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Wave flows induced by lifting of a rectangular beam partly immersed in shallow water

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Flows induced by vertical lifting of a rectangular beam partly immersed into shallow water in a rectangular prismatic channel with a horizontal bottom are studied within the framework of the long-wave approximation. The beam width coincides with the channel width and the lower and upper planes of the beam are parallel to the channel bottom. The lifting process in the general case consists of three stages. At the first stage, the lower surface of the beam is completely located in the liquid, which ascends following the beam under the action of hydrostatic pressure. At the second stage, the edges of the lower surface of the beam leave the water medium, the wetted part of the beam becomes smaller, and the liquid under this part of the beam move upward. At the beginning of the third stage, the beam is separated from water; as a result, liquid lifting that occurred at the second stage leads to the formation of two diverging waves. The liquid flow in the domain adjacent to the lower surface of the beam is calculated analytically, while the liquid flow outside this domain is obtained by means of numerical calculations by the CABARET (compact accurately boundary-adjusting high-resolution technique) scheme, which provides the second order of accuracy on smooth solutions.

Key words: gravity currents, shallow water flows

1. Introduction

Lifting of a body from the liquid surface and thereby induced flows are of interest for both theoretical studies and applications. Mathematical modelling of the lifting process and physical phenomena that have to be taken into account are now at the early stage of investigations. The processes that occur when a body enters a liquid were studied in Korobkin & Pukhnachov (1988). The weight of the liquid, capillary forces and forces of adhesion of liquid particles to the solid surface determine liquid separation from the surface of the lifting body. On the other hand, the action of adhesion forces is localized near the body surface and exerts a minor effect on the global flow of the liquid from whose surface the body is lifting. For flat surfaces with the horizontal size of the wetted part of the body being much greater than its immersion depth, a thin layer of the liquid stays on the lifting body surface owing to the forces of adhesion between the liquid and the body surface. However, this thin

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layer weakly affects the liquid flow in the main domain and the pressure distribution over the wetted part of the body during its lifting. Therefore, the size of the wetted part of the body, which decreases with time, can be approximately determined with adhesion forces being ignored. The limiting case of a flat body is a rectangular beam with a flat lower horizontal surface considered in this study.

Lifting of bodies from the surface of a deep liquid with the bottom effect being ignored has been studied theoretically, experimentally and numerically (Reis *et al.* 2010; Piro & Maki 2011; Tassin *et al.* 2013; Korobkin 2013). Reis *et al.* (2010) performed experimental investigations of lifting of a circular glass disk from the water surface. The goal of their study was to explain the process of lapping by felines. The law of vertical lifting of the disk followed the motion of cat's tongue recorded by a high-speed video camera. The analysis of the measured hydrodynamic forces made it possible to conclude that the main physical phenomena determined by these forces are the liquid inertia and weight. The liquid viscosity and surface tension exert minor effects on the loads and liquid flow induced by disk lifting from the liquid surface.

The entry into water with a constant negative acceleration and subsequent exit from water were studied for a wedge with a 10° rise of the bottom (Piro & Maki 2011) and for a parabolic contour (Tassin *et al.* 2013). The liquid flow and the load distribution along the wetted part of the body were determined by means of the numerical solution of the Navier–Stokes equations. It was shown that the hydrodynamic loads at the initial stage of the body entry into water are positive and sufficiently large compared to the loads that act on the body at the final stage of its entry into the water and at the stage when the body out of the water. As the body is gradually decelerated, the loads acquire negative values. The minimum (maximum in terms of the absolute values) negative loads are reached when the body velocity is equal to zero. Later on, as the body leaves the water, the magnitude of the negative loads increases and becomes equal to zero when the body is located at a certain height above the undisturbed level of the liquid. The negative loads last for a much longer time than the positive loads and may exceed the latter in magnitude. It was noted that the hydrodynamic forces at the stage of the entry into water substantially depend on the body shape. At the stage of the exit from water, these loads are mainly determined by the horizontal size of the wetted part of the body and weakly depend on the body shape.

Korobkin (2013) proposed a linearized model of lifting of a flat body from the free boundary of an infinitely deep liquid. This model is based on the main assumption that the velocity of motion of the boundary of the liquid–body contact region is proportional to the local velocity of the flow at the periphery of the contact region. This assumption is similar to that used in Bendgamin (1968) for modelling the outflow of an ideal incompressible liquid from a rectangular container. The hydrodynamic forces calculated by the model proposed in Korobkin (2013) for a wedge and a parabolic contour agree well with the results of numerical calculations reported in Piro & Maki (2011) and Tassin *et al.* (2013).

Problems similar to that discussed in this paper are encountered in investigations of the wave impact on the lower flat surface of offshore platforms. This problem has been studied numerically by the finite element method (Baarholm & Faltinsen 2004) and experimentally (Scolan *et al.* 2006). It was demonstrated that the impact of the wave on a body located above the liquid surface consists of two stages. At the first stage, the liquid impinges onto the body and the wetted surface of the body increases with time. At the second stage, the liquid in the wave under the body is shifted downward, the wetted part of the body decreases and the loads on the body

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are directed toward the liquid. Both numerical and experimental results show that the negative loads on the body impacted by the wave can exceed the positive loads in magnitude and last for a longer time.

Lifting of bodies partly immersed in a finite-depth liquid with allowance for the effect of the bottom was not considered previously. Therefore, the challenge of this work is to study wave flows induced by vertical lifting of a rectangular beam partly immersed in shallow water occupying a rectangular prismatic channel with a horizontal bottom. Such flows are modelled within the framework of the first approximation of the shallow water theory (Friedrichs 1948; Stocker 1957; Henderson 1966; Sturm 2001; Chaudhry 2008) ignoring the effects of friction, liquid viscosity and surface tension. The liquid flow in the domain adjacent to the lower beam surface is calculated analytically. The flow outside this domain is obtained by means of the CABARET (compact accurately boundary-adjusting high-resolution technique) scheme (Goloviznin 2005; Karabasov & Goloviznin 2009), which ensures second-order accuracy on smooth solutions.

2. Formulation of the problem

Let us assume that a rectangular prismatic channel with a horizontal bottom which is filled by an incompressible liquid with a partly immersed rectangular beam of length $2L$ whose width coincides with the channel width and whose upper and lower planes are parallel to the channel bottom (figure 1*a*). We assume that the channel is not limited by length. At the time $t < 0$, the liquid and the beam are at rest. Let us introduce a Cartesian coordinate system whose origin is on the channel bottom, the x axis is directed along the channel and the z axis is directed vertically upward, normal to the channel bottom. Let us assume that the z axis passes through the beam centre; therefore, the beam edges have the coordinates $-L$ and L on the x axis. The initial depth of the liquid h_0 outside the beam is assumed to be much smaller than the beam length

$$h_0/L = \varepsilon \ll 1, \tag{2.1}$$

and the beam is immersed in water to a depth $h_0 - h_1$, where h_1 is the distance between the channel bottom and the lower surface of the beam. The pressure on the free surface of the liquid at $|x| > L$ is considered equal to zero.

At the time $t \geqslant 0$, the beam moves vertically upward with a given acceleration $H''(t) \geqslant 0$, where $H''(t)$ is continuous function, and with a given initial velocity $H'(0) \geqslant 0$. We suppose that in the simulation of arising flow, we can neglect the influence of the liquid viscosity and surface tension. In particular this means that the liquid depth and the channel width are much greater than the thickness of viscous boundary layers on the side walls and bottom of the channel, and also on the wetted surface of the beam. It allows us to model a considered flow as a plane-parallel flow of an ideal incompressible liquid. Let us assume also that the following conditions are satisfied at beam lifting:

$$H'(t)/c_0 = O(\varepsilon), \quad H''(t)/g = O(\varepsilon^2), \tag{2.2a,b}$$

where $c_0 = \sqrt{gh_0}$, g is the free-fall acceleration. It follows from conditions (2.1) and (2.2) that the resultant liquid flow can be described within the framework of the first approximation of the one-dimensional shallow water theory (Friedrichs 1948; Stocker 1957; Henderson 1966; Sturm 2001; Chaudhry 2008), where h_0 and L is characteristic depth and wavelength of the flow.

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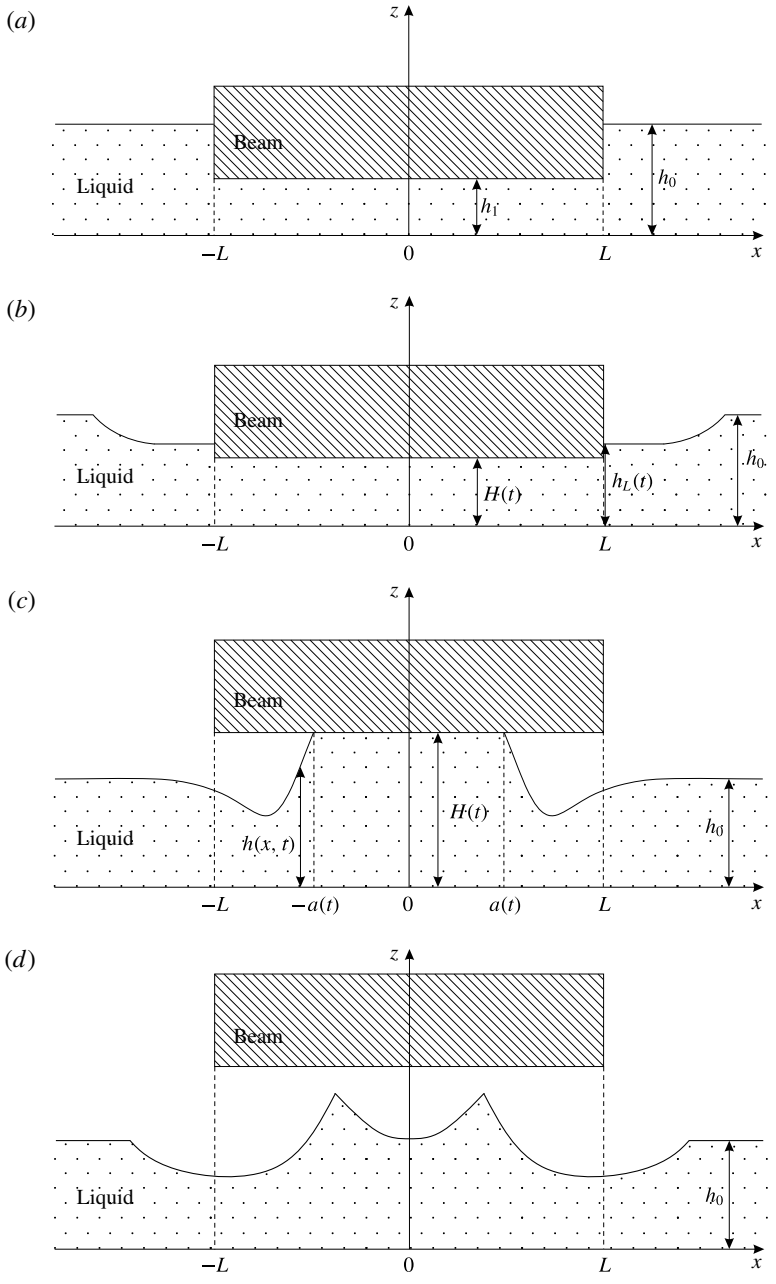


