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# BATHTUB FUNCTIONS AND BURN-IN

HENRY W. BLOCK*
*Department of Statistics*
*University of Pittsburgh*
*Pittsburgh, Pennsylvania 15260*

YI-KUAN JONG
*Department of Mathematics*
*Central Michigan University*
*Mt. Pleasant, Michigan 48858*

THOMAS H. SAVITS*
*Department of Statistics*
*University of Pittsburgh*
*Pittsburgh, Pennsylvania 15260*

Components which have bathtub shapes are reasonable candidates for burn-in in the sense that if these components are exposed to standard or elevated operating conditions for a short period of time, they tend to improve. In this paper we study the principle which states that burn-in should occur at or before the point at which a bathtub-shaped failure function starts increasing. To this end we develop three sign change results which characterize bathtub-shaped failure rates. These results are used to show that if a failure rate function is bathtub-shaped, then various other related functions have bathtub or upside-down bathtub shape. Finally, we give a framework for determining when the above principle holds.

## 1. INTRODUCTION

Burn-in is a widely used engineering methodology for eliminating weak components from a standard population. In industry, a standard population usually consists of various engineering systems composed of parts or components which are assembled to form a system. Both the components and the systems operate for a certain

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time period until they fail. Usually within any population of components, there are strong components with long lifetimes and weak components with short lifetimes. To ensure that only strong components reach customers, the components are tested under electrical or thermal conditions which approximate or exceed working conditions in field operation. In theory, weak components will fail during testing, leaving only strong components. Similar tests can be performed for systems after they are assembled in order to determine weak or strong subsystems or to uncover defects incurred during assembly. Although terminology for different tests, depending on whether systems or components are involved or depending on the specific field of application, is not consistent, we will use the term burn-in for all such tests. See Block and Savits [1] for a discussion of the various terms used to describe burn-in as well as a general review to the area. Kuo and Kuo [8] and Leemis and Beneke [11] also give reviews, but from more of an engineering point of view.

Different types of failures can occur in products which have been released for general usage. One of these is called early failure. It has been observed that many electrical and mechanical products exhibit early failure. By applying burn-in procedures, this early failure period can be minimized, preserving a company's credibility and increasing customer satisfaction.

The burn-in procedure is often curtailed according to engineering experience. This often corresponds to a time when a predetermined reliability goal is achieved—for example, when the mean residual life of a component is at a maximum. Another reason for ending burn-in is to minimize cost, which usually consists of a physical burn-in cost and a cost for customer satisfaction.

A related concern is to determine which component lifetimes are amenable to burn-in. In particular, which family of distributions will benefit from the burn-in procedure. Many authors believe that distributions which have a bathtub-shaped failure rate are most appropriate for burn-in. These distributions have failure rates which decrease, then remain constant, then increase.

In this paper one of our aims is to consider the intuitive principle: For distributions with bathtub-shaped failure rates, burn-in should occur at or before the point at which the failure rate function starts increasing. This Principle appears to be true for a wide variety of criteria and cost functions. The implication of this is that for a company accepting any of the criteria or cost functions, it would be a waste of resources to burn-in any component beyond the point specified in the principle.

In Section 4, we examine a large number of criteria and cost functions which are used to determine optimal burn-in. Then we describe a simple framework by which we can decide whether the above principle holds.

Another aim is to determine what implication a bathtub-shaped failure rate has on various functions related to the failure rate. For example, a function with bathtub-shaped failure rate has a mean residual life which is upside-down bathtub-shaped. Furthermore, it has a maximum value, and this value is related to the minimum value of the failure rate. Specifically, it precedes it. Consequently, according to the criterion of maximizing mean residual life, the optimal burn-in time satisfies the Principle.

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As a means of obtaining the above, three results are developed in Section 2 which are useful in characterizing distributions with bathtub-shaped failure rate. Two of these results use sign change properties and the third uses a local (three-point version) of the bathtub property. These results are in the spirit of Karlin's total positivity and have intrinsic interest. These results are used in Sections 3 and 4 to show that if the failure rate function is bathtub-shaped, various other related functions are bathtub- (or upside-down-)shaped. In particular, the new result is obtained that bathtub-shaped failure rates have upside-down percentile residual life functions.

