

FIGURE 3

We have $h^2 + s^2 = a^2$ and $h^2 + (a - s)^2 = b^2$ which yields $h = \frac{b}{2a}\sqrt{4a^2 - b^2}$ and $s = \frac{1}{2a}(2a^2 - b^2)$.

Substituting $a = 5$, $b = \sqrt{10}$, we obtain $h = 3$, $s = 4$, giving $3 : 4 : 5$ as the ratio of the lengths of the sides of the right-angle triangle DKC.

It follows that the central angle of the truncated octahedron, DCE, is the smallest angle of the $3 : 4 : 5$ triangle.

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109.13 Apollonius' theorem for *n*-gons

Let ABC be a triangle and M be the midpoint of BC as shown.

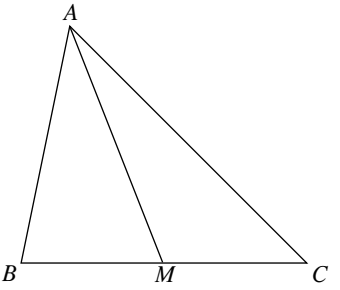


FIGURE 1

Apollonius' theorem states that $|AB|^2 + |AC|^2 = 2\left(|AM|^2 + |AM|^2\right)$. Substituting $|BM| = \frac{1}{2}|BC|$, we can derive an alternative form of Apollonius' theorem:

$$4|AM|^2 - 2\left(|AB|^2 + |AC|^2\right) + |BC|^2 = 0.$$

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Suppose now that $A_1, \ldots, A_n$ are distinct points on a plane forming an n-gon taken either to be convex or concave. Let M_1 be the midpoint of $A_{i+1}A_{i+2}$ for $i = 1, 2, \ldots, n-2$. Interpreting A_p as A_{p-n} when $p > n$, define $e_i = |A_iA_{i+1}|$ for $i = 1, 2, \ldots, n$; $m_i = |A_1M_i|$ for $i = 1, 2, \ldots, n-2$; and $d_i = |A_1A_{i+2}|$ for $i = 1, 2, \ldots, n-3$. Then $e_1, \ldots, e_n$ represent the lengths of the n-gon's edges; $m_1, \ldots, m_{n-2}$ represent the lengths of the medians connecting A_1 to the midpoints of its opposite edges; and $d_1, \ldots, d_{n-3}$ represent the lengths of the diagonals connecting A_1 to its opposite vertices. Using this notation, we can amend Figure 1 accordingly (see Figure 2) and express the alternative form of Apollonius' theorem as

$$4m_1^2 - 2\left(e_1^2 + e_3^2\right) + e_2^2 = 0.$$

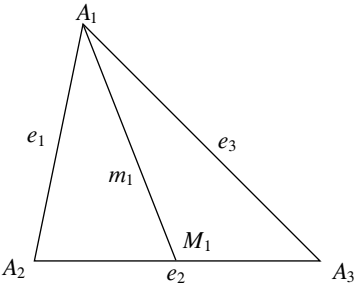


FIGURE 2

We now generalise this formula to n-gons.

Theorem: (Apollonius' theorem for n-gons) Let $A_1, \ldots, A_n$ be distinct points on a plane forming an n-gon. Then

$$4\sum_{i=1}^{n-2}(-1)^{i-1}m_i^2 - 2\left(e_1^2 + (-1)^{n-1}e_n^2\right) + \sum_{i=2}^{n-1}(-1)^i e_i^2 = 0.$$

Proof: Suppose the formula is true for $n = k$ and consider the case $n = k + 1$ as shown.

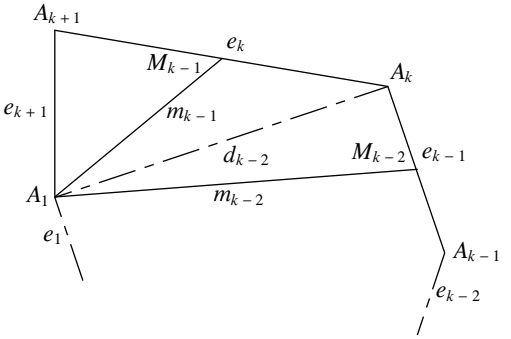


FIGURE 3

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Applying the inductive hypothesis to the k-gon $A_1, \ldots, A_k$ gives

$$4\sum_{i=1}^{k-2}(-1)^{i-1}m_i^2 - 2\left(e_1^2 + (-1)^{k-1}d_{k-2}^2\right) + \sum_{i=2}^{k-1}(-1)^i e_i^2 = 0, \qquad (1)$$

while applying Apollonius' theorem to the triangle $A_1A_kA_{k+1}$ gives

$$4m_{k-1}^2 - 2\left(d_{k-2}^2 + e_{k+1}^2\right) + e_k^2 = 0. \qquad (2)$$

Then

$$[\text{equation (1)}] + (-1)^k \times [\text{equation (2)}]$$

gives

$$4\sum_{i=1}^{k-1}(-1)^{i-1}m_i^2 - 2\left(e_1^2 + (-1)^k e_{k+1}^2\right) + \sum_{i}^{k}(-1)^i e_i^2 = 0.$$

Therefore, the formula holds also for $n = k + 1$ and hence, by mathematical induction, the formula holds for $n \geqslant 3$.