FIGURE 1. Initial position of the beam partly immersed in an undisturbed liquid (*a*). The first stage of the process in the time interval $(0, T_1)$: the edges of the beam lower surface are in the water medium (*b*). The second stage of the process in the time interval (T_1, T_2): the edges of the beam lower surface leave the water medium, while some part of the lower surface of the beam defined by the interval $|x| \leqslant a(t) < L$ still contacts the liquid (*c*). The third stage of the process at $t > T_2$: the beam is separated from the liquid (*d*).

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For the solution of the considered problem we use the standard dimensionless variables

$$x_* = \frac{x}{L}, \quad z_* = \frac{z}{h_0}, \quad t_* = \frac{c_0 t}{L}, \quad h_* = \frac{h}{h_0}, \quad u_* = \frac{u}{c_0}, \quad q_* = \frac{q}{h_0 c_0}, \quad p_* = \frac{p}{\rho c_0^2}, \tag{2.3a–g}$$

where $h(x, t)$, $u(x, t)$, $q = hu$ and ρ are the depth, velocity, flow rate and constant density of the liquid, $p(x, z, t)$ is the pressure in the liquid. To describe the beam lifting, we will introduce the following dimensionless variables

$$H_* = \frac{H}{h_0}, \quad H'_* = \frac{LH'}{h_0 c_0} = \frac{H'}{\varepsilon c_0} \quad \Rightarrow \quad H''_* = \frac{L^2 H''}{h_0 c_0^2} = \frac{H''}{g\varepsilon^2}, \tag{2.4a,b}$$

which are in agreement with conditions (2.2). Input parameters after being written in the dimensionless variables (2.3) become

$$L_* = (h_0)_* = g_* = \rho_* = (c_0)_* = 1, \quad (h_1)_* = h_1/h_0 < 1. \tag{2.5a,b}$$

Shallow water equations written in the dimensionless variables (2.3) have the form

$$h_t + q_x = 0, \quad q_t + (qu + h^2/2)_x = -h\widetilde{p}_x, \tag{2.6a,b}$$

where $\widetilde{p}(x, t) = p(x, h(x, t), t)$ is the pressure on the liquid surface. Here and everywhere further the asterisk at the dimensionless variables is omitted for brevity.

As the beam is lifting, the edges of its lower surface start to leave the water at a certain time instant $T_1 \geqslant 0$. At $T_1 > 0$, the first stage of the process occurs in the time interval $(0, T_1)$. At this stage (figure 1*b*), the lower surface of the beam is completely immersed in the liquid, and the distance $H(t)$ from the lower surface of the beam to the channel bottom is smaller than the depth $h_L(t) = h(\pm(L + 0), t)$, where $L = 1$, of the liquid adjacent to the end faces of the beam, i.e. $H(t) < h_L(t)$. We will assume that the hydrostatic pressure makes the liquid level $h(x, t)$ under the beam increase so that

$$h(x, t) = H(t) \quad \forall\, |x| \leqslant 1. \tag{2.7}$$

As a result, a liquid flow directed toward the beam centre is formed; outside the beam, this flow consists of depression waves of the initial level $h_0 = 1$. The first stage is finalized when the depth of the liquid adjacent to the end faces of the beam becomes smaller than or equal to the height of lifting of the lower surface of the beam, i.e. $h_L(T_1) \leqslant H(T_1)$. The first stage was studied in Kuznetsova & Ostapenko (2015) for the case of a channel of finite length.

Separation of the beam from the liquid occurs at a certain time $T_2 > T_1$. In the time interval (T_1, T_2), the second stage of the process takes place (figure 1*c*): the beam edges leave the water medium, while some part of the lower surface of the beam still contacts the liquid. We assume that this part of the lower surface of the beam is defined by the interval $|x| \leqslant a(t)$, where $a(t) < 1$ at $t \in (T_1, T_2)$. At the third stage (figure 1*d*), at $t > T_2$, the beam becomes separated from the water, then water movement occurs as a liquid flow with free upper surface.

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3. Liquid flow in the domain adjacent to the beam

At the first two stages, at $t \in (0, T_2)$, the flow domain is divided into two parts: subdomain $|x| \leqslant a(t)$ where the liquid is adjacent to the beam and the doubly connected subdomain $|x| > a(t)$ with the free upper boundary of the liquid, where $a(t) = 1$ at $t \in (0, T_1]$ and $a(t) \in (0, 1]$ at $t \in [T_1, T_2)$. In the first subdomain, where the liquid depth is $h = H(t)$, the first equation of system (2.6) yields

$$H'(t) + H(t)u_x = 0. \tag{3.1}$$

Integrating this equation with respect to x and taking into account the boundary condition $u(0, t) = 0$, which follows from the symmetry of the problem about the point $x = 0$, we obtain

$$u(x, t) = -xR(t), \quad R(t) = H'(t)/H(t). \tag{3.2a,b}$$

In particular, at the boundary point $x = a(t)$, we have

$$U(t) = u(a(t), t) = -a(t)R(t) = -a(t)H'(t)/H(t). \tag{3.3}$$

The second equation of system (2.6) is used to determine the pressure $\widetilde{p}(x, t)$ along the wetted lower surface of the beam. Substituting the values $h = H(t)$ and $u = -xR(t)$ into the left-hand side of this equation, we obtain

$$\widetilde{p}_x = -xG(t), \tag{3.4}$$

where

$$G(t) = R^2 - R' = \frac{2(H')^2 - HH''}{H^2}. \tag{3.5}$$

Integrating (3.4) with respect to x, we find

$$\widetilde{p}(x, t) = h_L(t) - H(t) + G(t)(1 - x^2)/2 \tag{3.6}$$

at the first stage and

$$\widetilde{p}(x, t) = G(t)(a^2(t) - x^2)/2 \tag{3.7}$$

at the second stage.

It follows from (3.7) that the pressure $\widetilde{p}(x, t)$ in the liquid–beam contact domain at the second stage satisfies the following conditions. This pressure $\widetilde{p}(x, t) > 0$ if the inequality is satisfied:

$$G > 0 \quad \Leftrightarrow \quad 2(H')^2 > HH''. \tag{3.8}$$

In this case, the beam decelerates lifting of the adjacent liquid. Under the condition

$$G = 0 \quad \Leftrightarrow \quad 2(H')^2 = HH'', \tag{3.9}$$

from which it follows that $p(x, t) = 0$, the beam does not prevent the free flow of the liquid ascending behind it. Finally, under the condition

$$G < 0 \quad \Leftrightarrow \quad 2(H')^2 < HH'', \tag{3.10}$$

which means that $\widetilde{p}(x, t) < 0$, the beam accelerates lifting of the adjacent liquid.

In §§ 4 and 5, we will study the liquid flow outside the beam at the first stage. The flow in the second stage essentially depend on the sign of function (3.5). We will study the case with $G = 0$ in §§ 6 and 7, the case with $G > 0$ in §§ 8 and 9, the case with $G < 0$ in the §§ 10–12. The correctness of constructed solutions will be considered in appendices A and B.

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4. Liquid flow outside the beam at the first stage

At the first stage the flow in the domain $|x| \leqslant 1$ is uniquely determined by formulas (3.2) and (3.6). To characterize the flow outside the beam at this stage (in view of the symmetry of the problem and of the fact that $p_x = 0$ at $|x| > 1$), it is sufficient for a homogeneous system of the shallow water equations

$$h_t + q_x = 0, \quad q_t + (qu + h^2/2)_x = 0 \tag{4.1a,b}$$

to solve the initial boundary value problem in the domain $x > 1, t > 0$ with the initial conditions

$$h(x, 0) = 1, \quad q(x, 0) = 0 \tag{4.2a,b}$$

and the boundary conditions

$$q(1, t) = Q(t) = U(t)H(t) = -H'(t). \tag{4.3}$$

The boundary condition (4.3) follows from the assumption that condition (2.7) is satisfied.

Problem (4.1)–(4.3) is well posed if the flow $h_L(t)$, $u_L(t) = Q(t)/h_L(t)$ on the boundary $x = L = 1$ obtained by solving this problem is subcritical or critical, i.e. satisfies the inequality

$$c_L(t) = \sqrt{h_L(t)} \geqslant |u_L(t)| = \frac{|Q(t)|}{h_L(t)} = \frac{H'(t)}{h_L(t)}. \tag{4.4}$$

In this case, the only characteristic of system (4.1) arrives on the boundary $x = 1$ from the domain $x > 1$. It is the r-characteristic bringing the value of the r-invariant

$$r_L = u_L - 2c_L = r_0 = -2c_0 = -2 \tag{4.5}$$

from the undisturbed part of the liquid.