## 2. PROPERTIES OF BATHTUB-SHAPED FUNCTIONS

In this section, we discuss some properties of general bathtub-shaped and upside-down bathtub-shaped functions. The bathtub-shaped functions are useful in modeling failure rate functions and the upside-down bathtub-shaped functions are important in describing standard functions of failure rates, such as mean residual life.

Three useful results are given which characterize these functions. Two of these results involve sign changes while a third gives a local (three-point) version of the bathtub property. These results are used in the next section to give a new result that the $\alpha$-percentile residual life function of a bathtub-shaped failure rate has upside-down bathtub shape.

We first present a definition of a bathtub-shaped function. Essentially, this is a function which is decreasing then increasing. This includes the case usually covered in other definitions where a constant middle section is often specified. Since we deal with functions which need not be continuous, some care needs to be taken in defining the change point, that is, the point where the monotonicity of the function changes.

DEFINITION 2.1: *A function $g(t)$ defined for $t \geq 0$ is said to have a* bathtub shape *if it is monotone or if it changes monotonicity once from decreasing to increasing in the following way. Either it changes monotonicity from decreasing to increasing at* 0 (*it must be discontinuous in this case*) *or there is a $t_0 > 0$ such that $g(t)$ is decreasing for $0 \leq t < t_0$, increasing for $t_0 < t < \infty$ and $g(t_0) \leq \max\{g(t_0+), g(t_0-)\}$. Further, $g(t)$ is said to have an* upside-down bathtub shape *if $-g(t)$ is bathtub-shaped. Since there may be many change points, we define the first change point by*

$$t_*(g) = \inf\{t \mid g(s) \text{ is increasing for all } t \leq s < \infty\} \qquad (\inf\phi = \infty)$$

*for a bathtub-shaped function and $t^*(g) = t_*(-g)$ for g with an upside-down bathtub shape. We say the function has degenerate bathtub shape if it is monotone.*

Because it is usually not easy to decide whether a functional form is bathtub-shaped, we develop criteria. Bathtub-shaped functions can be characterized by each of the following three theorems.

THEOREM 2.1: *A function $g(t)$, $t \geq 0$ is bathtub-shaped* (*upside-down bathtub-shaped*) *if and only if for all $0 \leq t_1 < t_2 < t_3$,*

$$g(t_2) \leq \max\{g(t_1), g(t_3)\} \qquad (g(t_2) \geq \min\{g(t_1), g(t_3)\}). \tag{2.1}$$

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PROOF: First, assume the condition does not hold. Then there exists $0 \le t_1 < t_2 < t_3$ such that $g(t_1) < g(t_2)$ and $g(t_3) < g(t_2)$. Consequently, $g$ cannot be bathtub. To prove the result in the other direction assume $g(t)$ is not bathtub. Define

$$a = \sup\{s | g(t) \text{ is decreasing for } 0 \le t \le s\}$$

and

$$b = \inf\{s | g(t) \text{ is increasing for } s \le t < \infty\},$$

where $\sup \phi = 0$ and $\inf \phi = \infty$. We cannot have that $b < a$, since in this case $g(t)$ would be bathtub (e.g., take $t_0$ between $a$ and $b$). Consequently, $a \le b$. If $a = b$, since $g$ is not bathtub, we must have $g(b) > \max\{g(b-), g(b+)\}$. From the definition of $g(b-)$ and $g(b+)$, it is easy to find $0 \le t_1 < t_2 = b < t_3$ such that $g(t_2) > \max\{g(t_1), g(t_3)\}$. If $a < b$, take $c = (a + b)/2$ if $b < \infty$ and $c = a + 1$ if $b = \infty$. By the definition of $a$ and $b$, $g$ is not decreasing for $t \le c$ and not increasing for $c \le t$. Thus, we can find $0 \le x_1 < x_2 \le c \le y_1 < y_2$ such that $g(x_1) < g(x_2)$ and $g(y_1) > g(y_2)$. Taking $t_1 = x_1$, $t_3 = y_2$, and $t_2$ to be either $x_2$ or $y_1$, depending on which of these $g$ is larger, we obtain the desired condition. ■

The following two criteria are given in terms of sign changes. We use the strict version of Karlin [7] in determining changes of sign, that is, we consider only strict changes not counting zeros.

