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Restricted orbit equivalence for ergodic $\mathbb{Z}^d$ actions I

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Abstract. In [**R1**] a notion of restricted orbit equivalence for ergodic transformations was developed. Here we modify that structure in order to generalize it to actions of higher-dimensional groups, in particular $\mathbb{Z}^d$-actions. The concept of a 'size' is developed first from an axiomatized notion of the size of a permutation of a finite block in $\mathbb{Z}^d$. This is extended to orbit equivalences which are cohomologous to the identity and, via the natural completion, to a notion of restricted orbit equivalence. This is shown to be an equivalence relation. Associated to each size is an entropy which is an equivalence invariant. As in the one-dimensional case this entropy is either the classical entropy or is zero. Several examples are discussed.

1. *Orderings and reorderings*

Let $(X, \mathcal{A}, \mu)$ be a Lebesgue probability space, and let $T = \{T_{\vec{v}}\}_{\vec{v}\in\mathbb{Z}^d}$ be a free ergodic $\mathbb{Z}^d$-action defined on $(X, \mathcal{A}, \mu)$. We write such a system as $(X, \mathcal{A}, \mu, T)$ or just as (X, T) if no confusion will arise. For $x \in X$ the orbit of x under the action of T is written as $\mathcal{O}(x) = \{T_{\vec{v}}(x); \vec{v} \in \mathbb{Z}^d\}$. The orbit relation of (X, T) is the set $\mathcal{O} = \{(x, x'); x' \in \mathcal{O}(x)\}$ in $X \times X$.

Dye's theorem tells us that any two such orbit relations on nonatomic Lebesgue spaces are equivalent. More precisely, there is a measure-preserving automorphism between any two such Lebesgue spaces mapping the orbit relation of one action to that of the other. This is true even when the dimensions of the actions differ, depending only on the amenability of the discrete groups $\mathbb{Z}^d$. In particular, we can regard all of our processes as sitting on the same space with the same orbit relation. It is worthwhile keeping this fact in mind as we proceed; but, as Dye's theorem is to be one consequence of our work

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here, we do not wish to explicitly use it. Instead, just consider the set of free actions of $\mathbb{Z}^d$ on X which have $\mathcal{O}$ as their orbit relation. Two such actions, then, can be thought of simply as two orderings of the equivalence classes of $\mathcal{O}$ and one can construct one action from another simply by reordering the orbits. We will carefully define these notions of an ordering and reordering. It is important to keep in mind in this context that a given action will arise in many different ways as an ordering of the classes of $\mathcal{O}$.

It is possible to give intrinsic conditions on the orbit relation $\mathcal{O}$ equivalent to its arising from a measure-preserving ergodic $\mathbb{Z}^d$-action. We do not do this, but rather simply assume there is *a priori* such an action T on $\mathcal{O}$.

Definition 1.1. An ordering is a measurable function

$$\alpha : \mathcal{O} \to \mathbb{Z}^d,$$

such that for a.e. $x \in X$, $\alpha(x, x) = \vec{0}$, $\alpha(x, *)$ is one-to-one, onto $\mathbb{Z}^d$ and such that

$$\alpha(x', x'') + \alpha(x, x') = \alpha(x, x''). \tag{1}$$

Fieldsteel and Friedman [**FF**], and Hasfura [**H**] refer to this as an affine structure on the orbit.

Condition (1) is called the cocycle condition. Notice that ergodicity of T can be interpreted as ergodicity of $\mathcal{O}$. We say $\mathcal{O}$ is ergodic if any measurable set invariant under the equivalence relation (if $x \in S$ and $(x, x') \in \mathcal{O}$ then $x' \in S$) is of measure 0 or 1.

LEMMA 1.1. *To say that α is an ordering of the ergodic equivalence relation $\mathcal{O}$ is equivalent to saying that there exists an ergodic $\mathbb{Z}^d$ action $T^\alpha = \{T^\alpha_{\vec{v}}\}_{\vec{v}\in\mathbb{Z}^d}$, where*

$$\alpha(x, x') = \vec{v} \quad \textit{iff} \quad T^\alpha_{\vec{v}}(x) = x'.$$

Proof. Certainly the existence of a free action $T^\alpha = \{T^\alpha_{\vec{v}}\}_{\vec{v}\in\mathbb{Z}^d}$ gives such an ordering. If the ordering exists, then defining $T^\alpha_{\vec{v}}(x)$ to be the unique $x' \in \mathcal{O}(x)$ with $\alpha(x, x') = v$ defines a measurable action. That it is measure preserving follows from the fact that $\mathcal{O}$ is, as does its ergodicity. □

We can view a pair of orderings α and β as inducing a family of permutations of $\mathbb{Z}^d$, one for each point $x \in X$. This is apparently not the best point of view to start from, as these permutations enjoy no natural equivariance with the underlying $\mathbb{Z}^d$ actions. Rather we will always view a reordering from α to β as built up asymptotically via a series of 'local' permutations of the orbit. Such local reorderings will indeed come from permutations of $\mathbb{Z}^d$ which will enjoy a natural equivariance with translations. Our aim is to axiomatize a metric-like notion of the size of such a family of permutations in order to measure how badly such a reordering has disrupted the structure of the orbit. These 'local' reorderings are in a sense trivial in that they come from naturally isomorphic systems.

Definition 1.2. Two orderings α and β are said to be cohomologous, or to differ by a coboundary if there exists a measurable $h : X \to \mathbb{Z}^d$ so that

$$\beta(x, x') = \alpha(x, x') - h(x) + h(x'). \tag{2}$$

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Coboundary changes, although seemingly trivial, are the building blocks of all our work. The origin of this approach can be seen from two perspectives. All the standard orbit equivalence theorems [**D**, **H**, **ORW**, **OW1**] and explicit cocycle constructions building from Dye's theorem are achieved via a pointwise convergent sequence of coboundary changes. Hence coboundaries are constructable and although 'trivial' can give limiting behavior that is nontrivial. Second, as Lemma 1.4 will show, a reordering to a cohomologous ordering looks locally on the orbit like a permutation of a finite block (when they are constructed this is precisely how they are constructed). Hence they provide a natural context for extending computations defined on finite permutations to computations on reorderings. In the next section we will see how to define a metric-like notion on permutations of finite blocks. This will extend to cohomologous orderings. Hence we can define the notion of a Cauchy sequence of cohomologous orderings and the limits of such will be the orderings equivalent to the original ordering. In this sense our work can be seen as simply axiomatizing the standard procedure taken in proving Dye-like theorems. Dye's theorem itself will be the least restrictive case of our general restricted orbit equivalence theorem. The most restrictive case will be Ornstein's isomorphism theorem, here for actions of $\mathbb{Z}^d$.

The following lemmas concerning cohomologous orderings are standard, but are included as our notation is slightly non-traditional and they indicate the structure of such orbit changes.

LEMMA 1.2. *If α and β are cohomologous orderings with*

$$\beta(x, x') = \alpha(x, x') - h(x) + h(x'),$$

then the map $\phi(x) = T^{\alpha}_{h(x)}(x)$ is an isomorphism from T^{β} to T^{α} which preserves orbits. On the other hand, if T^{α} and T^{β} are isomorphic by a map preserving orbits, then α and β are cohomologous.

Proof. This is a straightforward computation once we know that ϕ is one-to-one. Notice

$$\begin{aligned} \alpha(\phi(x), \phi(x')) &= \alpha(x', \phi(x')) + \alpha(x, x') + \alpha(\phi(x), x) \quad \text{by (1)} \\ &= h(x') + \alpha(x, x') - h(x) \\ &= \beta(x, x'). \end{aligned}$$

Thus if $\phi(x) = \phi(x')$ then $\beta(x, x') = \vec{0}$ and $x = x'$. □

When α and β are cohomologous the function h is not unique. Because the group $\mathbb{Z}^d$ is abelian, the function h is unique up to a constant. It is very important for our work to always fix a choice for this solution. We will express this choice not with h, but with the orbit-preserving isomorphism it generates.

Definition 1.3. We define the full group of $\mathcal{O}$ to be

$$\begin{aligned} FG(\mathcal{O}) = \{&\phi : X \to X; \phi \text{ is measure preserving, one-to-one, onto, invertible} \\ &\text{and } (x, \phi(x)) \in \mathcal{O}\}. \end{aligned}$$

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LEMMA 1.3. *For two orderings, α and β,*

$$\alpha(x, x') = \beta(x, x') + h(x) - h(x')$$

iff there is a $\phi \in FG(\mathcal{O})$, $\phi(x) = T^{\alpha}_{h(x)}(x)$ with $\beta(x, x') = \alpha(\phi(x), \phi(x'))$.

Proof. Just notice that ϕ is in $FG(\mathcal{O})$ iff it has the form $T^{\alpha}_{h(x)}(x)$ and is one-to-one and onto. □

If $\beta(x, x') = \alpha(\phi(x), \phi(x'))$, for $\phi \in FG(\mathcal{O})$ (which we abbreviate $\beta = \alpha\phi$), we write

$$\alpha\phi(x, x') = \alpha(\phi(x), \phi(x')). \tag{3}$$

In our work, cohomologous orderings will play much the same role that the rationals play in defining the reals. They provide an arithmetically trivial situation where computations become finite. In our case the reordering associated with a pair of cohomologous orderings will be local and hence describable by permutations of finite blocks. Our next lemma makes this more precise.

Let $B_N = \{\vec{v}; \vec{v} = (v_1, v_2, \ldots, v_d)$ and $|v_k| \leq N\}$, a box in $\mathbb{Z}^d$ centered at $\vec{0}$ with side length $2N + 1$.

LEMMA 1.4. *If $\alpha, \beta = \alpha\phi$ are orderings on the ergodic orbit relation $\mathcal{O}$, then for μ-a.e. $x \in X$*

$$\lim_{N\to\infty} \frac{|B_N \triangle \{\beta(x, T^{\alpha}_{\vec{v}}(x)); \vec{v} \in B_N\}|}{|B_N|} = \lim_{N\to\infty} \frac{|B_N \triangle \{\alpha(x, \phi(T^{\alpha}_{\vec{v}}(x))); \vec{v} \in B_N\}|}{|B_N|} = 0.$$

Proof. For $M > 0$ let

$$S_M = \{x \in X; \|h(x)\| > M\}.$$

Given $\varepsilon > 0$ select M so large that $\mu(S_M) < \varepsilon$. Fix an $x \in X$ and by the Birkhoff theorem,

$$\frac{|\{\vec{v} \in B_N; T^{\alpha}_{\vec{v}}(x) \in S_M\}|}{|B_N|} \xrightarrow{N} \mu(S_M) < \varepsilon.$$

Thus once N is sufficiently large,

$$|B_{N+2M} \triangle \{\beta(x, T^{\alpha}_{\vec{v}}(x)); \vec{v} \in B_N\}| \leq (2N + 4M + 1)^d - (2N + 1)^d + 2\varepsilon|B_N|.$$

Letting N go to infinity,

$$\overline{\lim_{N\to\infty}} \frac{|B_{N+2M} \triangle B_N|}{|B_N|} \leq 2\varepsilon$$

giving the first limit. As $\alpha(x, \phi(T^{\alpha}_{\vec{v}}(x))) = \beta(x, T^{\alpha}_{\vec{v}}(x)) + \beta(\phi^{-1}(x), x)$, the second limit also holds. □

Now for $\alpha, \phi \in FG(\mathcal{O})$ and $x \in X$ define a bijection of $\mathbb{Z}^d$,

$$f^{\alpha,\phi}_x = f_x : \mathbb{Z}^d \to \mathbb{Z}^d,$$

by

$$f_x(\vec{v}) = \vec{v} + h(T^{\alpha}_{\vec{v}}(x)) = \alpha(x, \phi(T^{\alpha}_{\vec{v}}(x))) = \vec{v} + \alpha(T^{\alpha}_{\vec{v}}(x), \phi(T^{\alpha}_{\vec{v}}(x))). \tag{4}$$

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Let $f_{x,N}^{\alpha,\phi}$ be the restriction of $f_x^{\alpha,\phi}$ to B_N.

Lemma 1.4 tells us that once N is large, $f_{x,N}^{\alpha,\phi}$ is approximately a permutation of B_N in that the fraction of B_N mapped outside of B_N is very small. In particular, we could modify $f_{x,N}^{\alpha,\phi}$ by mapping the vectors moved outside of B_N back into the holes left in B_N and obtain an actual permutation that agrees with $f_{x,N}^{\alpha,\phi}$ on most points.

It will be useful to have an explicit choice for this. Suppose S is some finite subset of $\mathbb{Z}^d$ and let $f : S \to \mathbb{Z}^d$ be some one-to-one function. Let

$$B_1 = \{\vec{v} \in S;\, f(\vec{v}) \notin S\} \quad \text{and} \quad B_2 = \{\vec{v} \in S;\, \vec{v} \notin \text{range}(f)\}.$$

Note that these two sets must have the same cardinality. Order both B_1 and B_2 lexicographically and let $\ell : B_1 \to B_2$ be the order-preserving bijection. Now define a permutation $\pi_f : S \to S$ by

$$\pi_f(\vec{v}) = \begin{cases} f(\vec{v}) & \text{if } f(\vec{v}) \in S \\ \ell(\vec{v}) & \text{if } f(\vec{v}) \notin S. \end{cases}$$

Returning to orderings, having defined the one-to-one map $f_{x,N}^{\alpha,\phi} : B_N \to \mathbb{Z}^d$, define the permutation

$$\pi_{x,N}^{\alpha,\phi} \quad \text{to be} \quad \pi_{f_{x,N}^{\alpha,\phi}}.$$

COROLLARY 1.5. *Given an ordering α and $\phi \in FG(\mathcal{O})$ for a.e. $x \in X$*

$$\lim_{N\to\infty} \frac{|\{\vec{v} \in B_N;\, f_{x,N}^{\alpha,\phi} \neq \pi_{x,N}^{\alpha,\phi}\}|}{|B_N|} = 0.$$

Proof. This limit is bounded by the second limit in Lemma 1.4. □

We will now develop a notion of the size of such permutations as a way of measuring the distortion of an orbit created by such a reordering. The axiomatics that will be developed here are different from those of the one-dimensional case [**R1**]. There, one took advantage of the well-ordering of the integers to define a size directly on orderings. In particular, two of the axioms were stated explicitly as conditions on the size of a reordering between two orderings. The critical observation was that any bijection from an interval of integers into the integers has associated with it a permutation of the interval that preserves the order on the image. It was observed soon after the publication of [**R1**] that there were equivalence relations, in particular α-equivalence [**FdJR**], that should come under the umbrella of restricted orbit equivalence, but do not because this 'push together' of the orbit is simply too violent an act on the orbit to preserve the delicate arithmetic structure of the equivalence relation. Furthermore, it is clear that no analog of this 'push together' exists in higher dimensions. Fieldsteel [**Fi**] made a fundamental observation that opened the door for our work here. He showed that, in the one-dimensional theory, a restricted orbit equivalence class was the m-closure of a cohomology class.

We will define what it means to measure the size of a permutation. As just seen in Corollary 1.5, a cohomologous reordering looks much like a permutation. In particular, we will see that it is not difficult to lift our definition of a size to measure the size of a

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cohomologous reordering. We will then use the m-closure of a cohomology class as the definition of a restricted orbit equivalence. Our fundamental problem in defining a size was to replace those (one-dimensional) axioms which directly concerned orderings with axioms which pertained only to permutations, in such a way that this m-closure was, in fact, an equivalence class.

2. *Sizes m and m-equivalence*

Let Π_n be the set of all permutations of the box B_n and let $\Pi = \cup_n \Pi_n$. We will define what it means for a function $m : \Pi \to [0, 1]$ to be called a 'size'. The idea is that we want to associate a number to the approximate permutations which, as we saw above, came from cohomologous orderings. This number should measure the asymptotic size of these permutations as N grows. Two orderings will then be called equivalent if one can get from one to the other via a sequence of cohomologous orderings which is Cauchy in the given size and converges pointwise to the other ordering. We now introduce the axioms of a size, define the associated relation and prove it to be an equivalence relation.

We begin with the six axioms required of m for it to be a size.

AXIOM 1. (Continuity near the identity.)
(a) $m(\mathrm{id}) = 0$ *for all identity permutations, and*
(b) *for all* $\varepsilon > 0$ *there is an* N *and* $\delta > 0$ *so that for* $n \geq N, \pi \in \Pi_n$, *and if*

$$\frac{|\{\vec{v} \in B_n; \pi(\vec{v}) \neq \vec{v}\}|}{|B_n|} < \delta,$$

then

$$m(\pi) < \varepsilon.$$

AXIOM 2. ('Lumpiness' of small permutations.) *For all* N_0 *and* $\varepsilon > 0$, *there is an* N_1 *and* $\delta > 0$ *so that if* $n \geq N_1$ *and* $\pi \in \Pi_n$ *with* $m(\pi) < \delta$, *then*

$$\frac{|\{\vec{v} \in B_n;\ \textit{for some } \vec{v}_0 \in B_{N_0}, \pi(\vec{v} + \vec{v}_0) \neq \pi(\vec{v}) + \vec{v}_0\}|}{|B_n|} < \varepsilon.$$

To make $\pi(\vec{v} + \vec{v}_0)$ *well defined we extend* π *outside* B_n *as the identity. In such contexts we will always make this choice.*

Axioms 1 and 2 are partial converses to one another. Axiom 1 tells us that if a permutation moves very few points, then it must be small. Axiom 2 tells us that if a permutation is small it must move most points in large rigid lumps.

AXIOM 3. (Continuity of inverses near zero.) *For all* $\varepsilon > 0$ *there is an* N *and* $\delta > 0$ *so that for* $n \geq N$ *and* $\pi \in \Pi_n$ *if* $m(\pi) < \delta$, *then* $m(\pi^{-1}) < \varepsilon$.

AXIOM 4. (Continuity of composition near zero.) *For all* $\varepsilon > 0$ *there is an* N *and* δ *so that for* $n \geq N$ *and* $\pi_1, \pi_2 \in \Pi_n$, *if* $m(\pi_2) < \delta$, *then*

$$m(\pi_2 \pi_1) < m(\pi_1) + \varepsilon.$$

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Axioms 3 and 4 simply tell us the kind of uniform continuity we will want of the group operations within each Π_n. There are two other natural operations within Π; namely, the concatenation of permutations on disjoint blocks, and the rearranging of blocks left invariant by a permutation. One of these builds a new permutation on a larger block from smaller ones, the other changes a given permutation to a new one. Our last two axioms concern continuity of such operations.

AXIOM 5. (Continuity of concatenation near zero.) *For all $\varepsilon > 0$ there is an N_0 and $\delta > 0$ so that if $n_1, n_2, \ldots, n_k > N_0$ and if $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \in \mathbb{Z}^d$ are such that:*

(a) *$A_i = B_{n_i} + \vec{v}_i$ are disjoint;*

(b) *for some n, $\cup_{i=1}^k A_i \subseteq B_n$ and $\dfrac{|\cup_{i=1}^k A_i|}{|B_n|} > 1 - \delta$;*

(c) *$\pi \in \Pi_n$ is such that for all i, $\pi(A_i) = A_i$ and setting $\pi_i \in \Pi_{n_i}$ to be $\pi_i(\vec{v}) = \pi(\vec{v} + \vec{v}_i) - \vec{v}_i$ we have*

(d) *$m(\pi_i) < \delta$ for all i; we then must have that $m(\pi) < \varepsilon$.*

To picture what Axiom 5 requires, notice that if B_n is almost completely covered by translates of various B_{n_i} each of which is fixed by π and if when π is viewed as a permutation of each translate it is small, then π itself must be small.

AXIOM 6. (Continuity of rearranging large lumps near zero.) *For all $\varepsilon > 0$ there is an N and $\delta > 0$ so that if $N_1 \geq N$ and $K_1, K_2, \ldots, K_s \subseteq B_n$ and*

(a) *the $K_i = B_{N_1} + \vec{v}_i$ are disjoint with*

$$\frac{|\cup_{i=1}^s K_i|}{|B_n|} > 1 - \delta;$$

(b) *there exist $\vec{u}_i \in B_{2n}$, $i = 1, \ldots, s$ so that $K_i + \vec{u}_i = K_i'$ are also disjoint and $\subseteq B_n$;*

(c) *the permutations $\pi, \pi' \in \Pi_n$ are such that $\pi(K_i) = K_i$, $\pi'(K_i') = K_i'$ and for $\vec{v} \in K_i$,*

$$\pi'(\vec{v} + \vec{u}_i) = \pi(\vec{v}) + \vec{u}_i;$$

(i.e. π acts on K_i exactly as π' acts on K_i'); and

(d) *$m(\pi) < \delta$.*

We then must also have $m(\pi') < \varepsilon$.

This last axiom says that smallness of a permutation is a local property. More precisely, if you have a small permutation (in m) that maps a lot of sets K_i to themselves, where the K_i are all translates of a large box B_{N_1}, and if you simply rearrange where these boxes K_i happen to lie in B_n then the permutation you obtain will still be small.

Our next step is to extend the definition of a size from finite permutations to reorderings by a coboundary. The size will be associated with a particular choice for h, the solution of the coboundary equation, which is the same as to say ϕ in the full group, and may change as h is changed. This may seem unnatural but in fact a choice of solution identifies an isomorphism between the two actions, hence pins down precisely which points in one action are being identified with which points in the other. In particular, it makes the machinery of sizes applicable to a wider range of equivalence relations.

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Suppose now that m is a size as defined by these six axioms. Suppose α is an ordering and $\phi \in FG(\mathcal{O})$. Let $x \in X$ and let $f_x^{\alpha,\phi}$ be the associated bijection of $\mathbb{Z}^d$. Recall that for μ-a.e. $x \in X$ for any $\varepsilon > 0$ there is an $N(x, \varepsilon)$ so that for all $n \geq N$ the permutation $\pi_{x,n}^{\alpha,\phi}$ agrees with $f_{x,n}^{\alpha,\phi}$ all but ε of the 'time', i.e. on all but a subset of B_n of density at most ε.