Formula (4.5) yields the equation for finding the depth h_L:

$$f(h_L) = 2\left(1 - \sqrt{h_L}\right) h_L = |Q|. \tag{4.6}$$

This equation has real solutions if the following inequalities are satisfied:

$$|Q| = H' \leqslant \max_h f(h) = f(h_c) = 8/27 \approx 0.296, \quad h_c = 4/9. \tag{4.7}$$

Using the Cardano formula, we obtain

$$h_L = \frac{1}{9}\left(1 + 2\cos\left(\frac{1}{3}\arccos\left(1 - \frac{27|Q|}{4}\right)\right)\right)^2 \geqslant h_c. \tag{4.8}$$

As the solution of problem (4.1)–(4.3) has a constant value (4.5) of the r-invariant, each s-characteristic leaving the boundary point $(1, \theta)$ extends with the constant speed

$$\lambda_s(\theta) = u_L(\theta) + c_L(\theta) = 3c_L(\theta) - 2, \tag{4.9}$$

and its current parameters remain invariable and equal to boundary values $h_L(\theta)$ and $u_L(\theta)$. In view of formulas (4.3), (4.8) and (4.9), it follows from the condition $H'' \geqslant 0$ that the functions $h_L(\theta)$ and $\lambda_s(\theta)$ decrease at $\theta \in [0, T_1]$. It means that the s-characteristics in problem (4.1)–(4.3) disperse in time (figure 2). Therefore, the solution of this problem is continuous and represents an s-depression wave of the initial liquid of depth $h_0 = 1$.

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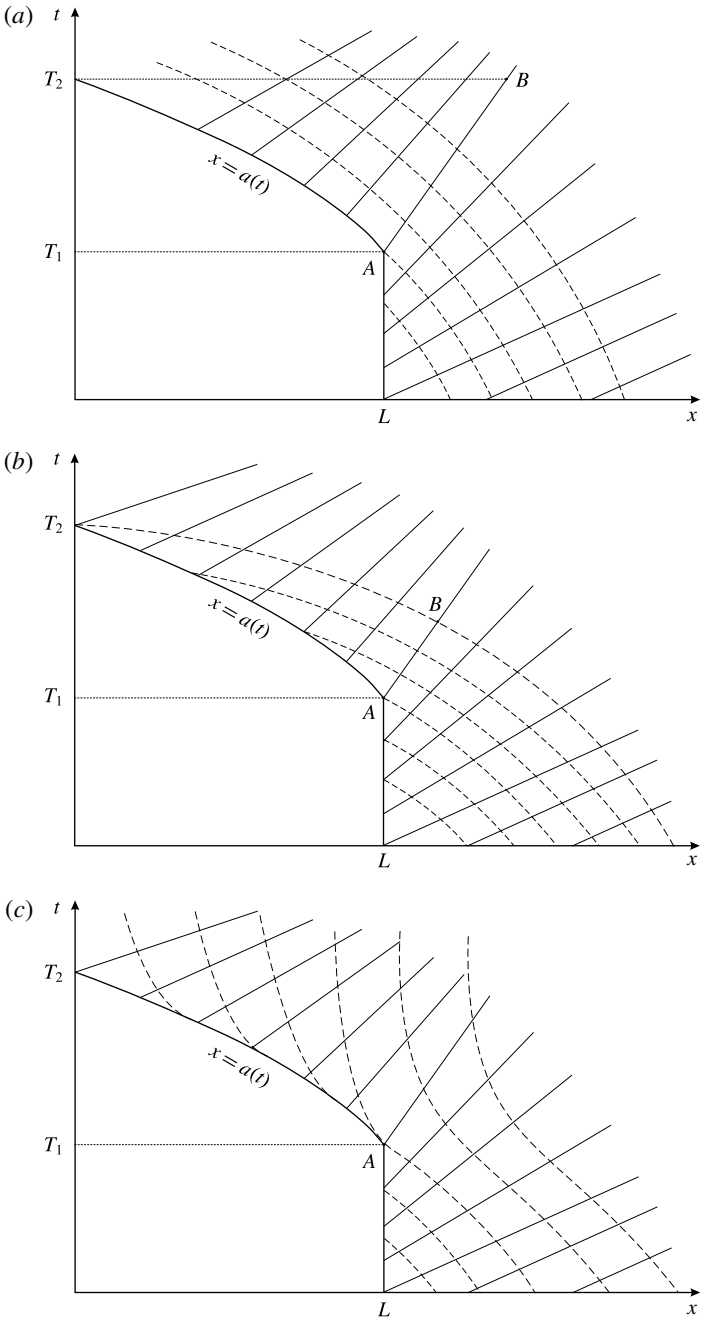


FIGURE 2. Fields of characteristics in the cases $G = 0$ (a), $G > 0$ (b) and $G < 0$ (c) at the first and second stages: the solid straight lines show the s-characteristics; the dashed lines show the r-characteristics.

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5. Lifting of the beam with a finite initial velocity

In solving problem (4.1)–(4.3), in the case with $H'(0) > 0$, the depth of the liquid adjacent to the beam at the point $x = L = 1$ instantaneously (i.e. at the time $t = 0$) decreases to a certain value $h_L(0) < 1$. In this case, for the inequality $T_1 > 0$, the following condition should be valid:

$$H(0) = h_1 < h_L(0). \tag{5.1}$$

Violation of this condition means that (within the shallow water theory) the depth of the liquid adjacent to the beam at the point $x = L + 0$ is smaller than or equal to the depth of the liquid under the beam immediately after the beginning of beam lifting. As a result, the first stage of the flow considered in this work will be absent.

If the initial depth of the liquid under the beam is $h_1 < h_c$, condition (5.1) is satisfied by virtue of inequality (4.8). If the initial depth is $h_1 \geqslant h_c$, condition (5.1) leads to an additional constraint on the initial velocity of beam lifting

$$H'(0) < 2h_1 \left(1 - \sqrt{h_1}\right). \tag{5.2}$$

As the liquid depth $H(t)$ under the beam at the first stage continuously and strongly monotonically increases, while the depth $h_L(t)$ of the liquid adjacent to the beam continuously and monotonically decreases, for the considered problem to be well posed in the case with $h_1 \geqslant h_c$, it is sufficient to satisfy the condition (5.2) at the initial time, which means that the first stage of the flow exists; in the case with $h_1 < h_c$, it is sufficient to satisfy the condition (4.7) in the time interval $(0, T_c)$, where $H(T_c) = h_c$ and $T_c \leqslant T_1$. If the condition (4.7) is violated in the time interval $(0, T_c)$, such flows are not considered in this paper.

If the beam lifting velocity H' is constant and satisfies well-posedness condition (4.7) at $h_1 \leqslant h_c$ or condition (5.2) of the first stage existence at $h_1 \geqslant h_c$, then the flow in the domain $x > 1$ at the first stage is a centred s-depression wave (Stocker 1957). In this case, the final time instant of the first stage is $T_1 = (h_L - h_1)/H'$.

For the description of wave flows at the second stage, first of all, it is necessary to define the law of motion of the boundary line $x = a(t)$ of the beam–liquid contact region. We will study various cases of such motion assuming that solutions of the considered problem are continuous on the boundary line.

6. Motion of the boundary line along the characteristic of the shallow water equations

Let us assume that beam lifting occurs under condition (3.9) at the second stage. Double integration of the second equation of this condition with initial data $H(T_1) = H_1$ and $H'(T_1) = H'_1$ yields

$$H'(t) = \frac{H'_1 H^2(t)}{H_1^2}, \quad H(t) = \frac{H_1^2}{H_1 - H'_1(t - T_1)}, \quad t \in (T_1, T_2),\ T_2 < T_1 + \frac{H_1}{H'_1}. \tag{6.1a,b}$$

Substituting (6.1) into formula (3.2), we obtain the liquid velocity in such a flow:

$$u(x, t) = -\frac{xH'(t)}{H(t)} = -\frac{xH'_1}{H_1 - H'_1(t - T_1)}. \tag{6.2}$$

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The depth $h = H(t)$ defined by the second formula of (6.1) and the velocity u defined by formula (6.2) represent the exact solution of the homogeneous system (4.1) defined in an infinite strip

$$|x| < +\infty, \quad t \in (T_1, T_2), \quad T_2 = T_1 + H_1/H_1'. \tag{6.3a,b}$$

This solution describes the process of filling of the entire space by the liquid at the final time $\Delta t = 1/H_1'$ owing to the inflow of the liquid from infinity.

In the considered simplest case, the beam at the second stage does not influence the flow of the liquid lifting behind it. At the beginning of the second stage, the temporal derivative of the flow depth has a discontinuity on the beam boundary $L = 1$:

$$h_t(L+0, T_1) = h_L'(T_1) \leqslant 0 < h_t(L-0, T_1) = H'(T_1). \tag{6.4}$$

This weak discontinuity of the exact solution lying on the boundary line $x = a(t)$ propagates along the r-characteristic of the homogeneous system (4.1) with the velocity

$$a' = \lambda_r = U - C = -aR - C, \quad C = \sqrt{H}. \tag{6.5}$$

It follows from formula (6.5) that, for obtaining the function $x = a(t)$, it is sufficient to integrate the ordinary differential equation

$$a' + aR + C = 0 \tag{6.6}$$

with the initial condition $a(T_1) = L$. In this case, however, it is more convenient to find the boundary line $x = a(t)$ by using the initial r-invariant value

$$r = U - 2C = -aR - 2C = r_0 = -2, \tag{6.7}$$

brought to the point (L, T_1) from the undisturbed part of the liquid.

