{\frac{1}{2}}. \end{aligned} \tag{5.12}$$

It is obvious that

$$\mathcal{J}_1^{(v)} \backslash \mathcal{P}_v\mathcal{M}_v \subset \mathcal{P}_v\left(\frac{x}{v^2}, \frac{x}{(v-1)^2}\right].$$

Hence, it follows from (5.3) that

$$|\mathcal{J}_1\backslash\mathcal{I}| \ll \sum_{H\le v\le K} \frac{v}{\log v}\frac{x}{v^3} \ll \frac{x}{H\log H}. \tag{5.13}$$

Moreover, on account of

$$\mathcal{J}_2\backslash\mathcal{J}_1 \subset \bigcup_{H\le v\le K} \left\{n\in[1,x] : n \text{ has at least two prime factors in } \mathcal{P}_v\right\}, \tag{5.14}$$

we get by using (5.3) again that

$$|\mathcal{J}_2\backslash\mathcal{J}_1| \ll \sum_{H\le v\le K} \sum_{p,q\in\mathcal{P}_v} \frac{x}{pq} \ll x \sum_{H\le v\le K} \left(\frac{|\mathcal{P}_v|}{(v-1)^2}\right)^2 \ll \frac{x}{H\log^2 H}. \tag{5.15}$$

Further, by Lemma 4.4, $z = (H-1)^2$, $w = K^2$, $P = P_K$, (5.5) and (5.4), we have

$$\sum_{n\in\mathcal{J}_3} |\lambda_{\pi\times\pi}(n)|^2 = \sum_{\substack{n\le x\\(n,P_K)=1}} |\lambda_{\pi\times\pi}(n)|^2 \ll x\frac{\log H}{\log K} \quad \text{and} \quad |\mathcal{J}_3| \ll x\frac{\log H}{\log K}. \tag{5.16}$$

Collecting (5.12)–(5.16), we obtain

$$S_{\mathcal{J}}(x) \ll_\pi x(\log x)^{\frac{1}{2}} \frac{1}{\sqrt{H\log H}} + x\frac{\log H}{\log K}. \tag{5.17}$$

By (5.10) and (5.17), we take $H = \log^2 x$, $K = \frac{1}{2}x^\delta$. Then we suppose that $|\alpha - a/q| \le 1/q^2$, $(a,q) = 1$ and $2 \le \log^3 x \le q \le x^{(1-\delta)}/(\log^3 x)$, we deduce that

$$S_{\lambda_{\pi\times\pi}}(x) = \sum_{n\le x} \lambda_{\pi\times\pi}(n)e(n\alpha) \ll_\pi x\frac{\log\log x}{(\log x)^{\frac{1}{2}}}. \tag{5.18}$$

https://doi.org/10.4153/S0008439521000242 Published online by Cambridge University Press

6 Proof of Theorem 1.6

For brevity, let $\alpha > 1$ be of finite type τ, $\tau \geq 1$ and $\gamma = \alpha^{-1}$. Put $\delta = \alpha^{-1}(1-\beta)$, $x_0 = [\alpha + \beta - 1]$ and $x = [\alpha X + \beta]$. Then by Lemma 2.3 and the definition of $\psi(x)$, we have

$$S(\alpha, \beta, X) = \sum_{n \leqslant X} \lambda_{\pi\times\pi}([\alpha n + \beta]) = \sum_{\substack{x_0 \leqslant n \leqslant x \\ 0 < \{\gamma n + \delta\} \leqslant \gamma}} \lambda_{\pi\times\pi}(n) = \sum_{x_0 \leqslant n \leqslant x} \lambda_{\pi\times\pi}(n)\psi(\gamma n + \delta).$$

Applying the properties of $\psi_\triangle(x)$, we obtain

$$S(\alpha, \beta, X) = \sum_{x_0 \leqslant n \leqslant x} \lambda_{\pi\times\pi}(n)\psi_\triangle(\gamma n + \delta) + O\Bigg(\sum_{\substack{x_0 \leqslant n \leqslant x \\ \{\gamma n + \delta\} \in \mathcal{I}}} |\lambda_{\pi\times\pi}(n)| \Bigg), \tag{6.1}$$

where $\mathcal{I} = [0, \triangle) \cup (\gamma - \triangle, \gamma + \triangle) \cup (1 - \triangle, 1)$ with $|\mathcal{I}| \ll \triangle$. Let $\mathcal{V}(\mathcal{I}, x_0, x) = \#\{x_0 \leqslant n \leqslant x \mid \{\gamma n + \delta\} \in \mathcal{I}\}$. Since the length of each interval above is at most $2\triangle$. Then by (2.1) and Lemma 2.2, we have

$$\mathcal{V}(\mathcal{I}, x_0, x) \ll_\alpha X^{1-1/(2\tau)} + |\mathcal{I}|X \ll_\alpha X^{1-1/(2\tau)} + \triangle X.$$

