J. London Math. Soc. (2) 70 (2004) 383–404 © 2004 London Mathematical Society
DOI: 10.1112/S0024610704005435

COMPLEXIFICATION OF LAMBDA LENGTH AS PARAMETER FOR SL(2, ℂ) REPRESENTATION SPACE OF PUNCTURED SURFACE GROUPS

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Dedicated to Professor Hiroki Sato on the occasion of his sixtieth birthday

Abstract

A coordinate-system called λ-lengths is constructed for an $\mathrm{SL}(2,\mathbb{C})$ representation space of punctured surface groups. These λ-lengths can be considered as complexification of R. C. Penner's λ-lengths for decorated Teichmüller spaces of punctured surfaces. Via the coordinates the mapping class group acts on the representation space as a group of rational transformations. This fact is applied to find hyperbolic 3-manifolds which fibre over the circle.

0. *Introduction*

Let $F_{g,m}$ denote an orientable compact surface of genus g with m boundary curves, and $F'_{g,m}$ denote the surface $F_{g,m}$ with one point p deleted. We assume that $2g-1+m>0$ and the m boundary curves are ordered. The fundamental group $G'_{g,m}$ of $F'_{g,m}$ is free on generators

$$a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_m,$$

where c_i represents the class of the ith boundary curve. Then

$$d = c_m^{-1} \ldots c_1^{-1} \left(\prod_{i=1}^{g} b_i a_i b_i^{-1} a_i^{-1} \right)$$

is the class of a loop which goes once around the puncture p.

Let $\hat{R}'_{g,m}$ denote the space of all faithful representations ρ of $G'_{g,m}$ into $\mathrm{SL}(2,\mathbb{C})$ such that $\rho(d)$ is parabolic and $\mathrm{tr}\rho(d) = -2$. The group $\mathrm{SL}(2,\mathbb{C})$ acts on $\hat{R}'_{g,m}$; for $A \in \mathrm{SL}(2,\mathbb{C})$ and $\rho \in \hat{R}'_{g,m}$, $(A,\rho) \longmapsto \rho^A$, where

$$\rho^A(g) = A^{-1}\rho(g)A \qquad \text{for } g \in G'_{g,m}.$$

Let $R'_{g,m}$ denote the orbit space. Let $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$. Then $R'_{g,m}(\alpha_1, \ldots, \alpha_m)$ means the subspace of $R'_{g,m}$ which consists of the classes of those ρ satisfying $\mathrm{tr}\rho(c_i) = -\alpha_i$ for $i = 1, \ldots, m$. This space may be empty for some choice of $\alpha_1, \ldots, \alpha_m$; see Remark 0.1. Two n-tuples $(A_1, \ldots, A_n)$ and $(B_1, \ldots, B_n)$ of matrices of $\mathrm{SL}(2,\mathbb{C})$ are said to be equivalent if there exists $C \in \mathrm{SL}(2,\mathbb{C})$ such that $A_i = C^{-1}B_iC$ for $i = 1, \ldots, n$. The equivalence class of $(A_1, \ldots, A_n)$ is denoted by $\{A_1, \ldots, A_n\}$. Then $R'_{g,m}(\alpha_1, \ldots, \alpha_m)$ is identified with the space of all $\{A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_m\}$

Received 11 April 2003; revised 9 December 2003.

2000 *Mathematics Subject Classification* 30F40 (primary), 57M05 (secondary).

Partially supported by Grant-in-Aid for Scientific Research (13440045), Ministry of Education, Science and Culture of Japan.

https://doi.org/10.1112/S0024610704005435 Published online by Cambridge University Press

such that $A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_m$ generate a free group of rank $2g+m+1$, $\alpha_i = -\mathrm{tr}C_i$ and

$$D^{-1} = [A_1, B_1] \ldots [A_g, B_g] C_1 \ldots C_m$$

is parabolic and $\mathrm{tr}D = -2$. It is possible to give a global coordinate-system to $R'_{g,m}$ by trace functions on some finite set of elements of $G'_{g,m}$ and there are plenty of publications related to this fact.

REMARK 0.1. If $\alpha_i = 2\cos q_i\pi/p_i$ for some indices i, where integers p_i and q_i are relatively prime, then $R'_{g,m}(\alpha_1, \ldots, \alpha_m)$ is empty. In such cases we define $R'_{g,m}(\alpha_1, \ldots, \alpha_m)$ to be the set of the classes of mappings $\rho : \{a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_m\} \longrightarrow \mathrm{SL}(2, \mathbb{C})$ such that $\rho(d)$ is parabolic and $\mathrm{tr}\rho(d) = -2$,

$$\rho\left(\left(\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1}\right) c_1 \ldots c_m d\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and ρ induces a faithful representation

$$G'_{g,m}(\alpha_1, \ldots, \alpha_m) \longrightarrow \mathrm{PSL}(2, \mathbb{C}),$$

where the group $G'_{g,m}(\alpha_1, \ldots, \alpha_m)$ is obtained by adding relations $c_j^{p_j} = 1$ to the group presentation of $G'_{g,m}$. Two such mappings ρ_1 and ρ_2 are equivalent if there exists $A \in \mathrm{SL}(2, \mathbb{C})$ such that $\rho_2(g) = A^{-1}\rho_1(g)A$ for $g \in \{a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_m\}$. Most of the arguments in the sequel apply also to this space except for $\mathcal{R}'_{0,2}(0,0)$. Note that the $(2,2,\infty)$-triangle group consists of Euclidean motions.

The purpose of this paper is to give a system of coordinates called λ-lengths. They are, in a sense, complexifications of the λ-lengths introduced by R. C. Penner in [**10**] to parametrize the decorated Teichmüller space of punctured surface groups. We illustrate our motivation by once-punctured torus groups: consider the space of $\{A, B\}$, where $A, B \in \mathrm{SL}(2, \mathbb{C})$ are such that $ABA^{-1}B^{-1}$ is parabolic and $\mathrm{tr}ABA^{-1}B^{-1} = -2$. It is well known that $\lambda_1 = \mathrm{tr}A$, $\lambda_2 = \mathrm{tr}B$ and $\lambda_3 = \mathrm{tr}AB$ satisfy

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - \lambda_1\lambda_2\lambda_3 = 0.$$

If none of λ_1, λ_2, λ_3 is zero, then these numbers determine matrices A and B up to conjugacy:

$$A = \begin{pmatrix} \dfrac{\lambda_1\lambda_2 + \lambda_3}{2\lambda_2} & \dfrac{\lambda_1^2\lambda_2^2 - 4\lambda_2^2 - \lambda_3^2}{8\lambda_1\lambda_2^2} \\ 2\lambda_1 & \dfrac{\lambda_1\lambda_2 - \lambda_3}{2\lambda_2} \end{pmatrix},$$

$$B = \begin{pmatrix} \dfrac{\lambda_1\lambda_2 + \lambda_3}{2\lambda_1} & \dfrac{-\lambda_1^2\lambda_2^2 + 4\lambda_1^2 + \lambda_3^2}{8\lambda_1^2\lambda_2} \\ -2\lambda_2 & \dfrac{\lambda_1\lambda_2 - \lambda_3}{2\lambda_1} \end{pmatrix}.$$

Therefore we can employ $(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{C}^*)^3$, where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, as coordinates for representation space of once-punctured torus group into $\mathrm{SL}(2, \mathbb{C})$. The changes of generators $S : \{A, B\} \longmapsto \{A, BA\}$ and $T : \{A, B\} \longmapsto \{ABA^{-1}, A^{-1}\}$ induce

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the mappings

$$S_*(\lambda_1,\lambda_2,\lambda_3)=(\lambda_1,\lambda_3,\lambda_1\lambda_3-\lambda_2),\quad T_*(\lambda_1,\lambda_2,\lambda_3)=(\lambda_2,\lambda_1,\lambda_1\lambda_2-\lambda_3).$$

Let $\sigma=TSTS^{-1}$. Then $\sigma_*=T_*S_*T_*S_*^{-1}$ fixes $\tau=(\frac{3-\sqrt{-3}}{2},\frac{3+\sqrt{-3}}{2},\frac{3-\sqrt{-3}}{2})$ and its complex conjugate. Let A and B be the matrices corresponding to τ. The fact that σ_* fixes τ means that $\{\sigma_*(A),\sigma_*(B)\}$ and $\{A,B\}$ are equivalent and thus there exists an $M\in\mathrm{SL}(2,\mathbb{C})$ such that

$$(ABA^{-3},A^2B^{-1}A^{-1})=\sigma(A,B)=(MAM^{-1},MBM^{-1}).$$

Actually,

$$M=\begin{pmatrix}1 & (3-\sqrt{-3})/6\\ 0 & 1\end{pmatrix}.$$

The group Γ generated by A, B and M is commensurable with the figure-eight knot group, because the conjugate of Γ by

$$R=\begin{pmatrix}1 & (9-\sqrt{-3})/24\\ 0 & 1\end{pmatrix},$$

contains $P=RAR^{-1}$ and $Q=RBR^{-1}$; hence it also contains

$$PQP^{-1}Q^{-1}=\begin{pmatrix}-1 & -1\\ 0 & -1\end{pmatrix},\quad Q(PQP^{-1}Q^{-1})Q^{-1}=\begin{pmatrix}-1 & 0\\ -12z & -1\end{pmatrix},$$

where $z=e^{\pi i/3}$. See [**6**, 4.4].

A useful property of Penner's λ-lengths is that they admit a rational representation of the mapping class group; see [**10**]. However, they are real analytic by nature because they are introduced to parametrize the Teichmüller space as a Fuchsian representation space of surface groups. This means that employing them we cannot pursue the above method to find a hyperbolic manifold fibred over the circle. We need to complexify λ-lengths to be able to find fixed points, if any, of the action of a mapping class and to find a hyperbolic 3-manifold. This is the main reason for our investigation of complexified λ-lengths. A λ-length here is a variant of the trace function. More precisely, it is the sum of two trace functions of canonical generators of $G'_{0,2}$ embedded into $G'_{g,m}$. We shall show that each ideal triangulation of $F'_{g,m}$ yields a system of λ-lengths which give global coordinates to $R'_{g,m}$ together with trace functions on the homotopy classes of boundary curves.

Using the Ptolemy equation (see Proposition 1.1), we will prove that the mapping class group $\mathcal{MC}'_{g,m}$ (fixing all boundary points) acts as a group of rational transformations on $\mathbb{C}^{6g-3+2m}$. Thus we obtain an $\mathcal{MC}'_{g,m}$ invariant holomorphic two-form in the same way as was done in [**11**]. We will also prove that any mapping class which is a composite of Dehn twists on mutually disjoint simple loops has no fixed points in $R'_{g,m}$.

1. *Trace identities*

The group $\mathrm{SL}(2,\mathbb{C})$ consists of matrices of the form:

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix},\qquad a,b,c,d\in\mathbb{C},\ ad-bc=1.$$

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We shall write $(a\ b\,|\,c\ d)$ for the above matrix. The matrix $(a\ b\,|\,c\ d)$ acts on the Riemann sphere $\hat{\mathbb{C}}$ by $z \longmapsto (az+b)/(cz+d)$. Matrices A and B of $\mathrm{SL}(2,\mathbb{C})$ satisfy the following trace identities.

(1) $\mathrm{tr}A = \mathrm{tr}A^{-1}$.

(2) $\mathrm{tr}AB + \mathrm{tr}AB^{-1} = \mathrm{tr}A\mathrm{tr}B$.

Equalities (1) and (2) induce the following identity for three matrices of $\mathrm{SL}(2,\mathbb{C})$.

(3) $\mathrm{tr}ABC = \mathrm{tr}A\mathrm{tr}BC + \mathrm{tr}B\mathrm{tr}CA + \mathrm{tr}C\mathrm{tr}AB - \mathrm{tr}A\mathrm{tr}B\mathrm{tr}C - \mathrm{tr}BAC$.