Define:

$$m_{n,x}(\alpha, \phi) = \min_{\pi \in \Pi_n} \max \left\{ m(\pi), \frac{|\{\vec{v} \in B_n; \pi_{x,n}^{\alpha,\phi}(\vec{v}) \neq \pi(\vec{v})\}|}{|B_n|} \right\} \tag{5}$$

and

$$m_x(\alpha, \phi) = \liminf_{n \to \infty} m_{n,x}(\alpha, \phi). \tag{6}$$

Finally, define

$$m(\alpha, \phi) = \operatorname*{ess\,inf}_{x \in X} m_x(\alpha, \phi). \tag{7}$$

We call $m(\alpha, \phi)$ the *m-size* of the reordering α to $\beta = \alpha\phi$ or the *m-distance between* α *and* β. This is of course bad usage as ϕ is not determined by α and β; but our choice will always be obvious. We will use a certain equivariance formula quite often. Notice that

$$f_x^{\beta,\phi\psi\phi^{-1}} = f_{\phi^{-1}(x)}^{\beta\phi,\psi}.$$

This immediately implies that

$$m(\beta\phi, \psi) = m(\beta, \phi\psi\phi^{-1}).$$

Each of our six axioms leads to a parallel fact about the m-size of a reordering. Notice that in defining the m-size, at both steps we took the smallest possible choice, a lim inf and an ess inf. Our first step will be to show that if instead we took a lim sup and an ess sup we would not change our theory. This fact arises naturally from Axiom 5, and in that sense, Theorem 2.1 lifts Axiom 5 to orderings.

THEOREM 2.1. *Set*

$$\hat{m}_x(\alpha, \phi) = \limsup_{n \to \infty} m_{n,x}(\alpha, \phi)$$

and now

$$\hat{m}(\alpha, \phi) = \operatorname*{ess\,sup}_{x \in X} \hat{m}_x(\alpha, \phi).$$

Given any $\varepsilon > 0$ there is a $\delta > 0$ so that if $m(\alpha, \phi) < \delta$ then $\hat{m}(\alpha, \phi) < \varepsilon$.

The first step in proving this is the following lemma.

LEMMA 2.2. *Suppose α is an ordering and $\phi \in FG(\mathcal{O})$. Given any $\varepsilon > 0$ there is a $\delta > 0$ so that if $m(\alpha, \phi) < \delta$ then for μ-a.e. $x \in X$, $m_x(\alpha, \phi) < \varepsilon$.*

Proof. Fix a value δ to be determined later and let

$$M = \{x; m_x(\alpha, \phi) < \delta\},$$

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a set of positive measure. As $\mathcal{O}$ is ergodic, for μ-a.e. $x \in X$, there is an $x' \in M$ so that $(x, x') \in \mathcal{O}$. Let x' be the 'closest' such point relative to the ordering α, and if there are more than one such, pick x' from among them. Define a permutation π_N of $B_N + \alpha(x, x')$ to be

$$\pi^{\alpha,\phi}_{x',N}(\vec{v} - \alpha(x, x')) + \alpha(x, x').$$

Let $a = a(x, x') = \|\alpha(x, x')\|_\infty$, the absolute value of the largest term in $\alpha(x, x')$. Now extend π_N to all of B_{N+a} by the identity map. It is an easy computation to show that

$$\lim_{N\to\infty} \frac{|\{\vec{v} \in B_{N+a}; \pi_N(\vec{v}) \neq \pi^{\alpha,\phi}_{x,N+a}(\vec{v})\}|}{|B_{N+a}|} = 0.$$

Axiom 5 tells us that if δ is small enough, once N is large enough,

$$m(\pi^{\alpha,\phi}_{x,N+a}) \leq \varepsilon/2$$

completing the result. □

Proof of Theorem 2.1. From Lemma 2.2 we know that for any δ_1, there is a δ so that if $m(\alpha, \phi) < \delta$, then $m_x(\alpha, \phi) < \delta_1$ for μ-a.e. $x \in X$. This means that for μ-a.e. $x \in X$ there is an increasing sequence of values $N_i(x)$ so that

$$m(\pi^{\alpha,\phi}_{x,N_i(x)}) < 2\delta_1.$$

We can also assume from Corollary 1.5 that

$$\frac{|\{\vec{v} \in B_{N_i(x)}; f^{\alpha,\phi}_{x,N_i(x)}(\vec{v}) \neq \pi^{\alpha,\phi}_{x,N_i(x)}(\vec{v})\}|}{|B_{N_i(x)}|} < \delta_1.$$

Given any value $N_0 \in \mathbb{N}$ we can assume all $N_i(x) > N_0$. Apply the backward Vitali lemma for free ergodic actions of $\mathbb{Z}^d$, in this case T^α. (In [**R2**], this theorem is proven for $d = 1$ and discussed in the exercises for $d > 1$.) We conclude that there is a subset $F \subseteq X$ and for each $x \in F$, a measurable choice $N(x) \in \{N_i(x)\}$, $N_0 < N(x) < N_1$ for some $N_1 \in \mathbb{N}$ so that the orbit slices $S(x) = \{T^\alpha_{\vec{v}}(x) : \vec{v} \in B_{N(x)}\}$ for $x \in F$ are pairwise disjoint and cover all but δ_1 of X.

Let

$$G = \bigcup_{x\in F} S(x).$$

As $\mu(G) > 1 - \delta_1$, by the Birkhoff Theorem, for μ-a.e. $x \in X$

$$\lim_{N\to\infty} \frac{|\{\vec{v} \in B_N; T^\alpha_{\vec{v}}(x) \in G\}|}{|B_N|} > 1 - \delta_1. \tag{8}$$

Fix a point x which satisfies (8). Let $V(x) = \{\vec{v} : T^\alpha_{\vec{v}}(x) \in F\}$. To each $\vec{v} \in V(x)$ set

$$N(\vec{v}) = N(T^\alpha_{\vec{v}}(x)) \quad \text{and} \quad \pi(\vec{v}) = \pi^{\alpha,\phi}_{T^\alpha_{\vec{v}}(x),N(\vec{v})}.$$

Now consider a value of N large enough so that

$$\frac{|\{\vec{v} \in B_N; T^\alpha_{\vec{v}}(x) \in G\}|}{|B_N|} > 1 - 2\delta_1, \quad \frac{|B_{N-N_1}|}{|B_N|} > 1 - \delta_1,$$

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and

$$\frac{|\{\vec{v} \in B_N;\, f_{x,N}^{\alpha,\phi}(\vec{v}) \neq \pi_{x,N}^{\alpha,\phi}(\vec{v})\}|}{|B_N|} < \delta_1.$$

Let $V(N,x) = V(x) \bigcap B_{N-N_1}$ and now consider the following permutation of B_N. For each $\vec{v} \in V(N,x)$, permute $B_{N(\vec{v})} + \vec{v}$ by $\pi(\vec{v})$ conjugated by translation by $\vec{v}$. Map the rest of the points of B_N by the identity. Call this permutation $\pi_{x,N}$. A computation shows that

$$\frac{|\{\vec{v} \in B_N; \pi_{x,N}(\vec{v}) \neq \pi_{x,N}^{\alpha,\phi}(\vec{v})\}|}{|B_N|} \leq 4\delta_1.$$

Hence $m_{N,x}(\alpha,\phi) \leq \max(m(\pi_{x,N}), 4\delta_1)$. Furthermore, by Axiom 5, if δ_1 was chosen small enough, and N_0 large enough,

$$m(\pi_{x,N}) \leq \varepsilon/2$$

completing the theorem. □

The following lemma lifts Axiom 1 to orderings.

LEMMA 2.3. *Let α be an ordering, $\phi \in FG(\mathcal{O})$ and m a size.*
(a) *If ϕ is the identity, then $m(\alpha,\phi) = 0$.*
(b) *For every $\varepsilon > 0$ there is a δ so that if $\mu\{x : \phi(x) \neq x\} < \delta$ then $m(\alpha,\phi) < \varepsilon$.*

Proof. Part (a) is clearly just the first half of Axiom 1. To see that (b) is just the second half, note that from the Birkhoff theorem and Corollary 1.5,

$$\lim_{N\to\infty} \frac{|\{\vec{v} \in B_N; \pi_{x,N}^{\alpha,\phi}(\vec{v}) \neq \vec{v}\}|}{|B_N|} < \delta.$$ □

Our next lemma lifts Axiom 2 to orderings.

LEMMA 2.4. *Let α be an ordering, $\phi \in FG(\mathcal{O})$ and m a size. For every $\varepsilon > 0$ and every $N_0 \in \mathbb{N}$ there is a $\delta > 0$ so that if $m(\alpha,\phi) < \delta$ then there exists a subset $A \subseteq X$ with $\mu(A) > 1 - \varepsilon$ and if $x \in A$ and $\vec{v} \in B_{N_0}$, then*

$$\alpha\phi(x, T_{\vec{v}}^{\alpha}(x)) = \vec{v}.$$

Proof. For N_0 fixed, let $A = \{x \in X;$ for all $\vec{v} \in B_{N_0}, \alpha\phi(x, T_{\vec{v}}^{\alpha}(x)) = \vec{v}\}$, a measurable set. Choose δ so small so that if $m(\alpha,\phi) < \delta$, then using Theorem 2.1, for μ-a.e. $x \in X$,

$$\hat{m}_x(\alpha,\phi) < \min\{(\text{half the } \delta \text{ of Axiom 2 with respect to } \varepsilon/2), |B_{N_0}|^{-1}\}.$$

Applying Corollary 1.5 and Axiom 2, we conclude that

$$\lim_{N\to\infty} \frac{|\{\vec{v} \in B_N; \pi_{T_{\vec{v}}^{\alpha}(x),N_0}^{\alpha,\phi} = \mathrm{id}\}|}{|B_N|} \geq 1 - \varepsilon/2,$$

and

$$\lim_{N\to\infty} \frac{|\{\vec{v} \in B_N;\, f_{T_{\vec{v}}^{\alpha}(x),N_0}^{\alpha,\phi} \neq \pi_{T_{\vec{v}}^{\alpha}(x),N_0}^{\alpha,\phi}\}|}{|B_N|} \leq \varepsilon/2.$$

By the Birkhoff theorem, $\mu(A) \geq 1 - \varepsilon$. □

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We now lift Axioms 3 and 4 with the following two lemmas. They will be used often in our proof that m-equivalence is an equivalence relation, so we will explicitly label the parameter δ by a subscript indicating the lemma where it is defined.

LEMMA 2.5. *Let α be an ordering, $\phi \in FG(\mathcal{O})$ and m a size. For every $\varepsilon > 0$ there exists $\delta = \delta_{2.5}(\varepsilon) > 0$ so that if $m(\alpha, \phi) < \delta$ then $m(\beta, \phi^{-1}) = m(\alpha\phi, \phi^{-1}) < \varepsilon$.*

Proof. Note that with $\beta = \alpha\phi$,

$$(f_x^{\alpha,\phi})^{-1} = f_{\phi^{-1}(x)}^{\beta,\phi^{-1}}.$$

Thus from Corollary 1.5,

$$\lim_{N\to\infty} \frac{|\{\vec{v} \in B_N; (\pi_{x,N}^{\alpha,\phi})^{-1}(\vec{v}) \neq \pi_{\phi^{-1}(x),N}^{\beta,\phi^{-1}}(\vec{v})\}|}{|B_N|} = 0. \tag{9}$$

From Theorem 2.1 we know that given any $\delta_1 > 0$ there is a δ so that if $m(\alpha, \phi) < \delta$ then for μ-a.e. $x \in X$, once N is large enough,

$$m(\pi_{N,x}^{\alpha,\phi}) < \delta_1.$$

Given any $\varepsilon > 0$, Axiom 3 tells us that if δ_1 is small enough and N large enough, then

$$m((\pi_{N,x}^{\alpha,\phi})^{-1}) < \varepsilon/2.$$

But now Axioms 1 and 4 and (9) above tell us that once N is large enough

$$m(\pi_{N,\phi^{-1}(x)}^{\beta,\phi^{-1}}) < \varepsilon$$

completing the result. □

LEMMA 2.6. *Suppose $\alpha, \beta = \alpha\phi$, and $\gamma = \beta\psi$ are orderings. For every $\varepsilon > 0$ there exists $\delta = \delta_{2.6}(\varepsilon) > 0$ such that if $m(\beta, \psi) < \delta$ then*

$$m(\alpha, \phi\psi) < m(\alpha, \phi) + \varepsilon.$$

Proof. To begin, note that

$$f_x^{\alpha,\phi\psi} = f_{\phi^{-1}(x)}^{\beta,\psi} \circ f_x^{\alpha,\phi}. \tag{10}$$

From here the argument is completed as for Lemma 2.5, by using Axioms 1 and 4 and Corollary 1.5 to replace the functions in (10) by permutations of large blocks, and then Axiom 3 once more to complete the argument. □

Theorem 2.1 is the natural lift to orderings of Axiom 5. We now construct the natural fact on orderings which parallels Axiom 6. Notice that this is the only one of our six facts which deals with orderings that are not necessarily cohomologous.

Definition 2.1. A sequence of orderings $\{\alpha_i\}$ is said to converge pointwise to the ordering β (written $\alpha_i \underset{\text{pw}}{\to} \beta$) if for a.e. $(x, x') \in \mathcal{O}$, $\lim_{i\to\infty} \alpha_i(x, x') = \beta(x, x')$. This is equivalent to saying for a.e. $x \in X$, and all vectors $\vec{v} \in B_1$, once i is large enough,

$$\beta(x, T_{\vec{v}}^{\alpha_i}(x)) = \alpha_i(x, T_{\vec{v}}^{\beta}(x)) = \vec{v}.$$

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The corresponding and very useful notion of convergence in probability is given by the metric

$$d_0(\alpha, \beta) = \mu(\{x;\ \text{there is vector } \vec{v} \in B_1 \text{ with } T^{\alpha}_{\vec{v}}(x) \neq T^{\beta}_{\vec{v}}(x)\}).$$

One easily calculates that if $\alpha_i \underset{\text{pw}}{\to} \beta$ then $d_0(\alpha_i, \beta) \to 0$ and if $\sum_i d_0(\alpha_i, \beta) \leq \infty$ then $\alpha_i \underset{\text{pw}}{\to} \beta$. And as is usual, if $d_0(\alpha, \beta) \to 0$ then along a subsequence of the α_i we have pointwise convergence to β.

If the orderings $\{\alpha_i\}$ are cohomologous to some α, i.e. $\alpha_i = \alpha\phi_i$, then we will say $\alpha \underset{\text{pw}}{\to} \beta$ via $\{\phi_i\}$.

Definition 2.2. Let α be any ordering. A sequence of orderings $\{\alpha_i\}$ with $\alpha_i = \alpha\phi_i$ for all i is said to be m-Cauchy if for every $\varepsilon > 0$ there exists an $I \in \mathbb{N}$ so that for all $i, j \geq I$, $m(\alpha_i, \phi_i^{-1}\phi_j) < \varepsilon$.

Notice that $m(\alpha_i, \phi_i^{-1}\phi_j) = m(\alpha, \phi_j\phi_i^{-1}) < \varepsilon$.

Definition 2.3. Let α and β be two orderings. Let $\{\alpha_i\}$ be a sequence of orderings with $\alpha_i = \alpha\phi_i$ for all i and $\alpha_0 = \alpha$. We say that $\alpha \underset{m}{\to} \beta$ via $\{\phi_i\}$ (or $\{\alpha_i\} \underset{m}{\to} \beta$ if the choice of ϕ_i is obvious) if the sequence $\{\alpha_i\}$ is m-Cauchy and converges pointwise to β.

The following lemma is our lifting of Axiom 6 to orderings, and is a fundamental tool in showing that m-equivalence is an equivalence relation.

LEMMA 2.7. *Let m be a size. Let α and β be any two orderings with $\alpha \underset{\text{pw}}{\to} \beta$ via $\{\phi_j\}$. Let $\phi' \in FG(\mathcal{O})$ and so $\alpha\phi' \underset{\text{pw}}{\to} \beta\phi'$ via $\{\phi'^{-1}\phi_j\phi'\}$. For every $\varepsilon > 0$ there exists $\delta = \delta_{2.7}(\varepsilon) > 0$ so that if, for all j, $m(\alpha_j, \phi') < \delta$ then $m(\beta, \phi') < \varepsilon$.*

Proof. We begin to develop some machinery which will show us how small δ must be. First consider the reordering from β to $\beta\phi'$. Fix a value $\eta > 0$ whose exact size we will set later. By Lemma 1.4, there is a set $G \subseteq X$ with $\mu(G) > 1 - \eta$ and an N_1 so large that for all $n \geq N_1$, and $x \in G$,

$$|B_n \triangle \{\beta(x, \phi' T^{\beta}_{\vec{v}}(x));\ \vec{v} \in B_n\}| < \eta |B_n|.$$

We also require N_1 be so large to apply Axiom 6 with 'epsilon' equal to $\varepsilon/2$.

Corollary 1.5 now tells us that for each $x \in G$ the bijection $f^{\beta,\phi'}_x$ of $\mathbb{Z}^d$ when restricted to B_{N_1} is 'essentially' a permutation of B_{N_1}. More precisely, we can construct a permutation $\pi^{\beta,\phi'}_{x,N_1}$ of B_{N_1} with

$$|\{\vec{v} \in B_{N_1};\ f^{\beta,\phi'}_{x,N_1}(\vec{v}) \neq \pi^{\beta,\phi'}_{x,N_1}(\vec{v})\}| < \eta |B_{N_1}|.$$

Build a Rokhlin tower for the $\mathbb{Z}^d$ system T^{β} with base set $F_1 \subseteq G$ of size B_{N_1} covering all but 2η in measure of X. (The strong Rokhlin lemma guarantees we can build such a tower covering all but η of X and independent of the partition $\{G, G^c\}$ and deleting that part of the base contained in G^c we get the desired tower.)

There are only finitely many permutations of B_{N_1} so we may partition F_1 into finitely many sets $\{F_{1,1}, \ldots, F_{1,(2N_1+1)^d}\}$, where $x \in F_{1,j}$ iff $\pi^{\beta,\phi'}_{x,N_1} = \pi_j$ a fixed element of Π_{N_1}. Set $a = \frac{1}{2}\min\{\mu(F_{1,j}) : \mu(F_{1,j}) > 0\} > 0$.

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Applying the Birkhoff theorem to the finite list of sets $F_{1,j}$ we can start to see how Axiom 6 will be used. The Birkhoff theorem tells us that for a.e. $x \in X$, once N_2 is large enough, for all j,

$$\left| \frac{1}{|B_{N_2-N_1}|} \sum_{\vec{v} \in B_{N_2-N_1}} 1_{F_{1,j}}(T_{\vec{v}}^{\beta}(x)) - \mu(F_{1,j}) \right| \leq \eta a.$$

We also require that

$$\frac{|B_{N_2-N_1}|}{|B_{N_2}|} > 1 - \eta a.$$

Examine the form of the map $f_{x,N_2}^{\beta,\phi'}$ restricted to the box B_{N_2}. For any $\vec{v} \in B_{N_2-N_1}$ with $x' = T_{\vec{v}}^{\beta}(x) \in F_{1,j}$, let $K(x') = B_{N_1} + \vec{v} \subseteq B_{N_2}$. We conclude for such an x' that

$$|\{\vec{u} \in K(x');\ f_{x,N_2}^{\beta,\phi'}(\vec{u}) \neq \pi_j(\vec{u} - \vec{v}) + \vec{v}\}| \leq \eta |B_{N_1}|.$$

We also know that

$$|\{\vec{u} \in B_{N_2};\ \text{for some } \vec{v} \in B_{N_2-N_1},\ x' = T_{\vec{v}}^{\beta}(x) \in F_1 \text{ and } \vec{u} \in K(x')\}| \geq 1 - 4\eta.$$

The sets $K(x')$ for $x' \in F_1$ are disjoint. Define a new permutation $\overline{\pi}_{x,N_2} \in \Pi_{N_2}$ as follows:

$$\overline{\pi}_{x,N_2}(\vec{u}) = \begin{cases} \pi_j(\vec{u} - \vec{v}) + \vec{v} & \text{if } T_{\vec{v}}^{\beta}(x) \in F_{1,j},\ \vec{u} \in K(T_{\vec{v}}^{\beta}(x)) \text{ and } \vec{v} \in B_{N_2-N_1} \\ \vec{u} & \text{otherwise.} \end{cases}$$

Notice that $\overline{\pi}_{x,N_2}$ is the kind of permutation Axiom 6 discusses. If we select a particular π_j then the number of disjoint boxes $K(x') \subseteq B_{N_2}$ with $x' \in F_{1,j}$ is $\mu(F_{1,j})(1 \pm 2\eta)|B_{N_2}|$; which is to say that in any of these permutations $\overline{\pi}_{x,N_2}$ with N_2 fixed, the number of occurences of a particular π_j is essentially the same, no matter which point x was chosen. We can compute

$$\begin{aligned} &|\{\vec{u} \in B_{N_2};\ f_{x,N_2}^{\beta,\phi'}(\vec{u}) \neq \overline{\pi}_{x,N_2}(\vec{u})\}| \\ &\quad \leq \quad |\{\vec{u};\ \text{for some } x' = T_{\vec{v}}^{\beta}(x) \in F_{1,j},\ \vec{v} \in B_{N_2-N_1},\ \vec{u} \in K(x') \\ &\qquad \text{and } f_{x',N_1}^{\beta,\phi'}(\vec{u} - \vec{v}) \neq \pi_j(\vec{u} - \vec{v})\}| \\ &\qquad + |\{\vec{u};\ \text{for no } T_{\vec{v}}^{\beta}(x),\ \vec{v} \in B_{N_2-N_1} \text{ is } \vec{u} \in K(T_{\vec{v}}^{\beta}(x))\}| \leq 5\eta. \end{aligned}$$

It follows that

$$m_{N_2,x}(\beta, \phi') \leq m(\overline{\pi}_{x,N_2}) + 5\eta.$$

We now turn our attention to the sequence α_j converging to β to find the other permutation of Axiom 6. As $\alpha_j \underset{\text{pw}}{\to} \beta$, for all N for μ-a.e. $x \in X$, once $j \geq J(N, x)$, for all $\vec{u} \in B_N$

$$T_{\vec{u}}^{\alpha_j}(x) = T_{\vec{u}}^{\beta}(x).$$

Choose J so large so that

$$\mu\{x \in X;\ J(N_1, x) > J\} < \eta a.$$

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Let $F_2 = \{x \in F_1 \mid J(N_1, x) \leq J\}$, a subset of all but a fraction η of F_1. This set is a subset of F_1 and so is a Rokhlin base for a tower of size B_{N_1} for T^β and restricted to this tower, T^β and T^{α_J} act identically. Thus for $x \in F_2$,

$$|\{\vec{u} \in B_{N_1};\, f_{x,N_1}^{\beta,\phi'}(\vec{u}) \neq f_{x,N_1}^{\alpha_J,\phi'}(\vec{u})\}| \leq \eta |B_{N_1}|$$

as the only points on which $f_{x,N_1}^{\beta,\phi'}$ and $f_{x,N_1}^{\alpha_J,\phi'}$ could possibly differ are those points which $f_{x,N_1}^{\beta,\phi'}$ maps outside of B_{N_1}.

