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Enumeration Schemes for Restricted Permutations

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Zeilberger's enumeration schemes can be used to completely automate the enumeration of many permutation classes. We extend his enumeration schemes so that they apply to many more permutation classes and describe the Maple package WilfPlus, which implements this process. We also compare enumeration schemes to three other systematic enumeration techniques: generating trees, substitution decompositions, and the insertion encoding.

1. Introduction

The ancestry of the problem of permutation class enumeration can be traced as far back as 1915 (MacMahon [25]). This problem has frequently been accomplished by charming arguments utilizing such diverse objects as Young tableaux, Dyck paths, and planar maps, to name only a few. Our concern herein is not with attractive proofs, but rather with systematic methods for solving the enumeration problem. We adopt a strict definition of systematic, insisting that the computations can be performed without any human interaction whatsoever. For the definition of enumeration, we follow Wilf [39] and insist only on a polynomial time (in n) algorithm to compute the number of length-n permutations in the class, referring to such an algorithm as a *Wilfian formula*.[1]

We begin with definitions. Two sequences of natural numbers are said to be *order isomorphic* if they have the same pairwise comparisons, so $9, 1, 6, 7, 2$ is order isomorphic to $5, 1, 3, 4, 2$. Every sequence w of natural numbers without repetition is order isomorphic

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[1] While our requirement for Wilfian formulas may seem somewhat lax, especially given that many permutation classes have known closed-form generating functions, it has been conjectured (*e.g.*, by Zeilberger, as recorded in [17]) that not all permutation classes have Wilfian formulas.

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to a unique permutation that we denote by $\mathrm{st}(w)$, so $\mathrm{st}(9,1,6,7,2) = 5,1,3,4,2$, which we shorten to 51342. We say that $\mathrm{st}(w)$ is the *standardization* of w. We further say that the permutation π *contains* the permutation β if π contains a subsequence that is order isomorphic to β, and in this case write $\beta \leqslant \pi$. For example, 391867452 contains 51342, as can be seen by considering the subsequence 91672. A permutation is said to *avoid* another if it does not contain it.

A *permutation class* is a lower-order ideal in the containment ordering, meaning that if π is contained in a permutation in the class, then π itself lies in the class. Permutation classes can be specified in numerous ways, for example as the set of permutations sortable by a particular machine (see, *e.g.*, Atkinson and Beals [4]), or as the set of finite restrictions of a bijection between two linearly ordered sets (these are the atomic representations of Atkinson, Murphy and Ruškuc [7]). However, permutation classes are most commonly specified in terms of the minimal permutations not lying in the class, which we call the *basis* of the class. For several reasons, this representation is the most natural in our situation: it is canonical, the simplest to implement, and every permutation class can be specified in this form.

By the minimality condition, bases are necessarily *antichains*, meaning that no element of a basis is contained in another. Although there are infinite antichains of permutations (see Atkinson, Murphy and Ruškuc [5] for constructions and references to earlier work), we restrict our attention to finitely based classes. Given a set of permutations B, we define $\mathrm{Av}(B)$ to be the set of permutations that avoid every permutation in B. Thus if $\mathscr{C}$ is a closed class with basis B then $\mathscr{C} = \mathrm{Av}(B)$, and for this reason the elements of a permutation class are often referred to as *restricted permutations*. We let $s_n(B)$ denote the number of permutations of length n in $\mathrm{Av}(B)$ and refer to $\sum_n s_n(B)x^n$ as the generating function of $\mathrm{Av}(B)$.

Rephrased in this terminology, our interest is in systematic methods to produce Wilfian formulas for $s_n(B)$, where B is a finite set of permutations. To date, four techniques with wide applicability have been introduced which satisfy this goal:

- generating trees,
- enumeration schemes,
- the substitution decomposition,
- the insertion encoding.

The aim of this paper is to extend the method of enumeration schemes so that it can enumerate a wider variety of permutation classes, and to compare this method to the other three. These extensions, implemented in the accompanying Maple package WilfPlus available from the author's homepage, are carried out in Section 6. Before that we briefly examine the other methods in Sections 2–4 and review enumeration schemes in Section 5. Section 7 contains examples of classes which still lie beyond the reach of enumeration schemes, while Section 8 shows numerous examples which can be handled.

Each of the four approaches for permutation class enumeration has a natural notion of a 'state', and in each case, if a class is such that only finitely many states are required, then these methods can systematically provide a Wilfian formula for the number of length-n

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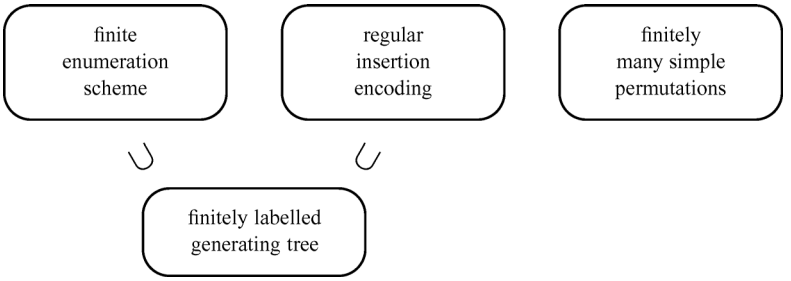


Figure 1. A depiction of the applicability of the four systematic enumeration techniques.

permutations in the class.[2] For generating trees, the states are the labels of the isomorphic generating tree. The classes possessing a generating tree with only finitely many labels are characterized in Vatter [36]; this characterization appears here as Theorem 2.1. For the insertion encoding, which associates a language to the permutation class, the natural notion of 'state' is a state in the accepting automaton for the associated language. The classes that require only finitely many states, *i.e.*, the classes that correspond to regular languages, were characterized by Albert, Linton and Ruškuc [2]; their result appears here as Theorem 3.1. For substitution decompositions, simple permutations play the role of states; Brignall, Ruškuc and Vatter [12] show how to decide whether a permutation class specified by a finite basis contains only finitely many simple permutations.

For enumeration schemes the translation of 'state' is 'ES$^+$-irreducible permutation' (or, for Zeilberger's original schemes, 'ES-irreducible permutation'). Should a class contain only finitely many such permutations then WilfPlus can automatically enumerate it. No characterization of these classes is known.[3] Moreover, unlike the other methods, there are subclasses of classes with finite enumeration schemes which do not themselves have finite enumeration schemes,[4] indicating that such a characterization may be too much to hope for.

The classes that these techniques can automatically enumerate are related as shown in Figure 1, which is to say, they are not very closely related at all (this is established via a series of examples in Section 7 and remarks in Sections 2 and 3). Care should be taken when reading one symbol in this diagram; while the inclusion from finitely labelled generating trees to finite enumeration schemes indicates an increase in the number of classes that can be counted, there is a corresponding decrease in information. If the generating tree associated with a permutation class can be finitely labelled, or if the class has a regular insertion encoding, then it has a rational generating function, while classes with only finitely many simple permutations have algebraic generating functions. It is not yet known what types of generating functions can arise from finite enumeration

[2] This is not to say that the these methods cannot produce Wilfian formulas when there are infinitely many states, just that there is no systematic procedure for doing so.

[3] Zeilberger [40] dismisses this by stating, 'if we know beforehand that we are guaranteed to succeed, then it is not research, but doing chores.'

[4] In fact, the set of all permutations, Av($\emptyset$), has a finite enumeration scheme (shown in Figure 4 on p. 153), while several examples of classes without finite enumeration schemes are given in Section 7.

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schemes, but they need not be algebraic. For example, Av(1234), which has a holonomic[5] but non-algebraic generating function (see Gessel [19]), has a finite enumeration scheme[6] (shown in Figure 5 on p. 153). It is natural to hope that finite enumeration schemes produce only holonomic sequences, but this remains conjectural.

Perhaps the greatest loss of information occurs with Wilf-equivalence. Two classes are said to be Wilf-equivalent if they are equinumerous. Clearly taking the reverse of a class yields a Wilf-equivalent class, as does taking the inverse, and these two operations generate a dihedral group on eight elements.[7] However, many examples of non-trivial Wilf-equivalences have been observed, ranging from the fact that each class defined by avoiding a single pattern of length three is Wilf-equivalent to every other, to the theorem of Atkinson, Murphy and Ruškuc [6] that Av(1342) is Wilf-equivalent to the infinitely based class

$$\mathrm{Av}(\{2(2m-1)416385\cdots(2m)(2m-3) : m = 2, 3, \dots\}).$$

If two classes both have finitely labelled generating trees, regular insertion encodings, or finitely many simple permutations then, since we can compute their generating functions from this information, we can decide whether they are Wilf-equivalent. For enumeration schemes this issue is not so clear. Occasionally, as with the enumeration schemes pictured in Figures 8(a) and 8(b) on p. 155, the Wilf-equivalence of two classes can be easily deduced from their enumeration schemes, but we give several examples in Section 8 where such deductions do not readily present themselves.