These identities yield other identities. One of the most interesting is the identity which we refer to as the Ptolemy equation.

PROPOSITION 1.1. *Let* $A, B, C, D \in \mathrm{SL}(2,\mathbb{C})$ *and* $x = (\mathrm{tr}A + \mathrm{tr}BCD)$, $y = (\mathrm{tr}B + \mathrm{tr}CDA)$, $z = (\mathrm{tr}C + \mathrm{tr}DAB)$, $w = (\mathrm{tr}D + \mathrm{tr}ABC)$, $u = (\mathrm{tr}AB + \mathrm{tr}CD)$, $v = (\mathrm{tr}BC + \mathrm{tr}AD)$. *If* $\mathrm{tr}ABCD = -2$, *then*

$$xz + yw = uv. \tag{1.1}$$

For the proof, see Appendix A. Later in Section 4 we use (1.1) to describe the action of the mapping class group on the representation spaces of punctured surface groups to $\mathrm{SL}(2,\mathbb{C})$ as Penner did in [**10**].

2. *Case of* $R'_{0,2}$

2.1.

Let $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. We shall identify $R'_{0,2}$ with the set of $\{A, B\}$, where A and B generate a free group of rank 2, AB is parabolic with $\mathrm{tr}AB = -2$. In this section we show that $\mathrm{tr}A + \mathrm{tr}B \neq 0$ for every $\{A, B\} \in R'_{0,2}$. The subgroup of $\mathrm{SL}(2,\mathbb{C})$ consisting of matrices commutative with $(-1\ -1\,|\,0\ -1)$ is

$$\mathcal{G}_\infty = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & c \\ 0 & -1 \end{pmatrix} : b, c \in \mathbb{C} \right\}.$$

LEMMA 2.1. *Let* $A, B \in \mathrm{SL}(2,\mathbb{C})$ *be such that* AB *is parabolic with* $\mathrm{tr}AB = -2$. *Let* p *be the fixed point of* AB. *Then* $\mathrm{tr}A + \mathrm{tr}B = 0$ *if and only if* $B(p) = p$.

Proof. We may assume that $AB = (-1\ -1|0\ -1)$. Assume first that $\infty = B(\infty) = A^{-1}(\infty)$. Then A and B are of the form:

$$A = \begin{pmatrix} a & \alpha \\ 0 & 1/a \end{pmatrix}, \quad B = \begin{pmatrix} b & \beta \\ 0 & 1/b \end{pmatrix}.$$

Since $AB = (-1\ -1\,|\,0\ -1)$, $ab = -1$ and thus $\mathrm{tr}A + \mathrm{tr}B = 0$.

Next assume that $B(\infty) = A^{-1}(\infty) \neq \infty$. By conjugating A and B by the matrix $(1\ B(\infty)|0\ 1)$, we assume that $B(\infty) = 0$. Then A and B are of the form

$$A = \begin{pmatrix} -a & 1/\lambda \\ -\lambda & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1/\lambda \\ -\lambda & -b \end{pmatrix}, \tag{2.1}$$

where $\lambda = -(\mathrm{tr}A + \mathrm{tr}B)$ and $\lambda \neq 0$. □

The proof of the previous lemma yields the following.

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COROLLARY 2.1. *For any pair of complex numbers $a, b \in \mathbb{C}$ with $a + b \neq 0$, there exists a unique class $\{A, B\}$ satisfying $\mathrm{tr}A = -a$, $\mathrm{tr}B = -b$ and $\mathrm{tr}AB = -2$.*

For the matrices in (2.1) it holds that $A(\infty) - B(\infty) = a/\lambda - 0$. Since this value is left unchanged if A and B are conjugated by an element of $\mathcal{G}_\infty$, we obtain the following.

COROLLARY 2.2. *Let $\lambda \in \mathbb{C}^*$. Then for any class $\{A, B\} \in R'_{0,2}$ satisfying $\mathrm{tr}AB = (-1\ -1\,|\,0\ -1)$ and $\lambda = -(\mathrm{tr}A + \mathrm{tr}B)$,*

$$A(\infty) - B(\infty) = -\mathrm{tr}A/\lambda. \tag{2.2}$$

2.2.

Let A and B satisfy the condition of Lemma 2.1. We assume that $D = AB$, where $D = (-1\ -1\,|\,0\ -1)$. If $B(\infty) = A^{-1}(\infty) = \infty$, then the $(2,1)$-entry of A is 0. The commutator $ADA^{-1}D^{-1}$ commutes with D. Hence, if A and B generate a free group of rank 2, then $\mathrm{tr}A + \mathrm{tr}B \neq 0$.

3. *Cases of $R'_{0,3}$*

In this section we deduce several properties of a triple of matrices A, B, C in $\mathrm{SL}(2,\mathbb{C})$ which generate a free group of rank 3 and such that ABC is parabolic with $\mathrm{tr}ABC = -2$.

3.1.

For the next proposition, see [**5**] and [**8**].

PROPOSITION 3.1. *Let $A, B, C \in \mathrm{SL}(2,\mathbb{C})$ and $D = (ABC)^{-1}$. Then $x = -\mathrm{tr}BC$, $y = -\mathrm{tr}CA$, $z = -\mathrm{tr}AB$, $a = -\mathrm{tr}A$, $b = -\mathrm{tr}B$, $c = -\mathrm{tr}C$ and $d = -\mathrm{tr}D$ satisfy*

$$x^2+y^2+z^2-xyz+(ad+bc)x+(bd+ca)y+(cd+ab)z+a^2+b^2+c^2+d^2+abcd-4 = 0.$$

If, in particular, $-d = \mathrm{tr}D = -2$, then $\lambda_1 = x + a$, $\lambda_2 = y + b$ and $\lambda_3 = z + c$ satisfy

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + a\lambda_2\lambda_3 + b\lambda_3\lambda_1 + c\lambda_1\lambda_2 = \lambda_1\lambda_2\lambda_3. \tag{3.1}$$

PROPOSITION 3.2. *Let $a, b, c \in \mathbb{C}$. If $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}^*$ satisfy (3.1), then there exists a unique class $\{A, B, C\}$ of matrices of $\mathrm{SL}(2,\mathbb{C})$ such that $\mathrm{tr}ABC = -2$ and that $a = -\mathrm{tr}A$, $b = -\mathrm{tr}B$, $c = -\mathrm{tr}C$, $\lambda_1 = a - \mathrm{tr}BC$, $\lambda_2 = b - \mathrm{tr}CA$, $\lambda_3 = c - \mathrm{tr}AB$.*

Proof. We will show that A, B and C are determined uniquely by the conditions $ABC = (-1\ -1\ |\ 0\ -1)$ and $C(\infty) = 0$. Let $\xi = BC(\infty)$ and $Q = (1\ \xi\,|\,0\ 1)$. Then, by (2.1),

$$AB = \begin{pmatrix} c-\lambda_3 & 1/\lambda_3 \\ -\lambda_3 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1/\lambda_3 \\ -\lambda_3 & -c \end{pmatrix},$$

and

$$A = Q\begin{pmatrix} -a & 1/\lambda_1 \\ -\lambda_1 & 0 \end{pmatrix}Q^{-1}, \quad BC = Q\begin{pmatrix} 0 & 1/\lambda_1 \\ -\lambda_1 & -\lambda_1 + a \end{pmatrix}Q^{-1}.$$

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Since $\mathrm{tr}A^{-1}(AB)+\mathrm{tr}CA=-\lambda_2$, we have $\xi=\lambda_2\lambda_1^{-1}\lambda_3^{-1}$. Therefore A, B and C must be of the form

$$A=\begin{pmatrix}-\lambda_2/\lambda_3-a & (\lambda_2^2+a\lambda_2\lambda_3+\lambda_3^2)/\lambda_1\lambda_3^2\\ -\lambda_1 & \lambda_2/\lambda_3\end{pmatrix},$$
$$B=\begin{pmatrix}-\lambda_1/\lambda_3-b & \lambda_2/\lambda_3^2\\ -(\lambda_1^2+b\lambda_1\lambda_3+\lambda_3^2)/\lambda_2 & \lambda_1/\lambda_3\end{pmatrix}, \tag{3.2}$$
$$C=\begin{pmatrix}0 & 1/\lambda_3\\ -\lambda_3 & -c\end{pmatrix}. \qquad \square$$

Note that $BC(\infty)-C(\infty)=A^{-1}(\infty)-C(\infty)=\lambda_2/\lambda_1\lambda_3-0$, if A,B and C are the matrices in (3.2).

COROLLARY 3.1. *Under the condition of Proposition* 3.2,

$$BC(\infty)-C(\infty)=\frac{\lambda_2}{\lambda_1\lambda_3},$$

provided that $ABC=(-1\ \ -1\mid 0\ \ -1)$.

3.2.

Suppose that matrices A,B and C of $\mathrm{SL}(2,\mathbb{C})$ are such that $D=ABC=(-1\ \ -1\,|\,0\ \ -1)$ and that they generate a free group Γ of rank 3. Then the stability subgroup Γ_∞ of ∞ is cyclic generated by D. Since Γ is free, $C\notin\Gamma_\infty$ and $BC\notin\Gamma_\infty$.

Let $A',B',C'\in\Gamma$ be such that $A'B'C'=D$, $C'(\infty)=C(\infty)$ and that $B'C'(\infty)=BC(\infty)$. If $\lambda_a=-(\mathrm{tr}A+\mathrm{tr}BC)$, $\lambda'_a=-(\mathrm{tr}A'+\mathrm{tr}B'C')$, $\lambda_b=-(\mathrm{tr}B+\mathrm{tr}CA)$, $\lambda'_b=-(\mathrm{tr}B'+\mathrm{tr}C'A')$, $\lambda_c=-(\mathrm{tr}C+\mathrm{tr}AB)$ and $\lambda'_c=-(\mathrm{tr}C'+\mathrm{tr}A'B')$, then Corollary 3.1 implies that $\lambda'_b/(\lambda'_c\lambda'_a)=\lambda_b/(\lambda_c\lambda_a)$. This identity can be verified also in another way. Since $\Gamma_\infty=\langle D\rangle$, there exist integers α,β and γ such that $A'=D^\alpha A$, $B'=BCD^\beta C^{-1}$ and $C'=CD^\gamma$. Since $A'B'C'=D$, $\alpha+\beta+\gamma=0$. Then $\mathrm{tr}A'+\mathrm{tr}B'C'=\mathrm{tr}D^\alpha A+\mathrm{tr}A^{-1}D^{1-\alpha}$. For each integer α define $f_\alpha=\mathrm{tr}D^\alpha A+\mathrm{tr}A^{-1}D^{1-\alpha}$. Then $f_0=\mathrm{tr}A+\mathrm{tr}BC$, $f_1=-(\mathrm{tr}A+\mathrm{tr}BC)$, and

$$\begin{aligned}f_\alpha&=(\mathrm{tr}D\mathrm{tr}D^{\alpha-1}A-\mathrm{tr}D^{\alpha-2}A)+(\mathrm{tr}A^{-1}D^{2-\alpha}\mathrm{tr}D^{-1}-\mathrm{tr}AD^{3-\alpha})\\&=-2f_{\alpha-1}-f_{\alpha-2},\end{aligned}$$

hence $\mathrm{tr}A'+\mathrm{tr}B'C'=f_\alpha=(-1)^\alpha(\mathrm{tr}A+\mathrm{tr}BC)$. We have shown that $\lambda'_a=(-1)^\alpha\lambda_a$. Similarly we have $\lambda'_b=(-1)^\beta\lambda_b$ and $\lambda'_c=(-1)^\gamma\lambda_c$; hence

$$\frac{\lambda'_b}{\lambda'_c\lambda'_a}=(-1)^{\beta-\alpha-\gamma}\frac{\lambda_b}{\lambda_c\lambda_a}=\frac{\lambda_b}{\lambda_c\lambda_a}.$$

4. *Complex λ-length and trace identities*

The purpose of this section is to establish trace identities for representations of a punctured surface group in $\mathrm{SL}(2,\mathbb{C})$ associated with an ideal triangulation of the surface.