Setting $F_{2,j} = F_2 \cap F_{1,j}$, $\mu(F_{2,j}) \geq \mu(F_{1,j})(1-\eta) > a$. For $x \in F_{2,j}$,

$$|\{\vec{u} \in B_{N_1};\, f_{x,N_1}^{\alpha_J,\phi'}(\vec{u}) \neq \pi_j(\vec{u})\}| \leq 2\eta |B_{N_1}|.$$

Apply the Birkhoff theorem to T^{α_J} to say that for μ-a.e. $x \in X$, once N_2 is large enough,

$$\left| \frac{1}{|B_{N_2-N_1}|} \sum_{\vec{v} \in B_{N_2-N_1}} 1_{F_{2,j}}(T_{\vec{v}}^{\alpha_J}(x)) - \mu(F_{2,j}) \right| < \eta a.$$

As we did with $f_{x,N_2}^{\beta,\phi'}$, examine the form of $f_{x,N_2}^{\alpha_J,\phi'}$ for N_2 sufficiently large. For any $\vec{v} \in B_{N_2-N_1}$ with $x' = T_{\vec{v}}^{\alpha_J}(x) \in F_{2,J}$, set $K'(x') = B_{N_1} + \vec{v} \subseteq B_{N_2}$.

Construct a new permutation $\overline{\pi}'_{x,N_2} \in \Pi_{N_2}$ as

$$\overline{\pi}'_{x,N_2} = \begin{cases} \pi_j(\vec{u} - \vec{v}) + \vec{v} & \text{if } T_{\vec{v}}^{\alpha_J}(x) \in F_{2,j},\ \vec{u} \in K'(T_{\vec{v}}^{\alpha_J}(x)) \text{ and } \vec{v} \in B_{N_2-N_1} \\ \vec{u} & \text{otherwise.} \end{cases}$$

As with $\overline{\pi}_{x,N_2}$ for any particular π_j, the number of disjoint boxes $K'(x') \subseteq B_{N_2}$ with $x' \in F_{2,J}$ is $\mu(F_{2,J})(1 \pm 2\eta)|B_{N_2}|$. By precisely the parallel argument used there we compute

$$|\{\vec{u} \in B_{N_2};\, f_{x,N_2}^{\alpha_J,\phi'}(\vec{u}) \neq \overline{\pi}'_{x,N_2}(\vec{u})\}| \leq 7\eta |B_{N_2}|.$$

In this computation one of the extra η's comes from the decrease in size of the tower over F_1 to that over F_2, which is the same as to say the decrease in the size of $F_{1,j}$ to $F_{2,j}$. The other is from the difference between $f_{x',N_1}^{\beta,\phi'}$ and $f_{x',N_1}^{\alpha_J,\phi'}$ on each $K'(x')$.

We must argue that given any $\gamma > 0$, if η is small enough then for a.e. $x \in X$ once N_2 is large enough

$$m_{N_2,x}(\alpha_J, \phi') \geq m(\overline{\pi}'_{x,N_2}) - \gamma.$$

Certainly, we know that the permutation π'_{x,N_2} is identical to the permutation $\pi_{x,N_2}^{\alpha_J,\phi'}$, except on perhaps a fraction 8η of B_{N_2}. The quantity $m_{N_2,x}(\alpha_J, \phi')$ is a certain minimum over all permutations in Π_{N_2}, achieved by some permutation π', and this minimum value is

$$\max \left\{ m(\pi'),\, \frac{|\{v \in B_{N_2};\, \pi_{x,N_2}^{\alpha_J,\phi'}(v) \neq \pi'(v)\}|}{|B_{N_2}|} \right\}.$$

We may think of this max as δ.

Apply Axiom 6 with respect to $\varepsilon/2$ to select δ' and N_3. Let $\gamma = \delta'/2$. Select δ_1 and N_4 so that (by Axiom 4) if $n \geq N_4$, $\pi \in \Pi_n$ and $m(\pi) < \delta_1$, then for all $\pi' \in \Pi_n$, we have $m(\pi\pi') < m(\pi') + \gamma$. Choose $\delta < \delta'/2$ and N_5 so that (by Axiom 1) if $n \geq N_5$ and if $\pi \in \Pi_n$ and if π equals the identity on all except a 2δ fraction of B_n, then $m(\pi) < \delta_1$.

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Require that $N_2 \geq \max\{N_3, N_4, N_5\}$ be sufficiently large and require that $8\eta < \delta$ be sufficiently small so that the permutation $\overline{\pi}'_{x,N_2}$ differs from the minimizing permutation π' on no more than a 2δ fraction of B_{N_2} so that

$$m(\overline{\pi}'_{x,N_2}) < m(\pi') + \gamma \leq m_{N_2,x}(\alpha_J, \phi') + \gamma,$$

as desired.

Thus

$$\delta \geq m(\alpha_J, \phi') \geq \operatorname*{ess\,inf}_{x\in X} \liminf_{N_2\to\infty} m(\overline{\pi}'_{x,N_2}) - \gamma \quad \text{or} \quad \operatorname*{ess\,inf}_{x\in X} \liminf_{N_2\to\infty} m(\overline{\pi}'_{x,N_2}) \leq \delta + \gamma.$$

Consider an x_2 with the structure leading to this estimate and then N_2 so large so that $m(\overline{\pi}'_{x_2,N_2}) < \delta + 2\gamma$. Now consider any x_1 and the two permutations $\overline{\pi}_{x_1,N_2}$ and $\overline{\pi}'_{x_2,N_2}$.

Select a maximal subset of vectors $V \subseteq B_{N_2-N_1} \times B_{N_2-N_1}$ that bijectively pairs as many points $(\vec{v}_1, \vec{v}_2)$ as possible with

$$T^{\beta}_{\vec{v}_1}(x_1) \in F_{1,j} \quad \text{iff} \quad T^{\alpha_J}_{\vec{v}_2}(x_2) \in F_{2,j} \text{ for some } j.$$

The blocks $K_1, K_2, \ldots, K_s$ of Axiom 6 will be the collection of translates of B_{N_1} of the form $K'(T^{\alpha_J}_{\vec{v}_2}(x_2))$, where $(\vec{v}_1, \vec{v}_2) \in V$.

We compute that if N_2 is large enough, for a.e. x_1 and x_2

$$\frac{|\bigcup_{i=1}^{s} K_i|}{|B_{N_2}|} > 1 - 14\eta.$$

For any $(\vec{v}_1, \vec{v}_2) \in V$, $\overline{\pi}'_{x_2,N_2}$ restricted to $K'(T^{\alpha_J}_{\vec{v}_2}(x_2))$ is identical to $\overline{\pi}_{x_1,N_2}$ restricted to $K(T^{\beta}_{\vec{v}_1}(x_1))$. Choosing δ small enough, and then γ and η small enough we can apply Axiom 6 to conclude $m(\overline{\pi}_{x_1,N_2}) < \varepsilon/2$ (recalling that B_{N_1} is large enough to apply Axiom 6 with error $\varepsilon/2$).

We finally conclude:

$$m(\beta, \phi') \leq \operatorname*{ess\,sup}_{x_1\in X} \liminf_{N_2\to\infty} m_{N_2,x_1}(\beta, \phi') \leq \varepsilon/2 + 5\eta < \varepsilon. \qquad \square$$

We prove the following corollary to Lemma 2.7.

COROLLARY 2.8. *Let m be a size. Let α and β be two orderings. For every $\varepsilon > 0$ there exists $\delta = \delta_{2.8}(\varepsilon) > 0$ ($\delta_{2.8}(1) = 1$) so that if:*

(1) $\alpha \underset{m}{\to} \beta$ *via* $\{\phi_j\}$ *and if*

(2) *for all* j, $m(\alpha, \phi_j) \leq \delta$,

then for any $\delta' > 0$ there exists J so that:

(1′) $\alpha \underset{m}{\to} \beta\phi_J^{-1}$ *via* $\{\phi_j\phi_J^{-1}\}_{j\geq J}$;

(2′) $m(\alpha, \phi_j\phi_J^{-1}) \leq \delta'$ *for all* $j \geq J$; *and*

(3′) $m(\beta, \phi_J^{-1}) \leq \varepsilon$.

Proof. The case with $\varepsilon = 1$ and $\delta = 1$ is trivial as $m(\beta, \phi) \leq 1$ always.

Let $\varepsilon > 0$. Choose $\eta > 0$ whose size will be specified later. Let $\delta' > 0$ be chosen. Since $\{\alpha_j\}$ is m-Cauchy, there exists J so that for all $j \geq J$, $m(\alpha, \phi_j\phi_J^{-1}) < \min\{\eta, \delta'\}$. Define (for $j \geq J$) $\alpha'_j = \alpha_j\phi_J^{-1}$. Then $\alpha \underset{m}{\to} \beta\phi_J^{-1}$ via $\{\phi_j\phi_J^{-1}\}$.

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We must only verify that $m(\beta, \phi_J^{-1}) < \varepsilon$.

Notice that

$$\alpha_j \overset{\phi_j^{-1}\phi_J}{\simeq} \alpha_J \overset{\phi_J^{-1}}{\simeq} \alpha \overset{\phi_j\phi_J^{-1}}{\simeq} \alpha_j'.$$

So

$$\alpha_j \overset{\phi_J^{-1}}{\simeq} \alpha_j'.$$

We have selected J so that $m(\alpha_j, \phi_j^{-1}\phi_J) < \eta$. Selecting δ (using Lemma 2.5) so that as $m(\alpha, \phi_J) < \delta$, we know that $m(\alpha_J, \phi_J^{-1}) < \eta$. Also, we have selected J so that $m(\alpha, \phi_j\phi_J^{-1}) < \eta$.

Fix a value $\gamma > 0$ and apply Lemma 2.6 twice to decide how small η must be to ensure that $m(\alpha_j, \phi_J^{-1}) < \gamma$ for all $j \geq J$. Finally, given $\varepsilon > 0$ select γ using Lemma 2.7 (with $\phi' = \phi_J^{-1}$) so that we may conclude $m(\beta, \phi_J^{-1}) < \varepsilon$. □

We now have the tools to define m-equivalence.

Definition 2.4. Let α and β be any two orderings. Let m be a size. We say that α is m-equivalent to β (at this point written $\alpha \underset{m}{\Longrightarrow} \beta$) if there exists a sequence of elements $\{\phi_i\}$ in $FG(\mathcal{O})$ so that $\alpha \underset{m}{\to} \beta$ via $\{\phi_i\}$.

We will prove that $\underset{m}{\Longrightarrow}$ is an equivalence relation allowing us to double the arrowhead. The fact that it is reflexive is obvious. The difficulty lies in proving that it is both symmetric and transitive. Corollary 2.8 (just proven) provides the tool.

THEOREM 2.9. *Let m be a size. Let α and β be any two orderings. If $\alpha \underset{m}{\Longrightarrow} \beta$ then $\beta \underset{m}{\Longrightarrow} \alpha$.*

Proof. We begin with a small observation:

$$\begin{aligned} d_0(\alpha\phi_i, \beta) &= \mu(\{x;\ \text{there is a vector } \vec{v} \in B_1 \text{ with } \phi_i^{-1} T_{\vec{v}}^{\alpha} \phi_i(x) \neq T_{\vec{v}}^{\beta}(x)\}) \\ &= \mu(\{y;\ \text{there is a vector } \vec{v} \in B_1 \text{ with } T_{\vec{v}}^{\alpha}(y) \neq \phi_i T_{\vec{v}}^{\beta} \phi_i^{-1}(y)\}) \\ &= d_0(\alpha, \beta\phi_i^{-1}). \end{aligned}$$

As $\alpha \underset{m}{\Longrightarrow} \beta$ there exists a sequence $\{\alpha\phi_i\}$ with $\phi_i \in FG(\mathcal{O})$ and $\alpha \underset{m}{\to} \beta$ via $\{\phi_i\}$. We can drop to a subsequence of the $\{\phi_i\}$ guaranteeing $\sum_i d_0(\alpha\phi_i, \beta) < \infty$. This then guarantees that $\beta \underset{\text{pw}}{\to} \alpha$ via $\{\phi_i^{-1}\}$. All that remains is to select a perhaps smaller subsequence of ϕ_i^{-1} that is m-Cauchy.

Let $\{\varepsilon_i\} \searrow 0$ to be specified later, but with $\varepsilon_1 = 1$. Remember $\delta_{2.8}$ of Corollary 2.8.

We can now state the zeroth step of an induction argument, with $\phi_{J_0} = \phi_0 = \text{id}$ and defining $\phi_{J_{-1}} = \text{id}$.

$(1)_0$ $\alpha \underset{m}{\to} \beta = \beta\phi_{J_0}^{-1}$ via $\{\phi_j\phi_{J_0}^{-1}\}_{j \geq J_0}$;

$(2)_0$ $m(\alpha, \phi_j\phi_{J_0}^{-1}) \leq \delta_{2.8}(\varepsilon_1) = 1$; and

$(3)_0$ $m(\beta, \phi_{J_0}^{-1}) = m(\beta\phi_{J_{-1}}^{-1}, \phi_{J_{-1}}\phi_{J_0}^{-1}) \leq \varepsilon_0 = 1$.

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Suppose inductively that we have found $J_0, J_1, \ldots, J_K$ and for all $k \leq K$:

$(1)_k$ $\alpha \underset{m}{\to} \beta\phi_{J_k}^{-1}$ via $\{\phi_j\phi_{J_k}^{-1}\}_{j\geq J_k}$;

$(2)_k$ $m(\alpha, \phi_j\phi_{J_k}^{-1}) \leq \delta_{2.8}(\varepsilon_{k+1})$; and

$(3)_k$ $m(\beta\phi_{J_{k-1}}^{-1}, \phi_{J_{k-1}}\phi_{J_k}^{-1}) \leq \varepsilon_k$.

Applying Corollary 2.8, with $\varepsilon = \varepsilon_{K+1}$ and $\delta = \delta_{2.8}(\varepsilon_{K+2})$ there exists J_{K+1} so that:

$(1)_{(K+1)}$ $\alpha \underset{m}{\to} \beta\phi_{J_K}^{-1}(\phi_{J_{K+1}}\phi_{J_K}^{-1})^{-1} = \beta\phi_{J_{K+1}}^{-1}$ via $\{\phi_j\phi_{J_{K+1}}^{-1}\}_{j\geq J_{K+1}}$;

$(2)_{(K+1)}$ $m(\alpha, (\phi_j\phi_{J_K}^{-1})(\phi_{J_{K+1}}\phi_{J_K}^{-1})^{-1}) = m(\alpha, \phi_j\phi_{J_{K+1}}^{-1}) \leq \delta_{2.8}(\varepsilon_{K+2})$; and

$(3)_{(K+1)}$ $m(\beta\phi_{J_K}^{-1}, \phi_{J_K}\phi_{J_{K+1}}^{-1}) \leq \varepsilon_{K+1}$.

Now consider the sequence $\{\beta_n\} = \{\beta\phi_{J_n}^{-1}\}$ which converges pointwise to α. All we need to show is that if the $\varepsilon_n \searrow 0$ rapidly enough then the β_n are m-Cauchy. Choose $\{\eta_i\}$ to be a summable sequence of positive numbers. Applying Lemma 2.6 inductively we can select ε_i so that for all $1 \leq i < j$,

$$m(\beta\phi_{J_i}^{-1}, \phi_{J_i}\phi_{J_j}^{-1}) \leq \sum_{k=i}^{j} \eta_k.$$

This forces $\{\beta\phi_{J_n}^{-1}\}$ to be m-Cauchy, which completes the proof. □

THEOREM 2.10. *Let m be a size. Let α, β and γ be any three orderings. If $\alpha \underset{m}{\Longrightarrow} \beta$ and $\beta \underset{m}{\Longrightarrow} \gamma$ then $\alpha \underset{m}{\Longrightarrow} \gamma$.*

Proof. By Theorem 2.9 we may assume there exist sequences $\{\phi_i\}$ and $\{\psi_i\}$ in $FG(\mathcal{O})$ so that $\beta \underset{m}{\to} \alpha$ via $\{\phi_i\}$ and $\beta \underset{m}{\to} \gamma$ via $\{\psi_i\}$.

Our first step will be to reconstruct the argument of Theorem 2.9, building a sequence of orderings $\gamma\psi_{I_n}^{-1}$ converging pointwise and in m to β, but also themselves each being a limit, pointwise and in m of sequences $\{\beta\psi_i\psi_{I_n}^{-1}\}_{i\geq I_n}$. We will then simply conjugate the sequence $\{\beta\psi_i\psi_{I_n}^{-1}\}_{i\geq I_n}$ by the term ϕ_{I_n} to be $\{(\beta\phi_{I_n})\phi_{I_n}^{-1}\psi_i\psi_{I_n}^{-1}\phi_{I_n}\}$ whose limit in i is $\gamma\psi_{I_n}^{-1}\phi_{I_n}$. We will control the Cauchy-ness of these latter sequences well enough so that using Lemma 2.6 and the observation beginning the proof of Theorem 2.9 we will be able to conclude that $\{\gamma\psi_{I_n}^{-1}\phi_{I_n}\}$ converges both pointwise and in m to α.

To carry out this program select $\varepsilon_i \searrow 0$, at this point thought of as free parameters. Require that

$$\delta_{2.5}(\delta_{2.10}(\varepsilon_{i+1})) \leq \delta_{2.10}(\varepsilon_{i+1}) \leq \min\{\delta_{2.8}(\varepsilon_{i+1}), \tfrac{1}{3}\delta_{2.7}(\varepsilon_i), \delta_{2.6}(\tfrac{1}{3}\delta_{2.7}(\varepsilon_i))\}.$$

To start the induction use Corollary 2.8 to select I_1 so that

(A_1) $\beta \underset{m}{\to} \gamma\psi_{I_1}^{-1}$ via $\{\psi_i\psi_{I_1}^{-1}\}_{i\geq I_1}$ and

(B_1) $m(\beta, \psi_i\psi_{I_1}^{-1}) < \delta_{2.5}(\delta_{2.10}(\varepsilon_2))$, whence

(B_1') $m(\beta\psi\psi_{I_1}^{-1}, \psi_{I_1}\psi_i^{-1}) < \delta_{2.10}(\varepsilon_2)$, from Lemma 2.5, and

(B_1'') $m(\beta\phi_{I_1}, \phi_{I_1}^{-1}\psi_i\psi_{I_1}^{-1}\phi_{I_1}) < \delta_{2.5}(\delta_{2.10}(\varepsilon_2))$, conjugating (B_1) by ϕ_{I_1}.

We initially move both α and γ by coboundaries so that the overall diameter of the sequences converging from β to them have such small diameters that we have

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(C_1) $m(\gamma, \psi_{I_1}^{-1}) < \varepsilon_1$ (from Corollary 2.8),
(D_1) $m(\beta, \phi_{I_1}) < \delta_{2.10}(\varepsilon_2)$ and
(E_1) $m(\beta\phi_{I_1}, \phi_{I_1}^{-1}\phi_i) < \delta_{2.10}(\varepsilon_3)$ for all $i \geq I_1$.

Now notice that

$$\beta\psi_i\psi_{I_1}^{-1} \overset{\psi_{I_1}\psi_i^{-1}}{\simeq} \beta \overset{\phi_{I_1}}{\simeq} \beta\phi_{I_1} \overset{\phi_{I_1}^{-1}\psi_i\psi_{I_1}^{-1}\phi_{I_1}}{\simeq} \beta\psi_i\psi_{I_1}^{-1}\phi_{I_1}.$$

Associated with each of these we have an inequality:

$m(\beta\psi_i\psi_{I_1}^{-1}, \psi_{I_1}\psi_i^{-1}) < \delta_{2.10}(\varepsilon_2) < \frac{1}{3}\delta_{2.7}(\varepsilon_1)$ which is condition (B_1'),
$m(\beta, \phi_{I_1}) < \delta_{2.6}(\frac{1}{3}\delta_{2.7}(\varepsilon_1))$ which is condition (D_1), and
$m(\beta\phi_{I_1}, \phi_{I_1}^{-1}\psi_i\psi_{I_1}^{-1}\phi_{I_1}) < \delta_{2.5}(\delta_{2.10}(\varepsilon_2)) < \delta_{2.6}(\frac{1}{3}\delta_{2.7}(\varepsilon_1))$ which is condition (B_1'').

We compute that

$$\begin{aligned} m(\beta\psi_i\psi_{I_1}^{-1}, \phi_{I_1}) &\leq m(\beta\psi_i\psi_{I_1}^{-1}, \psi_{I_1}\psi_i^{-1}\phi_{I_1}) + \tfrac{1}{3}\delta_{2.7}(\varepsilon_1) \\ &\leq m(\beta\psi_i\psi_{I_1}^{-1}, \psi_{I_1}\psi_i^{-1}) + \tfrac{2}{3}\delta_{2.7}(\varepsilon_1) \\ &\leq \delta_{2.7}(\varepsilon_1). \end{aligned}$$

But now Lemma 2.7 allows us to conclude that

(D_1^*) $m(\gamma\psi_{I_1}^{-1}, \phi_{I_1}) < \varepsilon_1$.

We will give the second step of the induction, as it will clarify the structure.