From (6.7), taking into account formula (3.2) we obtain

$$a(t) = \frac{2(1-C)}{R} = \frac{2H\left(1-\sqrt{H}\right)}{H'}. \tag{6.8}$$

Differentiating equation (6.8) and taking into account formulas (3.3) and (3.5), we obtain

$$a' = -\frac{2(1-C)R' + 2C'R}{R^2} = -\frac{aR' + 2C'}{R} = \frac{aG}{R} + U - C. \tag{6.9}$$

From here, taking into account conditions (3.9), we find the sought velocity of boundary line propagation (6.5). It follows from (6.8) that separation of the beam from the liquid occurs at the origin of the coordinate system $x = 0$, at the zero length of the contact region $a(T_2) = 0$, at the depth

$$h(0, T_2) = H(T_2) = 1. \tag{6.10}$$

After determining the position of the boundary line by (6.8), the flow in the external domain $x > a(t)$ is found by solving the initial boundary value problem for the system (4.1), with the initial data formed at $t = T_1$ and the boundary condition

$$h(a(t), t) = H(t), \tag{6.11}$$

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defined on the moving boundary $x = a(t)$. In the considered case, the liquid flow at the second stage can be also found by solving the initial boundary value problem for system (4.1) in the domain $x > 0$ with the no-penetration boundary condition

$$u(0, t) = q(0, t) = 0. \tag{6.12}$$

In such a calculation, the boundary line $x = a(t)$ is not allocated specially and is localized by the through computation.

Appendix A shows that the solution constructed in this section satisfies the correctness condition

$$h(x, t) < H(t) \quad \forall\, x \in (a(t), 1)\, \forall\, t \in (T_1, T_2) \tag{6.13}$$

of the considered problem.

7. Results of numerical simulations at $G = 0$

In view of the symmetry of the problem with respect to the origin $x = 0$, the numerical solution of the homogeneous system of the shallow water equations (4.1) is performed in the domain $[1, 4]$ at the first stage, in the domain $[a(t), 4]$ with a moving left boundary at the second stage and in the domain $[0, 4]$ at the third stage. The right boundary of the computational domain at $x = 4$ is subjected to the non-reflecting boundary condition

$$u(4, t) - 2\sqrt{h(4, t)} = r_0 = -2, \tag{7.1}$$

which means preservation of the r-invariant value brought from the undisturbed part $x > 1 + t$ of the liquid. The boundary condition (7.1) allows approximate modelling of the solution of an initial boundary value problem in a semi-infinite spatial interval.

The conditions imposed on the left boundaries of the computational domains are condition (4.3) at the first stage and the no-penetration condition

$$u(0, t) = q(0, t) = 0 \tag{7.2}$$

at the third stage. The conditions on the moving left boundary $x = a(t)$ at the second stage depend on the propagation velocity of this boundary line. In the considered case, when the boundary line $x = a(t)$ propagates along the r-characteristic of the shallow water system (4.1), only condition (6.11) is imposed on this line.

For the numerical solution of system (4.1), we used the explicit second-order CABARET scheme (Goloviznin 2005). The main advantages of this scheme are provided by the following property: in the linear case, this scheme is reversible in time and exact for two different Courant numbers $r = 0.5$ and $r = 1$. For this reason, the scheme possesses unique dissipative and dispersive properties (Karabasov & Goloviznin 2009). In view of the special flux correction the CABARET scheme is monotonic (Ostapenko 2012) at Courant numbers $r \leqslant 0.5$. Therefore, the value $r = 0.5$ is used in the computations described below. Moreover, the CABARET scheme has a compact stencil bounded by the size of one spatio-temporal cell of the difference grid. As a result, its accuracy does not decrease on severely non-uniform difference grids, which were used to construct the numerical algorithm at the second stage of the flow in the neighbourhood of the moving boundary $x = a(t)$ of the computational domain.

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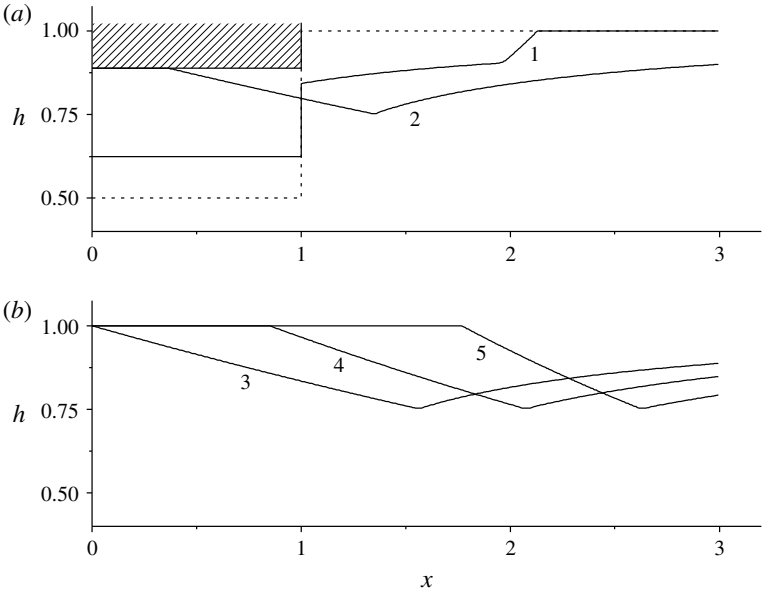


FIGURE 3. Wave profiles for five consecutive time instants for the case of beam lifting according to formula (7.3) with the initial velocity $H'(0) = 0.088$ at the initial depth of the liquid under the beam $h_1 = 0.5$; $t = 1.131$ (1), 2.475 (2), 2.828 (3), 3.677 (4) and 4.596 (5).

In all calculations described further, the beam is lifting in such a manner that condition (4.7) is satisfied at the first stage, which guarantees the correctness of problem (4.1)–(4.3). The dashed curves in the figures show the initial position of the liquid surface at $t = 0$. Let us assume that condition (3.9) holds during the first and second stages. In this case, it follows from (6.1) that the beam lifts by the law

$$h(t) = H(t) = \frac{h_1^2}{h_1 - tH'(0)}. \tag{7.3}$$

Figure 3 shows five consecutive instants of the results of numerical simulations of the considered problem for the case of beam lifting in accordance with law (7.3) for the initial velocity $H'(0) = 0.088$ and the initial depth of the liquid under the beam $h_1 = 0.5$. At the second stage, the calculations were carried out in two ways. In the first case, system (4.1) is solved in the domain $[a(t), 4]$ with a moving left boundary defined by formula (6.8); condition (6.11) is imposed on this boundary. In the second case, taking into account that the beam does not influence the liquid flow at $G = 0$ at the second stage, we solve system (4.1) in the domain $[0, 4]$ with the left boundary condition (7.2). The calculations performed on a sequence of consecutively refined grids show that the difference solutions obtained in these two ways converge to the same exact solution become visually indiscernible on a sufficiently fine grid. It is indirect confirmation of the correctness of formula (6.8) at $G = 0$.

The results of numerical simulations show that the first stage of the considered problem, when the lower surface of the beam is completely immersed in water and the depression wave propagates in the domain $x > 1$ (curve 1 in figure 3*a*), is finalized at the time $T_1 = 1.9$. The second stage, when the edges of the lower surface of the

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beam start to go out from the water (curve 2 in figure 3*a*), is finalized at the time $T_2 = 2.83$, when the beam separates from the liquid (curve 3 in figure 3*b*). At $t > T_2$, liquid lifting that occurred at the second stage leads to the formation of a wave of the liquid level increase (curves 4 and 5 in figure 3*b*) propagating in the positive direction of the x axis.

8. Motion of the boundary line with a subcritical velocity

Let us assume that the boundary line $x = a(t)$ in the second stage propagates with a subcritical velocity

$$\lambda_s = U + C > a' > \lambda_r = U - C \tag{8.1}$$

and the exact solution of the considered problem is continuous in the curvilinear triangle $\widetilde{\Delta} = ABT_2$ (figure 2*b*) bounded by the boundary line $x = a(t)$, the s-characteristic leaving the point $A_1 = (1, T_1)$ and the r-characteristic coming to the point T_2 on the t axis. In this case, the r-characteristics arriving on the boundary line $x = a(t)$ bring the initial value of the r-invariant from the undisturbed part of the liquid. As a result, (6.7) yields explicit formulas for the boundary line (6.8) and for the velocity of its propagation (6.9).

In view of (3.3), (3.5), (6.8) and (6.9), inequalities (8.1) can be rewritten as

$$2CR > aG > 0, \tag{8.2}$$

where the first inequality can be presented in the form

$$C(H')^2 > (1 - C)(2(H')^2 - HH''). \tag{8.3}$$

Since in the considered case $H' > 0$, $H'' \geqslant 0$ and $C = \sqrt{H} \leqslant 1$, inequality (8.3) is satisfied under the condition $H > 4/9$, which, in view of (4.8), follows from the constraint $H > h_L(T_1) \geqslant h_c = 4/9$. Thus, conditions (3.8) and (8.1) are equivalent at the second stage if the considered solution is continuous in the domain $\widetilde{\Delta}$.

If the liquid velocity U on the boundary line $x = a(t)$ is greater than the line propagation velocity (6.9), i.e. if condition (8.1) is taken into account, the velocity U satisfies the inequalities

$$U > a' = -\frac{aR' + 2C'}{R} > U - C, \tag{8.4}$$

the liquid outflows from the domain $|x| \leqslant a(t)$ adjacent to the beam. Taking into account (3.3), (3.5) and (6.8), we can present the first inequality of (8.4) in the form

$$2(1 - C)HH'' > (4 - 5C)(H')^2. \tag{8.5}$$

As $H'' \geqslant 0$ and $C \leqslant 1$ inequality (8.5) is obviously satisfied under the condition $H = C^2 > 16/25$. It follows from here that, if the liquid depth at the end of the first stage satisfies the condition $4/9 < H(T_1) < 16/25$ and the inequality opposite to (8.5) is valid at $t = T_1$, the liquid flows into the domain $|x| \leqslant a(t)$ at the beginning of the second stage.