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4.1. *Ideal arcs*

Let $F = F_{g,m}$, $F' = F'_{g,m}$ and $\mathrm{Int}(F)$ be the interior of F. Let $S^1 = \{e^{2\pi i\theta} : 0 \leqslant \theta \leqslant 1\}$ be the unit circle. An *ideal arc* is the image c of an injective continuous map

$$(S^1, 1, S^1 - \{1\}) \longrightarrow (\mathrm{Int}(F), p, F - \{p\}).$$

The image of $S^1 - \{1\}$ is called the *interior* of c. Two ideal arcs c, c' are *isotopic* if there exists a homeomorphism ϕ of F isotopic relative to $\{p\} \cup \partial F$ to the identity map such that $\phi(c) = c'$.

An *ideal arc system* Δ is a collection of non-isotopic ideal arcs with disjoint interiors, such that each component of $F - \Delta$ is either a disk or an annulus bordered by an ideal arc and a boundary curve of F. A disk component P of $F - \Delta$ is called a *polygon*. There is a characteristic map $D \longrightarrow F$ for P, where D is a convex Euclidean polygon, so that each vertex of D goes to p, each side of D goes to some ideal arc of Δ. Polygons are referred to as *triangles, quadrilaterals* etc. in Δ depending on the number of sides. An ideal curve system Δ can contain at most d ideal arcs, where

$$d = 6g + 2m - 3.$$

If Δ contains the maximal number of curves, then each disk component of $F - \Delta$ is a triangle. In this case we call Δ an *ideal triangulation* of F'. For an ideal triangulation Δ, there are m annuli and d_1 triangles in $F - \Delta$, where

$$d_1 = 4g + m - 2.$$

Definition 4.1. Let I be the unit interval. For a mapping $h : S^1 \times I \longrightarrow F$, we write $h_t(s) = h(s,t)$. Let c be an ideal arc in F. Choose an embedding $h : S^1 \times I \longrightarrow F$ such that $h_{1/2}$ is a characteristic map for c and denote by c_0, c_1 the images of S^1 under h_0 and h_1, respectively.

If ρ is a representation of $G'_{g,m}$ into $\mathrm{SL}(2,\mathbb{C})$, the trace function $\mathrm{tr}\rho$ is defined on the free homotopy classes of closed curves in $F' = F - \{p\}$ and depends only on the class of ρ. We call

$$\lambda(c,\rho) = -(\mathrm{tr}\rho(c_0) + \mathrm{tr}\rho(c_1))$$

the λ *length* of c with respect to $[\rho]$.

Let $\rho \in \hat{R}'_{g,m}$. Provided appropriate orientations of c_0 and c_1, the matrices $A = \rho(c_0)$ and $B = \rho(c_1)$ satisfy $\mathrm{tr}AB = -2$. By a suitable conjugation, we may assume that A and B are of the form (2.1). If ρ is a Fuchsian representation, then $\rho G'_{g,m}$ acts on $\mathbb{H}^2$ (the upper half-plane model of the 2-dimensional hyperbolic space). The horocycles $H_0 = \{x+i : -\infty < x < \infty\}$ and $H_1 = B(H_0)$ are based at the endpoints of the positive imaginary axis, which projects to a geodesic ideal arc on $\mathbb{H}^2/\rho G'_{g,m}$ homotopic to c. Since the $(2,1)$-entry of B is $-\lambda$, the hyperbolic distance between H_0 and H_1 is $2\log|\lambda|$. Therefore $|\lambda|$ coincides with the λ-length introduced in [**10**]. We note also that, due to the negative trace -2 of $\rho(d)$, in the Fuchsian case traces of A and B are simultaneously positive or negative [**12**, Section 33].

4.2. *Trace identities revisited*

Let $\Delta = (c_1, \ldots, c_d)$ be an ideal triangulation of $F' = F'_{g,m}$. We assume that $c_1, \ldots, c_m$ are on the border of m annuli in $F - \Delta$ and that c_i is freely homotopic

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in F to the ith boundary curve of F, which we denote by $\hat{c}_i$. Let $T_1, \ldots, T_{d_1}$ be triangles and $A_1, \ldots, A_m$ annuli in $F - \Delta$. For $i = 1, \ldots, d_1$, let c_{i1}, c_{i2}, c_{i3} be the ideal arcs in Δ which are the sides of T_i.

Following the terminology in [**7**], we call a triangle in an ideal triangulation of F' *untwisted* if its regular neighbourhood N in F is a three-holed sphere and *twisted* if N is a one-holed torus.

We define a function on the triangles in $F - \Delta$ by

$$\epsilon(T) = \begin{cases} +1 & \text{if } T \text{ is untwisted} \\ -1 & \text{if } T \text{ is twisted.} \end{cases}$$

Let $[\rho] \in R'_{g,m}$. For a triangle T with sides $a, b, c \in \Delta$, we define $s(T, \rho)$ to be

$$s(T, \rho) = \frac{\lambda(a, \rho)}{\lambda(b, c)\lambda(b, \rho)} + \frac{\lambda(c, \rho)}{\lambda(a, \rho)\lambda(c, \rho)} + \frac{\lambda(a, \rho)}{\lambda(b, \rho)\lambda(c_2, \rho)}.$$

PROPOSITION 4.1. *Let an ideal triangulation Δ of $F_{g,m}$ be given. Then with the notation above, the following identity holds for all $[\rho] \in R'_{g,m}$.*

$$\sum_{i=1}^{d_1} \epsilon(T_i) s(T_i, \rho) + \sum_{i=1}^{m} (-\mathrm{tr}\rho(\hat{c}_i))\lambda(c_i, \rho)^{-1} = 1. \tag{4.1}$$

Proof. Fix a Fuchsian representation $[\rho_0] \in \mathcal{R}'_{g,m}$ such that

$$D = \rho_0(d^{-1}) = \rho_0\left(\left(\prod_{j=1}^{g} a_j b_j a_j^{-1} b_j^{-1}\right) c_1 \ldots c_m\right) = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}.$$

Let $G = \rho_0(G'_{g,m})$. Let K denote the Nielsen convex region for G [**3**, 1.3]. We identify $G'_{g,m}$ and $F' = F'_{g,m}$ with G and K/G, respectively. Deforming by a isotopy, we may assume that all ideal arcs in Δ are geodesic.

First we consider the case where $m = 0$, that is $F = F_{g,0}$ is a surface without boundary. Then each component of $F - \Delta$ is a triangle. Choose d to be a small loop which goes once around p. Let $\hat{T}_1, \hat{T}_2, \ldots, \hat{T}_n, \hat{T}_{n+1} = \hat{T}_1$ be the triangles in Δ which d^{-1} meets in this order. We take a lift $\tilde{T}_j$ in $\mathbb{H}^2$ of $\hat{T}_j$ for $j = 1, 2, \ldots, n$ so that $\tilde{T}_j$ has an ideal vertex at ∞ and $\tilde{T}_{j+1}$ and $\tilde{T}_j$ share a side. Then $P = \tilde{T}_1 \cup \ldots \cup \tilde{T}_n$ is an ideal $(n+2)$-gon and two of its sides are identified by D, the difference of the largest and the smallest of the ideal vertices of P in $\mathbb{R}$ is one.

Let T be one of the triangles $\hat{T}_1, \ldots, \hat{T}_n$. Let N be a regular neighbourhood of T in F. Let A, B and C be canonical generators of the group G restricted to $N - \{p\}$ such that $ABC = (-1 - 1|0 - 1)$, and $C(\infty)$ and $BC(\infty)$ are vertices of $\tilde{T}$ other than ∞. Then $\lambda_1 = -(\mathrm{tr}A + \mathrm{tr}BC)$, $\lambda_2 = -(\mathrm{tr}B + \mathrm{tr}CA)$ and $\lambda_3 = -(\mathrm{tr}C + \mathrm{tr}AB)$ are λ-lengths of the sides of T and in particular λ_2 is the λ-length of the side opposite ∞.

If T is untwisted, then $N - \{p\}$ is a three-holed sphere with one puncture. Hence $BC(\infty) > C(\infty)$, see Figure 1(a). By Corollary 3.1

$$BC(\infty) - C(\infty) = \frac{\lambda_2}{\lambda_1 \lambda_3}. \tag{4.2}$$

If T is twisted, then $N - \{p\}$ is a one-holed torus with one puncture and hence $C(\infty) > BC(\infty)$, see Figure 1(b). By Corollary 3.1

$$C(\infty) - BC(\infty) = -\frac{\lambda_2}{\lambda_1 \lambda_3}. \tag{4.3}$$

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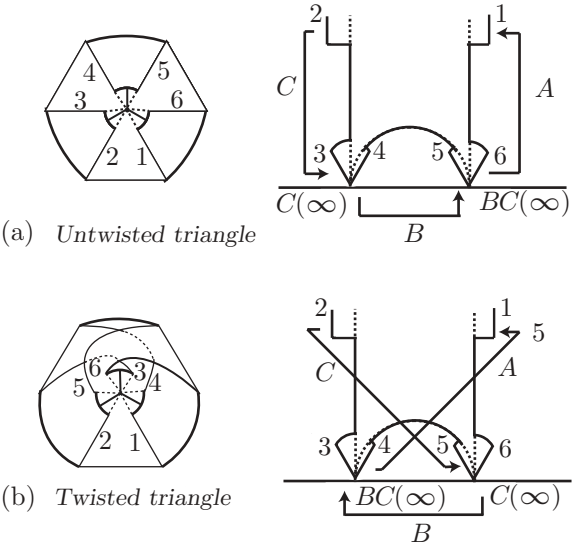


FIGURE 1. *Triangles.*

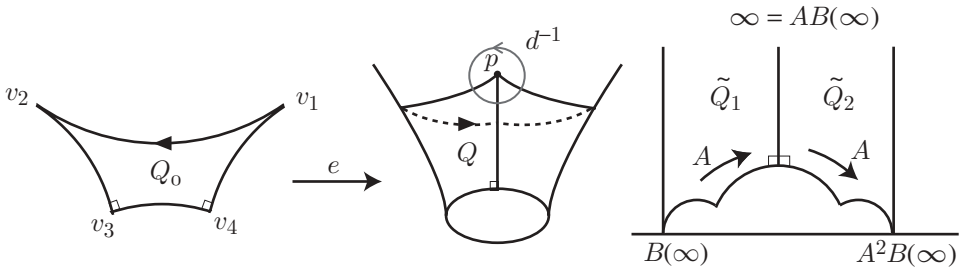


FIGURE 2.

Note that three ends of each triangle in Δ contribute three triangles to the sequence $\hat{T}_1, \ldots, \hat{T}_n$. Summing all terms as in (4.2), (4.3) over all triangles $T_1, \ldots, T_{d_1}$ yields the left-hand side of (4.1) for G. Since the difference of the largest ideal vertex and the smallest ideal vertex of P in $\mathbb{R}$ is one, we have (4.1).

Suppose that $F - \Delta$ contains annuli. We still assume that $F' = K/G$ and Δ is a triangulation by geodesic ideal arcs. Each annulus in $F - \Delta$ contains a unique simple geodesic arc joining the puncture and the boundary curve of F' which is perpendicular to the boundary curve. We denote Δ augmented with these m arcs again by Δ. Then Δ divides F into ideal triangles and geodesic quadrilaterals with angles 0, 0, $\pi/2$, $\pi/2$.