2. Generating trees

Generating trees were introduced by Chung, Graham, Hoggatt and Kleiman [15] and gained in popularity after West [37, 38] demonstrated their effectiveness as a method for permutation class enumeration. The closely related *ECO* (*enumerating combinatorial objects*) *method* (see Barcucci, Del Lungo, Pergola and Pinzani [8] for a survey) extends the notion of generating trees to other settings.

We say that the permutation σ of length n is a *child* of $\pi \in S_{n-1}$ if σ can be obtained by inserting n into π. This defines a rooted tree T on the set of all permutations. The *pattern-avoidance tree* of $\mathrm{Av}(B)$, denoted by $T(B)$, is then the subtree of T with nodes $\mathrm{Av}(B)$. For example, the first four levels of $T(132, 231)$ are shown in Figure 2.

A generating tree, on the other hand, is a rooted, labelled tree such that the labels of the children of each node are determined by the label of that node. Sometimes the labels

[5] A generating function is said to be *holonomic* (or synonymously in the univariate case, *D-finite*) if its derivatives span a finite-dimensional subspace over $\mathbb{C}(x)$. This is equivalent to the corresponding sequence s_n being holonomic (again synonymously in the univariate case, *P-recursive*), which means that there are polynomials $p_0, p_1, \dots p_k$ so that $p_k(n)s_{n+k} + p_{k-1}(n)s_{n+k-1} + \cdots + p_0(n)s_n = 0$.

[6] A more trivial example would be the class of all permutations.

[7] With the exception of substitution decompositions, these techniques are not invariant under the eight permutation class symmetries. To be precise, there are classes that cannot be handled with these methods, while their inverses can be handled easily. Thus the comment of Albert, Linton and Ruškuc [2] that 'this apparent asymmetry does represent a possible flaw of the insertion encoding in general' applies equally well to enumeration schemes and generating trees.

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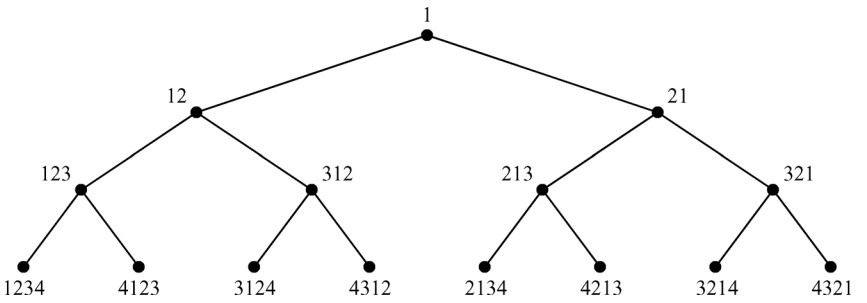


Figure 2. The first four levels of the pattern-avoidance tree $T(132, 231)$.

of the tree are taken to be natural numbers, but this is not necessary and frequently inconvenient. One specifies a generating tree by supplying the label of the *root* (also sometimes called the *axiom*) and a set of *succession rules* (also referred to as *inductive steps*). For example, the complete binary tree may be given by

$$\begin{aligned}&\text{Root: } (2)\\&\text{Rule: } (2) \rightsquigarrow (2)(2),\end{aligned}$$

where the notation $(2) \rightsquigarrow (2)(2)$ indicates that every node labelled (2) has two children, each also labelled (2).

In order to enumerate the permutation class $\mathrm{Av}(B)$, we want to find a generating tree isomorphic (as a rooted tree) to $T(B)$. For example, consider $T(132, 231)$. We may obtain a permutation in $\mathrm{Av}_n(132, 231)$ by inserting n either at the beginning or the end of any $\pi \in \mathrm{Av}_{n-1}(132, 231)$, but nowhere in between, so $T(132, 231)$ is isomorphic to the complete binary tree and thus to the generating tree given above. For a more complicated example we turn to $T(1234)$, first described by West [37]. This tree is isomorphic to the generating tree defined by

$$\begin{aligned}&\text{Root: } (2, 2)\\&\text{Rule: } (s, t) \rightsquigarrow (2, t+1)(3, t+1)\cdots(s, t+1)(s, s+1)(s, s+2)\cdots(s, t)(s+1, t+1).\end{aligned}$$

While verifying this isomorphism is not difficult (consider the lexicographically first ascent and the lexicographically first occurrence of 123 in the permutation), it is much harder to obtain the generating function for Av(1234) from this tree; for details see Bousquet-Mélou [11].

Let $T(B;\pi)$ denote the subtree of $T(B)$ that is rooted at π and contains all descendants of π. In an isomorphism between $T(B)$ and a generating tree, every permutation of $\mathrm{Av}(B)$ is assigned a label. Clearly two permutations π and σ may be assigned the same label if and only if $T(B;\pi)$ and $T(B;\sigma)$ are isomorphic (again, as rooted trees). Thus each pattern-avoidance tree $T(B)$ is isomorphic to a canonical generating tree whose labels correspond exactly to the isomorphism classes of $\{T(B;\pi) : \pi \in \mathrm{Av}(B)\}$.

In particular, $T(B)$ is isomorphic to a finitely labelled generating tree if and only if the set of all principal subtrees $\{T(B;\pi) : \pi \in \mathrm{Av}(B)\}$ contains only finitely many isomorphism classes. When this occurs, $\mathrm{Av}(B)$ has a rational generating function which may be routinely computed using the transfer matrix method (see Stanley's text [34, Section 4.7] for details).

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The finitely based classes for which this is possible are characterized by the following theorem.

Theorem 2.1. (Vatter [36]) *Let $\mathscr{C}$ be a finitely based permutation class. The pattern-avoidance tree of $\mathscr{C}$ is isomorphic to a finitely labelled generating tree if and only if $\mathscr{C}$ omits both a child of an increasing permutation and a child of a decreasing permutation.*

For example, $T(132, 231)$ satisfies the hypotheses of Theorem 2.1 because it omits both 132 (a child of the increasing permutation 12) and 231 (a child of 21). The Maple package FinLabel (described in [36] and available from the author's homepage) can find the generating functions for classes satisfying Theorem 2.1 completely automatically.

It is easy to see that the hypotheses of Theorem 2.1 are necessary.[8] The other direction is proved by showing that every sufficiently long permutation is 'GT-reducible'. Since GT-reducibility is a stronger condition than the ES-reducibility of enumeration schemes, every class with a finitely labelled generating tree has a finite enumeration scheme.

3. The insertion encoding

The insertion encoding, recently introduced by Albert, Linton and Ruškuc [2], is a correspondence between permutation classes and languages. With it, one may attack the enumeration problem with the tools of formal language theory. Roughly, this correspondence associates to each permutation a word describing how that permutation evolved. At each stage until the desired permutation has been constructed, at least one open *slot* (represented by a $\diamond$) exists in the intermediate *configuration*, and to proceed to the next configuration we insert a new maximal entry into one of these slots. This insertion can occur in four possible ways:

- the slot can be filled (replacing a $\diamond$ by n),
- the new entry can be inserted to the left of the slot (replacing a $\diamond$ by $n\diamond$),
- the new entry can be inserted to the right (replacing a $\diamond$ by $\diamond n$), or
- the slot can be divided into two slots with the new entry in between (replacing a $\diamond$ by $\diamond n \diamond$).

These operations are denoted by the symbols **f**, **l**, **r**, and **m**, respectively. Since each of these operations can be performed on any open slot at any stage, we subscript their symbols with the number of the slot they were applied to (read from left to right). For example, the permutation 31254 has the insertion encoding $\mathbf{m}_1\mathbf{l}_2\mathbf{f}_1\mathbf{r}_1\mathbf{f}_1$ because its evolution is:

$$\begin{gathered}\diamond\\ \diamond 1\diamond\\ \diamond 12\diamond\\ 312\diamond\\ 312\diamond 4\\ 31254\end{gathered}$$

[8] Suppose, without loss, that $\mathscr{C}$ contains every child of an increasing permutation. Thus the permutation $12\cdots n$ has $n+1$ children in the pattern-avoidance tree of $\mathscr{C}$, so no two of these nodes may share the same label.

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When the insertion encoding of a class forms a regular language then that class has a rational generating function that can be routinely computed. The only obstructions to having a regular insertion encoding turn out to be *vertical alternations* – permutations π of length $2n$ such that either $\pi(1), \pi(3), \ldots, \pi(2n-1) < \pi(2), \pi(4), \ldots, \pi(2n)$ or $\pi(1), \pi(3), \ldots, \pi(2n-1) > \pi(2), \pi(4), \ldots, \pi(2n)$.

Theorem 3.1. (Albert, Linton and Ruškuc [2]) *The insertion encoding of a finitely based class forms a regular language if and only if the class contains only finitely many vertical alternations.*

It follows from the Erdős–Szekeres theorem (every permutation contains a long monotone subsequence) that any class satisfying the hypotheses of Theorem 2.1, *i.e.*, that omits both a child of an increasing permutation and a child of a decreasing permutation, contains only finitely many vertical alternations, and thus Theorem 3.1 applies to every class with a finitely labelled generating tree.