An interesting corollary of Apollonius' theorem is that for any given triangle, the ratio of the sum of the squares of its edges to the sum of the squares of its medians is $4 : 3$ [1]. We now generalise this result to n-gons when n is odd.

Theorem: Let $A_1, \ldots, A_n$ be distinct points on a plane forming an n-gon. Interpreting M_p as M_{p-n} when $p > n$, define

$$S_i = |A_iM_i|^2 - |A_iM_{i+1}|^2 + |A_iM_{i+2}|^2 - \ldots + (-1)^{n-1}|A_iM_{i+n-1}|^2$$

for $i = 1, 2, \ldots, n$.

Then when is odd,

$$4\sum_{i=1}^{n}S_i = 3\sum_{i=1}^{n}e_i^2.$$

Proof: Let n be odd and interpret e_0 as e_n and e_p as e_{p-2} when $p > n$. Then by Apollonius' theorem for n-gons,

$$4S_i - 2\left(e_i^2 + e_{i-1}^2\right) + e_{i+1}^2 - e_{i+2}^2 + e_{i+3}^2 - e_{i+4}^2 + \ldots + e_{i+n-2}^2 = 0$$

for $i = 1, 2, \ldots, n$.

Therefore,

$$\begin{aligned}
&4S_1 - 2\left(e_1^2 + e_n^2\right) + e_2^2 - e_3^2 + e_4^2 - e_5^2 + \ldots + e_{n-1}^2 \\
&+ 4S_2 - 2\left(e_2^2 + e_1^2\right) + e_3^2 - e_4^2 + e_5^2 - e_6^2 + \ldots + e_n^2 \\
&+ 4S_3 - 2\left(e_3^2 + e_2^2\right) + e_4^2 - e_5^2 + e_6^2 - e_7^2 + \ldots + e_1^2 \\
&+ \ldots \\
&+ 4S_{n-2} - 2\left(e_{n-2}^2 + e_{n-3}^2\right) + e_{n-1}^2 - e_n^2 + e_1^2 - e_2^2 + \ldots + e_{n-4}^2 \\
&+ 4S_{n-1} - 2\left(e_{n-1}^2 + e_{n-2}^2\right) + e_n^2 - e_1^2 + e_2^2 - e_3^2 + \ldots + e_{n-3}^2
\end{aligned}$$

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$$+ 4S_n - 2\left(e_n^2 + e_{n-1}^2\right) + e_1^2 - e_2^2 + e_3^2 - e_4^2 + \ \ldots + e_{n-2}^2$$
$$= 0$$

which, by telescopic simplification, implies

$$4\sum_{i=1}^{n} S_i = 3\sum_{i=1}^{n} e_i^2.$$

Remark: When n is even, it is easy to show by a similar argument that

$$\sum_{i=1}^{n} S_i = 0.$$

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Reference

1. E. W. Weisstein, (2023). Triangle Median. Wolfram Mathworld. Available from: https://mathworld.wolfram.com/TriangleMedian.html [Accessed 10th July, 2023]

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109.14 Extensions of a Geometric Inequality

Let I denote the incentre of a triangle $A_1A_2A_3$. Let R be the cicumradius and r the inradius.Then

$$3Rr^5 \leqslant (IA_1 \times IA_2 \times IA_3)^2.$$

This result appears in [1] and can be improved.

Theorem 1: $IA_1 \times IA_2 \times IA_3 = 4Rr^2$.

Proof: From the definition of the incircle, it is easy to deduce that

$$\sin\frac{A_1}{2} = \frac{r}{IA_1}, \qquad \sin\frac{A_2}{2} = \frac{r}{IA_2}, \qquad \sin\frac{A_3}{2} = \frac{r}{IA_3},$$

so

$$IA_1 \times IA_2 \times IA_3 = \frac{r^3}{\sin\frac{1}{2}A_3 \sin\frac{1}{2}A_2 \sin\frac{1}{2}A_3}.$$

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