Let Q_0 denote a hyperbolic geodesic quadrilateral with angles 0, 0, $\pi/2$, $\pi/2$. Let the consecutive vertices be $v_1, \ldots, v_4$ in positive cyclic order, the angle at v_1, v_2 be 0 (so they are ideal vertices) and the angle at v_3 and v_4 be $\pi/2$.

Let Q be an arbitrary quadrilateral in Δ. There exists an isometric characteristic map $e : Q_0 \longrightarrow Q$ such that $e(v_1) = e(v_2) = p$ and that $e(v_3) = e(v_4)$. Thus the image of the edge $v_1 v_4$ under e coincides with that of $v_2 v_3$ and this is one of the augmented arcs in Δ. See Figure 2.

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We find two lifts $\tilde{Q}_1$ and $\tilde{Q}_2$ of Q and lifting maps $e_i : Q_0 \longrightarrow \tilde{Q}_i$ for $i = 1, 2$ so that $e_1(v_1) = e_2(v_2) = \infty$ and $e_1(v_4) = e_2(v_3)$ and so that $\tilde{Q}_1$ abuts on the edge $e_1(v_1)e_1(v_4)$ from the left and $\tilde{Q}_2$ does so from the right. Let N denote a regular neighbourhood of Q in F. Let A and B be canonical generators of the group G restricted to $N - \{p\}$, which is a two-holed sphere with one puncture, $AB = (-1 - 1|0 - 1)$ and A corresponds to the border curve $\partial Q \cap \partial F'$. Then the ideal vertex $e_1(v_2)$ of $\tilde{Q}_1$ is $B(\infty)$ and the ideal vertex $e_2(v_1)$ of $\tilde{Q}_2$ is $A^2B(\infty)$, and $A^2B(\infty) > B(\infty)$. By Corollary 2.2

$$A^2B(\infty) - B(\infty) = -\frac{\mathrm{tr}A}{\lambda}, \tag{4.4}$$

where $\lambda = -(\mathrm{tr}A + \mathrm{tr}B)$.

Going along a small loop d^{-1} around p, one passes two ends of a quadrilateral in Δ successively as in Figure 2. Each quadrilateral contributes a term like (4.4) to the sum in (4.1). By the monodromy $\rho(d^{-1}) = (-1 - 1|0 - 1)$, summing all terms as in (4.2), (4.3) and (4.4) over all ideal vertices of triangles and annuli in Δ yields again (4.1).

For a general $[\rho] \in R'_{g,m}$, choose a ρ with $\rho(d^{-1}) = (-1 - 1|0 - 1)$. We can regard ρ as a faithful representation of G into $\mathrm{SL}(2, \mathbb{C})$. Due to Corollaries 2.2 and 3.1, similar equations to (4.2), (4.3) and (4.4) hold when A, B, C are replaced by $\rho(A), \rho(B), \rho(C)$, respectively, and they yield (4.1). □

4.3. *Examples*

Case of $R'_{1,0}$. Let $A, B \in \mathrm{SL}(2, \mathbb{C})$ be such that $ABA^{-1}B^{-1}$ is parabolic and $\mathrm{tr}D = -2$. Suppose that A and B generate a free group. An ideal triangulation of the once-punctured torus gives rise to λ-lengths $\lambda_a = -(\mathrm{tr}A + \mathrm{tr}BA^{-1}B^{-1}) = -2\mathrm{tr}A$, $\lambda_b = -(\mathrm{tr}B + \mathrm{tr}AA^{-1}B^{-1}) = -2\mathrm{tr}B$, and $\lambda_c = -(\mathrm{tr}AB + \mathrm{tr}A^{-1}B^{-1}) = -2\mathrm{tr}AB$. In this case (4.1) can be read as

$$-2\left(\frac{\lambda_a}{\lambda_b\lambda_c} + \frac{\lambda_b}{\lambda_c\lambda_a} + \frac{\lambda_c}{\lambda_a\lambda_b}\right) = 1.$$

This is identical with $(\mathrm{tr}A)^2 + (\mathrm{tr}B)^2 + (\mathrm{tr}AB)^2 = \mathrm{tr}A\mathrm{tr}B\mathrm{tr}AB$.

Case of $R'_{1,1}$. Let $A, B, C, D \in \mathrm{SL}(2, \mathbb{C})$ satisfy $ABA^{-1}B^{-1}CD = 1$ and $\mathrm{tr}D = -2$. Then $\lambda_1 = -(\mathrm{tr}A + \mathrm{tr}BA^{-1}B^{-1}C)$, $\lambda_2 = -(\mathrm{tr}ABA^{-1} + \mathrm{tr}B^{-1}C)$, $\lambda_3 = -(\mathrm{tr}A^{-1} + \mathrm{tr}B^{-1}CAB)$, $\lambda_4 = -(\mathrm{tr}AB + \mathrm{tr}A^{-1}B^{-1}C)$, $\lambda_5 = -(\mathrm{tr}C + \mathrm{tr}ABA^{-1}B^{-1})$ and $c = -\mathrm{tr}C$ satisfy

$$\begin{aligned}\lambda_3\lambda_5\left(\lambda_1^2 + \lambda_2^2 + \lambda_4^2\right) + \lambda_1\lambda_5\left(\lambda_2^2 + \lambda_3^2 + \lambda_4^2\right)\\ = \lambda_2\lambda_4\left(\lambda_1^2 + \lambda_3^2 + \lambda_5^2\right) + (c - \lambda_5)\lambda_1\lambda_2\lambda_3\lambda_4.\end{aligned} \tag{4.5}$$

If $\lambda_i \neq 0$ for $i = 1, 2, 3, 4, 5$, this is nothing but (4.1) for this case. However, it can be seen that (4.5) holds for any A, B, C and D as above: for $\lambda_6 = -(\mathrm{tr}BA^{-1} + \mathrm{tr}AB^{-1}C)$ satisfies

$$\lambda_1\lambda_3 + \lambda_2^2 = \lambda_4\lambda_6$$

and

$$\lambda_4 + \lambda_6 = -a\lambda_2,$$

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where $a = -\mathrm{tr}A$, by (1.1) and (2) in Subsection 1.1. Apply Proposition 3.1 to A, B and C substituted by A, BAB^{-1} and C, respectively, to obtain

$$\lambda_1^2 + \lambda_3^2 + \lambda_5^2 + a\lambda_3\lambda_5 + a\lambda_1\lambda_5 + c\lambda_1\lambda_3 = \lambda_1\lambda_3\lambda_5.$$

These three equations yield (4.5).

If $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in \mathbb{C}^*$ satisfy (4.5), then we can reproduce the class of matrices $\{A, B, C\}$ in a unique manner. By the above computation we have

$$\mathrm{tr}A = -a = \frac{\lambda_4 + \lambda_6}{\lambda_2} = \frac{\lambda_4^2 + \lambda_2^2 + \lambda_1\lambda_3}{\lambda_2\lambda_4}.$$

Let $B' = BA^{-1}B^{-1}$, $\lambda_2' = -(\mathrm{tr}B' + \mathrm{tr}AC)$ and $\lambda_3' = \lambda_5$. Apply Proposition 3.2 to obtain matrix representations of A, B' and C, which are unique under the conditions $AB'C = (-1-1|0-1)$ and $C(\infty) = 0$.

Next, let $A_1 = ABA^{-1}$ and $B_1 = A^{-1}$. Note that $ABA^{-1}B^{-1} = A_1B_1A_1^{-1}B_1^{-1}$. Then, in order to obtain ABA^{-1} and hence B, proceed in the same way with $\lambda_1' = -(\mathrm{tr}A_1 + \mathrm{tr}B_1A_1^{-1}B_1^{-1}C)$, $\lambda_2' = -(\mathrm{tr}A_1B_1A_1^{-1} + \mathrm{tr}B_1^{-1}C)$, $\lambda_3' = -(\mathrm{tr}A_1^{-1} + \mathrm{tr}B_1^{-1}CA_1B_1)$, $\lambda_4' = -(\mathrm{tr}A_1B_1 + \mathrm{tr}A_1^{-1}B_1^{-1}C)$, $\lambda_5' = -(\mathrm{tr}C + \mathrm{tr}A_1B_1A_1^{-1}B_1^{-1})$ and $c = -\mathrm{tr}C$. These last numbers can be expressed by using $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$: $\lambda_1' = \lambda_2$, $\lambda_2' = \lambda_3$, $\lambda_4' = \lambda_6 = (\lambda_1\lambda_3 + \lambda_2^2)/\lambda_4$. For λ_3' we apply Proposition 1.1 to A, BA^{-1}, B^{-1}, C. Then $\lambda_1\lambda_3' + \lambda_5\lambda_6 = \lambda_2\lambda_3$. Now we obtain

$$A = \begin{pmatrix} \dfrac{\lambda_4^2\lambda_5 + \lambda_2^2\lambda_5 + \lambda_1\lambda_3\lambda_5 - \lambda_2\lambda_3\lambda_4}{\lambda_2\lambda_4\lambda_5} & \dfrac{\lambda_2\lambda_4\lambda_5^2 - \lambda_3\lambda_4^2\lambda_5 - \lambda_2^2\lambda_3\lambda_5 - \lambda_1\lambda_3^2\lambda_5 + \lambda_2\lambda_3^2\lambda_4}{\lambda_1\lambda_2\lambda_4\lambda_5^2} \\ -\lambda_1 & \dfrac{\lambda_3}{\lambda_5} \end{pmatrix},$$

$$B = \begin{pmatrix} \dfrac{\lambda_2}{\lambda_5} & \dfrac{\lambda_2^2\lambda_5 + \lambda_1\lambda_3\lambda_5 - \lambda_2\lambda_3\lambda_4}{\lambda_1\lambda_4\lambda_5^2} \\ \dfrac{\lambda_1\lambda_2 - \lambda_4\lambda_5}{\lambda_3} & \dfrac{-\lambda_2\lambda_4\lambda_5^2 + \lambda_1\lambda_2^2\lambda_5 + \lambda_3\lambda_4^2\lambda_5 + \lambda_1^2\lambda_3\lambda_5 - \lambda_1\lambda_2\lambda_3\lambda_4}{\lambda_1\lambda_3\lambda_4\lambda_5} \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 1/\lambda_5 \\ -\lambda_5 & -c \end{pmatrix}.$$

4.4.

Now we state the main theorem of this paper. Let $(a_1, \ldots, a_m) \in (\mathbb{C}^*)^m$. We fix an ideal triangulation $\Delta = (c_1, \ldots, c_d)$ of $F' = F'_{g,m}$ as described at the beginning of Subsection 4.2. Let $X_{g,m}(a_1, \ldots, a_m)$ be the set of all points $(\lambda_1, \ldots, \lambda_d)$ of $(\mathbb{C}^*)^d$ satisfying (4.1) with $\lambda(c_i)$ replaced by λ_i and with $-\mathrm{tr}\rho(\hat{c}_i)$ replaced by a_i.

THEOREM 4.1. *For each ideal triangulation Δ, the map*

$$\eta = \eta_\Delta : R'_{g,m}(a_1, \ldots, a_m) \longrightarrow X_{g,m}(a_1, \ldots, a_m)$$

defined by

$$\eta([\rho]) = (\lambda(c_1, \rho), \ldots, \lambda(c_d, \rho))$$

is injective.

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Proof. For the set of pairs (g,m) of non-negative integers satisfying $2g+m-1>0$, we define the notion $(g_1,m_1)<(g_2,m_2)$ to mean either $g_1<g_2$ or $m_1<m_2$ if $g_1=g_2$. Our proof proceeds by induction with respect to this order. The cases of $(0,2)$, $(0,3)$ and $(1,1)$ are done. See Corollary 2.1, Proposition 3.2 and Subsection 4.3. Let (g,m) be different from these three pairs and assume that the statement of the theorem is true for all $(g',m')<(g,m)$.

