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Abstract

We study the $SL(2,\mathbb{C})$ character variety of a Seifert-fibered homology 3-sphere from the point of view of gauge theory. Namely, we introduce a class of perturbations of the $SL(2,\mathbb{C})$ Chern–Simons functional and prove a localization result: the perturbed critical points either approach a compact subset of the $SL(2,\mathbb{C})$ character variety or else 'escape to infinity'. Furthermore, the Euler characteristic and Poincaré polynomial of the stable locus of the character variety are obtained by suitably counting the localizing critical points. As an application, we obtain formulas for the Euler characteristic and Poincaré polynomial of the stable locus of the $SL(2,\mathbb{C})$ character variety of a Seifert-fibered homology 3-sphere. In particular, we prove that the Euler characteristic equals the Milnor number (divided by 4) of any weighted-homogeneous isolated complete intersection singularity whose link is the given 3-manifold.

1. Introduction

In recent years, new invariants of manifolds of dimension 3 and 4 have been introduced, motivated by $SL(2,\mathbb{C})$ generalizations of the anti-self-duality equation [VW94, KW07, Wit12, Hay15]. Abouzaid and Manolescu [AM20, CM19] have defined a sheaf-theoretic model $HP^*(Y)$ for the $SL(2,\mathbb{C})$ Floer cohomology of a closed oriented 3-manifold Y based on the theory of perverse sheaves and derived critical loci [Bus14, Joy15, PTV$^+$13]; and Tanaka and Thomas [TT20, TT17] have defined Vafa–Witten invariants of complex projective surfaces using virtual localization. However, it remains an open problem to reinterpret these invariants in differential-geometric terms. The main obstacle here is the non-compactness of the relevant gauge-theoretic moduli spaces, due to the phenomenon of energy concentration ('bubbling') along singular sets of codimension 2 studied by Taubes and other authors [Tau13b, Tau13a, Tau17, HW15, MSW$^+$16, WZ21].

In this paper, we consider the three-dimensional case. For a 3-manifold Y (always assumed closed, oriented and connected) its $SL(2,\mathbb{C})$ *character scheme* (informally, the character 'variety') is the affine scheme $\mathscr{M} = \mathscr{M}(Y)$ obtained from the scheme of representations $\rho : \pi_1(Y) \to SL(2,\mathbb{C})$ as the affine geometric invariant theory (GIT) quotient by the conjugation $SL(2,\mathbb{C})$-action:

$$\mathscr{M} = \mathrm{Hom}(\pi_1(Y), SL(2,\mathbb{C}))//SL(2,\mathbb{C}).$$

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A representation ρ is *polystable* (or completely reducible) if every ρ-invariant line $L \subset \mathbb{C}^2$ admits a ρ-invariant complement, and is *stable* (or irreducible) if it admits no ρ-invariant line at all. The (closed) points of $\mathscr{M}$ are in correspondence with the orbits of polystable representations, and we denote by $\mathscr{M}^* \subset \mathscr{M}$ the open subset of stable representations.

From the point of view of gauge theory, $\mathscr{M}^*$ arises as the critical locus of the $SL(2,\mathbb{C})$ Chern–Simons functional on a suitable space of $SL(2,\mathbb{C})$ connections. In this paper, we consider the case when Y is a *Seifert-fibered homology sphere* and interpret the Euler characteristic and Poincaré polynomials of $\mathscr{M}^*$ as a *gauge-theoretic count of critical points* (Theorems A and B). As an application, we derive formulas for these (Corollary A) and prove that the Euler characteristic of $\mathscr{M}^*$ equals one-fourth the *Milnor number* of any weighted-homogeneous isolated complete intersection singularity (ICIS) whose link is Y (Corollary B).

1.1 A perturbation of the Chern–Simons functional

Let Y be a 3-manifold (always assumed closed, oriented and connected), equipped with a fixed $SL(2,\mathbb{C})$ bundle $E \to Y$ (necessarily trivial, since $\dim Y = 3$). The $SL(2,\mathbb{C})$ *Chern–Simons functional* is a natural function on the space of $SL(2,\mathbb{C})$ connections on E, whose value on a connection $\mathbb{A}$ is denoted by $CS(\mathbb{A}) \in \mathbb{C}$ (cf. Definition 2.7). The critical points of CS are the *flat* connections: $F_{\mathbb{A}} = 0$. We would like to extract a numerical invariant of Y by 'counting' flat $SL(2,\mathbb{C})$ connections—much as in the $SU(2)$ case [Tau90, Flo88, Don02]. However, the lack of a good compactness theory makes this more difficult. Instead, we propose to count (certain) critical points of a *perturbation* of the $SL(2,\mathbb{C})$ Chern–Simons functional, which we now describe. We shall later use this perturbation to obtain well-defined counts of $SL(2,\mathbb{C})$ connections on Seifert-fibered homology spheres (Theorems A and B).

Our starting point is to fix a Riemannian metric g on Y and a reduction of the structure group of E to $SU(2) \subset SL(2,\mathbb{C})$. An $SL(2,\mathbb{C})$ connection $\mathbb{A}$ is then uniquely expressed as $\mathbb{A} = A + i\Phi$ where A is an $SU(2)$ connection and Φ is a 1-form valued in the bundle $\mathfrak{su}(E)$ of skew-adjoint endomorphisms of E. Fixing an angle $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, we can consider the function $S_\theta(\mathbb{A}) = \mathrm{Re}(e^{-i\theta} CS(\mathbb{A})) \in \mathbb{R}$ whose critical points are also the flat connections.