Apply Corollary 2.8 to the sequence $\beta\phi_{I_1} \underset{m}{\to} \gamma\psi_{I_1}^{-1}\phi_{I_1}$ via $\{\phi_{I_1}^{-1}\psi_i\psi_{I_1}^{-1}\phi_{I_1}\}_{i\geq I_1}$ to select an I_2 so that

(A_2) $\beta\phi_{I_1} \underset{m}{\to} \gamma\psi_{I_2}^{-1}\phi_{I_1}$ via $\{\phi_{I_1}^{-1}\psi_i\psi_{I_2}^{-1}\phi_{I_1}\}_{I\geq I_2}$ and
(B_2) $m(\beta\phi_{I_1}, \phi_{I_1}^{-1}\psi_i\psi_{I_2}^{-1}\phi_{I_1}) < \delta_{2.5}(\delta_{2.10}(\varepsilon_3))$, whence
(B_2') $m(\beta\psi_i\psi_{I_2}^{-1}\phi_{I_1}, \phi_{I_1}^{-1}\psi_{I_2}\psi_i^{-1}\phi_{I_1}) < \delta_{2.10}(\varepsilon_3)$, from Lemma 2.5, and
(B_2'') $m(\beta\phi_{I_2}, \phi_{I_2}^{-1}\psi_i\psi_{I_2}^{-1}\phi_{I_2}) < \delta_{2.5}(\delta_{2.10}(\varepsilon_3))$, conjugating B_2 by $\phi_{I_1}^{-1}\phi_{I_2}$.

Also from Corollary 2.8 we know that

(C_2) $m(\gamma\psi_{I_1}^{-1}\phi_{I_1}, \phi_{I_1}^{-1}\psi_{I_1}\psi_{I_2}^{-1}\phi_{I_1}) < \varepsilon_2$.

Statement (E_1) allows us to conclude that

(D_2) $m(\beta\phi_{I_1}, \phi_{I_1}^{-1}\phi_{I_2}) < \delta_{2.6}(\frac{1}{3}\delta_{2.7}(\varepsilon_2))$.

Further require of I_2 that

(E_2) $m(\beta\phi_{I_2}, \phi_{I_2}^{-1}\phi_i) < \delta_{2.10}(\varepsilon_4)$ for all $i \geq I_2$.

Use conditions (B_2''), (B_2') and (D_2) as in step 1 of the induction to conclude

(D_2^*) $m(\gamma\psi_{I_2}^{-1}\phi_{I_1}, \phi_{I_1}^{-1}\phi_{I_2}) < \varepsilon_2$.

Finally, given $\eta_2 > 0$, select ε_2 sufficiently small so that conditions (C_2) and (D_2^*) imply, using Lemma 2.6, that

(F_2) $m(\gamma\psi_{I_1}^{-1}\phi_{I_1}, \phi_{I_1}\psi_{I_1}\psi_{I_2}^{-1}\phi_{I_2}) < \eta_2$.

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That is to say,

$$m(\gamma(\psi_{I_1}^{-1}\phi_{I_1}), (\psi_{I_1}^{-1}\phi_{I_1})^{-1}(\psi_{I_2}^{-1}\phi_{I_2})) < \eta_2.$$

This last statement tells us that the move from $\gamma\psi_{I_1}^{-1}\phi_{I_1}$ to $\gamma\psi_{I_2}^{-1}\phi_{I_2}$ is small. This is what we need to obtain inductively to show that the sequence of orderings $\gamma\psi_{I_n}^{-1}\phi_{I_n}$ converges in m to α. It automatically converges pointwise. To continue inductively, consider a sequence of values $\eta_n \searrow 0$ as free parameters, and select the values $\varepsilon_n \searrow 0$ rapidly enough to obtain (F_n) from (C_n) and (D_n^*). We now describe the general induction step.

Suppose we have selected I_{n-1} so that:

(A_{n-1}) $\beta\phi_{I_{n-2}} \underset{m}{\to} \gamma\psi_{I_{n-1}}^{-1}\phi_{I_{n-2}}$ via $\{\phi_{I_{n-2}}^{-1}\psi_i\psi_{I_{n-1}}^{-1}\phi_{I_{n-2}}\}_{i\geq I_{n-1}}$;

(B_{n-1}) $m(\beta\phi_{I_{n-2}}, \phi_{I_{n-2}}^{-1}\psi_i\psi_{I_{n-1}}^{-1}\phi_{I_{n-2}}) < \delta_{2.5}(\delta_{2.10}(\varepsilon_n))$;

(B'_{n-1}) $m(\beta\psi_i\psi_{I_{n-1}}^{-1}\phi_{I_{n-2}}, \phi_{I_{n-2}}^{-1}\psi_{I_{n-1}}\psi_i^{-1}\phi_{I_{n-2}}) < \delta_{2.10}(\varepsilon_n)$;

(B''_{n-1}) $m(\beta\phi_{I_{n-1}}, \phi_{I_{n-1}}^{-1}\psi_i\psi_{I_{n-1}}^{-1}\phi_{I_{n-1}}) < \delta_{2.5}(\delta_{2.10}(\varepsilon_n))$, conjugating ($\mathrm{B}_{n-1}$) by $\phi_{I_{n-2}}^{-1}\phi_{I_{n-1}}$;

(C_{n-1}) $m(\gamma\psi_{I_{n-2}}^{-1}\phi_{I_{n-2}}, \phi_{I_{n-2}}^{-1}\psi_{I_{n-2}}\psi_{I_{n-1}}^{-1}\phi_{I_{n-2}}) < \varepsilon_{n-1}$;

(E_{n-1}) $m(\beta\phi_{I_{n-1}}, \phi_{I_{n-1}}^{-1}\phi_i) < \delta_{2.10}(\varepsilon_{n+1})$ for all $i \geq I_{n-1}$; and

(D_{n-1}^*) $m(\gamma\psi_{I_{n-1}}^{-1}\phi_{I_{n-2}}, \phi_{I_{n-2}}^{-1}\phi_{I_{n-1}}) < \varepsilon_{n-1}$.

Just as in step 2 of the induction, given $\eta_{n-1} > 0$, if ε_{n-1} is sufficiently small, Lemma 2.6 together with conditions (C_{n-1}) and (D_{n-1}^*) directly imply that

(F_{n-1}) $m(\gamma(\psi_{I_{n-2}}^{-1}\phi_{I_{n-2}}), (\psi_{I_{n-2}}^{-1}\phi_{I_{n-2}})^{-1}(\psi_{I_{n-1}}^{-1}\phi_{I_{n-1}})) < \eta_{n-1}$.

Given η_n, if ε_n is small enough, Lemma 2.6 and statements (C_n) and (D_n^*) will imply (F_n). Thus all we need obtain are statements (A_n)–(D_n^*). Proceed as in step 2 of the induction.

Conjugate statement (A_{n-1}) by $\phi_{I_{n-2}}^{-1}\phi_{I_{n-1}}$ to get

(A'_{n-1}) $\beta\phi_{I_{n-1}} \underset{m}{\to} \gamma\psi_{I_{n-1}}^{-1}\phi_{I_{n-1}}$ via $\{\phi_{I_{n-1}}^{-1}\psi_i\psi_{I_{n-1}}^{-1}\phi_{I_{n-1}}\}_{i\geq I_{n-1}}$.

Apply Corollary 2.8 to (A'_{n-1}) to obtain I_n with

(A_n) $\beta\phi_{I_{n-1}} \underset{m}{\to} \gamma\psi_{I_n}^{-1}\phi_{I_{n-1}}$ via $\{\phi_{I_{n-1}}^{-1}\psi_i\psi_{I_n}^{-1}\phi_{I_{n-1}}\}_{i\geq I_n}$ and

(B_n) $m(\beta\phi_{I_{n-1}}, \phi_{I_{n-1}}^{-1}\psi_i\psi_{I_n}^{-1}\phi_{I_{n-1}}) < \delta_{2.5}(\delta_{2.10}(\varepsilon_{n+1}))$, whence

(B'_n) $m(\beta\psi_i\psi_{I_n}^{-1}\phi_{I_{n-1}}, \phi_{I_{n-1}}^{-1}\psi_{I_n}\psi_i^{-1}\phi_{I_{n-1}}) < \delta_{2.10}(\varepsilon_{n+1})$, from Lemma 2.5, and

(B''_n) $m(\beta\phi_{I_n}, \phi_{I_n}^{-1}\psi_i\psi_{I_n}^{-1}\phi_{I_n}) < \delta_{2.5}(\delta_{2.10}(\varepsilon_{n+1}))$, conjugating ($\mathrm{B}_n$) by $\phi_{I_{n-1}}^{-1}\phi_{I_n}$.

Also from Corollary 2.8 we know that

(C_n) $m(\gamma\psi_{I_{n-1}}^{-1}\phi_{I_{n-1}}, \phi_{I_{n-1}}^{-1}\psi_{I_{n-1}}\psi_{I_n}^{-1}\phi_{I_{n-1}}) < \varepsilon_n$.

Further, (E_{n-1}) allows us to conclude that

(D_n) $m(\beta\phi_{I_{n-1}}, \phi_{I_{n-1}}^{-1}\phi_{I_n}) < \delta_{2.6}(\frac{1}{3}\delta_{2.7}(\varepsilon_n))$.

Also require of I_n that

(E_n) $m(\beta\phi_{I_n}, \phi_{I_n}^{-1}\phi_i) < \delta_{2.10}(\varepsilon_{n+2})$ for all $i \geq I_n$.

Arguing precisely as in steps 1 and 2 at this point of the induction, we conclude that

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(D_n^*) $\quad m(\gamma\psi_{I_n}^{-1}\phi_{I_{n-1}}, \phi_{I_{n-1}}^{-1}\phi_{I_n}) < \varepsilon_n.$

Use (C_n) and (D_n^*) together with Lemma 2.6 to conclude that given η_n, we may choose ε_n sufficiently small so that

(F_n) $\quad m(\gamma(\psi_{I_{n-1}}^{-1}\phi_{I_{n-1}}), (\psi_{I_{n-1}}^{-1}\phi_{I_{n-1}})^{-1}(\psi_{I_n}^{-1}\phi_{I_n})) < \eta_n.$

This completes the induction.

Now consider the sequence of orderings $\{\gamma\psi_{I_n}^{-1}\phi_{I_n}\}_{n\geq 0}$ setting $\psi_{I_0} = \mathrm{id} = \phi_{I_0}$. Select a summable sequence of positive numbers ζ_i. Applying Lemma 2.6 inductively, using the sequence of statements (F_i), if the η_i decrease rapidly enough to zero, for all $2 \leq i < j$ we will have

$$m(\gamma\psi_{I_i}^{-1}\phi_{I_i}, (\psi_{I_i}^{-1}\phi_{I_i})^{-1}\psi_{I_j}^{-1}\phi_{I_j}) \leq \sum_{k=i}^{j-1}\zeta_k.$$

This forces the sequence $\{\gamma\psi_{I_n}^{-1}\phi_{I_n}\}_{n\geq 0}$ to be m-Cauchy. As $\gamma\psi_{I_n}^{-1} \underset{\text{pw}}{\to} \beta$ and $\beta\phi_{I_n} \underset{\text{pw}}{\to} \alpha$, it is clear that $\gamma\psi_{I_n}^{-1}\phi_{I_n} \underset{\text{pw}}{\to} \alpha$. Hence $\gamma \underset{m}{\Longrightarrow} \alpha$ completing the theorem. □

The previous two theorems tell us that m-equivalence is an equivalence relation and hence from now on we write $\alpha \underset{m}{\Leftrightarrow} \beta$.

Topologizing the m-equivalence classes. Knowing that $\underset{m}{\Leftrightarrow}$ is an equivalence relation, we can now extend m from the sets of cohomologous orderings to the full equivalence class, allowing us to topologize these classes. This point of view comes directly from Fieldsteel [**Fi**], who demonstrated that for the theory developed in [**R1**], the m-equivalence classes were not only the m-closure of the cohomology classes, but also satisfied the conclusion of the Baire category theorem. By lifting a size to measure the distortion between m-equivalent orderings, we see that a size, by our new axiomatics, satisfies the axioms of [**R1**] which are directly related to orderings. We write $\langle\alpha\rangle_m$ for $\{\beta : \alpha \underset{m}{\Leftrightarrow} \beta\}$.

Definition 2.5. For $\alpha \underset{m}{\Leftrightarrow} \beta$ we define

$$m(\alpha, \beta) = \inf\left\{\sup_i\left(m(\alpha, \phi_i); \alpha \underset{m}{\to} \beta \text{ via } \{\phi_i\}\right)\right\}.$$

Simple manipulations tell us that

$$m(\alpha, \beta) = \inf\left\{\liminf_i\left(m(\alpha, \phi_i); \alpha \underset{m}{\to} \beta \text{ via } \{\phi_i\}\right)\right\}.$$

If $\beta = \alpha\phi, \phi \in FG(\mathcal{O})$, clearly $m(\alpha, \beta) \leq m(\alpha, \phi)$. It is not at all obvious, and we conjecture it to be false for some sizes that

$$m(\alpha, \alpha\phi) = \inf\{m(\alpha, \phi'); \alpha\phi = \alpha\phi'\}.$$

We will see that using m as a notion of distance on $\langle\alpha\rangle_m$ (all that is missing from m to make it a metric are precise symmetry and the triangle inequality) the family

$$U_r = \{(\beta, \gamma); m(\beta, \gamma) < r\}, \quad r > 0$$

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is a uniformity [**Du**] in $\langle\alpha\rangle_m$. More importantly, we will see that in this topology the conclusion of the Baire category theorem holds, in that the countable intersection of open and dense sets is dense.

LEMMA 2.11. *Let m be a size. For $\alpha \underset{m}{\Leftrightarrow} \beta$, given any ε and N_0, there is a δ so that if $m(\alpha, \beta) < \delta$, there is a subset $A \subseteq X$, $\mu(A) > 1 - \varepsilon$ and for any $x \in A$ and $\vec{v} \in B_{N_0}$,*

$$\beta(x, T^{\alpha}_{\vec{v}}(x)) = \vec{v}.$$

Hence $m(\alpha, \beta) = 0$ if and only if $\alpha = \beta$.

Proof. Select δ according to Lemma 2.4 with respect to $\varepsilon/2$ and N_0. For this $\delta > 0$ we can choose a sequence ϕ_i with $\alpha \underset{m}{\to} \beta$ via $\{\phi_i\}$ and for all i, $m(\alpha, \phi_i) < \delta$. By Lemma 2.4 there is a subset A_i with $\mu(A_i) > 1 - \varepsilon/2$ and for all $x \in A$ and $\vec{v} \in B_{N_0}$, $\alpha\phi_i(x, T^{\alpha}_{\vec{v}}(x)) = \vec{v}$. But now as $\alpha\phi_i$ converges to β pointwise, once i is large enough, for a subset $A' \subseteq X$ with $\mu(A') > 1 - \varepsilon/2$, for all $\vec{v} \in B_{N_0}$, $\beta(x, T^{\alpha\phi_i}_{\vec{v}}(x)) = \vec{v}$. But this implies for $x \in A = A_i \cap A'$ that for all $\vec{v} \in B_{N_0}$,

$$\beta(x, T^{\alpha}_{\vec{v}}(x)) = \beta(x, T^{\alpha\phi_i}_{\vec{v}}(x)) = \vec{v}.$$

Letting ε tend to zero and N_0 to infinity, we conclude for a.e. $x \in X$, for all $x' \in \mathcal{O}(x)$ that $\alpha(x, x') = \beta(x, x')$. □

The following result is a corollary of the proofs of Theorems 2.9 and 2.10.

THEOREM 2.12. *For any $\varepsilon > 0$ there is a $\delta > 0$ so that for α, β, and γ all m-equivalent:*
(i) *if $m(\alpha, \beta) < \delta$ then $m(\beta, \alpha) < \varepsilon$; and*
(ii) *if $m(\beta, \gamma) < \delta$ then $m(\alpha, \gamma) < m(\alpha, \beta) + \varepsilon$.*

Proof. For (i) note that in the proof of Theorem 2.9, given $\varepsilon_0 > 0$, if δ is small enough the induction can be started with the statements

$(2)_0$ $m(\alpha, \phi_j\phi_{J_0}^{-1}) \leq \delta$ and
$(3)_0$ $m(\beta, \phi_{J_0}^{-1}) = m(\beta\phi_{J_{-1}}^{-1}, \phi_{J_{-1}}\phi_{J_0}^{-1}) \leq \varepsilon_0$.

But this says, given a sequence $\{\eta_i\}$, $\eta_i > 0, \sum \eta_i < \varepsilon$, we can select ϕ_{J_i} so that for $0 \leq i < j$,

$$m(\beta\phi_{J_i}^{-1}, \phi_{J_i}\phi_{J_j}^{-1}) \leq \sum_{k=i}^{j} \eta_k.$$

In particular,

$$m(\beta, \phi_{J_j}^{-1}) \leq \sum \eta_k < \varepsilon \quad \text{or} \quad m(\beta, \alpha) < \varepsilon.$$

For (ii), notice that statements (F_n) give, for $\Phi_j = \psi_{I_j}^{-1}\phi_{I_j}$,

$$m(\gamma\Phi_{n-1}, \Phi_{n-1}^{-1}\Phi_n) < \eta_n, \quad n \geq 2.$$

The first step (F_1) is not directly controlled in the proof. It is simply of size $m(\gamma, \psi_{I_1}^{-1}\phi_{I_1})$. But given any value $\eta_1 > 0$, if δ is small enough and $m(\alpha, \beta) < \delta$ and $m(\beta, \gamma) < \delta$ and we choose the sequences $\{\phi_i\}$ and $\{\psi_i\}$ for the proof to at most double these bounds, which are the infima over such sequences, we will have

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(F_1) $m(\gamma, \psi_{I_1}^{-1}\phi_{I_1}) < \eta_1$.

Now if we select a sequence $\zeta_i > 0, \sum \zeta_i < \varepsilon'$, and now select the $\eta_i > 0$ small enough we will obtain, via the argument proving Theorem 2.10, for all $1 \le i < j$,

$$m(\gamma\Phi_i, \Phi_i^{-1}\Phi_j) < \sum_{k=i+1}^{j} \zeta_i \quad \text{and hence} \quad (\gamma, \alpha) < \varepsilon'.$$

Using (i), if ε' were small enough, this would imply $m(\alpha, \gamma) < \delta_{2.6}(\varepsilon/2)/4$. Here we have assumed that $m(\alpha, \beta) < \delta$ which is not assumed in (ii), but in the sequence $\alpha\phi_i^{-1}$ converging pointwise and in m to β, there will be a term $\alpha\phi_i^{-1}$ for which both $m(\alpha\phi_i^{-1}, \beta) < \delta$ and $m(\alpha, \alpha\phi_i^{-1}) < m(\alpha, \beta) + \varepsilon/2$. Also use the fact that $m(\alpha\phi_i^{-1}, \gamma) < \delta_{2.6}(\varepsilon/2)/4$ together with Lemma 2.6 to see that the result follows. □

Lemma 2.11 and Theorem 2.12 tell us that the family $U_r \subset \langle\beta\rangle_m \times \langle\beta\rangle_m$ is a uniformity [**Du**] in $\langle\beta\rangle_m$ and hence generates a uniform topology τ on the m-equivalence class $\langle\beta\rangle_m$. We want to show that we obtain the Baire category theorem for this topology. This will be an essential piece of the proof of the equivalence theorem.

LEMMA 2.13. *For any ordering α there is a function*

$$d_{m,\alpha} : \cup_{k=1}^{\infty}(FG(\mathcal{O}))^k \to \mathbb{R}^+$$

so that if ϕ_j is a sequence of elements in $FG(\mathcal{O})$ satisfying

$$m(\alpha\phi_j, \phi_j^{-1}\phi_{j+1}) \le d_{m,\alpha}(\phi_1, \ldots, \phi_j),$$

then there exists β with $\alpha \underset{m}{\to} \beta$ via $\{\phi_j\}$.

Proof. Given $\alpha, \phi_1, \ldots, \phi_k$ choose $d_{m,\alpha}(\phi_i, \ldots, \phi_k)$ so small so that if

$$m(\alpha\phi_k, \phi_k^{-1}\phi_{k+1}) < d_{m,\alpha}(\phi_1, \ldots, \phi_k)$$

then Lemma 2.4 will imply there is a set $A_k \subseteq X, \mu(A_k) > 1 - 2^{-k}$ and for $x \in A_k$ and $\vec{v} \in B_k$

$$(\alpha\phi_{k+1})(x, T_{\vec{v}}^{\alpha}(x)) = (\alpha\phi_k)(x, T_{\vec{v}}^{\alpha}(x)).$$

This is not in fact a direct application of Lemma 2.4. But setting $\alpha' = \alpha\phi_k$, $\beta = \alpha\phi_{k+1} = \alpha'\phi_k^{-1}\phi_{k+1}$, for any N_0 and ε' there is a set $A', \mu(A') > 1 - 2^{-(k+1)}$ and for $x \in A'$, and $\vec{v} \in B_{N_0}$, by Lemma 2.4 directly,

$$\beta(x, T_{\vec{v}}^{\alpha'}(x)) = \alpha'(x, T_{\vec{v}}^{\alpha'}(x)) = \vec{v}.$$

If N_0 is sufficiently large, for a set $A'', \mu(A'') > 1 - 2^{-(k+1)}$ if $x \in A''$, we will have

$$\{T_{\vec{v}}^{\alpha}(x); \vec{v} \in B_k\} \subseteq \{T_{\vec{v}}^{\alpha'}(x); \vec{v} \in B_{N_0}\}.$$

Thus for $x \in A_k = A' \cap A''$, $\vec{v} \in B_k$

$$\beta(x, T_{\vec{v}}^{\alpha}(x)) = \alpha'(x, T_{\vec{v}}^{\alpha}(x)).$$

Assume $\{\phi_j\}$ satisfies $m(\alpha\phi_j, \phi_j^{-1}\phi_{j+1}) \le d_{m,\alpha}(\phi_1, \ldots, \phi_j)$. Then for any $x \in \underline{\lim} A_k = X$ a.s., for any $(x, x') \in \mathcal{O}$ the values $(\alpha\phi_k)(x, x')$ converge. In fact they

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are asymptotically constant. The limit value $\beta(x, x')$ is easily seen to be one-to-one from the orbit of x into $\mathbb{Z}^d$, but without further work is not necessarily onto, hence not necessarily an ordering.