We need only consider ideal triangulations Δ with the following property: an arc $\hat{c}$ in Δ divides $F'_{g,m}$ into two surfaces F_{g_1,m_1+1} and F_{g_2,m_2+1} where $(0,2)<(g_i,m_i+1)<(g,m)$ for $i=1,2$. If η_Δ is injective, then, since as we shall show later in Subsection 5.2, for any other triangulation Δ', $\eta_{\Delta'}\circ\eta_\Delta^{-1}$: $\eta_\Delta(R'_{g,m}(a_1,\dots,a_m))\longrightarrow\eta_{\Delta'}(R'_{g,m}(a_1,\dots,a_m))$ is injective, $\eta_{\Delta'}$ is also injective.

Thus we suppose that $\Delta=(c_1,\dots,c_d)$ and some $\hat{c}\in\Delta$ has the property stated above. As before $c_1,\dots,c_m$ are isotopic to boundary curves of $F_{g,m}$. Let N be a regular neighbourhood of $\hat{c}$ in $F_{g,m}$. Define two punctured surfaces $F'_1=F_{g_1,m_1+1}\cup N-\{p\}$ and $F'_2=F_{g_2,m_2+1}\cup N-\{p\}$. The restrictions $\Delta_1=(c'_1,\dots,c'_{d_1})$ and $\Delta_2=(c''_1,\dots,c''_{d_1})$ of Δ to F'_1 and F'_2 are ideal triangulations of F'_1 and F'_2, respectively. Here we assume that $(c'_1,\dots,c'_{m_1},c''_1,\dots,c''_{m_2})=(c_1,\dots,c_m)$ and $\hat{c}=c'_{m_1+1}=c''_{m_2+1}$. Finally, we set $\hat{c}_1=F_{g_2,m_2+1}\cap\partial N$ and $\hat{c}_2=F_{g_1,m_1+1}\cap\partial N$. Then $\hat{c}_i$ is a boundary curve of F'_i for $i=1,2$.

Let $(\lambda(c_1),\dots,\lambda(c_d))=\eta_\Delta(\rho)\in\eta_\Delta(R'_{g,m}(a_1,\dots,a_m))$. Thus $a_i=-\mathrm{tr}\rho(c'_i)$ for $i=1,\dots,m_1$ and $a_{m_1+i}=-\mathrm{tr}\rho(c''_i)$ for $i=1,\dots,m_2$. Let $T_1,\dots,T_s$ be triangles in Δ_1. Then $b_1=-\mathrm{tr}\rho(\hat{c}_1)$ can be computed using (4.1).

$$\sum_{i=1}^{s}\epsilon(T_i)s(T_i,\rho)+\sum_{i=1}^{m_1}a_i\lambda(c_i,\rho)^{-1}+b_1\lambda(\hat{c})^{-1}=1.$$

Then $\rho_1=\rho|_{\pi_1(F'_1,*)}\in R'_{g_1,m_1+1}(a_1,\dots,a_{m_1},b_1)$ and $\eta_{\Delta_1}(\rho_1)=(\lambda(c'_1),\dots,\lambda(c'_{d_1}))$ is the part of $(\lambda(c_1),\dots,\lambda(c_d))$ corresponding to Δ_1. (We choose the base point $*$ of the fundamental group $\pi_1(F'_1,*)$ from $N-\{p\}$.) Likewise $\rho_2=\rho|_{\pi_1(F'_2,*)}\in R'_{g_2,m_2+1}(a_{m_1+1},\dots,a_m,b_2)$, where $b_2=-\mathrm{tr}\rho(\hat{c}_2)$ is also computed using (4.1). Thus we have obtained from ρ a pair of representations ρ_1 and ρ_2, where

$$\rho_1\in R'_{g_1,m_1+1}(a_1,\dots,a_{m_1},b_1)\quad\text{and}\quad\rho_2\in R'_{g_2,m_2+1}(a_{m_1+1},\dots,a_m,b_2).$$

Let $\rho,\rho'\in R'_{g,m}(a_1,\dots,a_m)$ be such that $\eta_\Delta(\rho)=\eta_\Delta(\rho')$. The representations $\rho_1,\rho'_1\in R'_{g_1,m_1+1}(a_1,\dots,a_{m_1},b_1)$ and $\rho_2,\rho'_2\in R'_{g_2,m_2+1}(a_{m_1+1},\dots,a_m,b_2)$ obtained from ρ and ρ' as above satisfy $\eta_{\Delta_1}(\rho_1)=\eta_{\Delta_1}(\rho'_1)$ and $\eta_{\Delta_2}(\rho_2)=\eta_{\Delta_2}(\rho'_2)$. By the assumption, there exist $P,Q\in\mathrm{SL}(2,\mathbb{C})$ such that $\rho'_1=P^{-1}\rho_1P$ and $\rho'_2=Q^{-1}\rho_2Q$. Then on $\pi_1(N-\{p\},*)$, $\rho'=P^{-1}\rho P$ and $\rho'=Q^{-1}\rho Q$, which means that PQ^{-1} commutes with all matrices $\rho(g),g\in\pi_1(N-\{p\},*)$. Since $\rho(\pi_1(N-\{p\},*))$ is a non-elementary group, $Q=P$ or $Q=-P$. Therefore $\rho=P^{-1}\rho P$ on $\pi_1(F'_{g,m})$ and we conclude that η_Δ is injective. □

5. *The mapping class group*

5.1.

We consider four matrices A, B, C and D in $\mathrm{SL}(2,\mathbb{R})$ with $ABCD$ parabolic and $\mathrm{tr}ABCD=-2$ such that they are canonical generators of a rank 4 free Fuchsian group Γ. Let $P:\mathbb{H}^2\longrightarrow\mathbb{H}^2/\Gamma$ denote the canonical projection. We define $\lambda_A=-(\mathrm{tr}A+\mathrm{tr}BCD)$, $\lambda_B=-(\mathrm{tr}B+\mathrm{tr}ACD)$, $\lambda_C=-(\mathrm{tr}C+\mathrm{tr}ABD)$,

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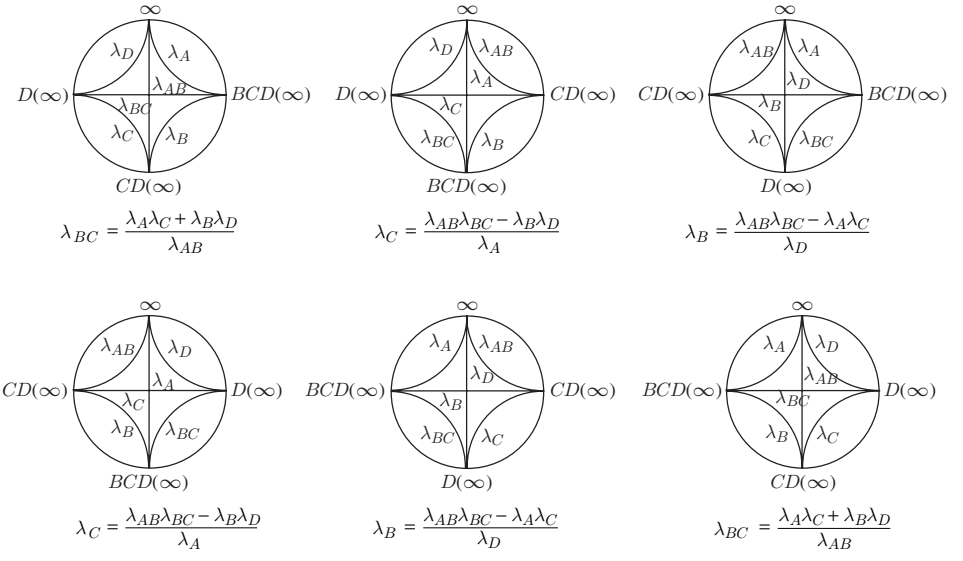


FIGURE 3.

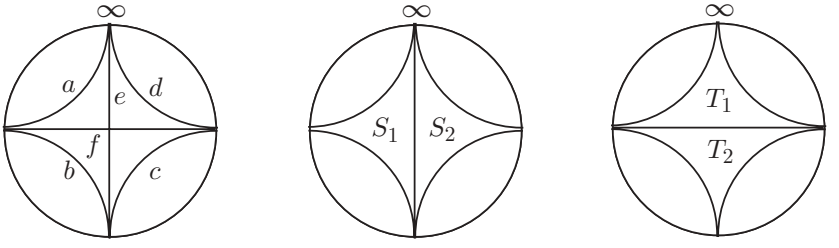


FIGURE 4.

$\lambda_D = -(\mathrm{tr}A + \mathrm{tr}ABC)$, $\lambda_{AB} = -(\mathrm{tr}AB + \mathrm{tr}CD)$ and $\lambda_{BC} = -(\mathrm{tr}BC + \mathrm{tr}AD)$. Then by Proposition 1.1, $\lambda_{AB}\lambda_{BC} = \lambda_A\lambda_C + \lambda_B\lambda_D$.

We assume that $ABCD = (-1\ -1|0\ -1)$. Then there are six possibilities for the order of the points $D(\infty)$, $CD(\infty)$ and $BCD(\infty)$ as depicted in Figure 3, where $\mathbb{H}^2 \cup \hat{\mathbb{R}}$ is identified with a closed disk. Sides of triangles are labelled by the λ-lengths with respect to Γ using their images under $P : \mathbb{H}^2 \longrightarrow \mathbb{H}^2/\Gamma$. For example, in the second figure the triangle with vertices ∞, $D(\infty)$, $BCD(\infty)$ and the triangle with vertices ∞, $D(\infty)$, $CD(\infty)$ are untwisted if identified with their images under P. On the other hand, the triangle with vertices ∞, $BCD(\infty)$, $CD(\infty)$ and the triangle with vertices $D(\infty)$, $BCD(\infty)$, $CD(\infty)$ are twisted.

Figure 4 is a copy of one of the six figures in Figure 3. We employ the notations as in Figure 4: $\lambda_a, \ldots, \lambda_f$ are again the λ-lengths of their corresponding sides. The conclusion is as follows.

$$\lambda_f = \frac{\epsilon_1\lambda_a\lambda_c + \epsilon_2\lambda_b\lambda_d}{\lambda_e}. \tag{5.1}$$

The pair of numbers (ϵ_1, ϵ_2) is defined as in Table 1. If S_1 and S_2 are of the same parity, that is they are both untwisted or both twisted, then $(\epsilon_1, \epsilon_2) = (1, 1)$. Otherwise, if S_1 and T_1 are of the same parity, then $(\epsilon_1, \epsilon_2) = (-1, 1)$; and if they are not of the same parity, then $(\epsilon_1, \epsilon_2) = (1, -1)$.

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TABLE 1.

S_1	S_2	T_1	T_2	(ϵ_1, ϵ_2)
Untwisted	Untwisted	Untwisted	Untwisted	$(1, 1)$
Twisted	Twisted	Twisted	Twisted	$(1, 1)$
Untwisted	Twisted	Twisted	Untwisted	$(1, -1)$
Twisted	Untwisted	Untwisted	Twisted	$(1, -1)$
Untwisted	Twisted	Untwisted	Twisted	$(-1, 1)$
Twisted	Untwisted	Twisted	Untwisted	$(-1, 1)$

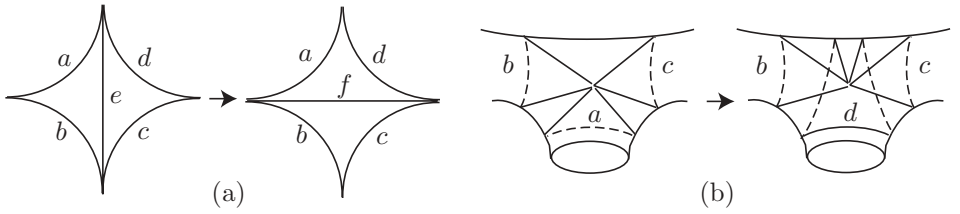


FIGURE 5. *Elementary moves.*

If ρ is a faithful representation of Γ in $\mathrm{SL}(2, \mathbb{C})$ with $\rho(ABCD) = (-1\ -1|0\ -1)$, then considering the points $\rho(D)(\infty)$, $\rho(CD)(\infty)$ etc., we see that (5.1) holds for λ-lengths with respect to ρ.