Thus, by Cauchy's inequality and Lemma 4.1, we see that the O-term in (6.1) is bounded by

$$\ll \Bigg(\sum_{x_0 \leqslant n \leqslant x} |\lambda_{\pi\times\pi}(n)|^2 \Bigg)^{\frac{1}{2}} \Bigg(\sum_{\substack{x_0 \leqslant n \leqslant x \\ \{\gamma n + \delta\} \in \mathcal{I}}} 1 \Bigg) \ll_{\pi\times\pi,\alpha} X^{\frac{1}{2}} (\log X)^{\frac{1}{2}} (X^{(1-\frac{1}{2\tau})} + \triangle X)^{\frac{1}{2}}.$$

Next, let $J \geq \triangle^{-1}$ be a large real number (to be specified later), and let ψ_J be the trigonometric polynomial given by

$$\psi_J(y) = \gamma + \sum_{j \leq J} \big(g_j e(jy) + h_j e(-jy) \big) \quad (y \in \mathbb{R}).$$

Using (2.3), we see that the estimate

$$\psi_\triangle(y) = \psi_J(y) + O\left(J^{-1}\triangle^{-1}\right)$$

holds uniformly for all $y \in \mathbb{R}$; therefore, let $y = \gamma n + \delta$

$$\sum_{n \leq x} \lambda_{\pi\times\pi}(n)\psi_\triangle(\gamma n + \delta) = \sum_{n \leq X} \lambda_{\pi\times\pi}(n)\psi_J(\gamma n + \delta) + O\left(J^{-1}\triangle^{-1}X\right),$$

https://doi.org/10.4153/S0008439521000242 Published online by Cambridge University Press

where we have used the bound $\sum_{n\leqslant X}|\lambda_{\pi\times\pi}(n)|\ll X$. Then by (2.2), (2.3), and Lemma 2.4, we deduce that

$$\begin{aligned}
S(\alpha,\beta,X) &= \sum_{x_0\leqslant n\leqslant x}\lambda_{\pi\times\pi}(n)\psi_{\triangle}(\gamma n+\delta)\\
&\quad + O_{\pi,\alpha}\left(X^{\frac{1}{2}}(\log X)^{\frac{1}{2}}(X^{(1-\frac{1}{2\tau})}+\triangle X)^{\frac{1}{2}}\right)\\
&= \sum_{x_0\leqslant n\leqslant x}\lambda_{\pi\times\pi}(n)\psi_J(\gamma n+\delta)+O(J^{-1}\triangle^{-1}X)\\
&\quad + O_{\pi,\alpha}\left(X^{\frac{1}{2}}(\log X)^{\frac{1}{2}}(X^{(1-\frac{1}{2\tau})}+\triangle X)^{\frac{1}{2}}\right)\\
&= \gamma\sum_{x_0\leqslant n\leqslant x}\lambda_{\pi\times\pi}(n)+\sum_{j\leq J}g_j e(\delta j)\sum_{x_0\leqslant n\leqslant x}\lambda_{\pi\times\pi}(n)e(\gamma jn)\\
&\quad + \sum_{j\leq J}h_j e(-\delta j)\sum_{x_0\leqslant n\leqslant x}\lambda_{\pi\times\pi}(n)e(-\gamma jn)\\
&\quad + O_{\pi,\alpha}\left(X^{\frac{1}{2}}(\log X)^{\frac{1}{2}}(X^{(1-\frac{1}{2\tau})}+\triangle X)^{\frac{1}{2}}+J^{-1}\triangle^{-1}X\right).
\end{aligned}\tag{6.2}$$

Due to $\pi\simeq\hat{\pi}$ and Landau's lemma, we have

$$\sum_{n\leq X}\lambda_{\pi\times\pi}(n)=c_\pi X+O_\pi(X^{\frac{3}{5}}),\tag{6.3}$$

where c_π is a positive constant.