Again using Lemma 2.4, require $d_{m,\alpha}(\phi_1, \ldots, \phi_k)$ to be so small that for a set A'_k, $\mu(A'_k) > 1 - 2^{-k}$, if $x \in A'_k$ and $\vec{v} \in B_k$,

$$\alpha\phi_{k+1}(x, T_{\vec{v}}^{\alpha\phi_k}(x)) = \vec{v}.$$

Now for $x \in \cap_{k=K}^{\infty} A'_k$, $\beta(x, x')$ will cover at least B_K and hence, for x also in $\underline{\lim}\, A'_k$, β is onto and is an ordering. Thus $\alpha \underset{\text{pw}}{\to} \beta$ via $\{\phi_i\}$. But now Lemma 2.6 tells us that if $m(\alpha\phi_k, \phi_k^{-1}\phi_{k+1})$ tends to zero fast enough (independent of α) then the sequence is m-Cauchy and so $\alpha \underset{m}{\to} \beta$ via $\{\phi_i\}$. □

THEOREM 2.14. *In the topological space $(\langle\beta\rangle_m, \tau)$ any countable intersection of open and dense sets is dense.*

Proof. Let O_i be such a collection, and we assume them to be nested. For $\alpha \in \langle\beta\rangle_m$ define $U_r(\alpha) = \{\gamma \in \langle\beta\rangle_m; m(\alpha, \gamma) < r\}$. We want to show that for any $\alpha \in \langle\beta\rangle_m$ and $r > 0$

$$(\cap O_i) \cap U_r(\alpha) \neq \emptyset.$$

Let $\phi_1 = \text{id}$ and $r_1 = r$. Now choose a $\phi_2 \in FG(\mathcal{O})$, and $r_2 > 0$ so that

$(1)_2$ $U_{r_2}(\alpha\phi_2) \subseteq U_{r_1/2}(\alpha\phi_1) \cap O_1$ and
$(2)_2$ $m(\alpha, \phi_2) < d_{m,\alpha}(\phi_1)$.

Choose $\phi_3 \in FG(\mathcal{O})$ and then $r_3 > 0$ so that

$(1)_3$ $U_{r_3}(\alpha\phi_3) \subseteq U_{r_2/2}(\alpha\phi_2) \cap O_2$ and
$(2)_3$ $m(\alpha\phi_2, \phi_2^{-1}\phi_3) < d_{m,\alpha}(\phi_1, \phi_2)$.

Having obtained $\phi_k \in FG(\mathcal{O})$ and then $r_k > 0$ satisfying

$(1)_k$ $U_{r_k}(\alpha\phi_k) \subseteq U_{r_{k-1}/2}(\alpha\phi_{k-1}) \cap O_{k-1}$ and
$(2)_k$ $m(\alpha\phi_{k-1}, \phi_{k-1}^{-1}\phi_k) < d_{m,\alpha}(\phi_1, \phi_2, \ldots, \phi_{k-1})$,

as $O_k \cap U_{r_k/2}(\alpha\phi_k) \cap U_{d_{m,\alpha}(\phi_1,\ldots,\phi_k)/2}(\alpha\phi_k) \neq \emptyset$ we can select a ϕ_{k+1} with

$$\alpha\phi_{k+1} \in U_{r_k/2}(\alpha\phi_k) \cap O_k \cap U_{d_{m,\alpha}(\phi_1,\ldots,\phi_k)/2}$$

and hence a value $r_{k+1} > 0$ with

$(1)_{k+1}$ $U_{r_{k+1}}(\alpha\phi_{k+1}) \subseteq U_{r_k/2}(\alpha\phi_k) \cap O_k$ and
$(2)_{k+1}$ $m(\alpha\phi_k, \phi_k^{-1}\phi_{k+1}) < d_{m,\alpha}(\phi_1, \ldots, \phi_k)$.

To do this just choose a term in a sequence with a small enough diameter going from $\alpha\phi_k$ to some element in $O_k \cap U_{r_k/2}(\alpha\phi_k) \cap U_{d_{m,\alpha}(\phi_1,\ldots,\phi_2)/2}(\alpha\phi_k)$. But now Lemma 2.13 tells us there is an α' with $\alpha \underset{m}{\to} \alpha'$ via ϕ_k. As $U_{r_{k+1}/2}(\alpha\phi_k) \subseteq U_{r_k/2}(\alpha\phi_{k-1})$, for $k \geq K$, $\alpha\phi_k \in U_{r_K/2}(\alpha\phi_K)$ and so for all K,

$$m(\alpha\phi_K, \alpha') \leq r_{K/2} \quad \text{which implies} \quad \alpha' \in \cap_K U_{r_K}(\alpha_K) \subseteq \cap_K O_K. \qquad \square$$

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3. *The $\overline{m}$-distance between names and m-entropy*

We will now develop the concept of m-entropy for an action T^α. This will follow almost exactly the path taken in [**R1**] and leads to the same final conclusion. There are two kinds of sizes, those for which m-entropy is always identically zero (entropy-free sizes) and those for which it is always the standard entropy (entropy-preserving sizes). As in [**R1**] we will give two definitions of m-entropy, one intrinsically defined by counting neighborhoods in an appropriate 'metric' ($\overline{m}$) on names, the other extrinsically defined as the infimum of entropies in an equivalence class. The equality of these two definitions is a kind of variational principle for m-entropy.

Before introducing entropy we begin by defining the $\overline{m}$-distance between names. Regard $\mathbb{N}$ as a set of labels for points or sets. A 'B_N-name' is a map

$$\mathcal{N} : B_N \to \mathbb{N}$$

which assigns to each vector $\vec{v} \in B_N$ a 'label' or 'symbol' in $\mathbb{N}$. Thus we can write the set of all B_N-names as $\{\mathbb{N}\}^{\mathbb{Z}^d}$. We will always work with finite subsets of this set of names whose ranges are bounded.

In these terms, a finite partition of X is a bounded map

$$P : X \to \mathbb{N}$$

which is to say $P \in L^\infty(X, \mathbb{N})$. In this context we set $\mathcal{P} = L^\infty(X, \mathbb{N})$.

If $P \in \mathcal{P}$ is a finite partition of X and α is an ordering, then the α, P, B_N-name of a point x is the B_N-name

$$\mathcal{N}_{x,N}^{P,\alpha}(\vec{v}) = P(T_{\vec{v}}^\alpha(x)).$$

We let $A_{x,N}^{P,\alpha} = \{x'; \mathcal{N}_{x',N}^{P,\alpha} = \mathcal{N}_{x,N}^{P,\alpha}\}$ denote the atom of $\bigvee_{\vec{v}\in B_N} T_{\vec{v}}^\alpha(P)$ containing x.

A permutation $\pi \in \Pi_N$ can be viewed as acting on a B_N-name by

$$\pi(\mathcal{N})(\vec{v}) = \mathcal{N}(\pi^{-1}(\vec{v})).$$

Now recall the familiar $\overline{d}$ (or normalized Hamming) metric on B_N-names given by

$$\overline{d}(\mathcal{N}_1, \mathcal{N}_2) = \frac{|\{\vec{v}; \mathcal{N}_1(\vec{v}) \neq \mathcal{N}_2(\vec{v})\}|}{|B_N|}.$$

Consider the following basic circumstance. Both α and $\beta = \alpha\phi$ are orderings, and $P \in \mathcal{P}$ is a finite partition. For a.e. $x \in X$ we have a sequence of permutations $\pi_{x,N}^{\alpha,\phi} \in \Pi_N$ approximating the action of $f_{x,N}^{\alpha,\phi}$ as it produces the reordering from α to β. That is

$$\frac{|\{\vec{v} \in B_N; \pi_{x,N}^{\alpha,\phi}(\vec{v}) \neq f_{x,N}^{\alpha,\phi}(\vec{v})\}|}{|B_N|} \underset{N}{\to} 0.$$

Consider the two B_N-names

$$\mathcal{N}_1 = \mathcal{N}_{x,N}^{P,\alpha} \quad \text{and} \quad \mathcal{N}_2 = \mathcal{N}_{\phi^{-1}(x),N}^{P,\beta}.$$

As

$$T_{f_{x,N}^{\alpha,\phi}(\vec{v})}^\beta(\phi^{-1}(x)) = T_{\vec{v}}^\alpha(x),$$

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we know

$$P(T^{\beta}_{f^{\alpha,\phi}_{x,N}(\vec{v})}(\phi^{-1}(x))) = P(T^{\alpha}_{\vec{v}}(x)),$$

and for $\vec{v} \in B_N$ with $\pi^{\alpha,\phi}_{x,N}(\vec{v}) = f^{\alpha,\phi}_{x,N}(\vec{v})$,

$$\mathcal{N}_2(\pi^{\alpha,\phi}_{x,N}(\vec{v})) = \mathcal{N}_1(\vec{v})$$

and hence

$$\lim_{N\to\infty} \overline{d}(\pi^{\alpha,\phi}_{x,N}(\mathcal{N}_1), \mathcal{N}_2) = 0.$$

This leads us to define a distance on names associated with a size m.

Definition 3.1. Suppose $\mathcal{N}_1$ and $\mathcal{N}_2$ are two B_N-names. We define

$$\overline{m}(\mathcal{N}_1, \mathcal{N}_2) = \inf\{\overline{d}(\pi(\mathcal{N}_1), \mathcal{N}_2) + m(\pi); \pi \in \Pi_N\}.$$

Notice that this fails to be a metric in exactly the same fashion as the m-distance between orderings does.

LEMMA 3.1. *Given any $\varepsilon > 0$ there is a $\delta > 0$ and an N_0 so that for all B_N-names $\mathcal{N}_1$, $\mathcal{N}_2$ and $\mathcal{N}_3$:*
(1) *$\overline{m}(\mathcal{N}_1, \mathcal{N}_1) = 0$ and $\overline{m}(\mathcal{N}_1, \mathcal{N}_2) \le \overline{d}(\mathcal{N}_1, \mathcal{N}_2)$;*
(2) *if $N \ge N_0$ and $\overline{m}(\mathcal{N}_1, \mathcal{N}_2) < \delta$ then $\overline{m}(\mathcal{N}_2, \mathcal{N}_1) < \varepsilon$;*
(3) *if $N \ge N_0$ and $\overline{m}(\mathcal{N}_2, \mathcal{N}_3) < \delta$ then $\overline{m}(\mathcal{N}_1, \mathcal{N}_3) < \overline{m}(\mathcal{N}_1, \mathcal{N}_2) + \varepsilon$.*

Proof. All of these are easy consequences of the axioms and the definition of $\overline{m}$. □

LEMMA 3.2. *Suppose $P \in \mathcal{P}$ is a finite partition, α is an ordering and $\phi \in FG(\mathcal{O})$, $\beta = \alpha\phi$. Then for μ-a.e. $x \in X$,*

$$\begin{bmatrix} \overline{\lim_N} \\ \underline{\lim_N} \end{bmatrix} \overline{m}(\mathcal{N}^{P,\alpha}_{x,N}, \mathcal{N}^{P,\beta}_{\phi^{-1}(x),N}) \le \begin{bmatrix} \hat{m}_x(\alpha,\phi) \\ m_x(\alpha,\phi) \end{bmatrix}.$$

Proof. As N grows, the permutations $\pi^{\alpha,\phi}_{x,N}$ push the $\overline{d}$ component of the computation of $\overline{m}$ to zero, and the m-component is $m_{N,x}(\alpha,\phi)$. □

Definition 3.2. We can write $\mathcal{P} = \cup\mathcal{P}(N_0)$, where

$$\mathcal{P}(N_0) = \{P; \|P\|_\infty \le N_0\},$$

a nested sequence of sets. We can topologize $\mathcal{P}$ with the metric

$$\triangle(P_1, P_2) = \frac{1}{2}\sum_j \mu(P_1^{-1}(j) \triangle P_2^{-1}(j)) = \mu(\{x; P_1(x) \ne P_2(x)\}).$$

Although $\mathcal{P}$ is not complete in this metric, each $\mathcal{P}(N_0)$ is.

Endow $\langle\alpha\rangle_m \times \mathcal{P} = \{(T^\beta, P); \beta \in \langle\alpha\rangle_m, P \in \mathcal{P}\}$ with the product topology of 'open balls' in the calculation $m \times \triangle$.

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This space is a space of 'processes', but the topology is not the weak*-topology. In particular, if we fix α and restrict P to a fixed $\mathcal{P}(N_0)$ then the entropy $h(T^\alpha, P)$ is a uniformly continuous function of P. This is not true of the weak*-topology and not in general true when α is allowed to vary.

Definition 3.3. Let $L = \{1, 2, \ldots, \infty\}$ be a one point compactification of $\mathbb{N}$, and let $\mathcal{M}$ be the set of all shift invariant borel probability measures on $\Sigma = L^{\mathbb{Z}^d}$. This is a compact, convex metric space when regarded in the weak*-topology as a subset of the dual of $C(\Sigma)$. It contains all $\mathcal{M}(N_0)$, the shift invariant measures on $\{1, 2, \ldots, N_0\}^{\mathbb{Z}^d}$. We are only interested in $\mathcal{M}_0 = \cup_N \mathcal{M}(N)$ which is convex, but no longer compact. We will always work within the compact subsets $\mathcal{M}(N_0)$.

There is a natural map

$$F : \langle\alpha\rangle_m \times \mathcal{P} \times X \to \Sigma$$

given by

$$F(\beta, P, x)(\vec{v}) = P(T^\beta_{\vec{v}}(x)).$$

Thus for each pair (α, P) we get a measure $\tilde{\mu}(\alpha, P) = F(\alpha, P, \cdot)_*(\mu) \in \mathcal{M}(\|P\|_\infty)$, the image of the measure μ under $F(\alpha, P, \cdot)$. The pull-back of the weak*-topology on $\mathcal{M}_0$ is what we refer to as the weak*-topology on $\langle\alpha\rangle_m \times \mathcal{P}$.

LEMMA 3.3. *The map $\tilde{\mu} : \langle\alpha\rangle_m \times \mathcal{P} \to \mathcal{M}_0$ is continuous.*

Proof. As a family of maps indexed by α, $\tilde{\mu}(\alpha, \cdot) : \mathcal{P} \to \mathcal{M}$ is a uniformly equicontinuous family. That is to say, for any $f \in C(\Sigma)$,

$$F_\alpha(P) = \int f \, d\tilde{\mu}(\alpha, P) : \mathcal{P} \to \mathbb{R}$$

is a uniformly equicontinuous family of functions. Hence it is only necessary to prove that for fixed P, we have continuity in α.

The proof of Corollary 2.11 tells us that if $m(\alpha_i, \alpha) \underset{i}{\to} 0$ then for any B_N, for an ever larger set of points $A_{i,N} \underset{i}{\to} X$, for $x \in X$ and $\vec{v} \in B_N$,

$$T^{\alpha_i}_{\vec{v}}(x) = T^{\alpha}_{\vec{v}}(x).$$

Thus for any cylinder set C in Σ,

$$\tilde{\mu}(\alpha_i, P)(C) \underset{i}{\to} \tilde{\mu}(\alpha, P)(C).$$

Hence for $f \in C(\Sigma)$,

$$\int f \, d\tilde{\mu}(\alpha_i, P) \underset{i}{\to} \int f \, d\tilde{\mu}(\alpha, P). \qquad \square$$

We can now give a first definition of m-entropy.

Definition 3.4. Letting

$$h(T^\alpha) = \sup\{h(T^\alpha, P) : P \in \mathcal{P}\}$$

be the standard measure-theoretic entropy of T^α, define

$$h^*_m(T^\alpha) = \inf\{h(T^\beta); \beta \in \langle\alpha\rangle_m\}.$$

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THEOREM 3.4. *The set of $\beta \in \langle\alpha\rangle_m$ with $h(T^\beta) = h_m^*(T^\alpha)$ is a dense G_δ.*

Proof. We know, for any $\varepsilon > 0$, that there is an $\alpha(\varepsilon) \in \langle\alpha\rangle_m$ with

$$h(T^{\alpha(\varepsilon)}) < h_m^*(T^\alpha) + \varepsilon.$$

As the set of $\beta = \alpha(\varepsilon)\phi$ is dense in $\langle\alpha\rangle_m$, $\{\beta; h(T^\beta) < h_m^*(T^\alpha) + \varepsilon\}$ is dense in $\langle\alpha\rangle_m$. Hence

$$\{\beta : h(T^\beta, P) < h_m^*(T^\alpha) + \varepsilon \text{ for all } P \in \mathcal{P}\}$$

is dense in $\langle\alpha\rangle_m$.

Letting σ be the action of translation by $\mathbb{Z}^d$ on Σ, we know that

$$h(\mu) = \text{ the entropy of } \sigma \text{ with respect to } \mu \in \mathcal{M}(N_0)$$

is upper semi-continuous on each $\mathcal{M}(N_0)$. Hence for any $c > 0$, $\{\mu; h(\mu) < c\}$ is an open set in $\mathcal{M}(N_0)$. Hence for each $P \in \mathcal{P}$, $P \in \mathcal{P}(N_0)$, the set

$$O(P, \varepsilon) = \{\beta; h(T^\beta, P) < h_m^*(T^\alpha) + \varepsilon\} = \tilde{\mu}^{-1}(\{\mu \in \mathcal{M}(N_0) : h(\mu) < h_m^*(T^\alpha) + \varepsilon\})$$

is open and dense in $\langle\alpha\rangle_m$.

Let $P_i \in \mathcal{P}$ be a countable collection of partitions dense in each $\mathcal{P}(N_0)$ and $\varepsilon_j \searrow 0$ and we conclude by Theorem 2.14 that $\cap_{i,j} O(P_i, \varepsilon_j)$ is a dense G_δ in $\langle\alpha\rangle_m$. For any β in this intersection we will have

$$h(T^\beta, P_i) \leq h_m^*(T^\alpha)$$

for all i. For each N_0, $h(T^\beta, P)$ is a continuous function of $P \in \mathcal{P}(N_0)$. We conclude that

$$h(T^\beta, P) \leq h_m^*(T^\alpha)$$

for all P, completing the result. □

In the final analysis the only real value of this result will be to show that for entropy-free sizes, in each equivalence class the transformations of zero-entropy are residual.

For our second definition of m-entropy we 'count names'.

Definition 3.5. Let T^α be fixed and $P \in \mathcal{P}$. An $N, \varepsilon, \overline{m}$-cover of (T^α, P) is a finite partition $\{A_i\}$ of a subset $A \subseteq X$ with $\mu(A) > 1 - \varepsilon$ so that for each i, if $x_1, x_2 \in A_i$, then

$$\overline{m}(\mathcal{N}_{x_1,N}^{P,\alpha}, \mathcal{N}_{x_2,N}^{P,\alpha}) < \varepsilon.$$

Set

$$H_m(T^\alpha, P, N, \varepsilon) = \frac{1}{|B_N|} \log_2 \inf\{|\{A_i\}|; \{A_i\} \text{ is an } N, \varepsilon, \overline{m}\text{-cover of } (T^\alpha, P)\}.$$

Now set

$$h_m^{**}(T^\alpha, P) = \lim_{\varepsilon \to 0} \underline{\lim}_{N \to \infty} H_m(T^\alpha, P, N, \varepsilon).$$

Finally, set

$$h_m^{**}(T^\alpha) = \sup_{P \in \mathcal{P}} (h_m^{**}(T^\alpha, P)).$$

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As usual in such definitions we get an actual limit in ε as an $N, \varepsilon, \overline{m}$-cover is automatically an $N, \varepsilon_1, \overline{m}$-cover for $\varepsilon_1 \geq \varepsilon$ and so the actual values cannot decrease as $\varepsilon \searrow 0$. Notice that for $\phi \in FG(\mathcal{O})$, $h_m^{**}(T^{\alpha\phi}) = h_m^{**}(T^{\alpha})$. Also notice that if we replace $\overline{m}$ by $\overline{d}$ in this definition we get Feldman's [**F**] approach to the classical entropy.

We now want to see that our two definitions of m-entropy agree. This will follow from two basic lemmas.

LEMMA 3.5. *Let (T^{α}, P) be a fixed process. Given any $\varepsilon > 0$, there is a $\phi \in FG(\mathcal{O})$ so that:*

(i) $m(\alpha, \phi) < \varepsilon$; *and*

(ii) $h(T^{\alpha\phi}, P) \leq h_m^{**}(T^{\alpha}, P) + \varepsilon$.

Proof. Fix $\delta > 0$ for the time being and set

$$H = \lim_{N \to \infty} H_m(T^{\alpha}, P, N, \delta) \leq h_m^{**}(T^{\alpha}, P).$$

Choose N_0 so that there is an $N_0, \delta, \overline{m}$-cover $\{A_i\}$ of (T^{α}, P) with

$$|\{A_i\}| \leq 2^{|B_{N_0}|(H+\delta)}.$$

Use the strong Rokhlin lemma to construct a Rokhlin tower of dimension B_{N_0} for T^{α} with base set F such that:

(i) $\mu(\cup_{\vec{v} \in B_{N_0}} T^{\alpha}_{\vec{v}}(F)) > 1 - \delta$; and

(ii) $F \perp \vee_{\vec{v} \in B_{N_0}} T^{\alpha}_{\vec{v}}(P)$.

By (ii) the distribution of α, P, B_{N_0}-names on F is identical to that on all of X. Hence there is a collection of disjoint sets $\{A_i'\}$ in F modeling precisely the $N_0, \delta, \overline{m}$-cover $\{A_i\}$ of (T^{α}, P).

For each A_i' choose a fixed name $\mathcal{N}_i$ so that for all $x \in A_i'$,

$$\overline{m}(\mathcal{N}_i, \mathcal{N}^{P,\alpha}_{x,N_0}) < \delta.$$

Hence for each $x \in A_i'$ we can select a permutation $\pi(x) \in \Pi_{N_0}$ so that:

(iii) $m(\pi(x)) < \delta$; and

(iv) $\overline{d}(\pi(x)(\mathcal{N}^{P,\alpha}_{x,N_0}), \mathcal{N}_i) < \delta$.

For $x \in F \setminus \cup A_i'$ set $\pi(x) = \mathrm{id}$. Now define $\phi \in FG(\mathcal{O})$ by

$$\phi(x) = \begin{cases} x & \text{if } x \notin \cup_{\vec{v}} T^{\alpha}_{\vec{v}}(F) \\ T^{\alpha}_{\pi(x')(\vec{v})}(x') & \text{if } x = T^{\alpha}_{\vec{v}}(x'), x' \in F, \vec{v} \in B_{N_0}. \end{cases}$$

Thus for $x \in F$,

$$\mathcal{N}^{P,\alpha\phi}_{\phi^{-1}(x),N_0} = \pi(x)(\mathcal{N}^{P,\alpha}_{x,N_0}).$$

The set $F_1 = \phi^{-1}(F)$ is a Rokhlin base for $T^{\alpha\phi}$ for a tower of size B_{N_0}. This tower, of course, consists of exactly the same set of points as the T^{α} tower with base F. We have just rearranged the orders of the points in a 'trajectory' through the tower.

