Proceedings of the Edinburgh Mathematical Society (2017) **60**, 387–390
DOI:10.1017/S0013091516000201

ON GROUPS WITH SLOW INTERSECTION GROWTH

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(Received 15 December 2014)

Abstract Intersection growth concerns the asymptotic behaviour of the index of the intersection of all subgroups of a group that have index at most n. In this paper we show that the intersection growth of some groups may not be a nicely behaved function by showing the following seemingly contradictory results: (a) for any group G the intersection growth function $i_G(n)$ is super linear infinitely often, and (b) for any non-decreasing unbounded function f there exists a group G such that the graph of i_G is below the one of f infinitely often.

Keywords: residually finite groups; growth in groups; intersection growth; residual finiteness growth

2010 *Mathematics subject classification:* Primary 20F69
Secondary 20E07; 20E26

1. Introduction

Let $\mathcal{G}$ be a class of subgroups of a group Γ. We define the *$\mathcal{G}$-intersection growth function* of Γ by letting $i_\Gamma^{\mathcal{G}}(n)$ be the index of the intersection of all $\mathcal{G}$-subgroups of Γ with index at most n. In symbols,

$$i_\Gamma^{\mathcal{G}}(n) := [\Gamma : \Lambda_\Gamma^{\mathcal{G}}(n)], \qquad \Lambda_\Gamma^{\mathcal{G}}(n) := \bigcap_{\substack{[\Gamma:\Delta]\leqslant n,\\ \Delta\in\mathcal{G}}} \Delta.$$

Here, $\mathcal{G}$ will always be either the class of all subgroups, the class $\lhd$ of normal subgroups or the class $\max\lhd$ of maximal normal subgroups of Γ, i.e. those subgroups that are maximal among normal subgroups. The corresponding intersection growth functions will then be written $i_\Gamma(n)$, $i_\Gamma^{\lhd}(n)$ and $i_\Gamma^{\max\lhd}(n)$.

Intersection growth was first defined by Biringer *et al.* in [**1**] where it was studied for free groups and some arithmetic groups. It was observed that it measures the Hausdorff dimension of Γ in a profinite metric and a connection has been drawn between intersection growth and the residual finiteness growth that will be mentioned at the end of this introduction.

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https://doi.org/10.1017/S0013091516000201 Published online by Cambridge University Press

The aim of this paper is to build examples of groups whose intersection growth behaves slowly at certain integers and much faster at others. We will assume all groups in this paper to be finitely generated. For a group Γ let $R(\Gamma)$ denote the intersection of all finite index subgroups of Γ. Obviously, $i_\Gamma^{\mathcal{G}}(n) = i_{\Gamma/R(\Gamma)}^{\mathcal{G}}(n)$, so we can assume that $R(\Gamma) = \{1\}$, i.e. *we will assume Γ to be finitely generated and residually finite.*

Observation 1.1. For any infinite residually finite group Γ one has that $i_\Gamma(n) \geqslant n$ for infinitely many positive integers n.

The main result of this paper states that, in some sense, the opposite is also true.

Theorem 1.2. *For any non-decreasing unbounded function $f\colon \mathbb{N} \to \mathbb{N}$ there exists a finitely generated infinite residually finite group Γ such that $i_\Gamma(n) < f(n)$ for infinitely many positive integers n.*

Remark 1.3. We notice that Observation 1.1 and Theorem 1.2 are still true if we replace $i_\Gamma(n)$ with either $i_\Gamma^{\lhd}(n)$ or $i_\Gamma^{\max\lhd}(n)$. It is immediate to deduce this for Theorem 1.2, because it is always true that $i_\Gamma^{\max\lhd}(n) \leqslant i_\Gamma^{\lhd}(n) \leqslant i_\Gamma(n)$ for all positive integers n. On the other hand, the proof for Observation 1.1 is an easy adaptation of the one we give below and we omit it to keep this paper short.

Observation 1.1 and Theorem 1.2 show that intersection growth functions may behave in a bad way. More precisely, these functions cannot be approximated via regular functions (polynomial, exponential, etc.) as they have different behaviour at different integers.

The main idea is to construct a group with a very limited number of finite quotients, and this is achieved by taking a suitable direct product of a family of simple groups.

Recall that for a residually finite finitely generated group $\Gamma = \langle X \rangle$ one can define the *residual finiteness growth function* $F_{\Gamma,X}(n)$ as the minimal natural number N such that any element of word length at most n with respect to X can be detected in a quotient Q of cardinality less than or equal to N (see Bou-Rabee [**2**] for an introduction). All the constructions in this paper can immediately be translated to get similar information about residual finiteness growth. See Bou-Rabee and Seward [**3**] for a similar result about building examples with arbitrarily large residual finiteness growth functions. We also mention the work of Kharlampovich *et al.* [**6**], who build finitely presented solvable examples with arbitrarily large residual finiteness growth functions.

Finally, we recall that for a group Γ one defines $r_\Gamma(n)$ to be the number of inequivalent finite irreducible n-dimensional representations of G (assuming that this number is finite). One defines $R_\Gamma(n) = \sum_{k=1}^{n} r_\Gamma(k)$ to be the *representation growth function* of Γ. Craven [**4**] showed analogues of Observation 1.1 and Theorem 1.2 for $R_\Gamma(n)$. Unlike Observation 1.1, Craven's upper bound Theorem G requires a more careful and involved proof.