Definition 1.1. For a parameter $\varepsilon \in \mathbb{R}$ we consider the *perturbation* of S_θ whose value at an $SL(2,\mathbb{C})$ connection $\mathbb{A} = A + i\Phi$ is

$$S^\varepsilon_\theta(\mathbb{A}) := S_\theta(\mathbb{A}) + \varepsilon \|\Phi\|^2_{L^2} + \varepsilon^2 f(\underline{\sigma}(\mathbb{A}))$$

where:

(1) $\|\Phi\|^2_{L^2} = -\int_Y \mathrm{Tr}(\Phi \wedge *\Phi)$ is the L^2 *norm squared* version of Φ;

(2) for some integer $N > 0$, $f : \mathbb{C}^N \to \mathbb{R}$ is a smooth function and $\underline{\sigma}(\mathbb{A}) = (\sigma_1(\mathbb{A}), \dots, \sigma_N(\mathbb{A}))$ is a function where each component $\sigma_i(\mathbb{A})$ is an $SL(2,\mathbb{C})$ *holonomy perturbation* (or 'regularized' Wilson loop operator) associated with a smooth solid torus embedding $q_i : S^1 \times D^2 \hookrightarrow Y$ and a non-negative 2-form μ_i on D^2 with integral 1 which vanishes near the boundary,

$$\sigma_i(\mathbb{A}) = \int_{D^2} \mathrm{Tr}(\mathrm{Hol}_{\mathbb{A}}(q(\cdot, z))) \cdot \mu_i(z).$$

To properly make sense of this perturbation we now describe the appropriate space of connections on which S^ε_θ acts. In our context, with a fixed Riemannian metric and a reduction to $SU(2)$, we say that $\mathbb{A}$ is *polystable* (or completely reducible) if $d^*_A \Phi = 0$, where d^*_A is the L^2-adjoint of the exterior covariant derivative, and that $\mathbb{A}$ is *stable* (or irreducible) if in addition its stabilizer $\mathscr{G}_{\mathbb{A}}$ under the action of the $SU(2)$ gauge group $\mathscr{G}$ is finite (it must then necessarily be

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$\{\pm I\}$, the center of $SU(2)$). It will be convenient to work with connections in the Sobolev class L^2_k and gauge transformations in L^2_{k+1} (where $k \geq 2$ shall suffice for our purposes). The space $\mathscr{B}^*$ of stable $\mathscr{G}$-orbits of the action can then be regarded as an 'infinite-dimensional' Kähler quotient (of the space of all $SL(2,\mathbb{C})$ connections by the $\mathscr{G}$-action) and is naturally a Hilbert Kähler manifold (cf. Corollary 2.6):

$$\mathscr{B}^* = \{\mathbb{A} = A + i\Phi \mid d_A^*\Phi = 0 \text{ and } \mathscr{G}_{\mathbb{A}} = \{\pm I\}\}/\mathscr{G}.$$

The function S^ε_θ has periods under the action of gauge transformation (except when $\theta = \pm\pi/2$ modulo 2π). Nevertheless, its derivative dS^ε_θ makes sense as a smooth 1-form on $\mathscr{B}^*$. The critical locus of $S_\theta = S^0_\theta$ on $\mathscr{B}^*$ is the moduli space of $\mathscr{G}$-orbits of *stable flat connections*

$$F_A - \Phi \wedge \Phi = 0, \quad d_A\Phi = 0, \quad d_A^*\Phi = 0,$$

which by a fundamental result of Donaldson and Corlette [Don87b, Cor88] is identified homeomorphically with the stable locus $\mathscr{M}^*$ of the character variety (with the analytic topology). Subsequently, we use the notation $\mathscr{M}^*$ to refer both to the critical locus of S_θ on $\mathscr{B}^*$ and to the stable locus of the $SL(2,\mathbb{C})$ the character variety of Y, and the notation $\mathscr{M}^{*,\varepsilon}_\theta$ for the critical locus of the perturbation S^ε_θ.

When it comes to 'counting', a crucial property is that the *Hessian* of S^ε_θ at an orbit $[\mathbb{A}]$ defines a self-adjoint Fredholm operator on the (L^2 completion of the) tangent space to $\mathscr{B}^*$ at $[\mathbb{A}]$. In addition, $\mathscr{D}^\varepsilon_{[\mathbb{A}],\theta}$ has a complete L^2-orthonormal system of eigenvectors, and its eigenvalues form a discrete set which is unbounded both from above and below (see Theorem 2.13). We say that $[\mathbb{A}] \in \mathscr{M}^{*,\varepsilon}_\theta$ is *non-degenerate* if $\mathscr{D}^\varepsilon_{[\mathbb{A}],\theta}$ is invertible; thus, $[\mathbb{A}]$ is an *isolated* point of $\mathscr{M}^{*,\varepsilon}_\theta$ by standard results in Fredholm theory, and we can try to count $[\mathbb{A}]$ with a sign defined using the spectral flow construction.

When $\theta = \pi/2$ and $f = 0$ in Definition 1.1, the moduli spaces $\mathscr{M}^{*,\varepsilon}_\theta$ have been studied by Taubes [Tau13b, Tau17, Tau24b, Tau24a]. They can be non-compact (even when $\varepsilon \neq 0$), and this is due to a phenomenon of concentration of energy in codimension 2. However, it is reasonable to expect that *their compactness theory should simplify for small* $\varepsilon \neq 0$. The heuristic for this is that $\mathscr{M}^{*,\varepsilon}_\theta$ should be closely related to the critical locus of $\|\Phi\|^2_{L^2} : \mathscr{M}^* \to \mathbb{R}$ (appropriately interpreted, since in general $\mathscr{M}^*$ has singularities), and when Y is a homology 3-sphere the latter function is *proper* [Tau13b]. (For instance, if $\mathscr{M}^*$ were non-singular and $\|\Phi\|^2_{L^2}$ happened to be an algebraic function on $\mathscr{M}^*$, then its critical locus would be compact.)

1.2 Localization results for Seifert-fibered spaces

We would like to understand the behavior of the moduli spaces $\mathscr{M}^{*,\varepsilon}_\theta$ for a small non-zero real parameter ε.