The tower covers all but δ in measure of X and the permutations $\pi(x)$ applied to reorder the trajectories through the tower by ϕ have $m(\pi(x)) < \delta$. This, Axiom 5 and

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the ergodic theorem applied to the characteristic function of the set of points in the tower tell us that if δ is chosen small enough (in terms of Axiom 5) then we will have

$$m(\alpha, \phi) < \varepsilon, \quad \text{verifying (i).}$$

The partition $\{A_i'\}$ pulls over to a partition $\{\phi^{-1}(A_i')\}$ of $\phi^{-1}(F)$, and for any $x \in \phi^{-1}(A_i')$,

$$\overline{d}(\mathcal{N}_{x,N_0}^{P,\alpha\phi}, \mathcal{N}_i) < \delta.$$

Standard estimates now tell us that if δ is chosen small enough then

$$h(T^{\alpha\phi}, P) < H + \varepsilon \leq h_m^{**}(T^\alpha, P) + \varepsilon, \quad \text{verifying (ii).} \qquad \square$$

COROLLARY 3.6.

$$h_m^*(T^\alpha) \leq h_m^{**}(T^\alpha).$$

Proof. Fix an $\varepsilon > 0$. From Lemma 3.5, for any $P \in \mathcal{P}$ there exists $\phi \in FG(\mathcal{O})$ with $m(\alpha, \phi) < \varepsilon$ and $h(T^{\alpha\phi}, P) \leq h_m^{**}(T^\alpha) + \varepsilon$.

For P fixed, upper semi-continuity of entropy in $\mathcal{M}(N_0)$ tells us there exists $\delta = \delta(\alpha\phi, \varepsilon)$ so that if $m(\alpha\phi, \beta) < \delta(\alpha\phi, \varepsilon)$ then $h(T^\beta, P) < h(T^{\alpha\phi}, P) + \varepsilon$.

Let P_i be a countable dense family of finite partitions. Using the above remark inductively we can select a sequence $\phi_i \in FG(\mathcal{O})$ with

(i) $m(\alpha\phi_i, \phi_i^{-1}\phi_{i+1}) < d_{\alpha,m}(\phi_1, \ldots, \phi_i)$;

(ii) $h(T^{\alpha\phi_i}, P_i) < h_m^{**}(T^{\alpha\phi_i}) + \varepsilon = h_m^{**}(T^\alpha) + \varepsilon$.

From (i) we know $\alpha\phi_i \underset{m}{\to} \beta$ via $\{\phi_i\}$ for some β. We also ask that our induction gives us

(iii) $m(\alpha\phi_i, \beta) < \delta(\alpha\phi_i, \varepsilon)$.

We can conclude that for all i,

$$h(T^\beta, P_i) \leq h(T^{\alpha\phi_i}, P_i) + \varepsilon \leq h_m^{**}(T^{\alpha\phi_i}) + 2\varepsilon = h_m^{**}(T^\alpha) + 2\varepsilon.$$

Hence $h(T^\beta) \leq h_m^{**}(T^\alpha) + 2\varepsilon$ and letting $\varepsilon \searrow 0$, we are done. $\square$

LEMMA 3.7. *Given any partition $P \in \mathcal{P}$ and $\varepsilon > 0$ there exists $\delta > 0$ so that if $m(\alpha, \beta) < \delta$ then*

$$\overline{\lim_{N\to\infty}} H_m(T^\alpha, P, N, \varepsilon) \leq h(T^\beta, P) + \varepsilon.$$

Proof. Since we have ϕ_i with $\alpha\phi_i \underset{m}{\to} \beta$ with $m(\alpha, \phi_i) < 2\delta$, the upper semi-continuity of entropy in $\mathcal{M}(N_0)$ implies $\overline{\lim} h(T^{\alpha\phi_i}, P) \leq h(T^\beta, P)$. Thus it is sufficient to prove the result under the hypotheses that $\beta = \alpha\phi$ is cohomologous to α and that $m(\alpha, \phi) < \delta$.

Consider the distribution of $\alpha\phi$, P, B_N-names. By the Shannon–McMillan theorem, once N is large enough, there is a subset A_N with $\mu(A_N) > 1 - \varepsilon/2$ so that in A_N there are at most

$$2^{|B_N|(h(T^{\alpha\phi}, P)+\varepsilon)}$$

distinct $\alpha\phi$, P, B_N-names.

If δ has been chosen small enough, then for a.e. $x \in X$ we know that:

(i) $\overline{\lim_{N\to\infty}} m(\pi_{x,N}^{\alpha,\phi}) < \varepsilon/4$; and

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(ii) $\lim_{N\to\infty} \overline{d}(\pi^{\alpha,\phi}_{x,N}(\mathcal{N}^{P,\alpha}_{x,N}), \mathcal{N}^{P,\alpha\phi}_{\phi^{-1}(x),N}) = 0.$
Once N is large enough then, on a subset $A'_N \subseteq A_N$ with $\mu(A'_N) > 1 - \varepsilon/2$,
(iii) $m(\pi^{\alpha,\phi}_{x,N}) < \varepsilon/2$ and
(iv) $\overline{d}(\pi^{\alpha,\phi}_{x,N}(\mathcal{N}^{P,\alpha}_{x,N}), \mathcal{N}^{P,\alpha\phi}_{\phi^{-1}(x),N}) < \varepsilon/2$ which tells us
(v) $\overline{m}(\mathcal{N}^{P,\alpha}_{x,N}, \mathcal{N}^{P,\alpha\phi}_{\phi^{-1}(x),N}) < \varepsilon.$

Thus if we partition A'_N according to the names $\mathcal{N}^{P,\alpha\phi}_{\phi^{-1}(x),N}$ we will obtain an $N, \varepsilon, \overline{m}$-cover of (T^α, P) containing at most $2^{|B_N|(h(T^{\alpha\phi},P)+\varepsilon)}$ sets. Hence, once N is large enough

$$H_m(T^\alpha, P, N, \varepsilon) \le h(T^{\alpha\phi}) + \varepsilon. \qquad \square$$

COROLLARY 3.8.

$$h^{**}_m(T^\alpha) \le h^*_m(T^\alpha).$$

Proof. Those β with $h(T^\beta) = h^*_m(T^\alpha)$ are dense in $\langle\alpha\rangle_m$. Hence from Lemma 3.7, for any $\varepsilon > 0$ and partition P we can choose such a β with $m(\alpha, \beta) < \delta$ and hence

$$\varlimsup_{N\to\infty} H_m(T^\alpha, P, N, \varepsilon) \le h(T^\beta, P) + \varepsilon \le h(T^\beta) + \varepsilon = h^*_m(T^\alpha) + \varepsilon.$$

Letting $\varepsilon \searrow 0$ and then taking the sup over P we get the result. $\square$

This completes the proof that $h^*_m = h^{**}_m$ but also gives us a useful corollary.

COROLLARY 3.9. *For any $P \in \mathcal{P}$,*

$$h^{**}_m(T^\alpha, P) = \varlimsup_{m(\alpha,\phi)\to 0} h(T^{\alpha\phi}, P).$$

Proof. Lemma 3.5 tells us

$$\varlimsup_{m(\alpha,\phi)\to 0} h(T^{\alpha\phi}, P) \le h^{**}_m(T^\alpha, P).$$

Lemma 3.7 tells us, for all $\varepsilon > 0$,

$$\varlimsup_{N\to\infty} H_m(T^\alpha, P, N, \varepsilon) \le \varlimsup_{m(\alpha,\phi)\to 0} h(T^{\alpha\phi}, P) + \varepsilon$$

(remember, w.l.o.g., $\beta = \alpha\phi$). Taking the limit as $\varepsilon \searrow 0$ completes the result. $\square$

From now on we set

$$h_m(T^\alpha) = h^*_m(T^\alpha) = h^{**}_m(T^\alpha)$$

and for each $P \in \mathcal{P}$,

$$h_m(T^\alpha, P) = h^{**}_m(T^\alpha, P).$$

THEOREM 3.10. *If $P \in \mathcal{P}$ is a generating partition under the action of T^α then*

$$h_m(T^\alpha) = h_m(T^\alpha, P).$$

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Proof. This argument follows the standard chain of steps going back to Kolmogorov and Sinai. First note that if P refines Q then an $N, \varepsilon, \overline{m}$-cover of (T^α, P) is automatically an $N, \varepsilon, \overline{m}$-cover of (T^α, Q) and hence

$$h_m(T^\alpha, P) \geq h_m(T^\alpha, Q).$$

Second, notice that Axiom 2 tells us that given $P \in \mathcal{P}$ and $N_0 > 0$, setting $P_{N_0} = \vee_{\vec{v} \in B_{N_0}} T^\alpha_{\vec{v}}(P)$, and given $\varepsilon > 0$, there exists $\delta > 0$ so that for any N sufficiently large, any $N, \delta, \overline{m}$-cover of (T^α, P) is automatically an $N, \varepsilon, \overline{m}$-cover of (T^α, P_{N_0}). (To regard P_{N_0} as an element of $\mathcal{P}$ we simply order the α, P, B_{N_0}-names of its elements lexicographically.) Hence

$$h_m(T^\alpha, P) \geq h_m(T^\alpha, P_{N_0}).$$

These two facts imply that for any $Q \subseteq \vee_{\vec{v} \in B_{N_0}} T^\alpha_{\vec{v}}(P)$, for any choice of N,

$$h_m(T^\alpha, P) \geq h_m(T^\alpha, Q).$$

Such Q form a dense subset of each $\mathcal{P}(M_0)$. As used several times already, for α and M_0 fixed, $h_m(T^\alpha, Q)$ is a continuous function of $Q \in \mathcal{P}(M_0)$ which completes the result. □

Even if P is not a generator it will generate some factor algebra under the action of T^α. A factor algebra for one ordering α of course need not be for others, so in this discussion we must be careful to specify the action relative to which the sub-σ-algebra is invariant. We need to understand how the structures of m-equivalence and m-entropy behave under projections to a factor.

THEOREM 3.11. *Suppose $(X, \mathcal{F}, \mu, T^{\overline{\alpha}})$ and $(Y, \mathcal{G}, \nu, S^\alpha)$ are two ergodic systems, where both X and Y are nonatomic Lebesgue spaces, $\Phi : X \to Y$ is measurable, measure preserving and $\Phi T^{\overline{\alpha}} = S^\alpha \Phi$ (i.e. $\overline{\alpha} = \alpha\Phi$). Let $\overline{\mathcal{O}}$ and $\mathcal{O}$ be the corresponding orbit relations.*

(i) *The map $\Phi^*(\beta) = \beta\Phi$ injects the orderings on $\mathcal{O}$ into those on $\overline{\mathcal{O}}$.*

(ii) *For $\phi \in FG(\mathcal{O})$, $\phi(y) = S^\alpha_{\vec{v}(y)}(y)$, the map $\Phi^*(\phi)(x) = T^{\overline{\alpha}}_{\vec{v}(\Phi(x))}(x)$ injects $FG(\mathcal{O})$ into $FG(\overline{\mathcal{O}})$.*

(iii) *If $\alpha\phi_i \underset{\text{pw}}{\to} \beta$, then*

$$\Phi^*(\alpha)\Phi^*(\phi_i) \underset{\text{pw}}{\to} \Phi^*(\beta).$$

(iv) *$m(\beta, \phi) = m(\Phi^*(\beta), \Phi^*(\phi))$ and $m(\beta_1, \beta_2) = m(\Phi^*(\beta_1), \Phi^*(\beta_2))$ and so if $\beta_1\phi_i \underset{m}{\to} \beta_2$ then $\Phi^*(\beta_1)\Phi^*(\phi_i) \underset{m}{\to} \Phi^*(\beta_2)$. Hence the map $\Phi^* : \langle\alpha\rangle_m \to \langle\overline{\alpha}\rangle_m$ is m-preserving.*

Proof. Since both X and Y are nonatomic probability spaces and $T^{\overline{\alpha}}$ and S^α are ergodic, the map Φ is a.s. a bijection when restricted to an individual orbit. Hence if β is an ordering of $\mathcal{O}$ then $\Phi^*(\beta)$ is an ordering of $\overline{\mathcal{O}}$ and if $\phi \in FG(\mathcal{O})$ then $\Phi^*(\phi) \in FG(\overline{\mathcal{O}})$. In both cases Φ^* is one-to-one, giving (i) and (ii),

For (iii) note that $\alpha\phi_i \underset{\text{pw}}{\to} \beta$ iff for a.e. $y \in Y$, and all $\vec{v} \in \mathbb{Z}^d$, if i is large enough, then

$$\alpha(\phi_i(y), \phi_i(S^\alpha_{\vec{v}}(y))) = \beta(y, S^\alpha_{\vec{v}}(y)).$$

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But this implies for a.e. $x \in X$,

$$\Phi^*(\alpha)(\Phi^*(\phi_i)(x), \Phi^*(\phi_i)(T_{\vec{v}}^{\overline{\alpha}}(x))) = \Phi^*(\beta)(x, T_{\vec{v}}^{\overline{\alpha}}(x))$$

and so

$$\Phi^*(\alpha)\Phi^*(\phi_i) \underset{\text{pw}}{\to} \Phi^*(\beta).$$

For (iv), just notice that $f_{\Phi(x)}^{\beta,\phi} = f_x^{\Phi^*(\beta),\Phi^*(\phi)}$ and hence

$$m(\beta, \phi) = m(\Phi^*(\beta), \Phi^*(\phi)) \quad \text{and} \quad m(\beta_1, \beta_2) = m(\Phi^*(\beta_1), \Phi^*(\beta_2)). \qquad \square$$

Let $\mathcal{H} = \Phi^{-1}(\mathcal{G})$ be the preimage algebra in $\mathcal{F}$. Then the range of Φ^* acting on orderings consists of those β for which the cocycles $f(x, \vec{v}) = \beta(x, T_{\vec{v}}^{\overline{\alpha}}(x))$ are $\mathcal{H}$-measurable. In this case we say β is $(\overline{\alpha}, \mathcal{H})$-measurable. Similarly, the set of maps $\Phi^*(FG(\mathcal{O}))$ consists of those $\overline{\phi} \in FG(\overline{\mathcal{O}})$ for which the function $\vec{v}(x)$ with $\overline{\phi}(x) = T_{\vec{v}(x)}^{\overline{\alpha}}(x)$ is $\mathcal{H}$-measurable. In this case we also say $\overline{\phi}$ is $(\overline{\alpha}, \mathcal{H})$-measurable.

Thus $\Phi^*(\langle\alpha\rangle_m)$ is the closure in $\langle\overline{\alpha}\rangle_m$ of $\{\overline{\alpha}\phi : \phi \text{ is } (\overline{\alpha}, \mathcal{H})\text{-measurable}\}$. As a last simple remark, notice that if $\mathcal{H}$ is a $T^{\overline{\alpha}}$-invariant factor algebra of $\mathcal{F}$ and β is $(\overline{\alpha}, \mathcal{H})$-measurable, then $\mathcal{H}$ is also T^{β}-invariant.

COROLLARY 3.12. *For any $P \in \mathcal{P}$ with $\mathcal{H}(\alpha, P) = \vee_{\vec{v}\in\mathbb{Z}^d} T_{\vec{v}}^{\alpha}(P)$ a nonatomic algebra, let $\Phi : X \to Y$ be the projection onto the factor action of T^{α} restricted to $\mathcal{H}(\alpha, P)$. Then*

$$h_m(T^{\alpha}, P) = h_m(\Phi T^{\alpha} \Phi^{-1}).$$

Proof. We know $\Phi(P)$ is a generating partition for $\Phi T^{\alpha}\Phi^{-1}$ acting on Y. To complete the argument, notice that an $N, \varepsilon, \overline{m}$-cover of (T^{α}, P) with a minimal number of elements can always be assumed to consist of sets that are $\vee_{\vec{v}\in B_N} T_{\vec{v}}^{\alpha}(P)$-measurable. Thus restricting the collection of $N, \varepsilon, \overline{m}$-covers to be $\mathcal{H}(\alpha, P)$-measurable does not change the computation of $h_m(T^{\alpha}, P)$. Hence

$$h_m(T^{\alpha}, P) = h_m(\Phi T^{\alpha}\Phi^{-1}, \Phi(P)) = h_m(\Phi T^{\alpha}\Phi^{-1}). \qquad \square$$

We need two last facts to reach the fundamental dichotomy that sizes are either entropy free or entropy preserving. Both become trivial once the dichotomy theorem is proven.

LEMMA 3.13. *If $P \in \mathcal{P}$ is such that*

$$h(T^{\alpha}) = h(T^{\alpha}, P),$$

then

$$h_m(T^{\alpha}) = h_m(T^{\alpha}, P).$$

Proof. For any $\varepsilon > 0$ there is a two set partition $Q(\varepsilon) = \{Q^{-1}(1), Q^{-1}(2)\}$ such that $\mu(Q^{-1}(1)) > 1 - \varepsilon$, and $Q(\varepsilon) \vee P$ is a generator for T^{α}. We order the sets of this span lexicographically, i.e. all the intersections of P atoms with $Q^{-1}(1)$ come first, listed in their P order, and then the intersections with $Q^{-1}(2)$. Thus

$$\lim_{\varepsilon\to 0} Q(\varepsilon) \vee P = P$$

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and hence

$$h_m(T^\alpha, P) = \lim_{\varepsilon \to 0} h_m(T^\alpha, Q(\varepsilon) \vee P) = h_m(T^\alpha). \qquad \square$$

We have used the relation between $\mathcal{N}_{x,N}^{\alpha,P}$ and $\mathcal{N}_{\phi^{-1}(x),N}^{\alpha\phi,P}$ numerous times. There is another way of expressing this relation. As $T^{\alpha\phi}$ and T^α are isomorphic, in particular

$$\phi^{-1} T^\alpha \phi = T^{\alpha\phi},$$

the processes $(T^\alpha, \phi(P))$ and $(T^{\alpha\phi}, P)$ are identical in that the measures $\tilde{\mu}(\alpha, \phi(P))$ and $\tilde{\mu}(\alpha\phi, P)$ are the same. Hence

$$\mathcal{N}_{\phi^{-1}(x),N}^{\alpha\phi,P} = \mathcal{N}_{x,N}^{\alpha,\phi(P)}.$$

Thus, in particular,

$$\underline{\lim}_{N\to\infty} \overline{m}(\mathcal{N}_{x,N}^{\alpha,P}, \mathcal{N}_{x,N}^{\alpha,\phi(P)}) \le m(\alpha,\phi) \quad \text{and} \quad \overline{\lim}_{N\to\infty} \overline{m}(\mathcal{N}_{x,N}^{\alpha,P}, \mathcal{N}_{x,N}^{\alpha,\phi(P)}) \le \hat{m}(\alpha,\phi).$$

The value of this observation is that it puts everything into the context of the one action T^α.

Recall the definition of the atoms of $\vee_{\vec{v}\in B_N} T_{\vec{v}}^\alpha(P)$ as

$$A_{x,N}^{\alpha,P} = \{x' \in X; \mathcal{N}_{x',N}^{\alpha,P} = \mathcal{N}_{x,N}^{\alpha,P}\}.$$

Also recall the Shannon–McMillan–Breiman theorem for actions of $\mathbb{Z}^d$ [**OW2**]. For a.e. $x \in X$

$$\lim_{N\to\infty} \frac{1}{|B_N|} \log_2(\mu(A_{x,N}^{\alpha,P})) = -h(T^\alpha, P).$$

LEMMA 3.14. *Suppose $P \in \mathcal{P}$ and*

$$\overline{\lim_{m(\alpha,\phi)\to 0}} h(T^\alpha, P | \mathcal{H}(\alpha, \phi(P))) = d.$$

Then

$$h_m(T^\alpha, P) \le h(T^\alpha, P) - d.$$

Proof. We can certainly assume $d > 0$ and hence that $h(T^\alpha, P) > 0$. Select ε_0 and $\delta > 0$. Later we will set an upper bound on δ in terms of ε_0 but, in particular, we want $\delta < \varepsilon_0/10$.

Choose $\phi \in FG(\mathcal{O})$ with $m(\alpha,\phi) < \delta/2$ and $h(T^\alpha, P | \mathcal{H}(\alpha, \phi(P))) > d - \varepsilon_0$.

For each N set

$$\begin{aligned}
G_N^1 &= \{x; -\log_2(\mu(A_{x,N}^{\alpha,P})) = |B_N|(h(T^\alpha, P) \pm \delta)\},\\
G_N^2 &= \{x; -\log_2(\mu(A_{x,N}^{\alpha,\phi(P)})) = |B_N|(h(T^\alpha, \phi(P)) \pm \delta)\}\\
G_N^3 &= \{x; -\log_2(\mu(A_{x,N}^{\alpha,P\vee\phi(P)})) = |B_N|(h(T^\alpha, P \vee \phi(P)) \pm \delta)\}\\
G_N^4 &= \{x; \overline{m}(\mathcal{N}_{x,N}^{\alpha,P}, \mathcal{N}_{x,N}^{\alpha,\phi(P)}) < \delta\}.
\end{aligned}$$

All of G_N^1–G_N^4 are $\vee_{\vec{v}\in B_N} T_{\vec{v}}^\alpha(P \vee \phi(P))$-measurable.

Let

$$G_N = \{x \in \cap_{j=1}^4 G_N^j : \mu(A_{x,N}^{\alpha,\phi(P)} \cap G_N^1 \cap G_N^3 \cap G_N^4) > (1-\delta)\mu(A_{x,N}^{\alpha,\phi(P)})\}.$$

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Thus x is in G_N if it belongs to all the G_N^j and, furthermore, its atom $A_{x,N}^{\alpha,\phi(P)}$ is all but a fraction δ covered by the intersection of the G_N^j.

The Shannon–McMillan theorem and Lemma 3.2 allows us to choose a value N_0 so that for all $N \geq N_0$, $\mu(G_N) > 1 - \delta$. We also require that N_0 is large enough so that

$$-\frac{2\log_2(\varepsilon_0)}{|B_{N_0}|} < \delta.$$

Thus if $n \geq N_0$ then for any atom $A \in \vee_{\vec{v}\in B_N} T_{\vec{v}}^{\alpha}(\phi(P))$ which intersects G_N nontrivially, there are at least

$$2^{|B_N|(d-\varepsilon_0-3\delta)}$$

atoms A′ of $\vee_{\vec{v}\in B_N} T_{\vec{v}}^{\alpha}(P \vee \phi(P))$ contained in A all of which are in $\cap_j G_N^j$.