4. The substitution decomposition

The substitution decomposition (also known as modular decomposition, disjunctive decomposition, and X-join) has proved to be a useful technique in a wide range of settings, ranging from game theory to combinatorial optimization (see Möhring and Radermacher [29] for extensive references). Permutation class enumeration is no exception.

An *interval* (also called a *block*, or in other contexts, *factor*, *clan*, or *convex subset*, to name a few) in the permutation π is a set of contiguous indices $I = [a, b]$ such that the set of images $\{\pi(i) : i \in I\}$ is also contiguous. Clearly every permutation of length n has n trivial intervals of length one and one trivial interval of length n. A permutation with only these trivial intervals is called *simple* (the analogous term in other contexts is often *prime* or *indecomposable*).

A class with only finitely many simple permutations has a recursive structure in which long permutations are built via substitution from smaller permutations (their intervals). Thus it is natural to expect these classes to have algebraic generating functions, and this intuition is borne out by the following theorem.

Theorem 4.1. (Albert and Atkinson [1]) *A permutation class with only finitely many simple permutations has an algebraic generating function.*

The canonical example of a class with only finitely many simple permutations is Av(132). By considering the entries to the left and to the right of the n in a permutation in $\mathrm{Av}_n(132)$ one simultaneously derives a decomposition of these permutations that leads immediately to the Catalan numbers and sees that this class contains only three simple permutations, 1, 12, and 21.

Another example of a class with only finitely many simple permutations is the class of separable permutations. This class, first introduced by Bose, Buss and Lubiw [10], is essentially the permutation analogue of series-parallel posets (see Stanley [34, Section 3.2]) and complement-reducible graphs (see Corneil, Lerchs and Burlingham [16]). To define

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separable permutations we first need two binary operations on permutations. Given two permutations $\pi \in S_m$ and $\sigma \in S_n$ we define their *direct sum*, written $\pi \oplus \sigma$, by

$$(\pi \oplus \sigma)(i) = \begin{cases} \pi(i) & \text{if } i \in [m], \\ \sigma(i-m)+m & \text{if } i \in [m+n] \setminus [m]. \end{cases}$$

Similarly, we define their *skew sum*, $\pi \ominus \sigma$, by

$$(\pi \ominus \sigma)(i) = \begin{cases} \pi(i)+n & \text{if } i \in [m], \\ \sigma(i-n) & \text{if } i \in [m+n] \setminus [m]. \end{cases}$$

A permutation is said to be *separable* if it can be built from the permutation 1 by repeated applications of $\oplus$ and $\ominus$. Bose, Buss and Lubiw [10] showed that the separable permutations can also be described as $\mathrm{Av}(2413, 3142)$. The enumeration of this class (which can now be seen to follow routinely from Theorem 4.1 and the fact that the only simple separable permutations are 1, 12, and 21) was first undertaken by West [37], who used generating trees to show that the separable permutations are counted by the large Schröder numbers.

5. Zeilberger's original enumeration schemes

Zeilberger [40] developed the notion of enumeration schemes and wrote the Maple package WILF to automate their discovery. Roughly, enumeration schemes are a divide and conquer technique which aims to partition a permutation class into smaller pieces from which recurrences can be derived.

Take $\pi \in S_k$, suppose that $n \geqslant k$, and let $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$. In Zeilberger's original formalization of enumeration schemes, we divide $\mathrm{Av}_n(B)$ into the sets

$$A_\pi(n; B; i_1, i_2, \ldots, i_k) = \{p \in \mathrm{Av}_n(B) : p(1) = i_{\pi(1)}, \ldots, p(k) = i_{\pi(k)}\}.$$

In words, $A_\pi(n; B; i_1, i_2, \ldots, i_k)$ is the set of B-avoiding length-n permutations that begin with the entries $i_1, i_2, \ldots, i_k$, in the order specified by π. For example,

$$A_{312}(9; B; 2, 3, 7) = \{723x_4x_5x_6x_7x_8x_9 \in \mathrm{Av}_n(B)\}. \tag{5.1}$$

In order to make enumeration schemes more closely resemble generating trees and the insertion encoding, we consider a symmetry of his approach, replacing permutations by their inverses. Thus we should specify the set of restrictions, B, a set of small entries of some length, π, and the positions in which the entries of π occur. Instead of specifying the positions of these small entries, however, we specify the gaps between them with a *gap vector*, $\mathbf{g}$. After performing these transformations, our version of (5.1) is

$$Z(B; 231; (1, 0, 3, 2)) = \{x_1 23 x_4 x_5 x_6 1 x_8 x_9 \in \mathrm{Av}_9(B)\},$$

and in general we are concerned with the sets

$$Z(B; \pi; \mathbf{g}) = \{p \in \mathrm{Av}_{k+\|\mathbf{g}\|}(B) : p(g_1 + 1) = \pi(1), \ldots, p(g_1 + \cdots + g_k + k) = \pi(k)\},$$

where k is the length of π and $\|\mathbf{g}\|$ denotes the sum of the components of $\mathbf{g}$. Thus $Z(B; \pi; \mathbf{g})$ is the set of all B-avoiding permutations of length $k + \|\mathbf{g}\|$ whose least k elements occur in the positions $g_1 + 1, g_1 + g_2 + 2, \ldots, g_1 + g_2 + \cdots + g_k + k$ and form a π-subsequence.

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Not all pairs $(\pi, \mathbf{g})$ result in a nonempty Z-set. Following Zeilberger, for a length-k permutation π we define

$$\mathscr{J}(\pi) = \{j \in [k+1] : Z(B;\pi;\mathbf{g}) = \emptyset \text{ for all } \mathbf{g} \text{ with } g_j > 0\}.$$

Thus $Z(B;\pi;\mathbf{g})$ is guaranteed to be empty if $\mathbf{g}$ does not *obey* $\mathscr{J}(\pi)$, meaning that $g_j \neq 0$ for some $j \in \mathscr{J}(\pi)$.

For example, consider the case $B = \{132\}$. Then $2 \in \mathscr{J}(12)$, because if $g_2 > 0$ then there is some entry between 1 and 2 in every permutation in $Z(B;\pi;\mathbf{g})$, giving a 132-pattern. In order to check that $\mathscr{J}(12) = \{2\}$ we need merely observe that 312 and 123 avoid 132. Our approach in this example can easily be generalized to compute $\mathscr{J}(\pi)$ for any π and B.

Proposition 5.1. *For any permutation π and basis B, $\mathscr{J}(\pi)$ can be computed by inspecting the B-avoiding children of π.*

Proof. Given i and j, consider the vector $\mathbf{h}$ for which $h_i = 0$ for all $i \neq j$ and $h_j = 1$. If $Z(B;\pi;\mathbf{h}) = \emptyset$ then $Z(B;\pi;\mathbf{g}) = \emptyset$ for all $\mathbf{g}$ with $g_j > 0$, so $j \in \mathscr{J}(\pi)$. If instead $Z(B;\pi;h) \neq \emptyset$ then $j \notin \mathscr{J}(\pi)$. □

For any $r \in [k]$, the set $Z(B;\pi;(g_1,\ldots,g_{k+1}))$ embeds naturally (remove the entry $\pi(r)$ and standardize) into

$$Z(B;\mathrm{st}(\pi - \pi(r));(g_1,\ldots,g_{r-1},g_r+g_{r+1},g_{r+2},\ldots,g_{k+1})), \tag{5.2}$$

where $\pi - \pi(r)$ denotes the word obtained from π by omitting the entry $\pi(r)$, so, for example, $51342 - 1 = 5342$. To make (5.2) easier to state, we define $d_r(\pi)$ to be $\mathrm{st}(\pi - \pi(r))$ and let

$$d_r((g_1,\ldots,g_{k+1})) = (g_1,\ldots,g_{r-1},g_r+g_{r+1},g_{r+2},\ldots,g_{k+1}).$$

Sometimes the embedding of $Z(B;\pi;\mathbf{g})$ into $Z(B;d_r(\pi);d_r(\mathbf{g}))$ is a bijection. If this is true for all gap vectors $\mathbf{g}$ that obey $\mathscr{J}(\pi)$, that is, that have $g_j = 0$ for all $j \in \mathscr{J}(\pi)$, then we say that $\pi(r)$ is *enumeration-scheme-reducible for π with respect to B*, or, for short, *ES-reducible*. (Zeilberger [40] refers to such entries as reversely deletable.) We also say that a permutation with an ES-reducible entry is itself ES-reducible, and a permutation without an ES-reducible entry is ES-irreducible.