In this paper, we focus on the case when Y is a *Seifert-fibered* 3-manifold: there is an orbifold principal S^1 bundle $S^1 \curvearrowright Y \xrightarrow{\pi} C$ over a closed oriented 2-orbifold C. By a *Seifert* metric g on the Seifert-fibered 3-manifold (Y,π) we shall mean a Riemannian metric on Y obtained as follows: choose an orbifold Riemannian metric g_C on C and a connection 1-form $i\eta \in \Omega^1(Y, i\mathbb{R})$ on the principal S^1 bundle π whose curvature is a constant multiple of the Riemannian volume form ω_C of (C, g_C) (i.e., $d\eta = \text{const} \cdot \pi^*\omega_C$); then $g = \eta \otimes \eta + \pi^* g_C$.

THEOREM A. *Let (Y,π,g) be a Seifert-fibered homology 3-sphere equipped with a Seifert metric g. Then there exist a compact subset $\mathscr{Z} = \mathscr{Z}(g) \subset \mathscr{M}^*$, a collection $q_1, \ldots, q_N : S^1 \times D^2 \hookrightarrow Y$ of disjoint smooth embeddings, and a smooth function $f : \mathbb{C}^N \to \mathbb{R}$ such that the following statement holds. Given any open set $\mathscr{U} \subset \mathscr{B}^*$ and a constant $C > 0$ such that:*

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– $\mathscr{Z} \subset \mathscr{U} \subset \{\|\Phi\|_{L^\infty} < C\}$;
– *the closure of* $\mathscr{U}$ *does not contain the trivial connection;*

there exists a constant $\varepsilon_0 = \varepsilon_0(g, q_i, f, \mathscr{U}, C) > 0$ *such that for* $0 < |\varepsilon| \leq \varepsilon_0$ *and* $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ *we have the following.*

(1) *If* $[\mathbb{A}] \in \mathscr{M}_\theta^{*,\varepsilon}$, *then either* $[(A, \Phi)] \in \mathscr{U}$ *or* $\|\Phi\|_{L^\infty} > C$. *Furthermore, points in* $\mathscr{M}_\theta^{*,\varepsilon} \cap \mathscr{U}$ *are non-degenerate critical points of* S_θ^ε *and comprise a finite set.*
(2) *The count of points of* $\mathscr{M}_\theta^{*,\varepsilon}$ *in* $\mathscr{U}$ *with signs given by the mod 2 spectral flow* $\mathbf{sf}_2(\mathscr{D}^t_{[\mathbb{A}],\theta}) \in \mathbb{Z}/2$ *of the path of Hessians* $\mathscr{D}^t_{[\mathbb{A}],\theta}$ *with* t *going from* $t = \varepsilon$ *to* $t = 0$ *recovers the Euler characteristic of the moduli space of stable flat* $SL(2, \mathbb{C})$ *connections on* Y*:*

$$\sum_{[\mathbb{A}] \in \mathscr{M}_\theta^{*,\varepsilon} \cap \mathscr{U}} (-1)^{\mathbf{sf}_2(\mathscr{D}^t_{[\mathbb{A}],\theta})} = \chi(\mathscr{M}^*).$$

Thus, as $\varepsilon \neq 0$ becomes small, the points on $\mathscr{M}_\theta^{*,\varepsilon}$ either tend to *localize* around the compact set $\mathscr{Z}$ or otherwise lie 'far away' as witnessed by the norm $\|\Phi\|_{L^\infty}$. Perturbed solutions of the second kind have recently been studied by Taubes [Tau24b, Tau24a] (without the ε^2 perturbation in Definition 1.1, i.e., setting $f = 0$).

Theorem A should be compared with Taubes' theorem [Tau90] describing the $SU(2)$ *Casson invariant* of a homology sphere as a count of critical points of a holonomy perturbation of the $SU(2)$ Chern–Simons functional. Taubes's theorem instead uses the spectral flow to the trivial connection. In contrast, a special feature of our situation is that the mod 2 spectral flow of the 'small' path $\mathscr{D}^t_{[\mathbb{A}],\theta}$ is well defined: because at $t = \varepsilon$ the operator is invertible, and at $t = 0$ the operator has *complex-linear* kernel since S_θ is the real part of the holomorphic functional CS.

1.3 The exact case

If $\theta = \pm\pi/2$ modulo 2π, then $S_\theta = \pm\mathrm{Im}(CS)$ is an *exact* functional: S_θ has no periods under the action of gauge transformations. Then S_θ defines an honest real-valued smooth function on $\mathscr{B}^*$ (as opposed to just a closed 1-form), and admits a simpler expression:

$$S_{\pi/2}(A + i\Phi) = (\mathrm{Im}CS)(A + i\Phi) = \int_Y \mathrm{Tr}(\Phi \wedge (F_A - \tfrac{1}{3}\Phi \wedge \Phi)).$$

In the exact setting, the family of Hessians $\mathscr{D}_\theta^\varepsilon$ has vanishing spectral flow along loops in $\mathscr{B}^*$, and we can assign an *integral* 'Morse index' to critical points to obtain a refined count. Below, we use $\mathcal{P}_T(M)$ (respectively, $\mathcal{P}^c_T(M)$) to denote the Poincaré polynomial (respectively, compactly supported Poincaré polynomial) of a given smooth manifold M.