THEOREM 2.2: *A right-continuous function $g(t)$, $t \ge 0$ is bathtub-shaped (upside-down bathtub-shaped) if and only if for any given $x > 0$, the function $h(t;x) = g(t + x) - g(t)$ has at most one sign change in $t$, and if there is a change, it is from negative to positive (positive to negative).*

PROOF: We prove only the bathtub-shaped result since the other is similar. Assume the sign change condition does not hold; that is, there exists an $x_0$ such that there is exactly one sign change for $h(t;x_0)$, but in the wrong direction, or that $h(t;x_0)$ has two or more sign changes. Right continuity is not required for this part of the proof.

*Case 1:* $h(t;x_0)$ has a sign change from positive to negative. We can then find a $u < v$ such that $g(u) < g(u + x_0)$ and $g(v) > g(v + x_0)$. Now apply Theorem 2.1 with $t_1 = u$, $t_3 = v + x_0$, and $t_2$ to be either $v$ or $u + x_0$ depending on which gives $g$ the larger value.

*Case 2:* $h(t;x_0)$ has at least two sign changes. We can then find three points $t_1 < t_2 < t_3$ such that $h(t;x_0)$ has three different signs, but two of these must be from + to − and so we can apply the argument of Case 1.

Now assume $g(t)$ is not bathtub. Then, by the previous result, there exists $0 \le t_1 < t_2 < t_3$ such that $g(t_1) < g(t_2)$ and $g(t_2) > g(t_3)$. By right continuity we can pick $t_2$ and $t_3$ so that $x_1 = t_2 - t_1$ and $x_2 = t_3 - t_2$ are rational. Without loss of generality, we assume $x_1 > x_2$. Now assume the condition holds. Then, for $t_1 \le t$, we have: (1) $g(t + x_1) - g(t) \ge 0$, and, for $t \le t_2$, we have: (2) $g(t + x_2) - g(t) \le 0$. Let $m$ and $n$

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be the smallest integers such that $x_1/x_2 = (n-1)/(m+1)$. It follows that $t_3 - nx_2 + mx_1 = t_1$. We will use (1) and (2) to deduce a contradiction.

Let $n_1$ be the largest $n$ such that $t_1 \le t_3 - nx_2$ ($n_1 \ge 2$ since $t_3 - 2x_2 > t_1$). Then, by (2),

$$g(t_2) = g(t_3 - x_2) \le g(t_3 - 2x_2) \le g(t_3 - n_1 x_2).$$

Since $n_1$ was chosen to be the largest, it is not hard to see that $t_3 - n_1 x_2 + x_1 \le t_3$. Now pick the largest $n_2$ such that $t_1 \le t_3 - n_2 x_2 + x_1$. Since $x_1 > x_2$, $n_2 > n_1$ and from (2), it follows that

$$g(t_2) \le g(t_3 - n_1 x_2 + x_1) \le g(t_3 - n_2 x_2 + x_1).$$

Continue for $m$ steps where $m$ is the integer chosen previously so that we have $t_3 - n_m x_2 + mx_1 \le t_3$. We can now pick the largest integer $n_{m+1}$ such that $t_1 \le t_3 - n_{m+1}x_2 + mx_1$. We have that $t_1 = t_3 - nx_2 + mx_1$. If $n_{m+1} > n$, then $t_3 - n_{m+1}x_2 + mx_1 < t_3 - nx_2 + mx_1 = t_1$ contrary to the construction. If $n_{m+1} < n$, $n_{m+1}$ is not the largest. Consequently, $n_{m+1} = n$. Our conclusion is that $g(t_2) \le g(t_3 - nx_2 + mx_1) = g(t_1)$, contrary to $g(t_2) > g(t_1)$. ■

*Notes.* (1) It is not hard to show that $g(t)$ is monotone—that is, it has a degenerate bathtub shape, if and only if for all $x > 0$, $h(t;x)$ has no sign change in $t$. (2) For another proof of the above theorem, see Jong [6].

The following theorem generalizes the well-known result that a function $g(t)$ is monotone if and only if for any real $c$, $g(t) - c$ has at most one sign change.

THEOREM 2.3: *A function is bathtub-shaped* (*upside-down bathtub-shaped*) *if and only if for all real c*, $g(t) - c$ *has at most two sign changes*, *and if there are two sign changes they are from positive to negative to positive* (*negative to positive to negative*).