It follows from (6.8) that, as in the case with $G = 0$, separation of the beam from the liquid occurs at depth (6.10), and at the time instant $T_2 = H^{-1}(1)$, where $H^{-1}(h)$

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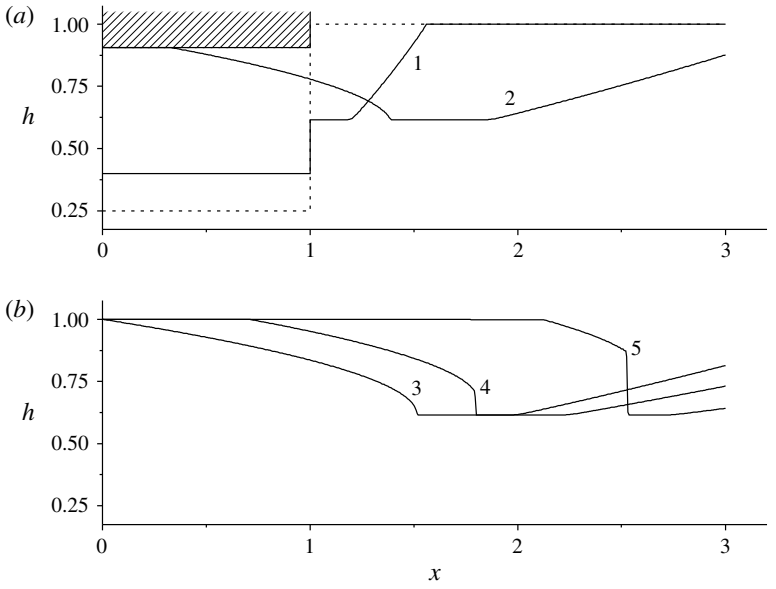


FIGURE 4. Wave profiles for five consecutive time instants for the case of beam lifting with the constant velocity $H'(0) = 0.265$ at the initial depth of the liquid under the beam $h_1 = 0.25$; $t = 0.566$ (1), 2.475 (2), 2.828 (3), 3.536 (4) and 4.950 (5).

is a function inverse to $H(t)$. In particular, if the beam lifts with a constant velocity during the first and second stages, we have

$$T_2 = \frac{1 - h_1}{H'(0)}. \tag{8.6}$$

As inequality (8.5) is satisfied in some time interval (t_2, T_2), where $t_2 \geqslant T_1$, the liquid always outflows from the domain $|x| \leqslant a(t)$ at the end of the second stage.

After determining the position of the boundary line by (6.8), the flow in the external domain $x > a(t)$ under condition (3.8) is found by solving the initial boundary value problem for the system (4.1) with the initial data formed at $t = T_1$ and the boundary condition (6.11). In the considered case with $G > 0$, in contrast to the case with $G = 0$, the validity of correctness conditions (6.13) on the entire segment $[a(t), 1]$ is verified by means of numerical calculations. At the same time, with taking into account that the solution

$$h(a(t), t) = H(t), \quad u(a(t), t) = U(t) \tag{8.7a,b}$$

is known on the boundary line $x = a(t)$, the condition (6.13) can be verified analytically in a certain neighbourhood of the boundary line (see appendix B).

9. Results of numerical simulations at $G > 0$

Figures 4–6 show the results of numerical simulations of the considered problem for the case of beam lifting in such a manner that condition (3.8) is satisfied and the exact solution is continuous in the domain $\widetilde{\Delta}$, which guarantees the well posedness of formula (6.8) for the boundary line $x = a(t)$ propagating with the subcritical

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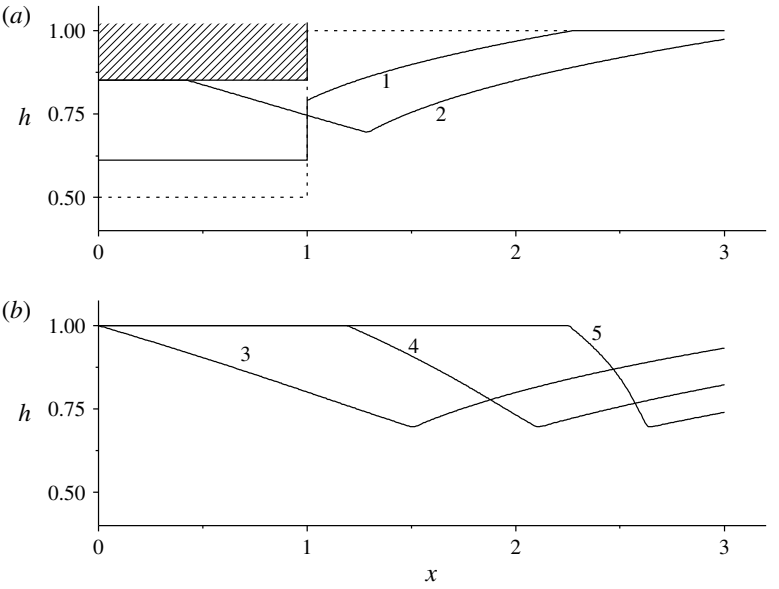


FIGURE 5. Wave profiles for five consecutive time instants for the case of beam lifting with the constant acceleration $H''(0) = 0.138$ at the initial depth of the liquid under the beam $h_1 = 0.5$; $t = 1.273$ (1), 2.263 (2), 2.701 (3), 3.889 (4) and 4.950 (5).

velocity (8.1) defined by (6.9). At the second stage, system (4.1) is solved in the domain $[a(t), 4]$ in two ways. In the first case, the left moving boundary of this domain is defined by formula (6.8), and condition (6.11) is imposed on this boundary. In the second case, two boundary conditions (8.7) are imposed on the boundary line $x = a(t)$; the position of this line is not defined by an exact formula; it is calculated in the course of solving system (4.1) in the domain $[a(t), 4]$. The calculations performed on a sequence of gradually refined grids show that the difference solutions obtained in these two ways converge to the same exact solution and become visually indiscernible on a sufficiently fine grid. It is indirect confirmation of the correctness of formula (6.8) for numerical simulations illustrated in figures 4–6.

In figure 4, the beam is lifting with a constant velocity $H'(0) = 0.265$ at the initial depth of the liquid under the beam $h_1 = 1/4$, which, by virtue of condition (4.7) and the inequality $h_1 < 4/9$, ensures the existence of the first stage of the flow. This stage, when the lower surface of the beam is completely immersed in water and the centred depression waves propagate in the domain $x > 1$ (curve 1 in figure 4*a*), is finalized at the time $T_1 = 1.38$. The second stage, when the edges of the lower surface of the beam start to go out from the water (curve 2 in figure 4*a*), is finalized, as predicted by formula (8.6), at the time $T_2 = 2.83$ when the beam separates from the liquid (curve 3 in figure 4*b*). At $t > T_2$, liquid lifting that occurred at the second stage leads to the formation of a shock wave moving in the positive direction of the x axis (curves 4 and 5 in figure 4*b*).

In figure 5, the beam is lifting with a constant acceleration $H_0'' = 0.138$ and zero initial velocity (which guarantees the existence of the first stage of the flow) at the initial depth $h_1 = 1/2$. The first stage, when the lower surface of the beam is completely immersed in water (curve 1 in figure 5*a*), is finalized at the time $T_1 = 1.68$ s, and the second stage shown by curve 2 in figure 5(*a*) is finalized at the

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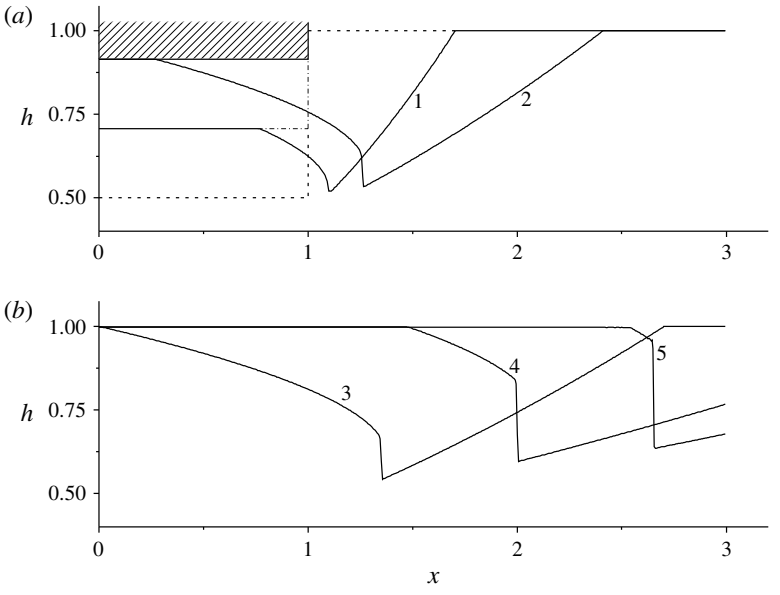


FIGURE 6. Wave profiles for five consecutive time instants for the case of beam lifting with the constant acceleration $H''(0) = 0.293$ at the initial depth of the liquid under the beam $h_1 = 0.5$; $t = 0.707$ (1), 1.414 (2), 1.711 (3), 3.182 (4) and 4.243 (5).

time $T_2 = 2.70$ s (curve 3 in figure 5*b*). At $t > T_2$, liquid lifting that occurred at the second stage leads to the formation of a wave of the liquid level increase (curves 4 and 5 in figure 5*b*).

In figure 6, the beam is lifting from the initial depth $h_1 = 1/2$ with a constant velocity

$$H'(0) = 2h_1\left(1 - \sqrt{h_1}\right) = 1 - 1/\sqrt{2} \approx 0.293, \tag{9.1}$$

which leads to violation of condition (5.2) necessary for the existence of the first stage of the flow. As a result, the second stage shown by curves 1 and 2 in figure 6(*a*) begins immediately after the beginning of beam lifting and ends at the time $T_2 = 1.71$ (curve 3 in figure 6*b*). At the third stage ($t > T_2$), liquid lifting that occurs at the second stage leads to the formation of a shock wave moving in the positive direction of the x axis (curves 4 and 5 in figure 6*b*). Let us note that in figures 4(*b*) and 6(*b*) the strong discontinuities are the shock waves (Friedrichs & Lax 1971; Lax 1972) modelling the hydraulic bores of the real flow.