Theorem B. *Suppose that* $\theta = \pm\pi/2$ *modulo* 2π. *Let* (Y, π, g) *be a Seifert-fibered homology 3-sphere equipped with a Seifert metric* g. *Then there exists a compact subset* $\mathscr{Z} = \mathscr{Z}(g) \subset \mathscr{M}^*$, *a collection* $q_1, \ldots, q_N : S^1 \times D^2 \hookrightarrow Y$ *of disjoint smooth embeddings, and a smooth function* $f : \mathbb{C}^N \to \mathbb{R}$ *such that the following holds. Let* $\mathscr{Z} = \cup_i \mathscr{Z}_i$ *denote the decomposition into connected components. Given an open subset* $\mathscr{U} \subset \mathscr{B}^*$ *which is a disjoint union of open sets* $\mathscr{U} = \sqcup_i \mathscr{U}_i$ *and a constant* $C > 0$, *such that for every* i *we have that:*

– $\mathscr{Z}_i \subset \mathscr{U}_i \subset \{\|\Phi\|_{L^\infty} < C\}$;
– *the closure of* $\mathscr{U}_i$ *does not contain the trivial connection;*

there exists a constant $\varepsilon_0 = \varepsilon_0(g, q_i, f, \mathscr{U}, C) > 0$ *such that for* $0 < |\varepsilon| \leq \varepsilon_0$ *we have the following.*

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(1) *If* $[\mathbb{A}] \in \mathscr{M}_\theta^{*,\varepsilon}$, *then either* $[(A,\Phi)] \in \mathscr{U}$ *or* $\|\Phi\|_{L^\infty} > C$. *Furthermore, points in* $\mathscr{M}_\theta^{*,\varepsilon} \cap \mathscr{U}$ *are non-degenerate critical points of* S_θ^ε *and comprise a finite set.*

(2) *For each* $[\mathbb{A}] \in \mathscr{M}_\theta^{*,\varepsilon} \cap \mathscr{U}_i$ *there exists a path* $[\mathbb{A}_t]$ *with* t *going from* $t=\varepsilon$ *to* $t=0$, *such that* $[\mathbb{A}_\varepsilon] = [\mathbb{A}]$, $[\mathbb{A}_0] \in \mathscr{Z}_i$, *and* $[\mathbb{A}_t] \in \mathscr{U}_i$ *for all* t; *fix one such path for each* $[\mathbb{A}] \in \mathscr{M}_\theta^{*,\varepsilon} \cap \mathscr{U}$. *Let* $\delta = \delta(\varepsilon) > 0$ *be a constant small enough so that the Hessian* $\mathscr{D}_\theta$ *at every point of* $\mathscr{Z}$ *has spectral gap greater than* δ, *as does the perturbed Hessian* $\mathscr{D}_\theta^\varepsilon$ *at every point in* $\mathscr{M}_\theta^{*,\varepsilon} \cap \mathscr{U}$. *For each* $[\mathbb{A}] \in \mathscr{M}_\theta^{*,\varepsilon} \cap \mathscr{U}$ *let* $\mathbf{sf}(\mathscr{D}^t_{[\mathbb{A}_t],\theta}) \in \mathbb{Z}$ *denote the spectral flow of the path of operators* $\mathscr{D}^t_{[\mathbb{A}_t]} + \delta$ *with* t *going from* $t=\varepsilon$ *to* $t=0$. *Then*

$$\sum_{[\mathbb{A}] \in \mathrm{Crit}(S_\varepsilon) \cap \mathscr{U}} T^{\mathbf{sf}(\mathscr{D}^t_{[\mathbb{A}_t],\theta}+\delta)} = \begin{cases} \mathcal{P}_T(\mathscr{M}^*(Y)) & \text{if } \varepsilon > 0, \\ \mathcal{P}^c_T(\mathscr{M}^*(Y)) & \text{if } \varepsilon < 0. \end{cases}$$

The sheaf-theoretic $SL(2,\mathbb{C})$ Floer cohomology $HP^*(Y)$ of Abouzaid and Manolescu [AM20] is an invariant of a 3-manifold taking the form of a finitely generated abelian group. The *sheaf-theoretic* $SL(2,\mathbb{C})$ *Casson invariant* is defined as its Euler characteristic

$$\lambda^P(Y) = \sum_{n \in \mathbb{Z}} (-1)^n \mathrm{rank}_{\mathbb{Z}} HP^n(Y) \in \mathbb{Z}.$$

For a Seifert-fibered homology sphere Y one has $\lambda^P(Y) = \chi(\mathscr{M}^*(Y))$ (see [AM20, Formula 35]), and thus Theorem A provides a gauge-theoretic interpretation of $\lambda^P(Y)$ in this case. By a variant of Theorem B, we can also interpret the Poincaré polynomial of $HP^*(Y)$ using gauge theory (see Corollary 5.2).

The set $\mathscr{Z}$ appearing in Theorems A and B has the following geometric significance. For a Seifert-fibered 3-manifold, the moduli space $\mathscr{M}^*(Y) = \mathrm{Crit}(S_\theta)$ is a smooth manifold: more precisely, it is Morse–Bott non-degenerate (Theorem 3.10). Then $\mathscr{Z}$ is given by the critical locus of the restricted function $\|\Phi\|^2_{L^2} : \mathscr{M}^*(Y) \to \mathbb{R}$, and it is also Morse–Bott non-degenerate. A crucial fact that enables the above localization phenomenon is that $\mathscr{Z}$ is compact (Proposition 4.20). This is verified by a direct calculation, based on an argument due to Hitchin [Hit87, Proposition 7.1].

In order to achieve non-degeneracy in Theorem A(1) we arrange our choices as follows. First, we show that the q_i and μ_i can be chosen so that the function $\underline{\sigma} : \mathscr{B}^* \to \mathbb{C}^N$ induces a smooth *embedding* of $\mathscr{Z}$ into $\mathbb{C}^N$ (Proposition 4.12). The function $f : \mathbb{C}^N \to \mathbb{R}$ is then chosen so as to restrict 'generically' onto $\mathscr{Z} \subset \mathbb{C}^N$ (see Theorem 4.17). In Theorem B(2) we must choose perturbations with additional care in order to achieve equality (in general, only the inequality $\geq$ holds). For this, we make use of the fact that $\|\Phi\|^2_{L^2} : \mathscr{M}^* \to \mathbb{R}$ is a *perfect* Morse–Bott function and that $\mathscr{Z}$ admits a *perfect* Morse function.