PROOF: We need only prove the result for the bathtub-shaped case. Assume the sign change condition does not hold. Then there must be a $c$ and $0 \le t_1 < t_2 < t_3$ such that $g(t) - c$ changes sign at these points from negative to positive to negative, that is,

$$g(t_2) - c > 0 > g(t_1) - c \quad \text{and} \quad g(t_2) - c > 0 > g(t_3) - c.$$

Consequently, $g(t_2) > \max\{g(t_1), g(t_3)\}$ and so by Theorem 2.1, $g$ is not bathtub.

Now assume $g$ is not bathtub. Then we can find $0 \le t_1 < t_2 < t_3$ such that $g(t_2) > \max\{g(t_1), g(t_3)\}$. Choosing $c$ between $g(t_2)$ and $\max\{g(t_1), g(t_3)\}$ gives that $g(t) - c$ changes sign from negative to positive to negative. This completes the proof. ■

## 3. PRESERVATION RESULTS

Optimization results in reliability are often based on functionals of the failure rate function. For example, in determining when a component should be burned in, one intuitive criterion is to burn-in the component until the time at which the failure rate function achieves its minimum. If the failure rate has a bathtub shape, then by ex-

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amination of the graph it is clear where the minimum occurs. Another approach involves finding a point where the mean residual life function is maximized. If the failure rate has a bathtub-shaped function, it is known that the mean residual life function is upside-down bathtub-shaped. Again, an examination of the graph identifies the maximum. It is also important to know the relationship between the two optimal points just discussed. In the following we determine results of this type. The proofs of these use the results of the preceding section.

We begin with the quantity $A_x(t)$, which is the area under the failure rate function from $t$ to $t+x$. It is clear by examining the graph of a bathtub-shaped failure rate function that this quantity first decreases, then increases. The theorem gives this result as well as compares the first change points.

THEOREM 3.1: *Let $r(t), t \geq 0$ be a bathtub-shaped failure rate function. Then $A_x(t) = \int_t^{t+x} r(s)\,ds$ is bathtub-shaped in $t$ for all $x > 0$. Moreover, $t_*(A_x) \leq t_*(r)$.*

PROOF: For any $x > 0$, $y > 0$, and $t \geq 0$,

$$\begin{aligned} A_x(t+y) - A_x(t) &= \int_{t+y}^{t+y+x} r(s)\,ds - \int_t^{t+x} r(s)\,ds \\ &= \int_t^{t+x} [r(s+y) - r(s)]\,ds \\ &= \int_0^{\infty} [r(s+y) - r(s)] I_{(t,\,t+x]}(s)\,ds \\ &= \int_0^{\infty} h(s;y) I_{(t,\,t+x]}(s)\,ds, \end{aligned} \tag{2.6}$$

where $h(s;y) = r(s+y) - r(s)$. Since $I_{(t,\,t+x]}(s)$ is $TP_2$ in $t$ and $s$ for each fixed $x > 0$, by [7, Chap. 5, Thm. 3.1] it follows that $A_x(t+y) - A_x(y)$ has the same sign change property as $r(s+y) - r(s)$. From Theorem 2.2 this gives the result. Since $r(t)$ is increasing for $t > t_*(r)$, it easily follows that $A_x(t)$ is increasing for the same range of $t$. Consequently, $t_*(A_x) \leq t_*(r)$. ■

COROLLARY 3.1: *Let $F$ have a bathtub-shaped failure rate function $r$. Let $F_t(x)$ denote the distribution function of an item that has survived a burn-in period of $t$ units of time. Then, the survival function $\bar{F}_t(x)$ has an upside-down bathtub shape in $t$ for each fixed $x \geq 0$; moreover, for $g_x(t) = \bar{F}_t(x)$, $t^*(g_x) \leq t_*(r)$. In particular, $F_t$ is stochastically decreasing in $t \geq t_*(r)$, that is, for each fixed $x \geq 0$, $\bar{F}_b(x) \geq \bar{F}_c(x)$ whenever $t_*(r) \leq b \leq c$.*

PROOF: Since

$$\bar{F}_t(x) = \exp\left\{-\int_t^{t+x} r(u)\,du\right\} = \exp\{-A_t(x)\},$$

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we see by Theorem 3.1 that $\bar{F}_t(x)$ is upside-down bathtub-shaped in $t$ for each $x \geq 0$ with $t^*(g_x) = t_*(A_x) \leq t_*(r)$. Consequently, $\bar{F}_t(x)$ is a decreasing function in $t \geq t_*(r)$ for fixed $x \geq 0$ and we are finished. ■

*Remark 3.1:* (1) The converse of Theorem 3.1 can also be shown to hold under the additional assumption of right-continuity on $r$ [6]. (2) If the failure rate function is bathtub-shaped, it can be shown by using a sign change argument (i.e., Thm. 2.3) that the function which is the average of the failure rate function, that is, $(1/t)\int_0^t r(s)\,ds$, is also bathtub-shaped.