For each such A select a name $\mathcal{N}(A) = \mathcal{N}_{x,N}^{\alpha,\phi(P)}$ for some $x \in A$. Any atom $A' \subseteq A$ as above is the intersection of an atom $A_1(A') \subseteq \vee_{\vec{v}\in B_N} T_{\vec{v}}^{\alpha}(P)$ and A, i.e. $A' = A_1(A') \cap A$.

If $x_1 \in A_1(A')$, even if $x \notin A'$ we will have

$$\overline{m}(\mathcal{N}_{x_1,N}^{\alpha,P}, \mathcal{N}(A)) < \delta.$$

Now choose δ sufficiently small, via Axiom 4, so that for N large enough, for any $x_1 \in A_1(A_1')$ and $x_2 \in A_1(A_2')$, where $A_1', A_2' \subseteq A$, we will have

$$\overline{m}(\mathcal{N}_{x_1,N}^{\alpha,P}, \mathcal{N}_{x_2,N}^{\alpha,P}) < \varepsilon_0.$$

We now want to construct an assignment, or function, L mapping the atoms A_1 of $\vee_{\vec{v}\in B_N} T_{\vec{v}}^{\alpha}(P)$ to atoms A of $\vee_{\vec{v}\in B_N} T_{\vec{v}}^{\alpha}(\phi(P))$, where:

(1) $A_1 \cap L(A_1) \subseteq G_N$ is not empty;

(2) for any $A \in \text{Range}(L)$, $\|L^{-1}(A)\| > 2^{(d-2\varepsilon_0)|B_N|}$; and

(3) $\mu(\{x; x \in A_1, A_1 \in \text{Domain}(L)\}) > 1 - \varepsilon_0$.

We construct L inductively by working through the possible image sets A which intersect G_N nontrivially. To begin, select any such A and for any $A' \subseteq G_N \cap A$ set $L(A_1(A')) = A$. This will satisfy (1) and (2). Suppose we have continued through various possible image sets A always maintaining (1) and (2), but at a certain point we still do not satisfy (3), i.e. the domain of L is still too small. Precisely,

$$\mu(\{x \in A_1; A_1 \notin \text{Domain}(L)\}) \geq \varepsilon_0.$$

As $\mu(G_N) > 1 - \delta$, there must exist an A with

$$\mu(A \cap G_N \cap \{x \in A_1; A_1 \notin \text{Domain}(L)\}) > (\varepsilon_0 - \delta)\mu(A).$$

The set $A \cap G_N \cap \{x \in A_1; A_1 \notin \text{Domain}(L)\}$ consists of 'good' atoms $A' \subseteq G_N$. Since they lie in G_N^3 and A lies in G_N^1, there must be at least

$$(\varepsilon_0 - \delta)2^{|B_N|(d-\varepsilon_0-3\delta)}$$

such atoms A' in A. Now $(\varepsilon_0 - \delta)2^{|B_N|(d-\varepsilon_0-3\delta)} > 2^{|B_N|(d-2\varepsilon_0)}$.

Extend L to the atoms $A_1(A')$ with $A' \subseteq A \cap G_N \cap \{x \in A_1; A_1 \notin \text{Domain}(L)\}$ by setting $L(A_1(A)) = A$. This process continues until (3) is satisfied as the set of atoms is finite.

Now consider the partition whose elements are the sets

$$S(A) = \{x; x \in A_1,\ L(A_1) = A\}.$$

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(1) If $x_1, x_2 \in S(A)$ then $\overline{m}(\mathcal{N}^{\alpha,P}_{x_1,N}, \mathcal{N}^{\alpha,P}_{x_2,N}) < \varepsilon_0$.
(2) $|\{S(A)\}| \leq 2^{|B_N|(h(T^\alpha,P)+\delta-d+2\varepsilon_0)}$ and
(3) $\mu(\cup_A S(A)) > 1 - \varepsilon_0$.

Thus $\{S(A)\}$ is an N, P, ε_0-cover of (T^α, P) and for N sufficiently large

$$H_m(T^\alpha, P, N, \varepsilon_0) \leq h(T^\alpha, P) - d + 3\varepsilon.$$

Thus $h_m(T^\alpha, P) \leq h(T^\alpha, P) - d$. □

COROLLARY 3.15. *Suppose α and $P_1, P_2 \in \mathcal{P}$ are such that both*

$$h_m(T^\alpha, P_1) = h(T^\alpha, P_1) \quad \textit{and} \quad h_m(T^\alpha, P_2) = h(T^\alpha, P_2).$$

Then we also have

$$h_m(T^\alpha, P_1 \vee P_2) = h(T^\alpha, P_1 \vee P_2).$$

Proof. As $h_m(T^\alpha, P_i) = h(T^\alpha, P_i), i = 1, 2$, we know

$$\lim_{m(\alpha,\phi)\to 0} h(T^\alpha, P_i | \mathcal{H}(\alpha, \phi(P_i))) = 0$$

from the previous lemma. Furthermore, from Corollary 3.9 we know

$$\underline{\lim}_{m(\alpha,\phi)\to 0} h(T^\alpha, \phi(P_i)) = h_m(T^\alpha, P_i) = h(T^\alpha, P_i)$$

and as

$$\overline{\lim}_{m(\alpha,\phi)\to 0} h(T^\alpha, \phi(P_i)) \leq h(T^\alpha, P_i)$$

we must have

$$\lim_{m(\alpha,\phi)\to 0} h(T^\alpha, \phi(P_i)) = h(T^\alpha, P_i).$$

This implies

$$\lim_{m(\alpha,\phi)\to 0} h(T^\alpha, \phi(P_i) | \mathcal{H}(\alpha, P_i)) = 0.$$

Thus

$$\begin{aligned} h_m(T^\alpha, P_1 \vee P_2) &= \underline{\lim}_{m(\alpha,\phi)\to 0} h(T^\alpha, \phi(P_1 \vee P_2)) \\ &= h(T^\alpha, P_1 \vee P_2) - \overline{\lim}_{m(\alpha,\phi)\to 0} [h(T^\alpha, \phi(P_1 \vee P_2) | \mathcal{H}(\alpha, P_1 \vee P_2)) \\ &\quad - h(T^\alpha, P_1 \vee P_2 | \mathcal{H}(\alpha, \phi(P_1 \vee P_2)))] \end{aligned}$$

which we know is zero, hence

$$h_m(T^\alpha, P_1 \vee P_2) = h(T^\alpha, P_1 \vee P_2).$$ □

Definition 3.6. A 'Bernoulli process' (T^α, P) is an ergodic action T^α and a partition P of X so that T^α acting on the factor algebra $\mathcal{H}(\alpha, P)$ is measurably isomorphic to the shift action on an i.i.d. collection of random variables indexed by $\mathbb{Z}^d$. Such processes are known to be completely classified up to isomorphism by their entropy (Ornstein's isomorphism theorem) and to sit as full entropy factors of any positive entropy ergodic $\mathbb{Z}^d$ action (Sinai's theorem for $\mathbb{Z}^d$) [**OW1**].

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We will use Sinai's and Ornstein's theorems for actions of $\mathbb{Z}^d$ in a very essential way in proving Theorem 3.17. An m-equivalence version of Sinai's and Ornstein's theorems are the principle goals of our work, appearing in the sequel to this paper. Because of our fundamental use of these theorems here, our work appears not to be self-contained. The size m_1 corresponding to isomorphisms, however, will be seen directly to be entropy preserving. Hence our versions of Sinai's and Ornstein's theorems will hold for it, without recourse to the arguments here which rely on them. Having obtained them, we can then use the results here to obtain the general m-equivalence versions of these theorems.

COROLLARY 3.16. *Suppose for some Bernoulli process* (T^β, P_0) *we have*

$$h_m(T^\beta, P_0) < h(T^\beta, P_0).$$

Then for all Bernoulli processes (T^α, P),

$$h_m(T^\alpha, P) < h(T^\alpha, P).$$

Proof. If $h(T^\beta, P_0) \leq h(T^\alpha, P)$ then by the isomorphism theorem we can find a generating partition for T^α of the form $(P_1 \vee B)$ where:
(1) the process (T^β, P_0) and (T^α, P_1) are identical, i.e. $\tilde{\mu}(\beta, P_0) = \tilde{\mu}(\alpha, P_1)$;
(2) (T^α, B) is an i.i.d. process; and
(3) $\mathcal{H}(\alpha, P_1) \perp \mathcal{H}(\alpha, B)$.
That is to say, we can write (T^α, P) as a direct product of (T^β, P_0) and an i.i.d. process of the correct entropy.

Now

$$\begin{aligned} h_m(T^\alpha, P) &= h_m(T^\alpha, P_1 \vee B) = \varlimsup_{m(\alpha,\phi)\to 0} (h(T^{\alpha\phi}, P_1 \vee B)) \\ &\leq \varlimsup_{m(\alpha,\phi)\to 0} (h(T^{\alpha\phi}, P_1) + h(T^{\alpha\phi}, B)) \\ &\leq \varlimsup_{m(\alpha,\phi)\to 0} (h(T^{\alpha\phi}, P_1)) + h(T^\alpha, B) \quad (\text{as } (T^\alpha, B) \text{ is i.i.d.}) \\ &= h_m(T^\beta, P_0) + h(T^\alpha, B) < h(T^\beta, P_0) + h(T^\alpha, B) = h(T^\alpha, P). \end{aligned}$$

If on the other hand $h(T^\beta, P_0) > h(T^\alpha, P)$, write (T^β, P_0) as the K-fold direct product of a single i.i.d. process (T^γ, P_2), where K is sufficiently large so that $h(T^\gamma, P_2) \leq h(T^\alpha, P)$. If $h_m(T^\gamma, P_2) = h(T^\gamma, P_2)$, then Corollary 3.15 tells us the same is true for the K-fold product, and hence for (T^β, P_0). This would of course be a conflict. Hence we know $h_m(T^\gamma, P_2) < h(T^\gamma, P_2)$ and we are reduced to the earlier case. □

THEOREM 3.17. *If for some ergodic system* $(S^\beta, Y, \mathcal{G}, \nu)$ *and some partition* Q *of* Y *we have*

$$h_m(S^\beta, Q) < h(S^\beta, Q),$$

then for all ergodic systems $(T^\alpha, X, \mathcal{F}, \mu)$ *and finite partitions* P *of* X *we have* $h_m(T^\alpha, P) = 0$ *and hence* $h_m(T^\alpha) = 0$.

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Proof. We know $h(S^\beta, Q) > 0$ and hence by Sinai's theorem there is a partition $P_0 \subseteq \mathcal{H}(\beta, Q)$ so that (S^β, P_0) is a Bernoulli system with $h(S^\beta, P_0) = h(S^\beta, Q)$. Lemma 3.13 tells us that

$$h_m(S^\beta, P_0) < h(S^\beta, P_0).$$

Corollary 3.16 tells us that for all Bernoulli processes (T^γ, P_1),

$$h_m(T^\gamma, P_1) < h(T^\gamma, P_1).$$

Suppose that for some ergodic system (T^α, P) we have

$$h_m(T^\alpha, P) > 0.$$

Restrict to the subalgebra $\mathcal{H}(\alpha, P)$ so that we may assume P is a generator and hence $0 < h_m(T^\alpha) < \infty$. Choose a $\gamma \in \langle\alpha\rangle_m$ with $h_m(T^\gamma) = h(T^\gamma) > 0$. Now by Sinai's theorem we can select a partition P_1 so that (T^γ, P_1) is Bernoulli and $h(T^\gamma, P_1) = h(T^\gamma) = h_m(T^\gamma)$. But now Lemma 3.13 tells us that $h_m(T^\gamma, P_1) = h(T^\gamma, P_1) > h_m(T^\gamma, P_1)$: a conflict. □

This completes our work here with m-entropy and allows us to divide sizes into two types.

Definition 3.7. If a size m is such that:
(1) $h_m(T^\alpha) = h(T^\alpha)$, then we call m an entropy-preserving size;
(2) $h_m(T^\alpha) = 0$, then we call m an entropy-free size.

4. *A few examples of sizes*

We give here three basic examples of sizes. We call them m_1, m_0, and m_K. As we will see, to be m_1-equivalent is simply to be cohomologous, hence corresponds to isomorphism. Any pair of orderings will be seen to be m_0-equivalent, hence this simply corresponds to orbit equivalence. These two are, respectively, the strongest and weakest sizes. The size m_K corresponds to what Hasfura refers to as id-Kakutani equivalence [**H**].

Example 1. Define

$$m_1(\pi) = \frac{|\{\vec{v} \in B_n; \pi(\vec{v}) \neq \vec{v}\}|}{|B_n|},$$

where $\pi \in \Pi_n$.

LEMMA 4.1. *m_1 is a size.*

Proof. Axiom 1 follows directly from the definition.

To see Axiom 2, notice that if a set $S \subseteq B_n$ satisfies

$$\frac{|S|}{|B_n|} > 1 - \frac{\varepsilon}{|B_{N_0}|}$$

then

$$|\{\vec{v} \in B_n;\ \text{for all } \vec{v}_0 \in B_{N_0}, \vec{v} + \vec{v}_0 \in S\}| > (1 - \varepsilon)|B_n|.$$

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Thus if $m_1(\pi) < \varepsilon/|B_{N_0}|$, we will have

$$\frac{|\{\vec{v} \in B_n;\ \text{for some } \vec{v}_0 \in B_{N_0}, \pi(\vec{v}+\vec{v}_0) \neq \vec{v}+\vec{v}_0 = \pi(\vec{v})+\vec{v}_0\}|}{|B_n|} < \varepsilon.$$

As earlier, to make $\pi(\vec{v}+\vec{v}_0)$ well defined we extend π as the identity outside of B_n.

To see Axioms 3 and 4 just notice that

$$m_1(\pi^{-1}) = m_1(\pi) \text{ and } m_1(\pi_1\pi_2) \leq m_1(\pi_1) + m_1(\pi_2).$$

Axioms 5 and 6 follow quite easily from the definition of m_1. □

LEMMA 4.2. *For any $\varepsilon > 0$, if $m_1(\alpha, \phi) < \varepsilon$ then*

$$\mu(\{x; \phi(x) = x\}) > 1 - 2\varepsilon.$$

Proof. Let $S = \{x; \phi(x) = x\}$ and now consider a point x for which:

(1) the ergodic theorem gives us

$$\frac{|\{\vec{v} \in B_n : T^{\alpha}_{\vec{v}}(x) \in S\}|}{|B_n|} \underset{n}{\to} \mu(S);$$

(2)

$$m_{1,x}(\alpha, \phi) \leq 2 \liminf_{n\to\infty} \frac{|\{\vec{v} \in B_n; \pi^{\alpha,\phi}_{x,n}(\vec{v}) \neq \vec{v}\}|}{|B_n|} < 2\varepsilon;$$

(3)

$$\frac{|\{\vec{v} \in B_n; f^{\alpha,\phi}_{x,n}(\vec{v}) \neq \pi^{\alpha,\phi}_{x,n}(\vec{v})\}|}{|B_n|} \underset{n}{\to} 0.$$

We know that the set of such x's has positive measure as (1) and (3) hold a.s. and (2) occurs on a set of positive measure. Now (1) is equivalent to saying

$$\frac{|\{\vec{v} \in B_n; f^{\alpha,\phi}_{x,n}(\vec{v}) \neq \vec{v}\}|}{|B_n|} \underset{n}{\to} 1 - \mu(S).$$

The result follows. □

COROLLARY 4.3. *If α and β are orderings with $m_1(\alpha, \beta) < \varepsilon < 1$ then there is a set $S \subseteq X$ with $\mu(S) > 1 - 2\varepsilon$ so that if $(x, x') \in \mathcal{O}$ and both are in S, then $\alpha(x, x') = \beta(x, x')$.*

Proof. Suppose $\alpha \underset{m}{\to} \beta$ via $\{\phi_i\}$, where $m_1(\alpha, \phi_i) < \varepsilon$. By dropping to a subsequence we can assume without loss of generality that

$$\sum_i^{\infty} m_1(\alpha\phi_i, \phi_i^{-1}\phi_{i+1}) < \varepsilon.$$

Thus there are sets $S_i \subseteq X$ with

$$\sum_i^{\infty} (1 - \mu(S_i)) < 2\varepsilon$$

and for $x \in S_i$, $\phi_i^{-1}\phi_{i+1}(x) = x$. Let $S = \cap_i S_i$. We have $\mu(S) > 1-2\varepsilon$ and, furthermore, for $x \in S$, for all ϕ_i, $\phi_i(x) = x$ and hence for $(x, x') \in \mathcal{O}$ and both in S we will have

$$\beta(x, x') = \lim_{i\to\infty} \alpha(\phi_i(x), \phi_i(x')) = \alpha(x, x').$$ □

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THEOREM 4.4. *Let α and β be two orderings. Then $\alpha \underset{m_1}{\Leftrightarrow} \beta$ iff there exists $\phi \in FG(\mathcal{O})$ with $\beta = \alpha\phi$. That is to say, α and β are m_1 equivalent iff they are cohomologous and, in particular, T^α and T^β are isomorphic.*

Proof. One direction is obvious, as cohomologous orderings are m-equivalent for all sizes m.

For the other direction, Corollary 4.3 tells us that as soon as $m_1(\alpha, \beta) < 1$ there is a subset S on which the two orderings agree in that between any two points of S both orderings assign the same vector. It is quite standard that this is equivalent to α and β being cohomologous, but for completeness we include the argument.

For $x \in X$, let $x' \in S$ be some point on the orbit of x and define a vector-valued function $h(x) = \alpha(x, x') - \beta(x, x')$. Notice that this definition is independent of the choice of $x' \in S$ and that for any $(x_1, x_2) \in \mathcal{O}$ we have

$$\beta(x_1, x_2) = \alpha(x_1, x_2) - h(x_1) + h(x_2),$$

so that the orderings are cohomologous. Clearly, if α and β are m_1-equivalent there exists an α_1 cohomologous to α with $m_1(\alpha_1, \beta) < 1$ and hence cohomologous to β. Hence α and β must be cohomologous. □

Example 2. For $\pi \in \Pi_n$ let

$$m_0(\pi) = \frac{|\{\vec{v} \in B_n;\ \text{there is a } \vec{v}_0 \in B_1 \text{ with } \pi(\vec{v} + \vec{v}_0) \neq \pi(\vec{v}) + \vec{v}_0\}|}{|B_n|}.$$

Once more, $\pi(\vec{v} + \vec{v}_0)$ is defined to be $\vec{v} + \vec{v}_0$ when outside of B_n.

THEOREM 4.5. *m_0 is a size.*

Proof. For Axiom 1, clearly $m_0(\mathrm{id}) = 0$ even taking into account the way π is extended outside of B_n. For part (b) we again note that if a subset $S \subseteq B_n$ satisfies

$$|S| > (1 - \varepsilon)\frac{|B_n|}{|B_{N_0}|},$$

then

$$|\{\vec{v} \in B_n;\ \text{for some } \vec{v}_0 \in B_{N_0},\ \vec{v} + \vec{v}_0 \notin S\}| < \varepsilon.$$

Thus if

$$\frac{|\{\vec{v} \in B_n; \pi(\vec{v}) \neq \vec{v}\}|}{|B_n|} < \varepsilon/|B_1|,$$

then

$$\frac{|\{\vec{v} \in B_n;\ \text{for some } \vec{v}_0 \in B_1,\ \pi(\vec{v} + \vec{v}_0) \neq \vec{v} + \vec{v}_0\}|}{|B_n|} < \varepsilon.$$

But this expression is greater than or equal to $m_0(\pi)$.

To see that Axiom 2 holds, note once more that B_{N_0} is the N_0-fold sum of elements of B_1. Using once more the above observation about sets $S \subseteq B_n$, if $m_0(\pi) < \varepsilon/|B_{N_0}|$ then we will have

$$\frac{|\{\vec{v} \in B_n;\ \text{for some } \vec{v}_0 \in B_{N_0}, \pi(\vec{v} + \vec{v}_0) \neq \pi(\vec{v}) + \vec{v}_0\}|}{|B_n|} < \varepsilon.$$

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To see that Axioms 3 and 4 hold just note that

$$m_0(\pi) = m_0(\pi^{-1}) \quad \text{and} \quad m_0(\pi_1\pi_2) \le m_0(\pi_1) + m_0(\pi_2).$$

Axiom 5 is easily seen to hold for m_0.

For Axiom 6, notice that under the hypotheses of the Axiom,

$$\frac{|\{\vec{v} \in K_i \text{ for some i; for all } \vec{v}_0 \in B_1, \vec{v} + \vec{v}_0 \in K_i \text{ and } \pi(\vec{v} + \vec{v}_0) = \pi(\vec{v}) + \vec{v}_0\}|}{|B_n|} > 1 - (|B_1| + 1)\delta.$$

This gives an upper bound for $m_0(\pi')$ of $(|B_1| + 1)\delta$. □

We now begin an argument that any two orderings are m_0-equivalent. Recall that we defined in Definition 2.1 a metric d_0 on orderings that gave a notion of convergence in probability for which $\underset{\text{pw}}{\to}$ was the corresponding notion of pointwise convergence. In particular, if $m_0(\alpha\phi_i, \beta) \to 0$, then for some subsequence $i(j)$ we will have $\alpha \underset{\text{pw}}{\to} \beta$ via $\phi_{i(j)}$.