Remark. From the classical condition (2.1) of the long-wave approximation it follows (Friedrichs 1948) that the spatial derivative of the liquid level should be much less than one ($|h_x| \ll 1$). This contradicts the fact that at the shocks, which describe the hydraulic bores, the spatial derivative of the liquid level is equal to infinity ($|h_x| = \infty$). Nevertheless, a comparison of exact discontinuous solutions of shallow water equations (4.1) with natural observations and with the results of laboratory experiments showed (Stocker 1957; Henderson 1966; Sturm 2001; Chaudhry 2008) that theoretical predictions agree fairly well with observable and experimental results in terms of the flow types, velocities of shock waves (hydraulic

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bores or jumps) and averaged depths of the flow behind their fronts. The paper by Ostapenko (2015) explains this paradox of the shallow water theory. In this paper constitutive conservation laws of the shallow water theory are obtained from the multidimensional integral conservation laws of mass and momentum, which describe the flow of an ideal incompressible fluid above a horizontal bottom. The restrictions on flow parameters arising in this case have the integral form and not directly related to the classic conditions of long-wave approximation (2.1). It proves the applicability of the shallow water theory for modelling of wave flows with hydraulic bores, with areas in which the horizontal velocity of the liquid changes its sign and is comparable with the vertical flow velocity, and also to simulate the initial stage of the formation of a depression waves where the wavelength is less than or comparable to the flow depth.

10. Energy conservation law in the domain adjacent to the beam

Let us assume that beam lifting in some time interval $(t_1, t_2) \subseteq (T_1, T_2)$ at the second stage occurs under condition (3.10). In this case, it follows from formulas (B 5)–(B 7) that the local correctness condition

$$h_x^+(t) = \frac{\Delta_1}{\Delta} = \frac{aHG}{(a' - \lambda_r)(a' - \lambda_s)} < 0 \quad \forall\, t \in (t_1, t_2) \tag{10.1}$$

implies propagation of the boundary line $x = a(t)$ with a supercritical velocity

$$a' < \lambda_r = U - C < \lambda_s = U + C \tag{10.2}$$

and does not imply its propagation with the subcritical velocity (8.1). If the boundary line propagates with the supercritical velocity (10.2), then the r- and s-characteristics of system (4.1) emanate from this line in the domain $x > a(t)$, and none of them arrive on the line $x = a(t)$ (figure 2*c*). This means that the external flow at $x > a(t)$ does not affect the boundary line motion within the framework of the shallow water theory. As a result, for definition of the boundary line position from (3.3) at $G < 0$, it is necessary to use one more additional relation, which should be derived from the fundamental physical law of the liquid flow in the domain $|x| \leqslant a(t)$ adjacent to the beam.

Therefore, we consider the energy conservation law in the two-dimensional spatial domain

$$S(t) = \{(x, z) : |x| \leqslant a(t),\, 0 \leqslant z \leqslant H(t)\}. \tag{10.3}$$

The conservation law will be written in dimensionless variables (2.3)–(2.5) in which the asterisk is omitted for the brevity. Within the long-wave approximation (2.1), (2.2), the total energy E of the liquid in this domain, which is the sum of the kinetic and potential energies, in view of (3.2) and (3.3) is determined by the formula

$$E = \int_{-a}^{a} \left(\int_0^H \left(\frac{u^2}{2} + z \right) \mathrm{d}z \right) \mathrm{d}x = aH \left(\frac{U^2}{3} + H \right). \tag{10.4}$$

The change in the total energy (10.4) results from its flux with the velocity

$$V_E = 2(U - a') \int_0^H \left(\frac{U^2}{2} + z \right) \mathrm{d}z = H(U - a') \left(U^2 + H \right) \tag{10.5}$$

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through the moving lateral boundaries $x = \pm a(t)$ of domain (10.3) and from the pressure work

$$W_1 = -2a' \int_0^H p(a, z, t)\,\mathrm{d}z = -2a' \int_0^H (H - z)\,\mathrm{d}z = -a'H^2, \tag{10.6}$$

on these lateral boundaries, where $p(x, z, t) = H(t) - z$ is the hydrostatic law of pressure. The change in the total energy (10.4) occurs also results from the pressure work $W_2 = -PH'$ on the top moving boundary $z = H(t)$ of domain (10.3). The total pressure force P is calculated by the formula

$$P(t) = \int_{-a(t)}^{a(t)} \widetilde{p}(x, t)\,\mathrm{d}x = \frac{2a^3(t)G(t)}{3}, \tag{10.7}$$

where the surface pressure $\widetilde{p}(x, t)$ is determined by (3.7). As a result, we have

$$W_2 = -PH' = -\frac{2a^3GH'}{3} = \frac{2a^3(R' - R^2)H'}{3}. \tag{10.8}$$

Part of the total energy (10.4) transforms with some speed ΔE to the energy of liquid vortices concentrated in small areas $|x \pm a(t)| < \delta$ of the boundary lines $x = \pm a(t)$ whose sizes 2δ are neglected within the long-wave approximation. Since the shallow water equations (4.1) represent the hyperbolic system of conservation laws with convex expansion (Friedrichs & Lax 1971; Lax 1972), an energy transition ΔE is similar to the losses of the total energy to shock waves in the classical shallow water theory (Stocker 1957; Ostapenko 2007; Chaudhry 2008). Taking this into consideration, we obtain the total energy conservation law in domain (10.3) from (10.4)–(10.8):

$$\frac{\mathrm{d}E}{\mathrm{d}t} + V_E + \Delta E = W_1 + W_2. \tag{10.9}$$

After conversions, (10.9) can be written in the form

$$\Delta E = H^2(U - a'), \tag{10.10}$$

which will be used for obtaining an additional relation that allows one to determine the velocity of the boundary line $x = a(t)$ under condition (3.10).

11. Motion of the boundary line under the condition $G < 0$

As the basic criterion that allows one to determine the position of the boundary line $x = a(t)$ under condition (3.10), we use the assumption that the minimum possible part of the total energy of the liquid transforms to the vortex energy ΔE when the liquid leaves domain (10.3). As a result we have from (10.10) under constraint (10.2) that the liquid flow occurs under the condition

$$\Delta E = \inf_{a' < \lambda_r} H^2(U - a') = H^2C = H^{5/2}. \tag{11.1}$$

Hence the boundary line $x = a(t)$ propagates with the critical velocity (6.5). The acceleration of such propagation, taking into account (3.3), (3.5) and (6.5), is calculated by the formula

$$a'' = (U - C)' = -(aR)' - C' = a(R^2 - R') + RC - C' = aG + C'. \tag{11.2}$$

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If the point $(a(t), t)$ of the boundary line $x = a(t)$ lies on the r-characteristic of system (4.1), where the r-invariant has a constant value r_1, the liquid and sound velocities $U = U(a(t), t)$ and $C = C(a(t), t)$ at this point satisfy the condition $U = 2C + r_1$, whereas the r-characteristic velocity at this point is calculated by the formula $\lambda_r = U - C = C + r_1$. From here, the r-characteristic acceleration on the boundary line is

$$\lambda_r' = (C + r_1)' = C'. \tag{11.3}$$

It follows from (11.2) and (11.3) that the boundary line propagating with the r-critical velocity at $G < 0$ is the envelope of the r-characteristics emanating from this line in the external domain $x > a(t)$ (figure 2*c*). This means that the flow in the domain $x > a(t)$ does not affect the liquid flow in the domain $|x| \leqslant a(t)$ adjacent to the beam. Formula (6.5) yields the ordinary differential equation (6.6) for the boundary line $x = a(t)$. Integrating this equation in the time interval (t_1, t_2), we obtain

$$a(t) = \frac{1}{H(t)} \left(a(t_1)H(t_1) - \int_{t_1}^{t} H^{3/2}(\theta)\, \mathrm{d}\theta \right), \quad t \in (t_1, t_2). \tag{11.4}$$

It follows from inequalities (10.1) and (10.2) that

$$\lim_{a' \to \lambda_r - 0} h_x^+(t) = -\infty. \tag{11.5}$$

Which means that, in the case of the boundary line (11.4) propagating with the critical velocity (6.5), the free liquid surface comes up to the bottom beam surface at a right angle within the long-wave approximation.

12. Results of numerical simulations at $G < 0$ at the beginning of the second stage

Figures 7–9 show the results of numerical simulations of the considered problem for the case of beam lifting with a constant acceleration and zero initial velocity in such a manner that condition (3.10) is satisfied at the beginning of the second stage. In figure 7, the beam lifts with the acceleration $H'' = 0.75$ from the initial depth $h_1 = 0.975$, which can be approximately interpreted as the beam lifting from the surface of the undisturbed liquid. In this case, inequality $G < 0$ is satisfied during the entire second stage (curves 1 and 2 in figure 7*a*), which ends at the time instant $T_2 = 0.905$, when the maximum liquid lifting $h(0, T_2) = 1.28$ is reached (curve 3 in figure 7*b*).

In figure 8, the beam lifts with acceleration $H'' = 2.5$ from the initial depth $h_1 = 0.75$. In this case, inequality $G < 0$ is satisfied only at the beginning of the second stage (curve 1 in figure 8*a*) in the time interval $(T_1, t_2) \subseteq (T_1, T_2)$, where $T_1 = 0.085$, $t_2 = 0.453$, $a(t_2) = 0.468$ and $T_2 = 0.792$. At the time instant t_2 (curve 2 in figure 8*a*), the function G changes its sign, and beam lifting in the time interval (t_2, T_2) occurs under the inequality $G > 0$ (curve 3 in figure 8*a*). At the end of the second stage (curve 4 in figure 8*b*), the maximum liquid lifting $h(0, T_2) = 1.53$ is reached.