In Haines and Singpurwalla [4] and Joe and Proschan [5], some properties of the $\alpha$-percentile residual life were discussed. The $\alpha$-percentile residual life $q_\alpha(t)$ is the $\alpha$-percentile of the remaining life given survival up to time $t$ and is defined below. The second set of authors showed, among other things, that the $\alpha$-percentile residual life is decreasing (increasing) if the distribution has increasing (decreasing) failure rate. Launer [9] showed that under certain conditions the maximum of the $\alpha$-percentile residual life always precedes the minimum of the failure rate function. In the following, we show that the $\alpha$-percentile residual life has an upside-down bathtub-shaped failure rate if the failure rate function has a bathtub shape.

DEFINITION 3.1: *For a lifetime distribution function $F$, for $0 < \alpha < 1$, the $\alpha$-percentile residual life function $q_\alpha(t)$ is defined by*

$$q_\alpha(t) = \inf\{x \geq 0 \,|\, \bar{F}(t+x) \leq (1-\alpha)\bar{F}(t)\}.$$

THEOREM 3.2: *If the failure rate function $r(t)$ of a lifetime distribution function $F$ has a bathtub shape, then the $\alpha$-percentile residual life function $q_\alpha(t)$ has an upside-down bathtub shape. Also, $t^*(q_\alpha) \leq t_*(r)$.*

PROOF: It is sufficient to show that, given $0 \leq t_1 < t_2 < t_3$,

$$q_\alpha(t_2) \geq \min\{q_\alpha(t_1), q_\alpha(t_3)\}.$$

Rewriting the definition of the $\alpha$-percentile residual life function, we have

$$q_\alpha(t) = \inf\{x > 0 \,|\, \bar{F}_t(x) \leq 1-\alpha\},$$

where $\bar{F}_t(x) = \bar{F}(t+x)/\bar{F}(t)$ is the residual survival function. By Corollary 3.1, we see that $g_x(t) = \bar{F}_t(x)$ is upside-down bathtub-shaped in $t$ for each $x \geq 0$ with $t^*(g_x) \leq t_*(r)$.

Given $0 \leq t_1 < t_2 < t_3$, let $x > 0$ be such that $\bar{F}_{t_2}(x) \leq 1-\alpha$. By Theorem 2.1, it follows that $\bar{F}_{t_2}(x) \geq \min\{\bar{F}_{t_1}(x), \bar{F}_{t_3}(x)\}$ and, consequently, either $\bar{F}_{t_1}(x) \leq 1-\alpha$ or $\bar{F}_{t_3}(x) \leq 1-\alpha$. In either case, we see that $\min\{q_\alpha(t_1), q_\alpha(t_3)\} \leq x$ and so $q_\alpha(t_2) \geq \min\{q_\alpha(t_1), q_\alpha(t_3)\}$, that is, $q_\alpha(t)$ is upside-down bathtub-shaped.

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To show that $t^*(q_\alpha) \le t_*(r)$, it suffices to show that $q_\alpha(t)$ is decreasing in $t \ge t_*(r)$. But this easily follows because, for $t_*(r) \le t_1 < t_2$,

$$\{x \ge 0 | \bar{F}_{t_1}(x) \le 1 - \alpha\} \subseteq \{x \ge 0 | \bar{F}_{t_2}(x) \le 1 - \alpha\}$$

using the fact that $\bar{F}_{t_1}(x) \ge \bar{F}_{t_2}(x)$ for all $x \ge 0$ whenever $t_*(r) \le t_1 < t_2$. ■

## 4. OPTIMAL BURN-IN

In this section we are interested in optimal burn-in policies. As mentioned in Section 1, burn-in is a widely used engineering protocol to weed out weak items. That is, items are put on test in shop for a time period $b$, called the burn-in period; those that survive are then released for field use. The burn-in time $b$ is determined by some criterion. Generally, we want to choose $b$ in an optimal manner.