1.4 Dimensional reduction results

Our next result describes $\mathscr{M}^*(Y)$ for a Seifert-fibered 3-manifold $Y \to C$ in terms of algebro-geometric data on the base 2-orbifold C.

THEOREM C. *Let* (Y,π,g) *be a Seifert-fibered 3-manifold equipped with a Seifert metric* $g = \eta \otimes \eta + \pi^* g_C$ *(in particular, the 2-orbifold* C *and the orbifold line bundle* $N := Y \times_{S^1} \mathbb{C}$ *are equipped with holomorphic structures induced by* g_C *and* $i\eta$, *respectively). Then the moduli space of stable flat* $SL(2,\mathbb{C})$ *connections* $\mathscr{M}^*(Y)$ *is a smooth manifold, and there is a natural diffeomorphism*

$$\mathscr{M}^*(Y) \cong \left(\bigsqcup_{E_0, \det E_0 = \mathscr{O}} \mathscr{M}^{*,\mathrm{Higgs}}_{E_0}(C) \right) \sqcup \left(\bigsqcup_{E_0, \det E_0 = N} \mathscr{M}^{*,\mathrm{Higgs}}_{E_0}(C) \right),$$

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where $\mathscr{M}^{*,\mathrm{Higgs}}_{E_0}(C)$ denotes the moduli space of stable Higgs bundles on the orbifold Riemann surface C in the (topological) rank-2 orbifold vector bundle $E_0 \to C$ with fixed determinant (given either by the trivial line bundle $\mathscr{O}$ or by N).

For a more precise version of this result see Theorem 3.10. The analog of Theorem C for the moduli space of irreducible $SU(2)$ flat connections on Y was proved by Furuta and Steer [FS92, Theorem 4.1]. The moduli spaces $\mathscr{M}^{*,\mathrm{Higgs}}_{E_0}(C)$ have been studied in [Hit87, NS95, BY96, BGR20]. (In the literature, a Higgs bundle over an orbifold Riemann surface C is more commonly called a *parabolic* Higgs bundle.) When Y is an integral homology sphere, the moduli spaces $\mathscr{M}^{*,\mathrm{Higgs}}_{E_0}(C)$ appearing in Theorem C are connected, simply connected, complete hyperkähler manifolds (of varying dimensions).

1.5 Formulas for Poincaré polynomial and Euler characteristic

For a Seifert-fibered homology sphere Y, the description of the moduli space $\mathscr{M}^*(Y)$ provided by Theorem C plays a key role in establishing that $\|\Phi\|^2_{L^2} : \mathscr{M}^*(Y) \to \mathbb{R}$ is a perfect Morse–Bott function with compact critical locus $\mathscr{Z}$, and in the explicit description of $\mathscr{Z}$ (see Proposition 4.20). In turn, this yields a relationship between the Poincaré polynomials of $\mathscr{M}^*(Y)$ and $\mathscr{Z}$. This is most simply expressed at the level of Euler characteristics, $\chi(\mathscr{M}^*(Y)) = \chi(\mathscr{Z})$, because the Morse–Bott indices happen to be even integers. Thus, we can use this to derive new formulas for the Euler characteristic and Poincaré polynomial of $\mathscr{M}^*(Y)$.

To describe these formulas, fix a Seifert fibration $Y \to C$ where C is a 2-sphere with n orbifold points with isotropy orders $\alpha_1, \dots, \alpha_n$ (where $\alpha_i \in \mathbb{Z}_{\geq 2}$ for each i). We introduce the following notation: given an integer vector $\underline{e} = (e, \beta_1, \dots, \beta_n)$ with $0 \leq \beta_i < \alpha_i$, define its *orbifold degree* by

$$\deg(\underline{e}) = e + \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i} \in \mathbb{Q}.$$

In particular, the *orbifold Euler characteristic* $\chi(C) \in \mathbb{Q}$ is given by minus the orbifold degree of the 'canonical' vector $K = (-2, \alpha_1 - 1, \dots, \alpha_n - 1)$, that is, $\chi(C) = -\deg(K)$.

Corollary A. *Let Y be a Seifert-fibered integral homology 3-sphere over the orbifold $C = S^2(\alpha_1, \dots, \alpha_n)$. Then the following assertions hold.*

(1) *The Poincaré polynomials of the moduli space of stable flat $SL(2,\mathbb{C})$ connections $\mathscr{M}^*(Y)$ and the moduli space of irreducible $SU(2)$ flat connections $\mathscr{N}^*(Y)$ are related by the following formula (where $\{x\} \in [0,1)$ denotes the fractional part of a non-negative number x):*

$$\mathcal{P}_T(\mathscr{M}^*(Y)) = \mathcal{P}_T(\mathscr{N}^*(Y)) + \sum_{\underline{e}} (T^2)^{-\chi(C) - \deg(\underline{e}) - 1 + \sum_{i=1}^{n} \{(\beta_i+1)/\alpha_i\}} \cdot \mathcal{P}_T(\mathbb{C}P^e),$$

where $\underline{e} = (e, \beta_1, \dots, \beta_n)$ ranges over all vectors of non-negative integers such that

$$\beta_i < \alpha_i \quad \text{and} \quad \deg(\underline{e}) < -\chi(C).$$

(2) *In particular (noting that the exponents in T^2 are non-negative integers), the Euler characteristics are related by*

$$\chi(\mathscr{M}^*(Y)) = \chi(\mathscr{N}^*(Y)) + \sum_{\underline{e}} \chi(\mathbb{C}P^e).$$

For a Seifert-fibered integral homology 3-sphere, the Poincaré polynomial of the moduli space of irreducible $SU(2)$ flat connections $\mathscr{N}^*(Y)$ was calculated by Bauer [Bau91] and by Furuta and Steer [FS92]. Orienting Y as the boundary of an isolated surface singularity, Fintushel and

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Stern [FS90] proved that the $SU(2)$ *Casson invariant* $\lambda(Y)$ (with its standard normalization, $\lambda(\Sigma(2,3,5)) = -1$) is given by $-(1/2)\chi(\mathscr{N}^*(Y))$. Several different formulas for $\lambda(Y)$ are given in [FS90, NW90, FMS90, KK91, Sav02].