5.2. *Elementary moves*

Let Δ be an ideal triangulation of $F'_{g,m}$. Let S_1 and S_2 be two triangles in Δ which share a common side e. Then e is a diagonal of the quadrilateral $Q = S_1 \cup S_2$. Let S_1 have sides a, b, e and S_2 have sides c, d, e, where a and c are opposite sides of Q. The other diagonal f divides Q into two triangles T_1 and T_2. We assume that T_1 has sides a, f, d and T_2 has sides f, b, c. By replacing e in Δ with f, we obtain a new ideal triangulation Δ'. We say that Δ' arises from Δ by an *elementary move* on e. See Figure 5(a) (in which we view the quadrilateral as a lift of Q in the universal covering space).

Suppose that a class of representations $[\rho]$ in $R'_{g,m}(a_1, \ldots, a_m)$ is given. Then $[\rho]$ defines lambda lengths λ_a, λ_b, λ_c, λ_d, λ_e, λ_f of a, b, c, d, e, f, respectively, which satisfy (5.1).

Let $a \in \Delta$ be an arc which separates a boundary curve of F'. Then a is a side of an untwisted triangle T in Δ. Let a, b, c be the sequence of the sides of T which agrees with the positive orientation. Replacing a with the arc d as depicted in Figure 5(b) produces another triangulation Δ'. We say also that Δ' arises from Δ by an elementary move on a.

First we consider a Fuchsian representation Γ with canonical generators A, B, C of the fundamental group of a regular neighbourhood of the triangle T with sides a, b, c (the tinted part in Figure 6(a)). We assume that A, B and C correspond to a, b and c, respectively and $ABC = (-1\ -1|0\ -1)$. Since T is untwisted, $C(\infty) < BC(\infty)$. Let $\tilde{c}$ be the hyperbolic line $\infty C(\infty)$ and $\tilde{b}, \tilde{a}$ be the hyperbolic lines $C(\infty)BC(\infty)$ and $BC(\infty)\infty$, respectively. Then the ideal triangle $\tilde{T}$ with sides $\tilde{a}, \tilde{b}$ and $\tilde{c}$ projects to T under the projection $P : \mathbb{H}^2 \longrightarrow \mathbb{H}^2/\Gamma$. Choose the lift $\tilde{d}^{-1}$ of d^{-1} such that

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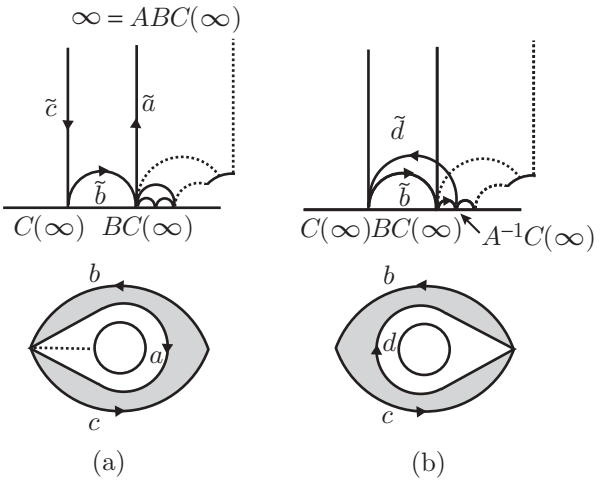


FIGURE 6.

it starts at the end of $\tilde{T}$ near $C(\infty)$. Then $\tilde{d}^{-1}$ ends at $A^{-1}C(\infty)$ and the triangle with sides b, d and c (the shaded part in Figure 6(b)) can lift to the ideal triangle with sides $\tilde{b}$, $\tilde{d}$ and $A^{-1}(\tilde{c})$, see Figure 6. We see that $A(A^{-1}C(\infty)) = C(\infty)$, $B(C(\infty)) = BC(\infty)$, $A^{-1}CA(BC(\infty)) = A^{-1}C(\infty)$ and $\mathrm{tr}(A^{-1}CA)BA = -2$. Now the λ-length of d is $\lambda_d = -\mathrm{tr}A - \mathrm{tr}A^{-1}CAB$. By (3) in Subsection 1.1 and with the notation of Proposition 3.1,

$$\begin{aligned}
&-\mathrm{tr}A - \mathrm{tr}A^{-1}CAB \\
&\quad = a - (-a\mathrm{tr}CAB + \mathrm{tr}AC\mathrm{tr}A^{-1}B + bc - \mathrm{tr}BC - ab\mathrm{tr}AC) \\
&\quad = a - (2a + (b - \lambda_b)(ab + \lambda_c - c) + bc + (\lambda_a - a) + ab(\lambda_b - b)) \\
&\quad = -c\lambda_b - b\lambda_c + \lambda_b\lambda_c - \lambda_a.
\end{aligned}$$

Thus by (3.1)

$$\lambda_d = -c\lambda_b - b\lambda_c + \lambda_b\lambda_c - \lambda_a = \frac{\lambda_b^2 + \lambda_c^2 + a\lambda_b\lambda_c}{\lambda_a}. \tag{5.2}$$

Since we have used only the trace identities for $\mathrm{SL}(2,\mathbb{C})$ to deduce the above equation, (5.2) holds for any faithful $\mathrm{SL}(2,\mathbb{C})$ representation of Γ preserving the traces of parabolic elements.

As shown in [**4**], any two triangulations $\Delta = (c_1, \ldots, c_d)$ and Δ' are related by a finite sequence of elementary moves. Therefore due to (5.1) and (5.2), $\eta_{\Delta'} \circ \eta_\Delta^{-1}$ is expressed as a birational mapping in variables $\lambda(c_1, \rho), \ldots, \lambda(c_d, \rho)$ and thus

$$\eta_{\Delta'} \circ \eta_\Delta^{-1} : \eta_\Delta(R'_{g,m}(a_1, \ldots, a_m)) \longrightarrow \eta_{\Delta'}(R'_{g,m}(a_1, \ldots, a_m))$$

is injective.

Each element τ of the mapping class group $\mathcal{MC}'_{g,m}$ fixing all boundary points sends Δ to another triangulation $\tau\Delta$ and hence defines a map $\tau_* : \eta_\Delta(R'_{g,m}(a_1, \ldots, a_m)) \longrightarrow \eta_{\tau\Delta}(R'_{g,m}(a_1, \ldots, a_m))$ by

$$\tau_*(\lambda(c_1, \rho), \ldots, \lambda(c_d, \rho)) = (\lambda(\tau(c_1), \rho), \ldots, \lambda(\tau(c_d), \rho)).$$

Therefore each mapping class τ acts on $\mathbb{C}^{6g+2m-3}$ as a rational mapping τ_*.

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PROPOSITION 5.1. *Via the λ-length coordinates, the action of the mapping class group $\mathcal{MC}'_{g,m}$ on $R'_{g,m}(a_1,\dots,a_m)$ produces an action on $\mathbb{C}^{6g+2m-3}$ as a group of birational mappings.*

5.3. *Holomorphic 2-form invariant under the mapping class group*

The description of Weil–Petersson 2-form in terms of λ-lengths defined in the Fuchsian representation space [**11**] extends easily to a holomorphic 2-form on $R'_{g,m}$.

Let λ_a, λ_b, λ_c, λ_d, λ_e and λ_f be as in the first part of Subsection 5.1. Then the sequence of a, b, e agrees with the positive orientation of S_1 and so does the sequence of e, c, d with the positive orientation of S_2. Consider the form

$$\frac{d\lambda_a\wedge d\lambda_b}{\lambda_a\lambda_b}+\frac{d\lambda_b\wedge d\lambda_e}{\lambda_b\lambda_e}+\frac{d\lambda_e\wedge d\lambda_a}{\lambda_e\lambda_a}+\frac{d\lambda_e\wedge d\lambda_c}{\lambda_e\lambda_c}+\frac{d\lambda_c\wedge d\lambda_d}{\lambda_c\lambda_d}+\frac{d\lambda_d\wedge d\lambda_e}{\lambda_d\lambda_e}.$$

By (5.1) this form equals

$$\begin{aligned}
&\frac{d\lambda_a\wedge d\lambda_b}{\lambda_a\lambda_b}+\frac{d\lambda_c\wedge d\lambda_d}{\lambda_c\lambda_d}+\left(\frac{d\lambda_b}{\lambda_b}-\frac{d\lambda_a}{\lambda_a}+\frac{d\lambda_d}{\lambda_d}-\frac{d\lambda_c}{\lambda_c}\right)\wedge\frac{d\lambda_e}{\lambda_e}\\
&=\left(\frac{d\lambda_b}{\lambda_b}-\frac{d\lambda_a}{\lambda_a}+\frac{d\lambda_d}{\lambda_d}-\frac{d\lambda_c}{\lambda_c}\right)\wedge\left(\frac{\epsilon_1\lambda_c d\lambda_a+\epsilon_1\lambda_a d\lambda_c+\epsilon_2\lambda_d d\lambda_b+\epsilon_2\lambda_b d\lambda_d}{\epsilon_1\lambda_a\lambda_c+\epsilon_2\lambda_b\lambda_d}\right)\\
&\quad+\left(\frac{d\lambda_b}{\lambda_b}-\frac{d\lambda_a}{\lambda_a}+\frac{d\lambda_d}{\lambda_d}-\frac{d\lambda_c}{\lambda_c}\right)\wedge\left(-\frac{d\lambda_f}{\lambda_f}\right)+\frac{d\lambda_c\wedge d\lambda_d}{\lambda_c\lambda_d}+\frac{d\lambda_a\wedge d\lambda_b}{\lambda_a\lambda_b}\\
&=\frac{d\lambda_a\wedge d\lambda_f}{\lambda_a\lambda_f}+\frac{d\lambda_f\wedge d\lambda_d}{\lambda_f\lambda_d}+\frac{d\lambda_d\wedge d\lambda_a}{\lambda_d\lambda_a}+\frac{d\lambda_f\wedge d\lambda_b}{\lambda_f\lambda_b}+\frac{d\lambda_b\wedge d\lambda_c}{\lambda_b\lambda_c}+\frac{d\lambda_c\wedge d\lambda_f}{\lambda_c\lambda_f}.
\end{aligned}$$

Note that the sequence of a, f, d agrees with the positive orientation of T_1 and so does the sequence of f, b, c with the positive orientation of T_2.

Let λ_a, λ_b, λ_c, λ_d be as in the second part of 5.1. Then Subsection (5.2) and a similar calculation as above yield

$$\frac{d\lambda_a\wedge d\lambda_b}{\lambda_a\lambda_b}+\frac{d\lambda_b\wedge d\lambda_c}{\lambda_b\lambda_c}+\frac{d\lambda_c\wedge d\lambda_a}{\lambda_c\lambda_a}=\frac{d\lambda_c\wedge d\lambda_b}{\lambda_b\lambda_c}+\frac{d\lambda_b\wedge d\lambda_d}{\lambda_b\lambda_d}+\frac{d\lambda_d\wedge d\lambda_c}{\lambda_d\lambda_c}.$$

Let Δ be an ideal triangulation of $F'_{g,m}$. Let $T_1,\dots,T_{d_1}$ be all triangles of Δ. Let the sequence of the sides a_i, b_i, c_i of T_i agree with the positive orientation of T_i. Then the above calculation shows that the 2-form on $\mathbb{C}^{6g+2m-3}$ defined by

$$\omega=\sum_{i=1}^{d_1}(d\log\lambda_{a_i}\wedge d\log\lambda_{b_i}+d\log\lambda_{b_i}\wedge d\log\lambda_{c_i}+d\log\lambda_{c_i}\wedge d\log\lambda_{a_i}) \tag{5.3}$$

is invariant under the elementary moves and therefore invariant under the action of the mapping class $\mathcal{MC}'_{g,m}$.