Many papers on this topic have appeared in the reliability literature (e.g., see the references quoted in the following examples). The general setup of many of these papers is to propose some objective function for optimization and then show that an optimal burn-in time $b^*$ does indeed exist. In addition, many of these papers conclude that if the underlying failure rate $r$ of the item under test is bathtub-shaped, then the optimal burn-in time $b^*$ occurs no later than the first change point $t_*(r)$ of $r$.

The above result is intuitively appealing and seems quite desirable. In fact, one is tempted to conjecture that any *reasonable* burn-in criterion should have this property. Toward this goal, we will describe a simple and unified framework which seems to capture the essence of this *meta-theorem.* Most of the previous results on optimal burn-in can be obtained as special cases.

*Framework:* Let $K(b)$ be the objective function we want to optimize. Suppose we can decompose $K$ as

$$K(b) = K_1(b) - K_2(b)$$

such that $K_1(b) \uparrow b \ge s$ and $K_2(b) \downarrow b \ge s$ for some $s \ge 0$. Then it is obvious that

$$\inf_{b \ge 0} K(b) = \inf_{b \le s} K(b),$$

that is, $K$ achieves its infimum no later than $s$.

*Remarks 4.1:*

1. It is helpful to think of $K_1$ as the cost of burn-in and $K_2$ as the gain obtained through burn-in.
2. A particular special case of interest occurs when $K_1$ is bathtub-shaped and $K_2$ is upside-down bathtub-shaped. In this case we can take $s = \max\{t_*(K_1), t^*(K_2)\}$.
3. If instead of minimizing the objective function $K$ we want to maximize it, we instead assume $K_1(b) \downarrow b \ge s$ and $K_2(b) \uparrow b \ge s$. Then,

$$\sup_{b \ge 0} K(b) = \sup_{b \le s} K(b).$$

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4. In maintenance theory, the objective function is more naturally expressed as $K(b) = K_1(b)/K_2(b)$. We can easily reduce this to the above by taking logarithms.

*Example 4.1:* Mi [12] considered the problem of maximizing the reliability of a burned-in item to survive a mission time $\tau$. For this problem we let $K_1(b) = \bar{F}_b(\tau)$ and $K_2(b) = 0$. According to Corollary 3.1, we know that $K_1$ is upside-down bathtub-shaped if the failure rate function $r$ of the item is bathtub-shaped. Consequently, $K = K_1 - K_2$ achieves its maximum no later than $t^*(K_1) \leq t_*(r)$.

*Example 4.2:* Another criterion of interest is to maximize the $\alpha$-percentile residual life function. This was considered by Launer [9]. Fixing $\alpha$, we let $K_1(b) = q_\alpha(b)$ and $K_2 = 0$. From Theorem 3.2, we know that $q_\alpha$ is upside-down bathtub-shaped with $t^*(q_\alpha) \leq t_*(r)$ whenever $r$ is bathtub-shaped. Consequently, it follows that the optimal burn-in time $b^*$ for this criterion occurs no later than the first change point of $r$.

*Remark 4.2:* We can easily incorporate costs into the above two examples as follows. Let $K_1(b)$ be any increasing cost function of the burn-in time $b$. Let $K_2(b)$ be a gain function proportional to either $\bar{F}_b(\tau)$ or $q_\alpha(b)$. Then we conclude that the optimal burn-in time which minimizes the cost function $K = K_1 - K_2$ satisfies $b^* \leq t_*(r)$ whenever $r$ is bathtub-shaped.

Another case which falls into our framework is the following. Let $\mathcal{G}$ be any class of life distributions which is closed under burn-in, that is, $G_b$ is in $\mathcal{G}$ whenever $G$ is. Here $G_b(x) = 1 - \bar{G}(b + x)/\bar{G}(b)$ is the distribution of an item which has survived the burn-in period $b$.