We also note that, using the results in [BY96, BC06], one can obtain other formulas for $\chi(\mathscr{M}^*(Y))$ which are different from those that we obtain in Corollary A.

1.6 The Milnor number and $SL(2,\mathbb{C})$ representations

The formula from Corollary A(2) can be used to give a singularity-theoretic interpretation of the Euler characteristic $\chi(\mathscr{M}^*(Y))$ of the stable locus of the $SL(2,\mathbb{C})$ character variety of a Seifert-fibered integral homology 3-sphere.

Let $(X,0) \subset \mathbb{C}^{2+k}$ be an ICIS of complex dimension 2, where $0 \in X$ is the singular point. The singularity $(X,0)$ is given by the vanishing locus of a polynomial map $f = (f_1, \dots, f_k) : (\mathbb{C}^{2+k}, 0) \to (\mathbb{C}^k, 0)$. The *Milnor fiber* of $(X,0)$ (see [Mil68, Ham71, Tra77]) is the compact 4-manifold with boundary given by

$$M = f^{-1}(z) \cap B_\varepsilon(\mathbb{C}^{2+k}, 0),$$

where $0 < \varepsilon \ll 1$, and z is a 'generic' point in $B_\delta(\mathbb{C}^k, 0)$ with $0 < \delta \ll \varepsilon$. The boundary of M is the *link* of the singularity $(X,0)$. A fundamental analytic invariant of the singularity $(X,0)$ is its *Milnor number* $\mu(X,0)$, which is defined as the second Betti number of the Milnor fiber: $\mu(X,0) = b_2(M)$.

An important class of singularities are the *weighted-homogeneous* ICISs, where each f_i is a weighted-homogeneous polynomial [Mil68] (e.g., these include the Brieskorn–Pham complete intersections). For such singularities, it is easily seen that the link Y is a Seifert-fibered 3-manifold [NR78]. By our next result, the analytic invariant $\mu(X,0)$ can be interpreted as a 'count' of $SL(2,\mathbb{C})$ representations of $\pi_1(Y)$.

COROLLARY B. *Let $(X,0)$ be a two-dimensional weighted-homogeneous ICIS, whose link Y is an integral homology 3-sphere. Then the Euler characteristic of the stable locus $\mathscr{M}^*(Y)$ in the $SL(2,\mathbb{C})$ character variety of Y is one-fourth the Milnor number of $(X,0)$:*

$$\chi(\mathscr{M}^*(Y)) = \frac{\mu(X,0)}{4}.$$

For the Brieskorn–Pham isolated hypersurface singularity $X = \{x^p + y^q + z^r = 0\}$, where p, q, r are pairwise coprime positive integers, the above identity was established by Boden and Curtis [BC06]. For this singularity, the link Y is the Brieskorn homology sphere $Y = \Sigma(p,q,r)$, and it is shown in [BC06] that the character variety $\mathscr{M}^*(Y)$ is finite and comprises $(p-1)(q-1)(r-1)/4$ elements, which gives one-fourth the Milnor number of $(X,0)$.

We obtain Corollary B as follows. First, we recall the well-known *Casson invariant conjecture*.

CONJECTURE 1.2 (Neumann and Wahl [NW90]). *Let $(X,0)$ be a two-dimensional ICIS, whose link Y is an integral homology 3-sphere. Then the $SU(2)$ Casson invariant $\lambda(Y)$ is one-eighth the signature of the Milnor fiber:*

$$\lambda(Y) = \frac{\sigma(M)}{8}.$$

The Casson invariant conjecture is known to hold for weighted-homogeneous ICISs [FS90, NW90] and singularities of splice type [NO09], but remains open in general.

Corollary B can be deduced from Corollary A(2) as follows. The term $\chi(\mathscr{N}^*(Y))$ can be rewritten as $\chi(\mathscr{N}^*(Y)) = -2\lambda(Y) = -\sigma(M)/4$ (see [FS90, NW90]). The key point (which we

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explain later) is to identify the second term $\sum_e \chi(\mathbb{C}P^e)$ as the *geometric genus* $p_g(X,0)$ of the singularity $(X,0)$, which is the content of a formula due to Pinkham [Pin77] and Dolgachev [Dol75] (also attributed to Demazure). In turn, there is the identity $b^+(M) = 2p_g(X,0)$ due to Durfee and Steenbrink [Dur78, Ste83]. Thus, Corollary A(2) yields the required identity:

$$\chi(\mathscr{M}^*(Y)) = -\frac{\sigma(M)}{4} + \frac{b^+(M)}{2} = \frac{\mu(X,0)}{4}.$$

The Euler characteristic $\chi(\mathscr{M}^*(Y))$ agrees with the sheaf-theoretic $SL(2,\mathbb{C})$ Casson invariant $\lambda^P(Y)$ for a Seifert-fibered homology sphere [AM20, Formula 35]. Thus, Corollary B provides evidence towards the following $SL(2,\mathbb{C})$ analog of the Casson invariant conjecture.

Conjecture 1.3. *Let $(X,0)$ be a two-dimensional ICIS, whose link Y is an integral homology 3-sphere. Then the sheaf-theoretic $SL(2,\mathbb{C})$ Casson invariant of Y is one-fourth the Milnor number of $(X,0)$:*

$$\lambda^P(Y) = \frac{\mu(X,0)}{4}.$$

Remarkably, Conjecture 1.3 implies that the Milnor number $\mu(X,0)$ of a two-dimensional ICIS (which is an *analytic* invariant of the singularity) should *only depend on its link Y* (or, equivalently, the *homeomorphism* type of the germ $(X,0)$). This is consistent with the prediction from the Casson invariant conjecture that $\sigma(M)$ only depends on Y, thanks to formulas of Laufer [Lau77] and Durfee and Steenbrink [Dur78, Ste83] which combined show that the quantity $2\mu(X,0) + 3\sigma(M)$ only depends on Y.