For example, suppose again that $B = \{132\}$ and consider the permutation 12. We have already observed that $\mathscr{J}(12) = \{2\}$. Now we claim that the entry 1 is ES-reducible. The gap vectors that obey $\mathscr{J}(12)$ are those of the form $(g_1, 0, g_3)$, and thus we would like to verify that the embedding of $Z(\{132\};12;(g_1,0,g_3))$ into $Z(\{132\};1;(g_1,g_3))$ is a bijection.

Take $p \in Z(\{132\};1;(g_1,g_3))$ and consider inverting this embedding. In this case, that amounts to inserting the element 1 into position $g_1 + 1$ and increasing all other entries of p by 1. Label the resulting permutation p'. For example, from $\mathbf{g} = (3,0,1)$ and $p = 52314$ we obtain $p' = 634125$.

We would like to show that p' avoids 132. To do this we consider all possible ways in which the new element 1 could participate in a 132-pattern. Clearly this entry must be the first entry in such a pattern. Now note that the 2 in p' cannot participate in this 132-pattern, because the 1 and 2 are adjacent. But then there is a 132-pattern in p' which uses the 2 instead of the 1, so p itself contains a 132-pattern, a contradiction.

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Table 1.

$\mathbf{g}$	$\lvert Z(\{132\};12;\mathbf{g})\rvert$	$\lvert Z(\{132\};1;d_1(\mathbf{g}))\rvert$
$(0,0,0)$	1	1
$(0,0,1)$	1	1
$(1,0,0)$	1	1
$(0,0,2)$	1	1
$(1,0,1)$	2	2
$(2,0,0)$	2	2

Thus we have shown that

$$|Z_n(\{132\};12;(g_1,g_2,g_3))| = \begin{cases} 0 & \text{if } g_2 > 0, \\ |Z_{n-1}(\{132\};1;(g_1,g_3))| & \text{if } g_2 = 0. \end{cases}$$

Although this example did not demonstrate it, detecting and verifying ES-reducibility by hand can be enormously tedious. Fortunately, it is also unnecessary. By adapting the approach used in [36], we arrive at the following test for ES-reducibility that can be routinely checked by computer.[9] In it we let $\|B\|_\infty$ denote the length of the longest permutation in B.

Proposition 5.2. *The entry $\pi(r)$ of the permutation π is ES-reducible if and only if*

$$|Z(B;\pi;\mathbf{g})| = |Z(B;d_r(\pi);d_r(\mathbf{g}))|$$

for all gap vectors $\mathbf{g}$ of length $|\pi|+1$ that obey $J(\pi)$ and satisfy $\|\mathbf{g}\| \leqslant \|B\|_\infty - 1$.

Proof. If $\pi(r)$ is ES-reducible then the claim follows by definition. To establish the other direction, suppose that $\pi(r)$ is not ES-reducible, and choose $\mathbf{g}$ and $p \in Z(B;d_r(\pi);d_r(\mathbf{g}))$ so that $\mathbf{g}$ obeys $\mathscr{J}(\pi)$ but p cannot be obtained from a permutation in $Z(B;\pi;\mathbf{g})$ by removing $\pi(r)$ and standardizing.

First form the (B-containing) permutation p' by incrementing each entry of p that is at least $\pi(r)$ by 1 and inserting $\pi(r)$ into position $g_1 + \cdots + g_r + r$. Thus p' is the permutation that would have mapped to p, except that p' contains a pattern from B and thus does not lie in $Z(B;\pi;\mathbf{g})$.

Now pick some $\beta \in B$ that is contained in p', and choose a specific occurrence of β in p'. Note that since $p = \mathrm{st}(p' - \pi(r))$ avoids B, this occurrence of β must include the entry $\pi(r)$. Let p'' denote the standardization of the subsequence of p' formed by all entries that are either in the chosen occurrence of β or in π (or in both), so p'' contains a β-pattern and lies in $Z(\emptyset;\pi;\mathbf{h})$ for some $\mathscr{J}(\pi)$-obeying $\mathbf{h}$ with $\|\mathbf{h}\| \leqslant \|B\|_\infty - 1$. On the other hand, $\mathrm{st}(p'' - \pi(r))$ avoids B, which implies that $|Z(B;d_r(\pi);\mathbf{h})| > |Z(B;\pi;\mathbf{h})|$, as desired. □

For example, consider the basis $B = \{132\}$ again. In order to show that the 1 in $\pi = 12$ is ES-reducible using this proposition, we first find that $\mathscr{J}(12) = \{2\}$ and then perform the 6 computations shown in Table 1.

[9] Zeilberger's approach in [40] used what he referred to as 'logical reasoning', and while it is no less rigorous than this approach, Proposition 5.2 has the advantage of lucidity.

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In a similar manner one can verify that for $B = \{132\}$, $\mathscr{J}(21) = \emptyset$ and the 1 in 21 is ES-reducible. This gives the following enumeration scheme for Av(132):

$$\begin{aligned} s_n(132) &= |Z(\{132\};\emptyset;(n))|, \\ |Z(\{132\};\emptyset;(g_1))| &= \sum_{i=0}^{g_1-1} |Z(\{132\};1;(i,g_1-i-1))| \qquad \text{for } g_1 \geqslant 1, \\ |Z(\{132\};1;(g_1,g_2))| &= \sum_{i=0}^{g_1-1} |Z(\{132\};21;(i,g_1-i-1,g_2))| \\ &\quad + \sum_{i=0}^{g_2-1} |Z(\{132\};12;(g_1,i,g_2-i-1))| \qquad \text{for } g_1,g_2 \geqslant 1. \end{aligned}$$

We have from our previous computations that

$$|Z_n(\{132\};21;(i,g_1-i,g_2))| = |Z_n(\{132\};1;(i,g_1+g_2-i-1))|;$$

and

$$|Z_n(\{132\};12;(g_1,i,g_2-i-1))| = \begin{cases} 0 & \text{if } i > 0, \\ |Z_{n-1}(\{132\};1;(g_1,g_2-i-1))| & \text{if } i = 0. \end{cases}$$

Substituting these into the above completes the enumeration scheme:

$$|Z(\{132\};1;(g_1,g_2))| = \sum_{i=0}^{g_1} |Z(\{132\};1;(i,g_1+g_2-i-1))| \qquad \text{for } g_1,g_2 \geqslant 1.$$

6. Extending enumeration schemes

We replace the sets $\mathscr{J}(\pi)$ in this section, giving us a more powerful version of enumeration schemes that can be found automatically with the Maple package WilfPlus. In order to motivate this change, we first consider a shortcoming of $\mathscr{J}(\pi)$. Let $B = \{1342, 1432\}$. It can be shown easily, even by hand, that 12 is ES-irreducible. To do so, first note that $\mathscr{J}(12) = \emptyset$, as witnessed by the permutations 312, 132, and 123. Now consider the set $Z(B;12;(0,2,0))$. This set is empty, but removing 1 gives the nonempty set $Z(B;1;(2,0))$ while removing 2 gives the nonempty set $Z(B;1;(0,2))$. Indeed, this reasoning generalizes to show that all permutations of the form $\ominus^m 12$ are ES-irreducible, so Av(1342, 1432) does not have a finite enumeration scheme in Zeilberger's original sense.

These schemes fail in the previous example for a very simple reason: $\mathscr{J}(12)$ is too coarse to capture the fact that $(0,2,0)$ is not a valid gap vector for a B-avoiding descendant of 12. We remedy this problem with the following definition.

Definition. The entry $\pi(r)$ of the length-k permutation π is said to be *ES$^+$-reducible* if

$$|Z(B;\pi;\mathbf{g})| = |Z(B;d_r(\pi);d_r(\mathbf{g}))| \tag{6.1}$$

whenever $Z(B;\pi;\mathbf{g})$ is nonempty. Further, we say that the permutation π is ES$^+$-reducible if it contains an ES$^+$-reducible entry, and ES$^+$-irreducible otherwise.

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Let us replace (for now) the set $\mathscr{J}$ by

$$\mathscr{G}(\pi) = \{\mathbf{g} : Z(B;\pi;\mathbf{g}) \neq \emptyset\}.$$

Recall that for $\pi(r)$ to be ES-reducible, it had to satisfy (6.1) for all $\mathbf{g}$ that contained 0s in the positions specified by $\mathscr{J}(\pi)$. Clearly if $\mathbf{g} \in \mathscr{G}(\pi)$ then $\mathbf{g}$ obeys $\mathscr{J}(\pi)$, but there can be gap vectors that obey $\mathscr{J}(\pi)$ and do not lie in $\mathscr{G}(\pi)$, as in our previous example with $B = \{1342, 1432\}$. Thus we have obtained a weaker condition by requiring the satisfaction of (6.1) less often.

The proof of Proposition 5.2 carries over to this context to give the analogous result on testing for ES$^+$-reducibility.

Proposition 6.1. *The entry $\pi(r)$ of the permutation π is ES$^+$-reducible if and only if*

$$|Z(B;\pi;\mathbf{g})| = |Z(B;d_r(\pi);d_r(\mathbf{g}))|$$

for all $\mathbf{g} \in \mathscr{G}(\pi)$ with $\|\mathbf{g}\| \leqslant \|B\|_\infty - 1$.