In figure 9, the beam lifts with the acceleration $H'' = 7.5$ from the initial depth $h_1 = 0.975$. In this case, inequality $G < 0$ is satisfied only at the beginning of the second stage (curve 1 in figure 9*a*) in the time interval $(T_1, t_2) \subseteq (T_1, T_2)$, where $T_1 = 0.003$, $t_2 = 0.294$, $a(t_2) = 0.497$ and $T_2 = 0.587$. At the time instant t_2 (curve 2 in figure 9*a*), the function G changes its sign, and beam lifting in the time interval

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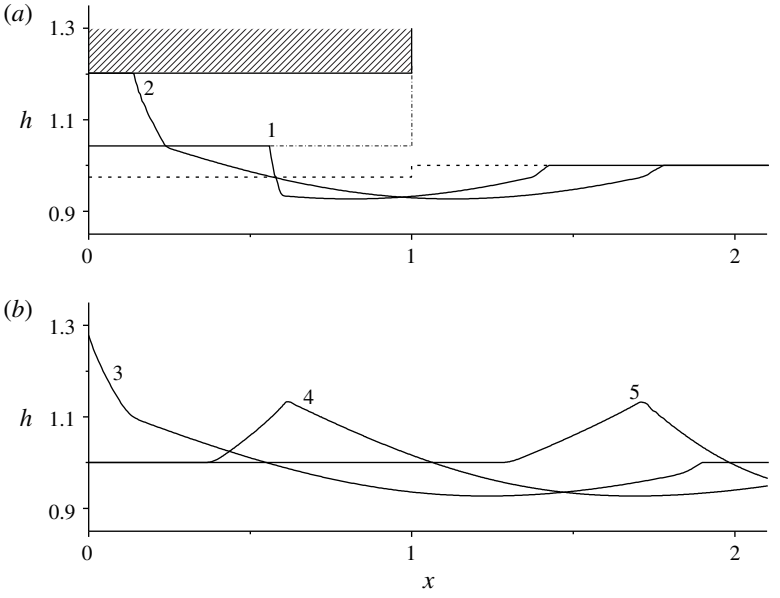


FIGURE 7. Wave profiles for five consecutive time instants for the case of beam lifting with the constant acceleration $H_0'' = 0.75$ and zero initial velocity at the initial depth of the liquid under the beam $h_1 = 0.975$; $t = 0.424$ (1), 0.778 (2), 0.905 (3), 1.414 (4) and 2.333 (5).

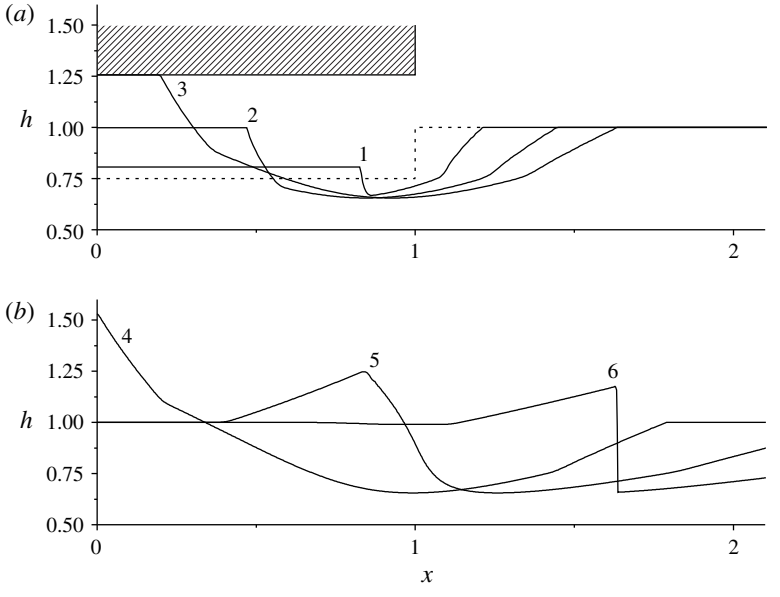


FIGURE 8. Wave profiles for six consecutive time instants for the case of beam lifting with the constant acceleration $H_0'' = 2.5$ and zero initial velocity at the initial depth of the liquid under the beam $h_1 = 0.75$; $t = 0.212$ (1), 0.453 (2), 0.636 (3), 0.792 (4), 1.414 (5) and 2.121 (6).

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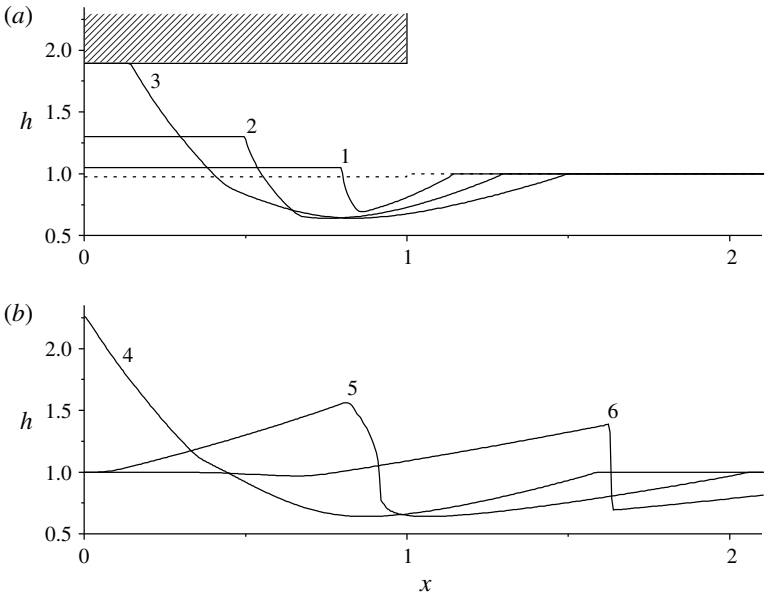


FIGURE 9. Wave profiles for six consecutive time instants for the case of beam lifting with the constant acceleration $H_0'' = 7.5$ and zero initial velocity at the initial depth of the liquid under the beam $h_1 = 0.975$; $t = 0.141$ (1), 0.294 (2), 0.495 (3), 0.587 (4), 1.061 (5) and 1.697 (6).

(t_2, T_2) occurs under the inequality $G > 0$ (curve 3 in figure 9*a*). At the end of the second stage (curve 4 in figure 9*b*), the maximum liquid lifting $h(0, T_2) = 2.27$ is reached. Let us note that in figures 8(*b*) and 9(*b*) the strong discontinuities are the shock waves modelling the hydraulic bores of the real flow.

At the second stage, two boundary conditions (8.7) for system (4.1) are imposed on the left moving boundary of the computational domain $[a(t), 4]$. If inequality $G < 0$ is satisfied in the time interval $(T_1, t_2) \subseteq (T_1, T_2)$, setting of the two boundary conditions on boundary line (11.4) is related to the fact that both characteristics pass from this line to the external domain $x > a(t)$. If the inequality $G > 0$ is satisfied in the time interval $(t_2, T_2) \subset (T_1, T_2)$, in view that the inequality $G < 0$ is satisfied in the previous time interval (T_1, t_2), the boundary line $x = a(t)$ at $t \in (t_2, T_2)$ is described by the equation $a(t) = -(2C + r_1)/R$ derived from the formula $U - 2C = -aR - 2C = r_1$, where r_1 is the r-invariant value brought along the r-characteristic coming from the domain $x > a(t)$ to the boundary line. Thus, the line can propagate both with the r-critical (6.5) and subcritical (8.1) velocities. It follows from formulas (11.2) and (11.3) that the boundary line $x = a(t)$ in the first case is the envelope of the r-characteristics coming to this line from the external domain $x > a(t)$. Unlike the situation studied in §§ 8 and 9, the r-invariant value r_1 in the considered case is unknown in advance. Therefore the only way to solve the problem is where the two boundary conditions (8.7) are imposed on the boundary line $x = a(t)$, and the position of this line is calculated in the course of solving system (4.1) in the domain $[a(t), 4]$.

The results of numerical calculations presented in figures 7–9 show that the maximum lifting of the liquid behind the beam can be appreciably greater than the initial liquid depth $h_0 = 1$ if condition (3.10) is satisfied at the beginning of the second stage. Thus, an increase in the acceleration of beam lifting leads to an increase in

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the maximum height of liquid lifting. The numerical calculations performed under the condition that the beam lifts with acceleration $H'' > 7.5$ show a further increasing of the maximum lifting of the liquid.

13. Conclusion

As we had to use the heuristic assumption (Korobkin 2013) that the velocity of the liquid–body contact boundary is proportional to the local velocity of the liquid flow in the theoretical modelling of lifting of a flat body from the free boundary of an infinitely deep liquid, the main challenge of this work was to obtain an exact law of motion of this boundary line within the framework of the long-wave approximation in solving the problem of lifting of a rectangular beam partly immersed in shallow water. To justify the initial motion of the liquid behind the beam during its lifting from the liquid surface, it is necessary to take into account the forces of adhesion of liquid particles with the solid body surface, which were neglected in this work. Therefore, to induce this initial motion of the liquid, we introduced the first stage at which the lower surface of the beam is immersed in the liquid to a finite depth.

In analysing the solution of the problem, we obtained constraint (4.7) on the beam lifting velocity at the first stage. If this condition is violated, initial boundary value problem (4.1)–(4.3) for the liquid flow in the external region $x > L$ becomes incorrect; such flows are not considered in this paper. We also obtained another constraint (5.2) on the initial velocity of beam lifting. If this condition is violated, the edges of the lower surface of the beam instantaneously leave the water medium in accordance with the shallow water theory (figure 6*a*). The latter does not contradict the requirement of the formation of initial motion of the liquid behind the beam because the shallow water model predicts that a horizontal flow of the liquid instantaneously arises for all finite initial velocities of beam lifting. This flow is directed toward the beam centre and induces vertical lifting of the liquid behind the beam.

The propagation of the beam–liquid boundary line $x = a(t)$ at the second stage is calculated without any heuristic assumptions about the relationship between this boundary line velocity and the local velocity of the liquid flow. Thus, the boundary line motion essentially depends on the sign of the function G defined by formula (3.5). Under the condition $G = 0$, the beam–liquid boundary line $x = a(t)$ propagates along the r-characteristic of the shallow water system (4.1). Under the condition $G > 0$, the boundary line can propagate with the r-critical (6.5) or subcritical (8.1) velocity. In the first case, it is the envelope of the r-characteristics coming to this line from the external domain $x > a(t)$. Under the condition $G < 0$, the boundary line propagates with the r-critical velocity (10.2) and is the envelope of the r-characteristics emanating from this line in the external domain $x > a(t)$.