Remark 1.4. After completing this work, the author was informed that in the case when $(X,0)$ is a Brieskorn–Pham complete intersection the identity $\chi(\mathscr{M}^*(Y)) = \mu(X,0)/4$ was also established in [GKS25, Theorem 5.3] by an essentially different method. As explained therein, in that case the identity can be deduced instead by combining Hamm's calculation of the spectrum of the monodromy of Brieskorn–Pham complete intersections [Ham72] with a different formula for $\chi(\mathscr{M}^*(Y))$ that follows from the work of Boden and Yokogawa [BY96, BC06].

1.7 Outline and comments

Section 2 introduces the Hilbert manifold $\mathscr{B}^*$ of orbits of stable $SL(2,\mathbb{C})$ connections and discusses the deformations of flat stable $SL(2,\mathbb{C})$ connections on a (closed, oriented and connected) 3-manifold based on this model of the configuration space. The main result we establish is the Fredholm property of the Hessians (Theorem 2.13).

Section 3 establishes the dimensional reduction theorem 3.10 (which implies Theorem C). The crucial results here are the vanishing results from Corollary 3.5 and Proposition 3.6, which are obtained by establishing suitable Bochner–Weitzenböck type formulas. In particular, from Theorem 3.10 we also deduce that $\mathscr{M}^*(Y) = \mathrm{Crit} S_\theta \subset \mathscr{B}^*$ is a Morse–Bott critical locus.

Section 4 studies the perturbed critical loci $\mathscr{M}_\theta^{*,\varepsilon}(Y)$ for a Seifert-fibered 3-manifold and proves Theorems A and B.

The proof is modeled on a finite-dimensional perturbation argument for Morse–Bott functions (Theorem 4.7) which should be of independent interest. In order to apply this argument in our 'infinite-dimensional' context several additional ingredients are needed: the existence of finite-dimensional 'Kuranishi' models for S_θ (Corollary 2.16); the Morse–Bott non-degeneracy of the critical manifolds of S_θ and of $\|\Phi\|_{L^2}^2 : \mathscr{M}^*(Y) \to \mathbb{R}$ (Theorem 3.10 and Proposition 4.20); a method for achieving non-degeneracy of the perturbed critical points, for which we employ the $SL(2,\mathbb{C}$ holonomy perturbations (the crucial transversality property is Proposition 4.12);

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and, finally, we need to ensure that basic linear analysis and compactness properties of the unperturbed moduli spaces go through in the perturbed setting, for which we need to establish various bounds on the $SL(2,\mathbb{C})$ holonomy perturbations and their derivatives (Proposition 4.9).

With these ingredients in place, we establish Theorems A and B by an extension of the finite-dimensional argument with several modifications.

Finally, § 5 contains the proofs of the formulas in Corollaries A and B, and the gauge-theoretic interpretation of the sheaf-theoretic $SL(2,\mathbb{C})$ Floer cohomology $HP^*(Y)$ for Seifert-fibered homology spheres.

2. Stable flat connections on 3-manifolds

This section sets up a framework for studying the deformations of flat $SL(2,\mathbb{C})$ connections over a closed oriented 3-manifold. The flat connections are the critical points of the Chern–Simons functional on the space of all connections and for our purposes, counting flat connections with gauge-theoretic techniques, we must make sense of the Hessian as a formally self-adjoint Fredholm operator. (This is well understood for compact structure groups, such as $SU(2)$; see, for example, [Tau90, Flo88, Don02].) In order to achieve this we work with a suitable space of connections obtained as an infinite-dimensional Kähler (symplectic) quotient of the space of connections.

2.1 Setup

We describe the basic setup that we use throughout this paper. We fix the following data:

(1) a closed, oriented, Riemannian 3-manifold (Y, g) (in fact, the 'three-dimensional' hypothesis will not play a role until § 2.3);

(2) a $U(2)$ bundle $E \to Y$, equipped with a $U(1)$ connection λ on its determinant line bundle $\Lambda = \Lambda^2 E$ (informally, we shall refer to E as a 'bundle with fixed determinant').

We often abuse notation by treating E sometimes as a hermitian vector bundle of rank 2, and at other times as a principal bundle with structure group $U(2)$. It should be clear from the context which viewpoint we are using.

2.1.1 *Spaces of connections and Sobolev completions.* We let $\mathscr{A}_E$ denote the space of $U(2)$ connections A on E which induce the fixed connection λ on its determinant Λ. Informally, we refer to A as a 'unitary' or $U(2)$ connection with 'fixed determinant'.

What we are really interested in is the space $\mathscr{A}_E^c$ of $GL(2,\mathbb{C})$ connections $\mathbb{A}$ on E inducing the fixed connection λ on its determinant Λ. Using the unitary structure on E, such a connection $\mathbb{A}$ can be decomposed as

$$\mathbb{A} = A + i\Phi,$$

where $A \in \mathscr{A}_E$ is a unitary connection with fixed determinant and $\Phi \in \Omega^1(Y, \mathfrak{g}_E)$ is a $\mathfrak{g}_E$-valued 1-form on Y. Here $\mathfrak{g}_E = \mathfrak{su}(E)$ is the *adjoint bundle* consisting of traceless, skew-adjoint endomorphisms of E. Thus, the space of $GL(2,\mathbb{C})$ connections with fixed determinant is $\mathscr{A}_E^c = \mathscr{A}_E \times \Omega^1(Y, \mathfrak{g}_E)$, and we typically refer to the connection $\mathbb{A}$ by its corresponding pair (A, Φ). Informally, we shall refer to $\mathbb{A}$ as a 'complex' or $GL(2,\mathbb{C})$ connection on E with 'fixed determinant'.