One can view $\mathscr{G}(\pi)$ as a lower-order ideal[10] of $\mathbb{N}^{|\pi|+1}$ under the product order, that is, where $(x_1, \ldots, x_k) \leqslant (y_1, \ldots, y_k)$ if and only if $x_i \leqslant y_i$ for all $i \in [k]$. We therefore carry our definitions about permutation classes over to this context. In particular, we say that the *basis* of $\mathscr{G}(\pi)$ is the set of minimal vectors not in $\mathscr{G}(\pi)$, and if B is a set of vectors then we write $\mathrm{Av}(B)$ to denote the set $\{\mathbf{g} : \mathbf{g} \not\geqslant \mathbf{b} \text{ for all } \mathbf{b} \in B\}$.

For example, let us compute the basis of $\mathscr{G}(12)$ when $B = \{1342, 1432\}$. As already observed, $(0, 2, 0)$ does not lie in $\mathscr{G}(12)$. This gap vector is minimal in $\mathbb{N}^3 \setminus \mathscr{G}(12)$ since $Z(B;12;(0,1,0))$ is nonempty. To show that the basis of $\mathscr{G}(12)$ is precisely $(0, 2, 0)$, it suffices to note that the permutation

$$3\ 4\ \cdots\ (g_1+2)\ 1\ (g_1+3)\ 2\ (g_1+4)\ (g_1+5)\ \cdots\ (g_1+g_2+3)$$

avoids B, so $(g_1, 1, g_2) \in \mathscr{G}(12)$ for all $g_1, g_2 \in \mathbb{N}$. Thus we have shown that $\mathscr{G}(12) = \mathrm{Av}((0, 2, 0))$.

Now that we have computed $\mathscr{G}(12)$, it is not hard to check that 2 is ES$^+$-reducible for 12. In order to do so we need to show that the embedding in question is a bijection for all gap vectors (g_1, g_2, g_3) with $g_2 \leqslant 1$.

Suppose to the contrary that the embedding is not onto and take $p \in Z(B;1;(g_1, g_2 + g_3))$ outside its range. In other words, the permutation p' obtained from p by inserting 2 into position $g_1 + g_2 + 2$ and incrementing all the entries of p of value at least 2 contains a 1342 or 1432-pattern. Since p avoids $\{1342, 1432\}$, this pattern must involve the entry 2. The 2 cannot play the role of the '2' in such a pattern, because then the 1 would be forced to play the role of the '1', and there can be at most one entry between the 1 and 2 since $g_2 \leqslant 1$. The only other possible role for the 2 is as the '1', but in this case we could substitute the 1, thereby finding a B-pattern in p, a contradiction.

Of course, we can replace $\mathscr{J}(\pi)$ by $\mathscr{G}(\pi)$ only if we are able to work with $\mathscr{G}(\pi)$. Because $\mathbb{N}^{|\pi|+1}$ is partially well-ordered by the product ordering,[11] the basis of $\mathscr{G}(\pi)$, which

[10] This means that $\mathbf{x} \in \mathscr{G}(\pi)$ whenever $\mathbf{x} \leqslant \mathbf{y}$ for some $\mathbf{y} \in \mathscr{G}(\pi)$.

[11] This fact, which is not difficult to prove, can be found many places, for example Nash-Williams [30].

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is by definition an antichain, must be finite. However, this basis may be quite large, or may contain vectors with large components. For example, another way to state the Erdős–Szekeres Theorem [18] is that

$$\mathscr{G}(\emptyset) = \mathrm{Av}(((j-1)(k-1)+1))$$

when $B = \{12\cdots j, k\cdots 21\}$. While this does not preclude effective computation of $\mathscr{G}(\pi)$, it does suggest that such computations could be time-consuming.

This problem can be circumvented by replacing $\mathscr{G}(\pi)$ by a different set of gap vectors. First we return to the way in which $\mathscr{G}(\pi)$ is used. By our definitions, if $\pi(r)$ is ES^+-reducible for π then

$$|Z(B;\pi;\mathbf{g})| = \begin{cases} 0 & \text{if } \mathbf{g} \notin \mathscr{G}(\pi), \\ |Z(B;d_r(\pi);d_r(\mathbf{g}))| & \text{otherwise.} \end{cases}$$

This equality shows that when enumerating the B-avoiding descendants of π with gap vector $\mathbf{g}$, we first check to see if $\mathbf{g}$ lies in $\mathscr{G}(\pi)$. If $\mathbf{g} \notin \mathscr{G}(\pi)$ then we can be sure that no such descendants exist. If $\mathbf{g} \in \mathscr{G}(\pi)$ and $\pi(r)$ is ES^+-reducible, then $Z(B;\pi;\mathbf{g})$ is in one-to-one correspondence with $Z(B;d_r(\pi);d_r(\mathbf{g}))$. Note that if $Z(B;\pi;\mathbf{g})$ and $Z(B;d_r(\pi);d_r(\mathbf{g}))$ are both empty then they are trivially in one-to-one correspondence, so we could instead use the recurrence

$$|Z(B;\pi;\mathbf{g})| = \begin{cases} 0 & \text{if } \mathbf{g} \notin \mathscr{G}(\pi) \text{ and } d_r(\mathbf{g}) \in \mathscr{G}(d_r(\pi)), \\ |Z(B;d_r(\pi);d_r(\mathbf{g}))| & \text{otherwise.} \end{cases}$$

This equality shows that instead of considering $\mathscr{G}(\pi)$, we can look at the larger set of gap vectors for which either $Z(B;\pi;\mathbf{g}) \neq \emptyset$ or $Z(B;d_r(\pi);d_r(\mathbf{g})) = \emptyset$. Because this set need not be an ideal,[12] we consider the largest lower-order ideal of $\mathbb{N}^{|\pi|+1}$ for which these conditions hold:

$$\mathscr{G}_r(\pi) = \{\mathbf{g} \in \mathbb{N}^{|\pi|+1} : Z(B;\pi;\mathbf{h}) \neq \emptyset \text{ or } Z(B;d_r(\pi);d_r(\mathbf{h})) = \emptyset \text{ for all } \mathbf{h} \leqslant \mathbf{g}\}.$$

Note that $\mathscr{G}(\pi) \subseteq \mathscr{G}_r(\pi)$: if $\mathbf{g} \in \mathscr{G}(\pi)$ then $Z(B;\pi;\mathbf{g}) \neq \emptyset$, so $Z(B;\pi;\mathbf{h}) \neq \emptyset$ for all $\mathbf{h} \leqslant \mathbf{g}$, so $\mathbf{g} \in \mathscr{G}_r(\pi)$. With this observation we have

$$|Z(B;\pi;\mathbf{g})| = \begin{cases} 0 & \text{if } \mathbf{g} \notin \mathscr{G}_r(\pi), \\ |Z(B;d_r(\pi);d_r(\mathbf{g}))| & \text{otherwise,} \end{cases}$$

if $\pi(r)$ is ES^+-reducible for π and B.

This new set has several advantages over $\mathscr{G}(\pi)$. For one, its basis may be considerably smaller, thus simplifying the scheme. More importantly, the following bound on basis elements implies that $\mathscr{G}_r(\pi)$ can be found automatically.

[12] An example of this occurs with $B = \{231, k\cdots 21\}$. In Proposition 8.1 we observe that the set of $\mathbf{g} \in \mathbb{N}^3$ for which $Z(B;21;\mathbf{g}) \neq \emptyset$ is $\mathrm{Av}((0,1,0),(k-2,0,0))$ while the set of $\mathbf{g} \in \mathbb{N}^2$ for which $Z(B;1;\mathbf{g}) \neq \emptyset$ is $\mathrm{Av}((k-1,0))$, so the set of $\mathbf{g} \in \mathbb{N}^3$ for which either $Z(B;21;\mathbf{g}) \neq \emptyset$ or $Z(B;1;d_1(\mathbf{g})) = \emptyset$ is $\{(g_1,g_2,g_3) : g_1+g_2 \geqslant k-1\} \cup \{(g_1,0,g_3) : g_1 \leqslant k-3\}$.