LEMMA 4.6. *For an ordering α and $\phi \in FG(\mathcal{O})$*

$$m_0(\alpha, \phi) \le d_0(\alpha, \alpha\phi).$$

Proof. Let

$$S = \{x_0; \text{ for all } \vec{v}_0 \in B_1,\ T^{\alpha}_{\vec{v}_0}(x_0) = T^{\alpha\phi}_{\vec{v}_0}(x_0)\}.$$

Then

$$d_0(\alpha, \alpha\phi) = 1 - \mu(S).$$

For any $x \in X$,

$$f^{\alpha,\phi}_x(\vec{v} + \vec{v}_0) = f^{\alpha,\phi}_x(\vec{v}) + \vec{v}_0 \text{ for all } \vec{v}_0 \in B_1 \text{ iff } x_0 = T^{\alpha}_{\vec{v}}(x) \text{ is in } S.$$

Thus for a.e. $x \in X$,

$$\lim_{n\to\infty} \frac{|\{\vec{v} \in B_n; \text{ for some } \vec{v}_0 \in B_1, \pi^{\alpha,\phi}_{x,n}(\vec{v} + \vec{v}_0) \ne \pi^{\alpha,\phi}_{x,n}(\vec{v}) + \vec{v}_0\}|}{|B_n|}$$

exists and is $1-\mu(S) = d_0(\alpha, \alpha\phi)$. But the essential infimum of this limit is $m_0(\alpha, \phi)$. □

COROLLARY 4.7. *If $\alpha\phi_i \underset{\text{pw}}{\to} \beta$, then $\alpha \underset{m_0}{\Leftrightarrow} \beta$.*

Proof. We know $\alpha\phi_i \underset{\text{pw}}{\to} \beta$ implies $d_0(\alpha\phi_i, \beta) \to 0$. By Lemma 4.6, this implies that $\alpha\phi_i$ is m_0-Cauchy. □

It only remains to show that $\{\alpha\phi; \phi \in FG(\mathcal{O})\}$ is d_0-dense in the set of all orderings of $\mathcal{O}$. This will require a few very basic facts about orderings and ε-invariance.

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Definition 4.8. A finite subset $K \subseteq \mathbb{Z}^d$ is said to be ε-invariant if for each $\vec{v} \in B_1$ we have

$$\frac{|(K+\vec{v})\triangle K|}{|K|} < \varepsilon.$$

As the n-fold sum of elements of B_1 is precisely B_n, if K is $(\varepsilon/|B_n|)$-invariant, then for all $\vec{v} \in B_n$,

$$\frac{|(K+\vec{v})\triangle K|}{|K|} < \varepsilon.$$

LEMMA 4.9. *For any two orderings α and β and for a.e. $x \in X$, for any $\varepsilon > 0$ there exists N so that for all $n \geq N$,*

$$\{\beta(x, T^{\alpha}_{\vec{v}}(x)); \vec{v} \in B_n\}$$

is ε-invariant.

Proof. Let

$$V(x) = \{\alpha(x, T^{\beta}_{\vec{v}_0}(x)); \vec{v}_0 \in B_1\}.$$

Let

$$S(b) = \{x; |\vec{v}| < b \text{ for all } \vec{v} \in V(x)\}.$$

As $b \to \infty$, $S(b) \to X$, and hence we can choose $b_0 \in \mathbb{N}$ such that $S = S(b_0)$ satisfies

$$\mu(S) > 1 - \varepsilon/2.$$

For a.e. $x \in X$, choose N so that for $n \geq N - b_0$,

$$\frac{|\{\vec{v} \in B_n; T^{\alpha}_{\vec{v}}(x) \in S\}|}{|B_n|} > \varepsilon/2 \quad \text{and} \quad \frac{|B_n|}{|B_{n+b_0}|} > 1 - \varepsilon/2.$$

Now for $n \geq N$,

$$\begin{aligned} &|\cap_{\vec{v}_0 \in B_1} \{\beta(x, T^{\alpha}_{\vec{v}}(x)) + \vec{v}_0; \vec{v} \in B_n\}| \\ &\quad = |\{\vec{v} \in B_n; \text{ for all } \vec{v}_0 \in B_1, \alpha(x, T^{\beta}_{\vec{v}_0}(T^{\alpha}_{\vec{v}}(x))) \in B_n\}| \geq (1-\varepsilon)|B_n|. \end{aligned}$$ □

LEMMA 4.10. *Given any $\varepsilon > 0$ there exists $\delta > 0$ so that for any $S \subseteq \mathbb{Z}^d$ that is δ-invariant with $|S| = |B_n|$, there exists a bijection*

$$f : S \to B_n$$

with

$$|\{\vec{v} \in S; \text{ for some } \vec{v}_0 \in B_1, f(\vec{v}+\vec{v}_0) \neq f(\vec{v}) + \vec{v}_0\}| < \varepsilon|B_n|.$$

Proof. Choose N_0 so that

$$|B_{N_0-1}| > (1-\varepsilon/2)|B_{N_0}|$$

and choose $\delta < \varepsilon/|B_{(2N_0+1)}|$, which implies that

$$|\{\vec{v} \in S; \text{ for some } \vec{v}_0 \in B_{(2N_0+1)}, \vec{v}+\vec{v}_0 \notin S\}| < \frac{\varepsilon|S|}{2}.$$

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Let $L = (2N_0 + 1)\mathbb{Z}^d$ be the lattice of points $(2N_0 + 1)$ apart in $\mathbb{Z}^d$. The 'boxes' $\{\vec{u} + B_{N_0}; \vec{U} \in L\}$ partition $\mathbb{Z}^d$. If $\vec{v} \in S \cap (\vec{u} + B_{N_0})$ but $\vec{u} + B_{N_0} \not\subseteq S$, then for some $\vec{v}_0 \in B_{(2N_0+1)}$, $\vec{v} + \vec{v}_0 \notin S$. Hence

$$|\{\vec{v} \in S : \vec{v} \in \vec{u} + B_{N_0} \subseteq S, \vec{u} \in L\}| > (1 - \varepsilon/2)|S|.$$

Since $|B_n| = |S| > 1/\delta$, by decreasing δ we can obtain

$$\frac{|B_{n-N_0}|}{|B_n|} > 1 - \varepsilon/2.$$

Thus

$$|\{\vec{v} \in B_n; \vec{v} \in \vec{u} + B_{N_0} \subseteq B_n, \vec{u} \in L\}| > (1 - \varepsilon/2)|S|.$$

To define f, map by translation as many boxes $(\vec{u}+B_{N_0}) \subseteq S$ to some $(\vec{u}'+B_{N_0}) \subseteq B_n$ as possible ($\vec{u}$ and $\vec{u}' \in L$). Thus for such a $\vec{v} \in \vec{u} + B_{N_0}$,

$$f(\vec{v}) = \vec{v} - \vec{u} + \vec{u}'.$$

This will define f on at least $(1 - \varepsilon/2)|S|$ points. Map the rest of S bijectively to the remainder of B_n in any way.

If $\vec{v} \in \vec{u} + B_{(N_0-1)}$, where $\vec{u} + B_{N_0} \xrightarrow{f} \vec{u}' + B_{N_0}$, then for all $\vec{v}_0 \in B_1$, $\vec{v} + \vec{v}_0 \in \vec{u} + B_{N_0}$ and so

$$f(\vec{v} + \vec{v}_0) = \vec{v} + \vec{v}_0 - \vec{u} + \vec{u}' = f(\vec{v}) + \vec{v}_0.$$

The number of such points in S is at least $(1 - \varepsilon/2)^2|S| > (1 - \varepsilon)|S|$. □

THEOREM 4.11. *For any orderings α and β and $\varepsilon > 0$ there exists $\phi \in FG(\mathcal{O})$ with*

$$d_0(\alpha\phi, \beta) < \varepsilon.$$

Proof. Let δ be chosen using Lemma 4.10 with 'epsilon' chosen to be $\varepsilon/3$. Apply Lemma 4.9 to choose N sufficiently large so that

$$\mu(\{x; \{\beta(x, T^\alpha_{\vec{v}}(x)); \vec{v} \in B_N\} \text{ is } \delta\text{-invariant}\}) > 1 - \varepsilon/6.$$

The sets

$$S(x) = \{\beta(x, T^\alpha_{\vec{v}}(x)); \vec{v} \in B_N\}$$

are from the countable collection of finite subsets of $\mathbb{Z}^d$. Select a finite collection $S_1, S_2, \ldots, S_t$ so that

$$\mu(\{x; S(x) = S_i \text{ for some } i \le t \text{ and } S_i \text{ is } \delta\text{-invariant }\}) > 1 - \varepsilon/3.$$

From Lemma 4.10, for each $i \le t$ there is an $f_i : S_i \to B_N$ satisfying its conclusion.

Let $G_i = \{x; S(x) = S_i\}$ and use the strong Rokhlin lemma to construct a tower of dimension B_N with base set F satisfying

$$\mu\left(\bigcup_{\vec{v} \in B_N} T^\alpha_{\vec{v}}(F)\right) > 1 - \varepsilon/3.$$

and

$$G_i \perp F, \quad 1 \le i \le t.$$

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We define $\phi \in FG(\mathcal{O})$ as follows. If $x = T^{\alpha}_{\vec{v}}(x')$ with $x' \in F \cap G_i$ and $\vec{v} \in B_N$ then set

$$\phi(x) = T^{\alpha}_{f_i(\beta(x',x))}(x').$$

Notice that what this does is to take the slice through the tower $\{T^{\alpha}_{\vec{v}}(x'); \vec{v} \in B_N\}$ and simply permutes its points. This permutation takes place in three steps. First the slice is mapped to S_i by sending $x \to \beta(x', x)$. This is then mapped back to B_N by f_i and then re-embedded in the slice by $T^{\alpha}_{\cdot}(x')$.

Notice that for $x' \in F \cap G_i$ we have

$$\begin{aligned}&|\{\vec{v} \in B_N;\ \text{for all } \vec{v}_0 \in B_1, \alpha(\phi(T^{\alpha}_{\vec{v}}(x')), \phi(T^{\alpha}_{\vec{v}+\vec{v}_0}(x'))) = \beta(T^{\alpha}_{\vec{v}}(x'), T^{\alpha}_{\vec{v}+\vec{v}_0}(x'))\}|\\ &\quad = |\{\vec{v} \in B_N;\ \text{for all } \vec{v}_0 \in B_1, f_i(\vec{v}+\vec{v}_0) = f(\vec{v}) + \vec{v}_0\}| > (1-\varepsilon/3)|B_N|.\end{aligned}$$

For all other points $x \in X$ set $\phi(x) = x$. It now follows that

$$\begin{aligned} d_0(\alpha\phi, \beta) &= \mu(\{x;\ \text{for all } \vec{v}_0 \in B_1, T^{\alpha}_{\vec{v}_0}(x) = T^{\beta}_{\vec{v}_0}(x)\})\\ &\geq \sum_{i=1}^{t} \mu(G_i \cap F)|\{\vec{v} \in B_N;\ \text{for all } \vec{v}_0 \in B_1, f_i(\vec{v}+\vec{v}_0) = f_i(\vec{v}) + \vec{v}_0\}|\\ &\geq (1-\varepsilon)^2 \sum_{i=1}^{t} \mu(G_i) > 1-\varepsilon. \end{aligned}$$

□

COROLLARY 4.12. *For all orderings α and β, $\alpha \underset{m_0}{\Leftrightarrow} \beta$.*

Our third example ties in our work here with that of Hasfura on id-Kakutani equivalence. Our definition of the size m_K is precisely his. In [**H**] he has proven the corresponding equivalence theorem. Lemma 4.15 will show why this size is intrinsically easier to work with than the general situation, as it tells you that a permutation that is m_K-small moves most points by only a small fraction of the size of the box.

Definition 4.13. A set $S \in B_n$ is called an ε-fixing set for $\pi \in \Pi_n$ if:
(1) $|S| \geq (1-\varepsilon)|B_n|$; and
(2) for all $\vec{u}, \vec{v} \in S$,

$$|(\pi(\vec{u}) - \pi(\vec{v})) - (\vec{u} - \vec{v})| \leq \varepsilon|\vec{u} - \vec{v}|.$$

As always $|\vec{u}| = \max(|u_i|)$ is the L^{∞} norm.

Example 3. For $\pi \in \Pi_n$ define

$$m_K(\pi) = \min(\{\varepsilon; \pi \text{ has a } \varepsilon\text{-fixing set}\}).$$

THEOREM 4.14. *m_K is a size.*

Proof. Clearly $m_K(\text{id}) = 0$, and if

$$|\{\vec{v} \in B_n; \pi(\vec{v}) \neq \vec{v}\}| < \delta|B_n|,$$

then $\{\vec{v} \in B_n; \pi(\vec{v}) = \vec{v}\}$ is a δ-fixing set for π, giving Axiom 1.

For Axiom 2, if $\delta < \varepsilon/|B_{N_0}|$ and $m_K(\pi) < \delta$, then in the δ-fixing set S for π we will have

$$|\{\vec{v} \in S;\ \text{for all } \vec{v}_0 \in B_{N_0}, \vec{v} + \vec{v}_0 \in S\}| > (1-\varepsilon)|B_n|.$$

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Assuming also that $\delta < 1/|B_{N_0}|$, for $\vec{v}_0 \in B_{N_0}$, if

$$|\pi(\vec{v}) - \pi(\vec{v} + \vec{v}_0) + \vec{v}_0| < \delta|\vec{v}_0|,$$

then in fact $\pi(\vec{v}) - \pi(\vec{v} + \vec{v}_0) + \vec{v}_0$ must be 0 which is to say $\pi(\vec{v} + \vec{v}_0) = \pi(\vec{v}) + \vec{v}_0$ and

$$\frac{|\{\vec{v} \in B_n; \text{ for some } \vec{v}_0 \in B_{N_0}, \pi(\vec{v} + \vec{v}_0) \neq \pi(\vec{v}) + \vec{v}_0\}|}{|B_n|} < \varepsilon.$$

For Axiom 3, note that if $\delta < 1$ then

$$\left|\frac{a}{b} - 1\right| < \delta \quad \text{implies} \quad \left|\frac{b}{a} - 1\right| < \frac{\delta b}{a} < \left(\frac{\delta}{1-\delta}\right).$$

Thus for $\varepsilon > 0$, if $m_K(\pi) < \delta = \varepsilon/(1+\varepsilon)$, then

$$m_K(\pi^{-1}) < \varepsilon.$$

For Axiom 4, suppose $m_K(\pi_1) = d < 1$ and S_1 is a d-fixing set for π_1. Suppose $m_K(\pi_2) < \varepsilon/2$, and S_2 is an $\varepsilon/2$-fixing set for π_2. Let

$$S = \{\vec{v} \in S_1; \pi_1(\vec{v}) \in S_2\}$$

so that

$$|S| > (1 - (d + \varepsilon))|B_n|.$$

For $\vec{v}_1, \vec{v}_2 \in S$ we compute

$$\begin{aligned} |(\pi_2\pi_1(\vec{v}_1) - \pi_2\pi_1(\vec{v}_2)) - (\vec{v}_1 - \vec{v}_2)| &\leq |(\pi_2\pi_1(\vec{v}_1) - \pi_2\pi_1(\vec{v}_2)) - (\pi_1(\vec{v}_1) - \pi_1(\vec{v}_2))| \\ &\quad + |(\pi_1(\vec{v}_1) - \pi_1(\vec{v}_2)) - (\vec{v}_1 - \vec{v}_2)| \\ &< \frac{\varepsilon}{2}|\pi_1(\vec{v}_1) - \pi_1(\vec{v}_2)| + d|\vec{v}_1 - \vec{v}_2| \\ &< \frac{\varepsilon}{2}(1+d)|\vec{v}_1 - \vec{v}_2| + d|\vec{v}_1 - \vec{v}_2| \\ &< (\varepsilon + d)|\vec{v}_1 - \vec{v}_2|. \end{aligned}$$

Thus S is an $(\varepsilon + d)$-fixing set for $\pi_2\pi_1$.

To obtain Axioms 5 and 6 a small geometric lemma is helpful.

LEMMA 4.15. *For any $\varepsilon > 0$ there exists $\delta > 0$ so that if $\pi \in \Pi_n$ has a δ-fixing set S, then for all $\vec{v} \in S$,*

$$|\pi(\vec{v}) - \vec{v}| < \varepsilon n.$$

Proof. To start, require $\delta < |B_{[4/\varepsilon]}|^{-1}$ so that if $\pi \in \Pi_n$ has a δ-fixing set, then either:
(1) $[\varepsilon n/8] \geq 1$; or since $n < 8/\varepsilon$,
(2) $\pi = \text{id}$ and the result is true.

In case (1) let $n_0 = [\varepsilon n/8]$ and as $n_0 \geq \varepsilon n/16$,

$$\frac{|B_{n_0}|}{|B_n|} > (\varepsilon/16)^d$$

is bounded away from zero uniformly in n.

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Requiring $\delta < (\varepsilon/16)^d$, if S is a δ-fixing set for π, then for any $\vec{v} + B_{n_0} \subseteq B_n$, we must have $S \cap (\vec{v} + B_{n_0}) \neq \emptyset$.

Consider the two boxes

$$K_1 = B_{n_0} + (n - n_0, n - n_0, \ldots, n - n_0)$$
$$K_2 = B_{n_0} + (n_0 - n, n_0 - n, \ldots, n_0 - n).$$

Let $\vec{v}_1 \in S \cap K_1$, $\vec{v}_2 \in S \cap K_2$. As $|(\pi(\vec{v}_1) - \pi(\vec{v}_2)) - (\vec{v}_1 - \vec{v}_2)| < \delta|\vec{v}_1 - \vec{v}_2|$, on each coordinate $j = 1, \ldots, d$ we have

$$|(\pi(\vec{v}_1) - \pi(\vec{v}_1))_j - (\vec{v}_1 - \vec{v}_2)_j| \leq 2n\delta.$$

Now $(\vec{v}_1 - \vec{v}_2)_j \in [2n - 2n_0, 2n]$. But as both $\pi(\vec{v}_1)$ and $\pi(\vec{v}_2)$ are in B_n this can only happen if $\pi(\vec{v}_1)_j \in [n(1 - \varepsilon/4), n]$ and so

$$|\vec{v}_1 - \pi(\vec{v}_1)| < \frac{\varepsilon}{4}n.$$

Now for any other $\vec{v} \in S$,

$$|(\vec{v}_1 - \vec{v}) - (\pi(\vec{v}_1) - \pi(\vec{v}))| < \delta|\vec{v} - \vec{v}_1| < 2n\delta < \frac{\varepsilon}{8}n.$$

Thus

$$|\vec{v} - \pi(\vec{v})| < \varepsilon n. \qquad \square$$

Completion of the proof of Theorem 4.14. To see that m_K satisfies Axiom 5, choose $\delta < \varepsilon/4$ sufficiently small and N_0 sufficiently large so that if S_i is the δ-fixing set inside each A_i then:

(1) by Lemma 4.15, for each $\vec{v} \in S_i$,

$$|\pi(\vec{v}) - \vec{v}| < \frac{\varepsilon^2 n_i}{4d}; \text{ and}$$

(2) for $n_i' = [n_i(1 - \varepsilon/4d)]$ we have

$$\frac{|B_{n_i'}|}{|B_{n_i}|} > 1 - \varepsilon/2.$$

Let $S = \cup_i (S_i \cap B_{n_i'})$ and we compute

$$|S| \geq \sum_i (|S_i| - (1 - \varepsilon/2)|B_{n_i}|) > (1 - \varepsilon)|B_n|.$$

Now if $\vec{v}_1$, $\vec{v}_2 \in S$ and are in the same S_i, then

$$|(\vec{v}_1 - \vec{v}_2) - (\pi(\vec{v}_1) - \pi(\vec{v}_2))| < \delta|\vec{v}_1 - \vec{v}_2| < \varepsilon|\vec{v}_1 - \vec{v}_2|.$$

If they are in distinct S_i's, say $\vec{v}_1 \in S_i$ and $\vec{v}_2 \in S_j$ then

$$|\vec{v}_1 - \vec{v}_2| \geq \frac{\varepsilon}{4d}(n_i + n_j)$$

and as

$$|\pi(\vec{v}_1) - \vec{v}_1| \leq \varepsilon(\varepsilon/4d)n_i \quad \text{and} \quad |\pi(\vec{v}_2) - \vec{v}_2| \leq \varepsilon(\varepsilon/4d)n_j$$

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we conclude that

$$|(\vec{v}_1 - \vec{v}_2) - (\pi(\vec{v}_1) - \pi(\vec{v}_2))| \leq \varepsilon |\vec{v}_1 - \vec{v}_2|.$$

Hence S is an ε-fixing set for π.

For m_K, Axiom 6 can be seen to be a consequence of Axiom 5. Suppose $m_K(\pi) < \delta < 1/2$ and $K_1, \ldots, K_s$ are all translates of some B_{N_1}, whose union covers all but $\delta|B_n|$ of B_n, and $\pi(K_i) = K_i$. Let S be the δ-fixing set for π. Let $K_{i_1}, \ldots, K_{i_t}$ be a listing of those K_i that satisfy

$$|K_i \cap S| \geq \sqrt{\frac{\delta}{1-\delta}}|K_i|.$$

As

$$\sum_i |K_i \cap S| \geq \left(1 - \left(\frac{\delta}{1-\delta}\right)\right) \sum_i |K_i|$$

we must have

$$\sum_{j=1}^{t} |K_{i_j}| \geq \left(1 - \sqrt{\frac{\delta}{1-\delta}}\right)(1-\delta)|B_n| \geq (1 - 3\sqrt{\delta})|B_n|.$$

On each K_{i_j}, the set $S \cap K_{i_j}$ is a $2\sqrt{\delta}$-fixing set for π restricted to K_{i_j}, and so if $3\sqrt{\delta}$ is smaller than the 'delta' of Axiom 5, we can conclude that

$$m_K(\pi') < \varepsilon. \qquad \square$$

For each of the three examples we have discussed there is a corresponding 'equivalence theorem' saying that within a certain intrinsically defined class of transformations, m-entropy is a complete invariant of m-equivalence. For m_0 this is Dye's theorem [**D**], saying that any two ergodic actions of $\mathbb{Z}^d$ on nonatomic spaces are orbit equivalent. For m_1 this is Ornstein's isomorphism theorem [**OW1**] for Bernoulli actions of $\mathbb{Z}^d$. For m_K this is Hasfura's result that any two loosely-Bernoulli actions of $\mathbb{Z}^d$ of the same entropy are id-Kakutani equivalent [**H**]. In the sequel to this paper we will develop the necessary machinery and prove an m-equivalence theorem for an arbitrary size m. Specifically, we will define what it means for a $\mathbb{Z}^d$ process to be m-finitely determined and we will prove that any two m-finitely determined processes of equal m-entropy are m-equivalent. This will subsume all three of the examples discussed above. In [**K**] the author discusses five characterizations of a $\mathbb{Z}^d$ Bernoulli shift as finitely determined, extremal, weakly extremal, strongly extremal and very weakly Bernoulli. In the sequel to this paper, we will provide alternate characterizations to that of m finitely determined, as in [**K**], to provide further insight into the nature of these processes.

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