If the inequality $G \geqslant 0$ is satisfied and the exact solution is continuous in the second stage, the boundary line $x = a(t)$ is defined by the exact formula (6.8). In this case, when the beam decelerates lifting of the adjacent liquid at $G > 0$ or does not prevent its free flow at $G = 0$, the maximum liquid depth behind the beam coincides with the initial liquid depth outside the beam (figures 3–6). If the inequality $G < 0$ is satisfied in some time interval $(t_1, t_2) \subseteq (T_1, T_2)$, the boundary line in this time interval is defined by formula (11.4). If the inequality $G > 0$ is satisfied in some time interval $(t_3, t_4) \subseteq (T_1, T_2)$ and the opposite inequality $G < 0$ is satisfied in some previous time interval $(t_1, t_2) \subseteq (T_1, T_2)$, where $t_2 < t_3$, it is impossible to obtain an exact formula for the boundary line. In this case, the position of the boundary line $x = a(t)$ is defined numerically in the course of solving system (4.1) in the domain $[a(t), 4L]$. If the

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inequality $G < 0$ is satisfied in some time interval $(T_1, t_2) \subseteq (T_1, T_2)$ in the beginning of the second stage, whereby the beam accelerates lifting of the adjacent liquid in this time interval, the maximum lifting of the liquid behind the beam can be appreciably greater than the initial liquid depth outside the beam (figures 7–9).

It is further planned to compare the solutions obtained in this work with results of laboratory experiments and with numerical simulations performed on the basis of hydrodynamic models of higher levels.

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Appendix A. Well posedness of the solution at $G = 0$

Let us show that the solution constructed in § 6 satisfies the correctness condition (6.13) of the considered problem. If beam lifting at the second stage occurs under condition (3.9), the r-invariant of system (4.1) is constant and equal to the initial value $r_0 = -2$ in some right neighbourhood

$$U_\delta^+ = \{(x, t) : a(t) < x < a(t) + \delta,\ T_1 < t < T_2\} \tag{A 1}$$

of the boundary line (6.8). By virtue of this fact, the solution of system (4.1) is continuous in neighbourhood (A 1), and each s-characteristic of the solution leaving the boundary line at $\theta \in (T_1, T_2)$ propagates with a constant velocity

$$\lambda_s(\theta) = U(\theta) + C(\theta) = 3C(\theta) - 2 \tag{A 2}$$

in this vicinity.

Let us show that neighbourhood (A 1) contains a curvilinear triangle

$$\Delta = \{(x, t) : a(t) < x < \lambda_s(T_1)t + 1,\ T_1 < t < T_2\} \tag{A 3}$$

limited by the boundary line $x = a(t)$, the s-characteristic leaving the point $A_1 = (1, T_1)$ and the level line $t = T_2$ (triangle ABT_2 in figure 2a). For the inclusion $\Delta \subset U_\delta^+$, it is sufficient to prove that the exact solution is continuous in domain (A 3). As the function $C(\theta)$ at $\theta \in (T_1, T_2)$ is monotonously increasing, the s-characteristics (A 2) leaving the boundary line at $\theta \in (T_1, T_2)$ eventually converge, forming an s-compression wave. The evolution of this compression wave leads to a gradient catastrophe, which results in the formation of a shock wave describing the hydraulic bore of a real flow. For the solution continuity in the domain Δ, it is sufficient that the point (x_c, t_c) of this gradient catastrophe lies in figure 2(a) above the level line $t = T_2$, i.e. that the inequality

$$t_c > T_2 \tag{A 4}$$

is satisfied.

For determining the coordinates of the point (x_c, t_c), we consider two rectilinear s-characteristics

$$x = \lambda_s(\theta)(t - \theta) - a(\theta), \quad x = \lambda_s(\theta_\tau)(t - \theta_\tau) - a(\theta_\tau), \tag{A 5a,b}$$

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leaving the boundary line $x = a(t)$ in time instants $\theta, \theta_\tau \in [T_1, T_2]$, where $\theta_\tau = \theta + \tau$. As these characteristics cross at a point with the coordinates

$$x_\theta^\tau = \lambda_s(\theta) t_\theta^\tau - a(\theta), \quad t_\theta^\tau = \frac{\lambda_s(\theta_\tau)\theta_\tau - \lambda_s(\theta)\theta + a(\theta_\tau) - a(\theta)}{\lambda_s(\theta_\tau) - \lambda_s(\theta)}, \tag{A 6a,b}$$

the coordinates of the point (x_c, t_c) are found by the formulas

$$x_c = x_c(\theta_c) = \lambda_s(\theta_c) t_c - a(\theta_c), \tag{A 7}$$

$$t_c = \min_{\theta \in [T_1, T_2]} \lim_{\tau \to 0} t_\theta^\tau = \min_{\theta \in [T_1, T_2]} \psi(\theta) = \psi(\theta_c) \tag{A 8}$$

where $\theta_c \in [T_1, T_2]$,

$$\psi(\theta) = \frac{(\lambda_s(\theta)\theta)' - a'(\theta)}{\lambda_s'(\theta)} = \theta + \frac{\lambda_s(\theta) - a'(\theta)}{\lambda_s'(\theta)}. \tag{A 9}$$

Taking into account (6.1), (6.5) and (A 2), we have

$$\psi(\theta) = \theta + \frac{\lambda_s(\theta) - \lambda_r(\theta)}{3C'(\theta)} = \theta + \frac{4H(\theta)}{3H'(\theta)} = \frac{4}{3}\left(\frac{H_1}{H_1'} + T_1\right) - \frac{\theta}{3}. \tag{A 10}$$

As $\theta_c \leqslant T_2$ and function (A 10) monotonously decreases, the validity of condition (A 4) follows from the inequalities

$$t_c = \psi(\theta_c) \geqslant \psi(T_2) = \frac{4}{3}\left(\frac{H_1}{H_1'} + T_1\right) - \frac{T_2}{3} > T_2. \tag{A 11}$$

The last inequality is equivalent to the condition of existence of solution (6.1). Thus, the continuity of the exact solution in domain (A 3) is proved. As a result, the flow parameters are constant on each s-characteristic lying in domain (A 3), and the liquid depth (strictly monotonously increasing on the boundary line) satisfies the condition

$$h_x(x, t) < 0 \quad \forall\, (x, t) \in \Delta, \tag{A 12}$$

from which the correctness condition (6.13) follows.

Appendix B. About the well posedness of the solution at $G > 0$

If the boundary line $x = a(t)$ propagates with the subcritical velocity (8.1), the solution (8.7) known on this line can be unambiguously continued into the external domain $x > a(t)$. In particular, it is possible to determine one-sided temporal and spatial derivatives of the functions $h(x, t)$ and $u(x, t)$, for which the following abbreviated notations are used:

$$f_t^+ = f_t(a(t), t + 0), \quad f_x^+ = f_x(a(t) + 0, t). \tag{B 1a,b}$$

Differentiating relations (8.7) with respect to t at $x = a(t) + 0$, we obtain

$$h_t^+ + a' h_x^+ = H', \quad u_t^+ + a' u_x^+ = U'. \tag{B 2a,b}$$

Eliminating the temporal derivatives h_t^+ and u_t^+ from system (B 2) by using the first equation of system (4.1) and the differential corollary

$$u_t + u u_x + h_x = 0 \tag{B 3}$$

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of this system, we obtain the following algebraic equations for determining the spatial derivatives:

$$(a' - U)h_x^+ - Hu_x^+ = H', \quad -h_x^+ + (a' - U)u_x^+ = U'. \tag{B 4a,b}$$

Solving system (B 4) by Cramer's rule and taking into account (3.3) and (3.5), we obtain the expressions for the spatial derivatives

$$h_x^+ = \frac{\Delta_1}{\Delta}, \quad u_x^+ = \frac{\Delta_2}{\Delta}, \tag{B 5a,b}$$

where the determinants are calculated by the formulas

$$\Delta = (a' - U)^2 - C^2 = (a' - \lambda_r)(a' - \lambda_s), \tag{B 6}$$

$$\begin{aligned} \Delta_1 &= (a' - U)H' + HU' = (a' + aR)H' - H(aR)' \\ &= a(RH' - HR') = aH(R^2 - R') = aHG, \end{aligned} \tag{B 7}$$

$$\Delta_2 = (a' - U)U' + H' = H' - (a' + aR)(aR)'. \tag{B 8}$$

From (B 5)–(B 7), taking into account the inequality $G > 0$, we obtain

$$\Delta(t) < 0, \quad \Delta_1(t) > 0 \quad \Rightarrow \quad h_x^+(t) < 0 \quad \forall\, t \in (T_1, T_2), \tag{B 9a,b}$$

which guarantees the well posedness (6.13) of the resultant solution in a certain right neighbourhood $[a(t), a(t) + \delta]$ of the boundary line.

Using formulas (6.9) and (8.1), we obtain

$$a' - \lambda_r = a' - U + C = \frac{aG}{R}, \quad a' - \lambda_s = a' - U - C = \frac{aG - 2CR}{R}. \tag{B 10a,b}$$

Substituting expression (B 10) in (B 6), we obtain from first (B 5) the following explicit formula for the spatial depth derivative:

$$h_x^+ = \frac{H'}{a' - \lambda_s} = \frac{RH'}{aG - 2CR} = \frac{(H')^3}{2[(1 - C)(2(H')^2 - HH'') - C(H')^2]}. \tag{B 11}$$

Within the framework of the shallow water theory (B 11) defines the angle between the free surface of the liquid and the lower surface of the lifting beam. In particular, if the beam lifts with a constant velocity, we have

$$h_x^+ = \frac{H'}{2(2 - 3C)}. \tag{B 12}$$

In the limiting case $G \to 0$ $(a' \to \lambda_r)$, when the boundary line propagates along the r-characteristic, we obtain

$$h_x^+ = \frac{HR}{\lambda_r - \lambda_s} = -\frac{H'}{2\sqrt{H}}. \tag{B 13}$$

In the end of the second stage from (B 12) and (B 13) we have $h_x^+(T_2) = -H'(T_2)/2$.

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