PROPOSITION 4.1: *Let $D_i : [0,\infty) \times \mathcal{G} \to (-\infty,\infty)$ for $i = 1,2$. Suppose that for some fixed $s \geq 0$,*

(a) *$D_1(t,G) \uparrow t \geq s$ ($D_2(t,G) \downarrow t \geq s$) for each fixed $G \in \mathcal{G}$;*
(b) *$D_1(t,H) \leq D_1(t,G)$ and $D_2(t,H) \geq D_2(t,G)$ for each $t \geq 0$ whenever $G \leq_{st} H$ (i.e., $\bar{G}(u) \leq \bar{H}(u)$ for all $u \geq 0$).*

Assume $F \in \mathcal{G}$ has a bathtub failure rate function $r$ and set $K = K_1 - K_2$, where $K_1(b) = D_1(b, F_b)$ and $K_2(b) = D_2(b, F_b)$. Then the optimum value $b^*$ which minimizes $K$ occurs no later than $t_*(r)$.

PROOF: According to Corollary 3.1, we know that $F_b$ is stochastically decreasing in $b \geq s = t_*(r)$. It is then easy to show that $K_1(b) \uparrow b \geq s$ while $K_2(b) \downarrow b \geq s$ and the result follows. ■

We illustrate the above with some examples.

*Example 4.3:* A reasonable criterion for burn-in is to maximize the mean residual life. Watson and Wells [15] and Lawrence [10] have discussed this problem in detail for particular life distributions. Park [14] and Mi [13] considered the problem when

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the failure rate function has a bathtub shape. In our context, we let $\mathcal{G}$ denote the class of life distributions having a finite mean. Set $D_1 = 0$ and

$$D_2(t,G) = \int_0^\infty \bar{G}(u)\,du.$$

Clearly $D_i$ satisfy the conditions of Proposition 4.1. Since

$$\mu(b) = \int_0^\infty \bar{F}_b(t)\,dt = -K(b),$$

maximizing $\mu(b)$ is equivalent to minimizing $K(b)$, and so it follows that $b^* \le t_*(r)$ for a bathtub failure rate function $r$.

*Remark 4.3:* Actually we know that the mean residual life function $\mu(b)$ is upside-down bathtub-shaped whenever $r$ is bathtub-shaped; moreover, $t^*(\mu) \le t_*(r)$ (see, e.g., Block, Savits, and Singh [2]). Hence, we could have used the method of Examples 4.1 and 4.2 instead.

*Example 4.4:* A variation of a criterion considered by Mi [12] is the following. We assume that we have a stockpile of burned-in items available for field use. Whenever one item fails, we immediately replace it with another burned-in item. We assume its cost is given by $c(b)$ which is an increasing function of $b$. Our objective function is the total expected cost of failed items during a mission time $\tau$. Letting $N_b$ denote the associated renewal process generated by the failure distribution $F_b$, our goal is thus to choose a burn-in time $b^*$ which minimizes the expected cost $c(b)E[N_b(\tau)]$.

Let $\mathcal{G}$ denote the class of life distributions. It is well known that if $N$ is a renewal process generated by $G \in \mathcal{G}$, then we can express $E[N(t)]$ as

$$E[N(t)] = \sum_{k=1}^{\infty} G^{(k)}(t),$$

where $G^{(k)}$ is the $k$-fold convolution of $G$ with itself. It is thus easy to check that if $G \le_{st} H$, then $G^{(k)} \le_{st} H^{(k)}$ for all $k \ge 1$. Hence, $D_1(t,G) = c(t)\sum_{k=1}^{\infty} G^{(k)}(\tau)$ and $D_2(t,G) = 0$ satisfy the conditions of Proposition 4.1. We thus conclude that $b^* \le t_*(r)$ for a bathtub failure rate function $r$.

*Remark 4.4:* One of the important cost functions in the burn-in literature is due to Clarotti and Spizzichino [3]. It can be expressed as

$$K(b) = c_0 F(b) + c_1[F(b+\tau) - F(b)] - c_2\bar{F}(b),$$

where $0 < c_0 < c_1 < c_2$. In Jong [6], it was shown that for a bathtub failure rate $r$, the optimal burn-in time $b^*$ which minimizes $K(b)$ satisfies either $b^* \le t_*(r)$ or $b^* = \infty$. Moreover, $b^* = \infty$ if and only if either of the following conditions hold:

(a) $t_*(r) = \infty$ and $v \equiv (c_1 - c_0)/(c_1 + c_2) \ge \exp\{-\lim_{b\to\infty} A_\tau(b)\}$;
(b) $t_*(r) < \infty$ and $v > \exp\{-A_\tau[t_*(A_\tau)]\}$.

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Thus, in some sense, this would imply that the above cost function is not a *reasonable* cost function for burn-in.

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