2. Proofs of the results

Proof of Observation 1.1. Since Γ is infinite and residually finite for every n, there exists a finite index subgroup $H_N < \Gamma$ of index $[\Gamma : H_N] = N > n$. This clearly implies that $\Lambda_\Gamma(N) \leqslant H_N$, i.e. $i_\Gamma(N) \geqslant N$. □

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Observation 2.1. Let $\{n_i\}$ be a strictly increasing sequence of positive integers such that $n_1 \geqslant 5$. If $\mathfrak{G} := \prod_{i=1}^{\infty} \mathrm{Alt}(n_i)$, then

$$i_{\mathfrak{G}}(n_k - 1) = \prod_{i<k} |\mathrm{Alt}(n_i)|.$$

Proof. The group $\mathfrak{G}$ is profinite and endowed with the product topology. Notice that for profinite groups it is natural to define intersection growth with respect to finite index closed subgroups. In this case, by Nikolov and Segal [**8**] every subgroup of finite index is closed, but this is not used in our argument. We observe the following:

(1) if $H \leqslant \mathfrak{G}$ has index less than or equal to $n_k - 1$, then $H \cap \mathrm{Alt}(n_k)$ has index less than or equal to $n_k - 1$ inside $\mathrm{Alt}(n_k)$;

(2) $\mathrm{Alt}(n_k)$ has no non-trivial subgroups of index less than or equal to $n_k - 1$, but it has subgroups of index n_k.

Therefore, if $H \leqslant \mathfrak{G}$ is a closed subgroup of index less than or equal to $n_k - 1$, then $H \cap \mathrm{Alt}(n_i) = \mathrm{Alt}(n_i)$ for any $i \geqslant k$. This implies that

$$H \geqslant \overline{\bigoplus_{i \geqslant k} \mathrm{Alt}(n_i)} = \prod_{i \geqslant k} \mathrm{Alt}(n_i),$$

and therefore $\Lambda_{\mathfrak{G}}(n_k - 1) \geqslant \prod_{i \geqslant k} \mathrm{Alt}(n_i)$. It is very easy to see the opposite inclusion (since the group $\prod_{i \geqslant k} \mathrm{Alt}(n_i)$ can be realized as the intersection of point stabilizers of the actions of $\mathfrak{G}$ via $\mathrm{Alt}(n_i)$ on n_i points for $i \leqslant k$), which concludes the proof. □

This observation shows that if the sequence $\{n_k\}$ grows very quickly, then, in some sense, the function $i_{\mathfrak{G}}$ is very small at certain values.

Remark 2.2. The group $\mathfrak{G}$ constructed in Observation 2.1 is not finitely generated, but it is finitely generated as a profinite group.

The idea is to construct a finitely generated group whose profinite completion is the same as the group $\mathfrak{G}$ constructed in Observation 2.1. Since $\mathfrak{G}$ is finitely generated as a profinite group, it is very easy to find a finitely generated dense subgroup $\Gamma \subseteq \mathfrak{G}$. Such an embedding will give a surjective map $\pi\colon \hat{\Gamma} \to \mathfrak{G}$, however this map is often not injective because Γ might have finite index subgroups that are not visible in $\mathfrak{G}$ [**7**, **9**] and cannot be used to obtain information about i_Γ. If the map π is an isomorphism, it is very easy to see that $i_\Gamma = i_{\mathfrak{G}}$ (see [**1**]).

For some time it was not known if it is possible to find Γ such that the map π is an isomorphism. This question was settled in [**5**, Theorem 1.2], which we restate here in the weaker form that we need.

Theorem 2.3 (Kassabov and Nikolov [5]). *For any strictly increasing sequence $\{n_k\}$, the Cartesian product $\prod_{k=1}^{\infty} \mathrm{Alt}(n_k)$ is a profinite completion of a finitely generated residually finite group.*

We are now ready to prove Theorem 1.2.

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Proof of Theorem 1.2. We choose $n_1 = 5$ and, inductively, we define n_k in the following way: given n_k, let n_{k+1} be an integer large enough such that $f(n_{k+1} - 1) \geqslant \prod_{i \leqslant k} |\mathrm{Alt}(n_i)|$ and $n_{k+1} > n_k$.

The chosen sequence allows us to construct the group $\mathfrak{G}$ of Observation 2.1. By Theorem 2.3 we have that $\mathfrak{G} = \hat{\Gamma}$ for a finitely generated residually finite group Γ. As was observed above, $i_\Gamma = i_{\mathfrak{G}}$ and so

$$i_\Gamma(n_k - 1) = \prod_{i<k} |\mathrm{Alt}(n_i)|.$$

By construction, $i_\Gamma(n_k - 1) < f(n_k - 1)$ for any $k > 1$ and we are done. □

Acknowledgements. We are grateful to Khalid Bou-Rabee for helpful mathematical conversations. M.K. was partly supported by NSF Grants DMS-0900932 and DMS-1303117. F.M. gratefully acknowledges the Fondation Mathématique Jacques Hadamard (Grant ANR-10-CAMP-0151-02, FMJH, Investissement d'Avenir) for the support received during the development of this work.

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