6. *Parabolic transformations*

In this section we consider mapping classes called 'parabolic' according to the classification by L. Bers [**2**].

6.1.

Let $\rho\in R'_{0,3}(a,b,c)$, where $(a,b,c)\in(\mathbb{C}^*)^3$, and let A, B, C be a system of canonical generators of $\Gamma=\rho(G'_{0,3})$ with $a=-\mathrm{tr}A$, $b=-\mathrm{tr}B$ and $c=-\mathrm{tr}C$. We employ the notation of Proposition 3.1 and refer to Figure 7. The composite of two

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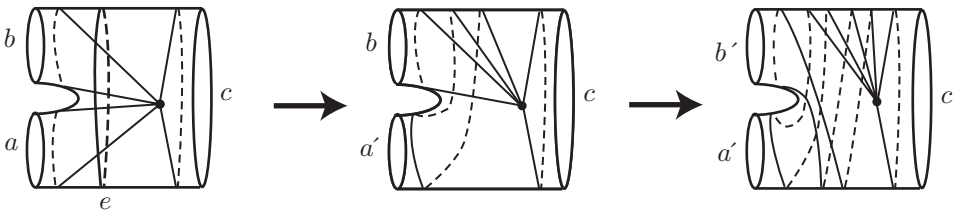


FIGURE 7.

elementary moves on arcs a and b is a Dehn twist τ about the loop e. Let λ'_a and λ'_b be the λ-lengths of a' and b' with respect to Γ, respectively. Then by (5.2)

$$\lambda'_a = -\lambda_a + \lambda_b\lambda_c - c\lambda_b - b\lambda_c, \quad \lambda'_b = -\lambda_b + \lambda'_a\lambda_c - c\lambda'_a - a\lambda_c. \tag{6.1}$$

Since $\rho \in R'_{g,m}(a,b,c)$, by Lemma 2.1 none of λ_a, λ_b, λ'_a and λ'_b equals zero. The rational representation of τ is $\tau_*(\lambda_a,\lambda_b,\lambda_c) = (\lambda'_a,\lambda'_b,\lambda_c)$. By (6.1) we have $\boldsymbol{x_1} = P\boldsymbol{x} + \lambda_c\boldsymbol{v}$, where

$$\boldsymbol{x_1} = \begin{pmatrix}\lambda'_a\\ \lambda'_b\end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix}\lambda_a\\ \lambda_b\end{pmatrix}, \quad \boldsymbol{v} = -\begin{pmatrix}1 & 0\\ -(\lambda_c - c) & 1\end{pmatrix}^{-1}\begin{pmatrix}b\\ a\end{pmatrix}$$

and

$$P = \begin{pmatrix}1 & 0\\ -(\lambda_c - c) & 1\end{pmatrix}^{-1}\begin{pmatrix}-1 & \lambda_c - c\\ 0 & -1\end{pmatrix} = \begin{pmatrix}-1 & \lambda_c - c\\ -(\lambda_c - c) & (\lambda_c - c)^2 - 1\end{pmatrix}.$$

If τ_*^n had a fixed point $(\lambda_a,\lambda_b,\lambda_c)$ in $R'_{0,3}(a,b,c)$ for some integer $n \geqslant 1$, then, for $\boldsymbol{x} = {}^t(\lambda_a,\lambda_b)$,

$$(I + P + P^2 + \ldots + P^{n-1})((P-I)\boldsymbol{x} + \lambda_c\boldsymbol{v}) = \boldsymbol{0}, \tag{6.2}$$

where I denotes the unit matrix. Assume that $(P-I)\boldsymbol{x}+\lambda_c\boldsymbol{v} = \boldsymbol{0}$. If $\det(P-I) \neq 0$, then ${}^t(\lambda_a,\lambda_b) = -\lambda_c(P-I)^{-1}\boldsymbol{v}$ and (3.1) reduces to $\mathrm{tr}[A,B] = (\mathrm{tr}A)^2 + (\mathrm{tr}B)^2 + (\mathrm{tr}AB)^2 - (\mathrm{tr}A)(\mathrm{tr}B)(\mathrm{tr}AB) - 2 = 2$. This implies that A and B share a common fixed point and hence A and B do not generate a free group of rank 2. If $\det(P-I) = 0$, we obtain by an easier computation (or by using continuity of trace function) that $\mathrm{tr}[A,B] = 2$. We conclude that $(P-I)\boldsymbol{x} + \lambda_c\boldsymbol{v} \neq \boldsymbol{0}$. In this case (6.2) is impossible unless P is diagonalizable and $\mathrm{tr}P = (\lambda_c - c)^2 - 2 = \xi + \xi^{-1}$ for some nth root of unity ξ. Then $\mathrm{tr}AB = -\lambda_c + c = \pm(\xi^{1/2} + \xi^{-1/2})$. This contradicts ρ being a faithful representation. Therefore τ_*^n has no fixed points in $R'_{0,3}(a,b,c)$ for all integers $n \geqslant 1$.

6.2.

Let $\rho \in R'_{1,1}(c)$, where $c \in \mathbb{C}^*$ and let A, B, C be a system of canonical generators of $\Gamma = \rho(G'_{1,1})$ with $c = -\mathrm{tr}C$. We employ the notation of Subsection 4.5 (case of $\mathcal{R}'_{1,1}$) and refer to Figure 8. The elementary move on b is a Dehn twist τ about the loop e. Let λ'_4 be the λ-length of b' with respect to Γ. Since the two adjacent triangles of b are both twisted, we obtain by (5.1) and (4.5) (regarded as a quadratic equation in λ_2)

$$\lambda'_4 = -\lambda_2 + p\lambda_4, \quad p = \frac{\lambda_1^2 + \lambda_3^2 + \lambda_5^2 - \lambda_1\lambda_3\lambda_5 + c\lambda_1\lambda_3}{(\lambda_1 + \lambda_3)\lambda_5}. \tag{6.3}$$

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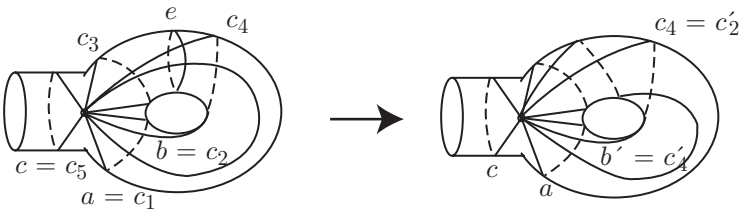


FIGURE 8.

The rational representation of τ is $\tau_*(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5) = (\lambda_1, \lambda_2', \lambda_3, \lambda_4', \lambda_5)$, where $\lambda_2' = \lambda_4$. Then by (6.3),

$$\begin{pmatrix} \lambda_2' \\ \lambda_4' \end{pmatrix} = P \begin{pmatrix} \lambda_2 \\ \lambda_4 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 \\ -1 & p \end{pmatrix}.$$

If τ_*^n had a fixed point in $R'_{1,1}(c)$ for some integer $n \geqslant 1$, then $p = \pm(\xi^{n/2} + \xi^{-n/2})$ for some nth root of unity ξ. We rewrite the second term of (6.3) as

$$\lambda_1^2 + \lambda_3^2 + \lambda_5^2 - p\lambda_3\lambda_5 - p\lambda_1\lambda_5 + c\lambda_1\lambda_3 = \lambda_1\lambda_3\lambda_5.$$

Then, by Proposition 3.1, this means that $p = -\mathrm{tr}A$. This is a contradiction since ρ is a faithful representation. Therefore τ_*^n has no fixed points in $R'_{1,1}(c)$ for all integers $n \geqslant 1$.

6.3.

We assume that $(g, m) \neq (0, 2)$, so $F = F_{g,m}$ is not an annulus. As before $F' = F - \{p\}$. Let $a_1, \ldots, a_s$ be a collection of disjoint and non-isotopic simple loops on $F'_{g,m}$. We assume that none of a_i are isotopic to a boundary curve. Let τ_i be a Dehn twist about a_i. We consider $\tau = \tau_1^{n_1} \ldots \tau_s^{n_s}$ for a tuple of integers $(n_1, \ldots, n_s) \in \mathbb{Z}^s - \{(0, \ldots, 0)\}$.

We may assume that a_1 is not separated from the puncture p by any other loop in $\{a_2, \ldots, a_s\}$. Then there exists a simple arc b_1 in $F - \bigcup_{i=1}^s a_i$ which goes from p to a point q of a_1. Let N be a regular neighbourhood of a_1. N is an annulus and $N - a_1$ has two components N_1 and N_2 both of which are annuli. By choosing N sufficiently small, we can assume that N_1 meets b_1 and N_2 is disjoint from b_1. Since a_1 is not isotopic to a boundary curve and F is not an annulus, there exists a pair of pants S, that is a three-holed sphere embedded in $F - (a_1 \cup \ldots \cup a_s)$, such that a_1 is a boundary curve of S and $N_2 \subset S$.

If S can be chosen so that S contains both components of $N - a_1$, then the three boundary components of S are a_1, a_1 and the other one which we denote by c. There is a simple arc b_2 from p to q in S so that $b = b_1 b_2^{-1}$ is a simple loop based at p and meets a_1 transversally at q. We choose a simple arc γ in $S - (a_1 \cup b)$ from p to a point of c so that γ does not meet $a_1 \cup b$ except at p. We choose simple loops c_1, c_2, c_3, c_4 and c_5 based at p and homotopic to $b_1 a_1 b_1^{-1}$, b, $b_2 a_1 b_2^{-1}$, $b_1 a_1 b_2^{-1}$ and $\gamma c \gamma^{-1}$, respectively, so that any two of them do not meet except at p. Then we add more loops $c_6, \ldots, c_d$ so that $\Delta = (c_1, \ldots, c_d)$ defines an ideal triangulation of F'.

We consider the λ-length coordinates associated to Δ for the space $R'_{g,m}$. The action of τ^n restricted to $(c_1, c_2, c_3, c_4, c_5)$ is the same as that of $\tau_1^{nn_1}$. Therefore as in Subsection 6.2 we conclude that τ_*^n has no fixed points in $R'_{g.m}$.

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In the remaining cases S can be chosen so that its boundary components (denoted by b and c) other than a_1 are not isotopic to a_1. Choose simple arcs β and γ in S so that β goes from q to a point of b and γ goes from q to a point of c and they are disjoint except at q. We choose simple loops c_1, c_2, c_3 based at p and homotopic to $b_1a_1b_1^{-1}$, $(b_1\beta)b(b_1\beta)^{-1}$ and $(b_1\gamma)c(b_1\gamma)^{-1}$, respectively, so that any two of them do not meet except at p. Then we add more loops $c_4,\dots,c_d$ so that $\Delta=(c_1,\dots,c_d)$ defines an ideal triangulation of F'. As in the previous case we consider λ-length coordinates for the space $R'_{g,m}$ associated to Δ. The action of τ^n restricted to (c_1,c_2,c_3) is the same as that of $\tau_1^{nn_1}$. As in Subsection 6.1 we conclude that τ_*^n has no fixed points in $R'_{g.m}$.

Now we have obtained a simple proof of part of the well-known result of W. Thurston (cf. [**9**, 6.4]).