Put $R=\log^3 X$. We claim that if X is sufficiently large, then for every j in the above sum there is a reduced fraction a/q such that $|j\gamma-a/q|\leq q^{-2}$ and $R\leq q\leq X^{1-\delta}/R$. Assuming this is true for the moment, Theorem 1.4 implies that

$$\sum_{x_0\leqslant n\leqslant x}\lambda_{\pi\times\pi}(n)e(\gamma jn)\ll\frac{x\log\log x}{(\log x)^{1/2}}\quad(0<|j|\leq J).$$

Therefore, it follows from (6.2), (6.3), and Theorem 1.4 that

$$\begin{aligned}
S(\alpha,\beta,X) &= c_\pi X+O_\pi\left(X^{\frac{3}{5}}\right)+X\frac{\log\log X}{(\log X)^{\frac{1}{2}}}\left\{\sum_{j\leqslant\triangle^{-1}}j^{-1}+\sum_{j>\triangle^{-1}}\triangle^{-1}j^{-2}\right\}\\
&\quad + O_{\pi,\alpha}\left(X^{\frac{1}{2}}(\log X)^{\frac{1}{2}}(X^{(1-\frac{1}{2\tau})}+\triangle X)^{\frac{1}{2}}+J^{-1}\triangle^{-1}X\right)\\
&= c_\pi X+O_\pi\left(X^{\frac{3}{5}}\right)+X\frac{\log\log X}{(\log X)^{\frac{1}{2}}}\log\triangle^{-1}\\
&\quad + O_{\pi,\alpha}\left(X^{\frac{1}{2}}(\log X)^{\frac{1}{2}}(X^{(1-\frac{1}{2\tau})}+\triangle X)^{\frac{1}{2}}+J^{-1}\triangle^{-1}X\right).
\end{aligned}$$

To balance the error terms, we choose $\triangle=(\log X)^{-2}, J=\triangle^{-3/2}=(\log X)^3$, thus obtaining the bound stated in the theorem.

To prove the claim (see [10]), let j be an integer with $0<|j|\leq J=(\log X)^3$, and let $r_i=a_i/q_i$ be the ith convergent in the continued fraction expansion of $j\gamma$. since γ is

https://doi.org/10.4153/S0008439521000242 Published online by Cambridge University Press

of finite type τ, $\tau \geq 1$ for every $\varepsilon > 0$ there is a constant $C = C(\gamma, \varepsilon)$ such that

$$C\left(|j|q_{i-1}\right)^{-(\tau+\varepsilon)} < \| \gamma|j|q_{i-1} \| \leq |\gamma|j\,|q_{i-1} - a_{i-1}| \leq q_i^{-1}.$$

Put $\varepsilon = \tau$, and let j be the least positive integer for which $q_j \geq R$ (note that $j \geq 2$). Then

$$R \leq q_j \ll \left(|j|q_{i-1}\right)^{2\tau} \leq (JR)^{2\tau} = \log^{12\tau} X.$$

It follows that $R \leq q_j \leq X^{1-\delta}/R$ if X is sufficiently large, depending only on α. This finishes the proof of Theorem 1.6.

Acknowledgment The author is grateful to the referee for many very useful comments and suggestions. The author would like to thank Professor Guangshi Lü for his encouragement and advice.