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Proposition 6.2. *Suppose that $\pi(r)$ is an ES$^+$-reducible entry in π (with respect to the permutation class with basis B). Then each basis vector $\mathbf{b}$ of $\mathscr{G}_r(\pi)$ satisfies $\|\mathbf{b}\| \leqslant \|B\|_\infty - 1$.*

Proof. Suppose to the contrary that $\mathscr{G}_r(\pi)$ has a basis vector $\mathbf{b}$ with $\|\mathbf{b}\| \geqslant \|B\|_\infty$. By the minimality of $\mathbf{b}$, we have that $Z(B;\pi;\mathbf{b}) = \emptyset$ and $Z(B;d_r(\pi);d_r(\mathbf{b})) \neq \emptyset$. We also have that $Z(B;\pi;\mathbf{a}) \neq \emptyset$ or $Z(B;d_r(\pi);d_r(\mathbf{a})) = \emptyset$ for every $\mathbf{a} < \mathbf{b}$ because these vectors lie in $\mathscr{G}_r(\pi)$. The latter case can be eliminated easily: because $Z(B;d_r(\pi);d_r(\mathbf{b}))$ is nonempty, $Z(B;d_r(\pi);d_r(\mathbf{a}))$ must also be nonempty for all $\mathbf{a} \leqslant \mathbf{b}$. Therefore $Z(B;\pi;\mathbf{a}) \neq \emptyset$ for all $\mathbf{a} < \mathbf{b}$.

Now, since $\pi(r)$ is ES$^+$-reducible and $Z(B;\pi;\mathbf{a}) \neq \emptyset$ for all $\mathbf{a} < \mathbf{b}$, we know that there is a bijection between $Z(B;\pi;\mathbf{a})$ and $Z(B;d_r(\pi);d_r(\mathbf{a}))$ for all $\mathbf{a} < \mathbf{b}$. Thus we have the following diagram:

$$\begin{array}{ccc} Z(B;\pi;\mathbf{b}) = \emptyset & & Z(B;d_r(\pi);d_r(\mathbf{b})) \neq \emptyset \\ \downarrow & & \downarrow \\ Z(B;\pi;\mathbf{a}) \neq \emptyset & \longleftrightarrow & Z(B;d_r(\pi);d_r(\mathbf{a})) \neq \emptyset \end{array}$$

The rest of the proof is similar to the proof of Proposition 5.2. Choose a permutation $p \in Z(B;d_r(\pi);d_r(\mathbf{b}))$ and form p' by incrementing each entry of p that is at least $\pi(r)$ by 1 and inserting $\pi(r)$ into position $g_1 + \cdots + g_r + r$. Choose a specific occurrence of some $\beta \in B$ in p'. Since p avoids B, this occurrence of β must involve the entry $\pi(r)$. Let p'' denote the standardization of the subsequence of p' given by the entries from π together with the entries from the chosen occurrence of β. Therefore p'' contains a β-pattern and lies in $Z(\emptyset;\pi;\mathbf{a}) \setminus Z(B;\pi;\mathbf{a})$ for some $\mathbf{a} < \mathbf{b}$ with $\|\mathbf{a}\| \leqslant \|B\|_\infty - 1$. However, this is a contradiction because $d_r(p'')$ avoids B and thus lies in $Z(B;d_r(\pi);d_r(\mathbf{a}))$, and we have assumed that d_r is a bijection between $Z(B;\pi;\mathbf{a})$ and $Z(B;d_r(\pi);d_r(\mathbf{a}))$. □

We conclude this section by writing out the enumeration scheme that we have derived for Av(1342, 1432):

$$\begin{aligned} s_n(1342,1432) &= |Z(\{1342,1432\};\emptyset;(n))|, \\ |Z(\{1342,1432\};\emptyset;(g_1))| &= \sum_{i=0}^{g_1-1} |Z(\{1342,1432\};1;(i,g_1-i-1))| \qquad \text{for } g_1 \geqslant 1, \\ |Z(\{1342,1432\};1;(g_1,g_2))| &= \sum_{i=0}^{g_1-1} |Z(\{1342,1432\};21;(i,g_1-i-1,g_2))| \\ &\quad + \sum_{i=0}^{g_2-1} |Z(\{1342,1432\};12;(g_1,i,g_2-i-1))| \\ &\qquad\qquad \text{for } g_1, g_2 \geqslant 1 \\ &= \sum_{i=0}^{g_1-1} |Z(\{1342,1432\};1;(i,g_1+g_2-i-1))| \\ &\quad + 2|Z(\{1342,1432\};1;(g_1,g_2-1))| \qquad \text{for } g_1, g_2 \geqslant 1. \end{aligned}$$

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7. A collection of failures

In this section we collect numerous negative results needed to verify the lack of inclusions in Figure 1. The class Av(123) shows immediately that classes can have finite enumeration schemes without having regular insertion encodings. An example of a class with a regular insertion encoding but without a finite enumeration scheme is given by Av(1234, 4231). It follows routinely from Theorem 3.1 that this class has a rational insertion encoding, but the following proposition shows that it does not have a finite enumeration scheme.

Proposition 7.1. *For all k, the permutation $k\cdots 21$ is ES$^+$-irreducible for* Av(1234, 4231).

Proof. Let $\pi = k\cdots 21$. First we show that the entries $\pi(r) = k - r + 1$ for $r \in [k-1]$ are ES$^+$-irreducible. Let $\mathbf{g}$ denote the vector in $\mathbb{N}^{k+1}$ which is identically zero except for g_r and g_{r+1}, which are both 1 (these two components correspond to the gaps on either side of $\pi(r)$). Now observe that there are two permutations in $Z(\{1234, 4231\}; d_r(\pi); d_r(\mathbf{g}))$:

$$k(k-1)\cdots(k-r+2)(k+1)(k+2)(k-r)\cdots 21, \quad \text{and}$$
$$k(k-1)\cdots(k-r+2)(k+2)(k+1)(k-r)\cdots 21,$$

while $Z(\{1234, 4231\}; \pi; \mathbf{g})$ contains only one permutation,

$$k(k-1)\cdots(k-r+2)(k+1)(k-r+1)(k+2)(k-r)\cdots 21.$$

Thus $\mathbf{g} \in \mathscr{G}_r(\pi)$ but $Z(\{1234, 4231\}; \pi; \mathbf{g})$ and $Z(\{1234, 4231\}; d_r(\pi); d_r(\mathbf{g}))$ are not in one-to-one correspondence, so $\pi(r)$ is not ES$^+$-reducible for any $r \in [k-1]$.

To show that $\pi(k) = 1$ is not ES$^+$-reducible, it suffices to consider the gap vector $\mathbf{g} = (0, 0, \ldots, 0, 3, 0)$, for which $|Z(\{1234, 4231\}; \pi; \mathbf{g})| < |Z(\{1234, 4231\}; d_k(\pi); d_k(\mathbf{g}))|$, finishing the proof. □

The substitution decomposition approach appears, at least on the surface, completely independent from the other three methods. The class Av(321, 2341, 3412, 4123) has a finitely labelled generating tree by Theorem 2.1 (and thus it also has a finite enumeration scheme and a regular insertion encoding), but it contains infinitely many simple permutations.[13]

Moreover, the separable permutations defined in Section 4, which contain only three simple permutations, possess neither a finite enumeration scheme nor a regular insertion encoding. To see that they do not possess a regular insertion encoding, we need only note that they have an algebraic, but not rational, generating function (so for this purpose Av(132) would work just as well).

To establish that the separable permutations do not have a finite enumeration scheme, thereby completing our list of negative examples, we show that every unimodal permutation – that is, every permutation which consists of an increasing sequence followed by a decreasing sequence – is ES$^+$-irreducible for Av(2413, 3142). (This set forms a permutation class itself, with basis $\{213, 312\}$.) Therefore, not only do the separable permutations

[13] These simple permutations can be defined as the standardizations of even-length initial segments of the sequence $4, 1, 6, 3, 8, 5, \ldots, 2k+2, 2k-1, \ldots$.

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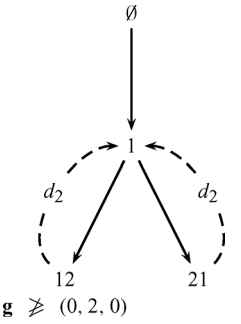


Figure 3. The enumeration scheme for Av(1342, 1432).

not have a finite enumeration scheme, but for each n there are 2^{n-1} ES$^+$-irreducible permutations. Moreover, one can observe (either from the definition or the basis) that the class of separable permutations is invariant under the eight permutation class symmetries, so none of them offer any simplification.

Proposition 7.2. *Every* $\pi \in \mathrm{Av}(213, 312)$ *is ES*$^+$*-irreducible for* $\mathrm{Av}(2413, 3142)$.