PROPOSITION 6.1. *Let $\tau\in\mathcal{MC}'_{g,m}$. If $\tau^n=\tau_1^{n_1}\dots\tau_s^{n_s}$ for some disjoint family of simple loops $a_1,\dots,a_s$ as above and integers $n,n_1,\dots,n_s$ with $(n_1,\dots,n_s)\neq(0,\dots,0)$, then τ_* has no fixed points in $R'_{g,m}$.*

REMARK 6.1. If we consider the action of a mapping class on $\mathbb{C}^d$, then it may have a fixed point. For example, consider the case of $R'_{0,3}(a,b,c)$ and use the notations as in Subsection 6.1. Then τ_* induced by a Dehn twist fixes points $(-bc/2,-ac/b,c)$ whenever $a^2+b^2=4$.

7. *Example*

The twice-punctured torus group is generated by matrices A,B and C of $\mathrm{SL}(2,\mathbb{C})$ with C and $D=(ABA^{-1}B^{-1}C)^{-1}$ parabolic. We assume that $D^{-1}=\left(\begin{smallmatrix}-1 & -1\\ 0 & -1\end{smallmatrix}\right)$.

We consider the following mapping classes which, as outer automorphisms of the group, act on the canonical generators in the following manner.

$$\begin{aligned}&\varphi_1:(A,B,C)\longmapsto(A,BA,C),\\ &\varphi_2:(A,B,C)\longmapsto(ABA^{-1},A^{-1},C),\\ &\varphi_3:(A,B,C)\longmapsto(A,BA^{-1}B^{-1}CB,BA^{-1}B^{-1}CBAB^{-1}).\end{aligned}$$

We define λ-lengths by $\lambda_1=-(\mathrm{tr}A+\mathrm{tr}BA^{-1}B^{-1}C)$, $\lambda_2=-(\mathrm{tr}B+\mathrm{tr}B^{-1}C)$, $\lambda_3=-(\mathrm{tr}A+\mathrm{tr}AC)$, $\lambda_4=-(\mathrm{tr}AB+\mathrm{tr}A^{-1}B^{-1}C)$ and $\lambda_5=-(\mathrm{tr}C+\mathrm{tr}ABA^{-1}B^{-1})$. Assume that $\mathrm{tr}C=-2$. Then these numbers satisfy (4.5) with c replaced by 2. The rational mappings induced by the above mapping classes are

$$\varphi_{1*}(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)=\left(\lambda_1,\lambda_4,\lambda_3,\frac{\lambda_4^2+\lambda_1\lambda_3}{\lambda_2},\lambda_5\right),$$

$$\varphi_{2*}(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)=\left(\lambda_2,\lambda_3,\frac{-\lambda_2^2\lambda_5-\lambda_5\lambda_1\lambda_3+\lambda_4\lambda_2\lambda_3}{\lambda_4\lambda_1},\frac{\lambda_2^2+\lambda_1\lambda_3}{\lambda_4},\lambda_5\right),$$

and

$$\begin{aligned}&\varphi_{3*}(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5)\\ &=\left(\lambda_1,\frac{-\lambda_4\lambda_5\lambda_2+\lambda_1\lambda_2^2+\lambda_1^2\lambda_3}{\lambda_4\lambda_3},\frac{-\lambda_4^2\lambda_5\lambda_1+\lambda_4\lambda_5^2\lambda_2+\lambda_4\lambda_1^2\lambda_2-\lambda_5\lambda_1\lambda_2^2-\lambda_5\lambda_1^2\lambda_3}{\lambda_4\lambda_2\lambda_3},\right.\\ &\quad\left.\frac{-\lambda_4\lambda_5+\lambda_1\lambda_2}{\lambda_3},\frac{(\lambda_4^2\lambda_5-\lambda_4\lambda_1\lambda_2+\lambda_4\lambda_5\lambda_1\lambda_2+\lambda_5\lambda_2^2+\lambda_5\lambda_1\lambda_3-\lambda_4\lambda_2\lambda_3)^2}{\lambda_4^2\lambda_5\lambda_2^2}\right).\end{aligned}$$

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Consider the mapping class $\psi = \varphi_1\varphi_3^{-1}\varphi_2$. The rational map $\psi_* = \varphi_{1*}\varphi_{3*}^{-1}\varphi_{2*}$ fixes the point

$$\tau_0 = (-4, -4, -4+8\sqrt{-1}, -8, -8-8\sqrt{-1})$$

and its complex conjugate. The matrices corresponding to τ_0 are

$$A = \begin{pmatrix} 3 & -1-3\sqrt{-1} \\ 1 & -\sqrt{-1} \end{pmatrix}, \quad B = \begin{pmatrix} -\sqrt{-1} & -1+\sqrt{-1} \\ -1+2\sqrt{-1} & 3 \end{pmatrix},$$

$$C = \begin{pmatrix} -1 & 0 \\ 2+2\sqrt{-1} & -1 \end{pmatrix}.$$

Actually, if A', B' and C' denote the matrices in the second case in Subsection 4.3 with $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ replaced by $-4, -4, -4+8\sqrt{-1}, -8, -8-8\sqrt{-1}$ and $c = 2$, respectively, then $A = P^{-1}A'P$, $B = P^{-1}B'P$ and $C = P^{-1}C'P$, where

$$P = \begin{pmatrix} 1/2 & (1-\sqrt{-1})/8 \\ 0 & 2 \end{pmatrix}.$$

Let

$$M = \begin{pmatrix} -1 & -1+\sqrt{-1} \\ 0 & -1 \end{pmatrix}.$$

Then

$$\begin{aligned}(ABA^{-1}, C^{-1}B^2A^{-1}, C^{-1}BCB^{-1}C) &= \psi(A, B, C) \\ &= (M^{-1}AM, M^{-1}BM, M^{-1}CM).\end{aligned}$$

The projection to $\mathrm{PSL}(2, \mathbb{C})$ of the group Γ generated by A, B, C and M, which is denoted by Γ again, is a subgroup of $\mathrm{PSL}(2, \mathcal{O}_1)$, where $\mathcal{O}_1$ denotes the ring of integers in $\mathbb{Q}(\sqrt{-1})$. Hence Γ is a Kleinian group. It is known that $\mathbb{H}^3/\Gamma$ is the Whitehead link complement.

The above method can be applied to finding periodic mapping classes. For an example, we consider $\varphi_1\varphi_3\varphi_2$. It can be seen that $\varphi_{1*}\varphi_{3*}\varphi_{2*}$ has period 4 and fixes the point $(-8, -8, -24, -16, 32)$. Let A', B' and C' denote the matrices in the second case in Subsection 4.3 with $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, c)$ replaced by $(-8, -8, -24, -16, 32, 2)$. Let $Q = (2\sqrt{2} \ -3\sqrt{2}/16 \,|\, 2\sqrt{2} \ \sqrt{2}/16)$. Then $A = QA'Q^{-1}$, $B = QB'Q^{-1}$ and $C = QC'Q^{-1}$ are

$$A = \begin{pmatrix} 5 & -1 \\ 6 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}.$$

Therefore the group Γ generated by A, B and C is a subgroup of $\Gamma_0(2)$. The fact that $\varphi_1\varphi_3\varphi_2$ has period 4 implies that the surface $\mathbb{H}^2/\Gamma$ admits an isometry of order 4.

Appendix A. *Proof of Proposition* 1.1

Let $x = (\mathrm{tr}A + \mathrm{tr}BCD)$, $y = (\mathrm{tr}B + \mathrm{tr}CDA)$, $z = (\mathrm{tr}C + \mathrm{tr}DAB)$, $w = (\mathrm{tr}D + \mathrm{tr}ABC)$, $u = (\mathrm{tr}AB + \mathrm{tr}CD)$, $v = (\mathrm{tr}BC + \mathrm{tr}AD)$. We assume that $\mathrm{tr}ABCD = -2$.

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Then

$$\begin{aligned}
uv &= \mathrm{tr}ABC\mathrm{tr}CB + \mathrm{tr}BA\mathrm{tr}DA + \mathrm{tr}BC\mathrm{tr}DC + \mathrm{tr}CD\mathrm{tr}AD\\
&= \mathrm{tr}ABCB + \mathrm{tr}BADA + \mathrm{tr}BCDC + \mathrm{tr}CDAD + 2\mathrm{tr}AC^{-1} + 2\mathrm{tr}BD^{-1}\\
&= (\mathrm{tr}ABC\mathrm{tr}B - \mathrm{tr}ABCB^{-1}) + (\mathrm{tr}DAB\mathrm{tr}A - \mathrm{tr}DABA^{-1}) + (\mathrm{tr}BCD\mathrm{tr}C\\
&\quad - \mathrm{tr}BCDC^{-1}) + (\mathrm{tr}CDA\mathrm{tr}D - \mathrm{tr}CDAD^{-1}) + 2\mathrm{tr}AC^{-1} + 2\mathrm{tr}BD^{-1}\\
&= (\mathrm{tr}ABC\mathrm{tr}B - \mathrm{tr}A\mathrm{tr}C + \mathrm{tr}ABC^{-1}B^{-1})\\
&\quad + (\mathrm{tr}DAB\mathrm{tr}A - \mathrm{tr}B\mathrm{tr}D + \mathrm{tr}ABA^{-1}D^{-1}) + (\mathrm{tr}BCD\mathrm{tr}C - \mathrm{tr}BCDC^{-1})\\
&\quad + (\mathrm{tr}CDA\mathrm{tr}D - \mathrm{tr}CDAD^{-1}) + 2(\mathrm{tr}A\mathrm{tr}C - \mathrm{tr}AC + \mathrm{tr}B\mathrm{tr}D - \mathrm{tr}BD)\\
&= \mathrm{tr}ABC\mathrm{tr}B + \mathrm{tr}DAB\mathrm{tr}A + \mathrm{tr}BCD\mathrm{tr}C + \mathrm{tr}CDA\mathrm{tr}D + \mathrm{tr}A\mathrm{tr}C + \mathrm{tr}B\mathrm{tr}D\\
&\quad + \mathrm{tr}ABDD^{-1}C^{-1}B^{-1} + \mathrm{tr}ABCC^{-1}A^{-1}D^{-1} - \mathrm{tr}BCDC^{-1} - \mathrm{tr}CDAD^{-1}\\
&\quad - 2\mathrm{tr}AC - 2\mathrm{tr}BD\\
&= \mathrm{tr}ABC\mathrm{tr}B + \mathrm{tr}DAB\mathrm{tr}A + \mathrm{tr}BCD\mathrm{tr}C + \mathrm{tr}CDA\mathrm{tr}D + \mathrm{tr}A\mathrm{tr}C + \mathrm{tr}B\mathrm{tr}D\\
&\quad + \mathrm{tr}DAB\mathrm{tr}BCD + \mathrm{tr}ABC\mathrm{tr}CDA\\
&\quad - \mathrm{tr}BCDABD - \mathrm{tr}ABCDAC - \mathrm{tr}BCDC^{-1} - \mathrm{tr}CDAD^{-1} - 2\mathrm{tr}AC - 2\mathrm{tr}BD\\
&= xz + yw\\
&\quad - \mathrm{tr}BCDA\mathrm{tr}BD + \mathrm{tr}BCDAD^{-1}B^{-1} - \mathrm{tr}ABCD\mathrm{tr}AC + \mathrm{tr}ABCDC^{-1}A^{-1}\\
&\quad - \mathrm{tr}BCDC^{-1} - \mathrm{tr}CDAD^{-1} - 2\mathrm{tr}AC - 2\mathrm{tr}BD.
\end{aligned}$$

Since $\mathrm{tr}ABCD = -2$, the last term equals $xz + yw$.

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https://doi.org/10.1112/S0024610704005435 Published online by Cambridge University Press

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https://doi.org/10.1112/S0024610704005435 Published online by Cambridge University Press