References

[1] A. G. Abercrombie, W. D. Banks, and I. E. Shparlinski, *Arithmetic functions on Beatty sequences*. Acta Arith. **136**(2009), 81–89.
[2] W. D. Banks and I. E. Shparlinski, *Character sums with Beatty sequences on burgess-type intervals*. Cambridge University Press, Cambridge, MA, 2009, 15–21.
[3] L. Barthel and D. Ramakrishnan, *A nonvanishing result for twists of L-functions of* $GL(n)$. Duke Math. J. **74**(1994), 681–700.
[4] A. S. Besicovitch, *Sets of fractional dimensions. IV: on rational approximation to real numbers*. J. Lond. Math. Soc. **9**(1934), 126–131.
[5] D. Bump, *Automorphic forms and representations*. Cambridge Studies in Advanced Mathematics, 55, Cambridge University Press, Cambridge, UK, 1997.
[6] H. Davenport, *On certain exponential sums*. J. Reine Angew. Math. **169**(1933), 158–176.
[7] P. Deligne, *La conjecture de Weil I*. Inst. Hautes tudes Sci. Publ. Math. **48**(1974), 273–308.
[8] J. Friedlander and H. Iwaniec, *Opera de cribro*. American Mathematical Society Colloquium Publications, 57, American Mathematical Society, Providence, RI, 2010.
[9] S. Gelbart and H. Jacquet, *A relation between automorphic representations of* $GL(2)$*and*$GL(3)$. Ann. Sci. École Norm. Sup. **11**(1978), 471–552.
[10] A. M. Güloğlu and C. W. Nevans, *Sums of multiplicative functions over a Beatty sequence*. Bull. Aust. Math. Soc. **78**(2008), 327–334.
[11] R. R. Hall and G. Tenenbaum, *Divisors*. Cambridge Tracts in Mathematics, 90, Cambridge University Press, Cambridge, MA, 1988.
[12] E. Hecke, *Theorie der Eisensteinsche Reihen höherer Stufe und ihre Anwendung auf Funktionentheorie und Arithmetik*. Abh. Math. Abh. Math. Semin. Univ. Hamb. 5(1927), 199–224.
[13] H. Iwaniec, *Topics in classical automorphic forms*. Graduate Studies in Mathematics, 17, American Mathematical Society, Providence, RI, 1997.
[14] H. Iwaniec, *Spectral methods of automorphic forms*. Graduate Studies in Mathematics, 53, American Mathematical Society, Providence, RI, 2002.
[15] H. Iwaniec and E. Kowalski, *Analytic number theory*. American Mathematical Society Colloquium Publications 53, American Mathematical Society, Providence, RI, 2004.
[16] H. Jacquet and J. A. Shalika, *On Euler products and the classification of automorphic representations I*. Amer. J. Math. **103**(1981), 499–558.
[17] V. Jarník, *Diophantische Approximationen und Hausdorffsches Mass*. Mat. Sb. **36**(1929), 371–382.
[18] Y. J. Jiang and G. S. Lü, *The generalized Bourgain-Sarnak-Ziegler criterion and its application to additively twisted sums on*$GL(m)$. Sci. China Math. **63**(2020). https://doi.org/10.1007/s11425-020-1717-1.
[19] A. Khintchine, *Einige Sätzeüber Kettenbrüche, mit Anwendungen auf die Theorie der Diophantischen Approximationen*. Math. Ann. **92**(1924), 115–125.
[20] H. H. Kim, *Functoriality for the exterior square of* $GL(4)$ *and the symmetric fourth of* $GL(2)$. J. Amer. Math. Soc. **16**(2003), 139–183. With appendix 1 by Dinakar Ramakrishnan and appendix 2 by Kim and Peter Sarnak.

https://doi.org/10.4153/S0008439521000242 Published online by Cambridge University Press

[21] H. H. Kim, *A note on Fourier coefficients of cusp forms on* $GL(n)$. Forum Math. **18**(2006), 115–119.
[22] H. H. Kim and F. Shahidi, *Cuspidality of symmetric powers with applications*. Duke Math. J. **112**(2004), 177–197.
[23] H. D. Kloosterman, *Asymptotische Formeln für die Fourierkoeffizienten ganzer Modulformen*. Abh. Math. Abh. Math. Semin. Univ. Hamb. **5**(1927), 337–352.
[24] L. Kurpers and H. Niederreiter, *Uniform distribution of sequences. Pure and Applied Mathematics*, Wiley-Interscience, New York, NY, 1974.
[25] H. X. Lao, *Oscillations of coefficients of primitive cusp form over some special sequences*. Ramanujan J. **19**(2009), 339–350.
[26] G. S. Lü, *On sums involving coefficients of automorphic* L*-functions*. Proc. Amer. Math. Soc. **137**(2009), 2879–2887.
[27] G. S. Lü, *Estimates for coefficients of certain L-functions*. Monatsh. Math. **181**(2016), 657–674.
[28] G. S. Lü, *Exponential sums with Fourier coefficients of automorphic forms*. Math. Z. **289**(2018), 267–278.
[29] W. Z. Luo, Z. Rudnick, and P. Sarnak, *On Selberg's eigenvalue conjecture*. Geom. Funct. Anal. **5**(1995), 387–401.
[30] S. Miller, *Cancellation in additively twisted sums on* $GL(n)$. Amer. J. Math. **128**(2006), 699–729.
[31] K. F. Roth, *Rational approximations to algebraic numbers*. Mathematika **2**(1955), 1–20.
[32] Z. Rudnick and P. Sarnak, *Zeros of principal* L*-functions and random matrix theory*. Duke Math. J. **81**(1996), 269–322.
[33] Q. F. Sun, *Cancellation of cusp forms coefficients over Beatty sequences on* $GL(m)$. Canad. Math. Bull. **54**(2011), 757–762.
[34] M. Technau and A. Zafeiropoulos, *Metric results on summatory arithmetic functions on beatty sets*. Acta Arith. **197**(2021), 93–104.
[35] I. M. Vinogradov, *The method of trigonometrical sums in the theory of numbers*. Dover Publications, Mineola, NY, 2004.
[36] A. Weil, *On some exponential sums*. Proc. Natl. Acad. Sci. **34**(1948), 204–207.

School of Mathematics, Shandong University, Jinan 250100, China
e-mail: qma@mail.sdu.edu.cn

https://doi.org/10.4153/S0008439521000242 Published online by Cambridge University Press