Proof. Let π denote a unimodal permutation of length k. First we show that $\pi(r)$ is not ES$^+$-reducible for any $2 \leqslant r \leqslant k$. If $\pi(r-1) > \pi(r)$, consider the gap vector $\mathbf{g}$ with all components 0 except for $g_r = g_{r+1} = 1$. Then $Z(\{2413, 3142\}; \pi; \mathbf{g})$ contains exactly one permutation, because to avoid 2413, the smaller of these new entries must be to the left of the larger one. However, $Z(\{2413, 3142\}; d_r(\pi); d_r(\mathbf{g}))$ contains 2 permutations (the most possible). The case where $\pi(r-1) < \pi(r)$ can be handled similarly with the gap vector that is 0 except for $g_{r-1} = g_r = 1$. Finally, $\pi(1)$ is not ES$^+$-reducible because $Z(\{2413, 3142\}; \pi; (2, 1, 0, \ldots, 0))$ contains at most 5 permutations (the new entries cannot form a 132-pattern, because that would give rise to a 2413-pattern), while $Z(\{2413, 3142\}; d_1(\pi); (3, 0, \ldots, 0))$ contains 6 permutations. □

8. An assortment of enumeration schemes successes

Here we present several examples of finite enumeration schemes. In these presentations, we adopt the following pictorial representation. If π is ES$^+$-irreducible, so $|Z(B; \pi; \mathbf{g})|$ is computed by summing over the B-avoiding children of π, then we draw a solid arrow from π to each of its children. If the entry $\pi(r)$ if ES$^+$-reducible in π then we draw a dashed arrow from π to $d_r(\pi)$, label this arrow with d_r, and indicate the basis of $G_r(\pi)$ beneath π. For example, the enumeration scheme for Av(1342, 1432) is shown in Figure 3.

We define the *depth* of an enumeration scheme to be the least integer k so that every permutation of length at least k is ES$^+$-reducible. Thus the scheme shown in Figure 3 is of depth 2, while Figure 4 shows two enumeration schemes of depth 1. The scheme on the right of this figure is for the class $\mathrm{Av}(M_{a,b})$ where $M_{a,b} = \{\beta \in S_{a+b+1} : \beta(a+1) = 1\}$. These were first counted by Mansour [26]. The scheme for Av(1234), which generates sequence A005802 in the OEIS [31], is shown in Figure 5.

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Figure 4. The enumeration schemes for the set of all permutations (left) and for Av($M_{a,b}$) (right).

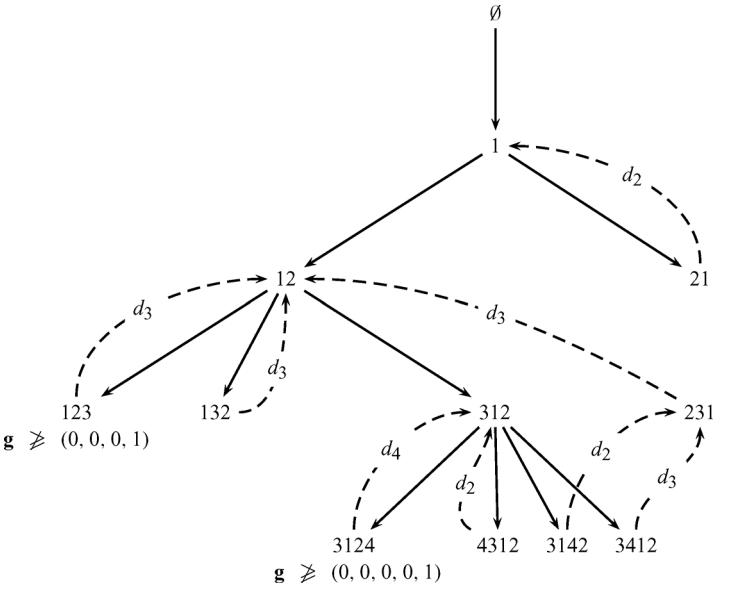


Figure 5. The enumeration scheme for Av(1234).

The 321-hexagon-avoiding permutations, which can be defined as

$$\mathrm{Av}(321, 46718235, 46781235, 56718234, 56781234),$$

have the scheme shown in Figure 6. This class was first introduced by Billey and Warrington [9], who showed how to compute the Kazhdan–Lusztig polynomials for them. Stankova and West [33] proved that the number, s_n, of these permutations of length n satisfies

$$s_n = 6s_{n-1} - 11s_{n-2} + 9s_{n-3} - 4s_{n-4} - 4s_{n-5} + s_{n-6}$$

for all $n \geqslant 7$, which gives sequence A058094 in the OEIS [31]. (Hence this complicated scheme produces only a sequence with a rational generating function.) Later, Mansour and Stankova [28] counted 321-($2k$)-gon-avoiding permutations for all k.

The *freely braided permutations*, introduced by Green and Losonczy [20], are the class

$$\mathrm{Av}(3421, 4231, 4312, 4321).$$

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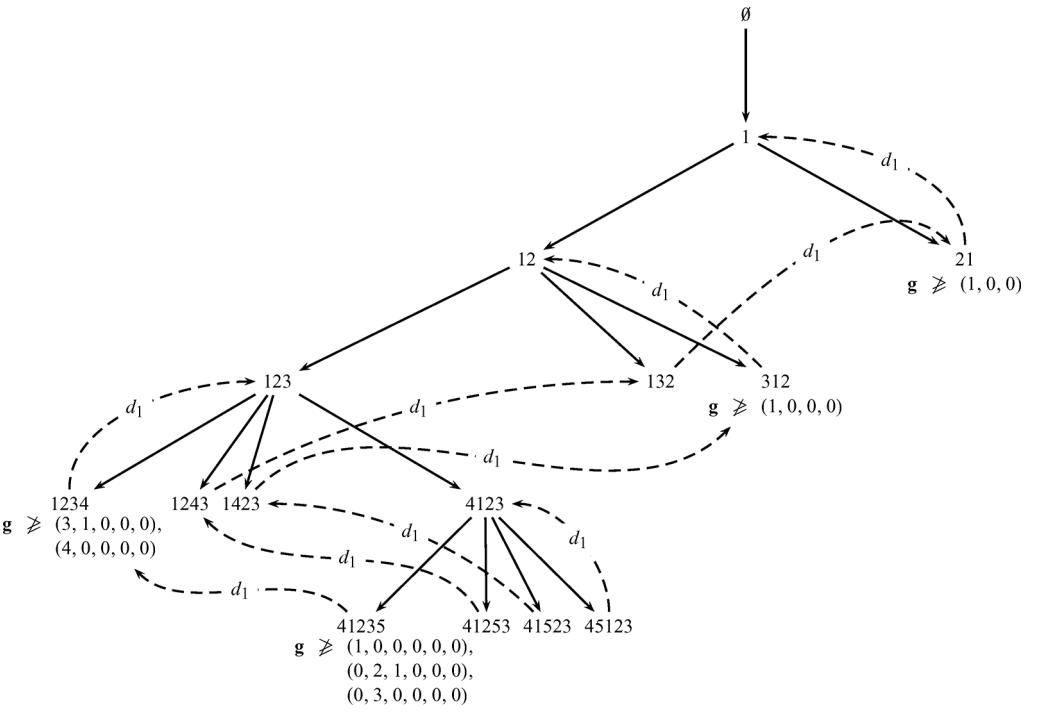


Figure 6. The enumeration scheme for the 321-hexagon-avoiding permutations.

Mansour [27] found that the freely braided permutations have the generating function

$$\frac{1-3x-2x^2+(1+x)\sqrt{1-4x}}{1-4x-x^2+(1-x^2)\sqrt{1-4x}}$$

(sequence A108600 in the OEIS [31]). They also have an enumeration scheme of depth 3, shown in Figure 7.

Chow and West [14] showed, using generating trees, that the classes in Figure 8 are Wilf-equivalent, that their generating functions are rational,[14] and that these generating functions can be expressed as quotients of Chebyshev polynomials. For any fixed k, WilfPlus can automatically and rigorously derive enumeration schemes for these classes. This makes it easy to make conjectures for the general form of these enumeration schemes for all k, although they must be verified by hand. We carry this out for one of these classes below.

Proposition 8.1. *The enumeration scheme for* $\mathrm{Av}(231, k\cdots21)$ *is as shown in Figure 8*(*b*).

[14] This fact can be verified quite quickly: by inverting the permutations in $\mathrm{Av}(321, 23\cdots k1)$ one obtains the class $\mathrm{Av}(321, k12\cdots(k-1))$, which satisfies the hypotheses of Theorem 2.1 (and thus also of Theorem 3.1). Note, however, that while symmetries of these classes always have finitely labelled generating trees, and thus also regular insertion encodings, the complexity of their generating trees and insertion encodings increases with k, while the complexity of their enumeration schemes stays fixed.

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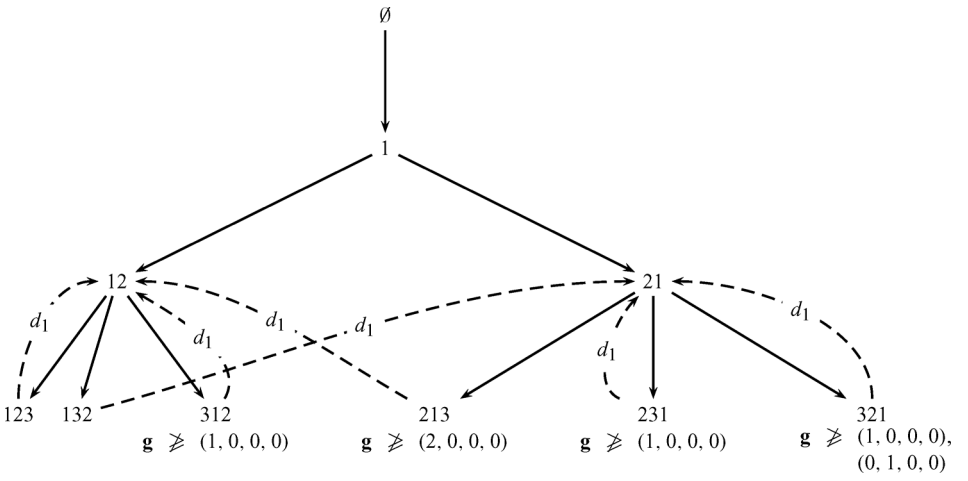


Figure 7. The enumeration scheme for the freely braided permutations, Av(3421, 4231, 4312, 4321).

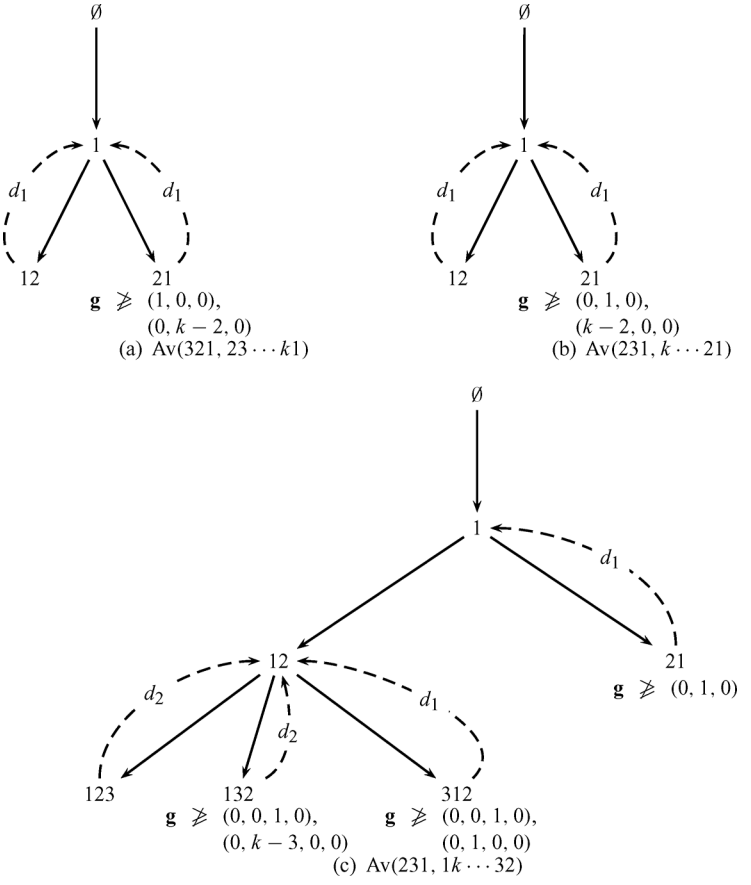


Figure 8. Enumeration schemes for three Wilf-equivalent classes.

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Proof. Let $B = \{231, k\cdots21\}$. First we verify the computations of $G_r(\pi)$ given in the diagram. For 12, the diagram shows that $\mathscr{G}_1(12) = \mathbb{N}^3$. In any permutation in $Z(B;12;(g_1,g_2,g_3))$, the entries before the 2 (other than the 1) must be in decreasing order from left to right as any ascent before the 2 would give rise to a 231-pattern. Then, since these permutations must avoid $k\cdots21$, there can be at most $k-2$ new entries before the 2, so $(g_1,g_2,g_3) \notin \mathscr{G}(12)$ whenever $g_1+g_2 \geqslant k-1$. The B-avoiding permutations

$$g_1+g_2+2,\ldots,g_2+4,g_2+3,1,g_2+2,\ldots,4,3,2,g_1+g_2+3,g_1+g_2+4,\ldots,g_1+g_2+g_3+2$$

then show that $\mathscr{G}(12)$ is $\mathrm{Av}(\{(g_1,g_2,g_3) : g_1+g_2 \geqslant k-1\})$. It can be shown similarly that $\mathscr{G}(1) = \mathrm{Av}((k-1,0))$. This shows that $\mathscr{G}_1(12) = \mathbb{N}^3$ because $Z(B;1;d_1(\mathbf{g}))$ is empty whenever $Z(B;12;\mathbf{g})$ is empty.

The computation for 21 is similar. One first checks that

$$\mathscr{G}(21) = \mathrm{Av}((0,1,0),(k-2,0,0)).$$

Since $\mathscr{G}(1) = \mathrm{Av}((k-1,0))$, the set of gap vectors $\mathbf{g}$ for which either $Z(B;21;\mathbf{g}) \neq \emptyset$ or $Z(B;1;d_1(\mathbf{g})) = \emptyset$ is $\{(g_1,g_2,g_3) : g_1+g_2 \geqslant k-1\} \cup \{(g_1,0,g_3) : g_1 \leqslant k-3\}$. This implies that $\mathscr{G}_1(21) = \mathrm{Av}((0,1,0),(k-2,0,0))$. Now we must verify the ES^+-reducible entries. The 12 case is clear because the only way 1 can participate in a 231 or $k\cdots21$-pattern is as the last entry, and in either case the 2 could play the same role. For 21, first note that the 2 also cannot participate in any 231 or $k\cdots21$-pattern if $\mathbf{g} \in \mathscr{G}_1(21)$: it cannot be the '2' in a 231 because there can be nothing between the 2 and the 1, and it cannot be the '2' in a $k\cdots21$-pattern because our restrictions on $\mathbf{g}$ prevent there from being enough entries to the left of the 2 to accommodate such a pattern. This shows that the 2 can only possibly be the minimal entry in a forbidden pattern, but then the 1 could play the same role. □

In addition to the examples already mentioned, there are several other interesting classes avoiding a pair of permutations of length four.[15]

- Kremer and Shiu [23] proved that there are four classes of this form counted by $(4^{n-1}+2)/3$. These all have finite enumeration schemes: $\mathrm{Av}(1234,2143)$ and $\mathrm{Av}(1432,2341)$ have schemes of depth 3, $\mathrm{Av}(2341,4321)$ has a scheme of depth 4, and $\mathrm{Av}(2143,4123)$ has a scheme of depth 6.
- Kremer [22] (see also Stanley [35, Exercise 6.39.l]) completed the characterization of the classes counted by the large Schröder numbers. Up to symmetry, there are ten such classes, seven of which have finite enumeration schemes, as shown in Table 2.
- A permutation is said to be *skew-merged* if it is the union of an increasing subsequence and a decreasing subsequence. Stankova [32] was the first to prove that the skew-merged permutations are given by $\mathrm{Av}(2143,3412)$. Atkinson [3] showed that the generating function for this class is

$$\frac{1-3x}{(1-2x)\sqrt{1-4x}}$$

(sequence A029759 in the OEIS [31]). This class has an enumeration scheme of depth four.

[15] The characterization of the Wilf-equivalences between these classes has recently been completed by Le [24].

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Table 2.

Class	Finite enumeration scheme?
Av(1342, 2341)	Yes, of depth 3
Av(1342, 1432)	Yes, of depth 2, shown in Figure 3
Av(2341, 2413)	No
Av(2413, 3142)	No, these are the separable permutations considered in Proposition 7.2
Av(2431, 3241)	No
Av(3241, 3421)	Yes, of depth 4
Av(3241, 4231)	Yes, of depth 2 (Knuth [21] proved that these are precisely the permutations that can be sorted by an input-restricted deque)
Av(3412, 3421)	Yes, of depth 3
Av(3421, 4321)	Yes, of depth 2
Av(3421, 4231)	Yes, of depth 4

9. Conclusion

We have developed an extension of Zeilberger's enumeration schemes and provided the mechanisms for their automatic generation, a task implemented in the accompanying Maple package WilfPlus, which is available from the author's homepage. As the examples in Section 7 demonstrate, there remain considerable differences in the applicability of enumeration schemes, substitution decompositions, and the insertion encoding. This obviously suggests the following question.

Question 9.1. *Is there a systematic method of permutation class enumeration which is applicable to all classes with finite enumeration schemes, all classes with regular insertion encodings, and all classes with only finitely many simple permutations?*

Additionally, one would like such a method to be invariant under the eight permutation class symmetries, but Question 9.1 is probably demanding enough in the form above.

We conclude by collecting two of the questions raised earlier.

Question 9.2. *Is every sequence produced by a finite enumeration scheme holonomic?*

As demonstrated by the examples in Section 8, another interesting question is the equivalence problem:

Question 9.3. *Is it decidable whether two finite enumeration schemes produce the same sequence?*

Brlek, Duchi, Pergola and Rinaldi [13] consider a related issue, the equivalence problem for generating trees.

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