Let $\mathscr{G}_E$ be the group of $U(2)$ bundle automorphisms of E which induce the identity on its determinant Λ, and $\mathscr{G}_E^c$ be the group of $GL(2,\mathbb{C})$ bundle automorphisms of E which induce

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the identity on Λ (again, we refer to these as 'unitary' or 'complex' gauge transformations with 'fixed determinant'). The groups $\mathscr{G}_E$ and $\mathscr{G}_E^c$ act on $\mathscr{A}_E$ and $\mathscr{A}_E^c$, respectively, in a natural way (our convention is that they act on the left): for example, a unitary gauge transformation u with fixed determinant acts on the complex connection (A, Φ) by

$$u \cdot (A, \Phi) = (A - d_A u \cdot u^{-1}, u\Phi u^{-1}).$$

For later purposes, it will be convenient to make $\mathscr{A}_E, \mathscr{A}_E^c$ and $\mathscr{G}_E, \mathscr{G}_E^c$ into Hilbert manifolds and Hilbert Lie groups, respectively, by considering their completions with respect to suitable Sobolev norms. Namely, from now on we will only consider unitary or complex connections with regularity in L_k^2, and unitary or complex gauge transformations with regularity in L_{k+1}^2. For our purposes, choosing any $k \geq 2$ will suffice. We denote Sobolev-completed spaces simply by $\mathscr{A}_E, \mathscr{A}_E^c, \mathscr{G}_E, \mathscr{G}_E^c$ unless we need to make reference to the Sobolev structure, in which case we shall use the notation $\mathscr{A}_{E,k}, \mathscr{A}_{E,k}^c, \mathscr{G}_{E,k+1}, \mathscr{G}_{E,k+1}^c$.

If the determinant (Λ, λ) is trivial as a bundle with connection, then E has structure group $SU(2)$ and $\mathscr{A}_E, \mathscr{A}_E^c$ consist of the $SU(2)$ or $SL(2, \mathbb{C})$ connections. We allow for non-trivial (Λ, λ) in order to also study also the $PSU(2) = SO(3)$ connections and $PSL(2, \mathbb{C})$ connections, via the following standard construction.

2.1.2 *The 'U(2) model' for SO(3) and PSL(2, ℂ) connections.* The $U(2)$ connections on E can be regarded as $PSU(2) = SO(3)$ connections by the following procedure (see [BD95, Part II] and [Don02, § 5.6] for further details).

Associated to a $U(2)$ bundle E there is a $PSU(2) = SO(3)$ bundle $P = E \times_{U(2)} SO(3)$, whose corresponding rank-3 real vector bundle over Y is the adjoint bundle $\mathfrak{g}_E = \mathfrak{su}(E)$. Conversely, any $SO(3)$ bundle P lifts to a $U(2)$ bundle E which is unique up to replacing E by its tensor product $E \otimes L$ with a $U(1)$ bundle L. (Indeed, on a 3-manifold the $U(2)$ bundles are classified by $c_1(E) = c_1(\Lambda) \in H^2(Y, \mathbb{Z})$ and the $SO(3)$ bundles by $w_2(P) \in H^2(Y, \mathbb{Z}/2)$; a choice of lift of P corresponds to an integral lift of $w_2(P)$, and this is unique modulo even classes in $H^2(Y, \mathbb{Z})$.) Of course, a similar relationship holds if we replace $U(2)$ and $SO(3)$ by their complexifications $GL(2, \mathbb{C})$ and $PSL(2, \mathbb{C})$.

A $U(2)$ connection with fixed determinant on E induces an $SO(3)$ connection on P, and conversely an $SO(3)$ connection on P lifts to a $U(2)$ connection on E with fixed determinant which is unique up to tensoring (E, A) by a flat real line bundle. This gives a bijection between $\mathscr{A}_P/\mathscr{G}_P$ and the orbits of the action of $H^1(Y, \mathbb{Z}/2)$ on $\mathscr{A}_E/\mathscr{G}_E$. The same relationship holds between the $GL(2, \mathbb{C})$ connections with fixed determinant and the $PSL(2, \mathbb{C})$ connections.

This gives a fairly concrete model for studying the spaces of $SO(3)$ (or $PSL(2, \mathbb{C})$) connections, effectively translating the complexity in the $SO(3)$ stabilizers to that of the $SU(2)$ stabilizers and the stabilizers of the action of $H^1(Y, \mathbb{Z}/2)$ on $\mathscr{A}_E/\mathscr{G}_E$.

2.2 Orbit spaces

We now discuss the Kähler quotient of the space $\mathscr{A}_E^c$ of complex connections.

2.2.1 *The Kähler structure of $\mathscr{A}_E^c$.* The space $\mathscr{A}_E^c$ carries a canonical complex structure,

$$J(\dot{A}, \dot{\Phi}) = (-\dot{\Phi}, \dot{A}), \tag{2.1}$$

and a canonical symplectic form ω_J,

$$\omega_J((\dot{A}_1, \dot{\Phi}_1), (\dot{A}_2, \dot{\Phi}_2)) = -\int_Y \mathrm{Tr}(\dot{A}_1 \wedge *\dot{\Phi}_2 - \dot{\Phi}_1 \wedge *\dot{A}_2). \tag{2.2}$$

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The pair (ω_J, J) is compatible, making $\mathscr{A}_E^c$ into a Hilbert *Kähler manifold*. The complex gauge group $\mathscr{G}_E^c$ is a Hilbert *complex Lie group* which acts *holomorphically* on $(\mathscr{A}_E^c, J)$. In turn, the unitary gauge group $\mathscr{G}_E$ preserves the whole Kähler structure.

2.2.2 *Moment maps and stability.* Furthermore, the unitary gauge group $\mathscr{G}_E$ acts on $(\mathscr{A}_E^c, \omega_J)$ in a *Hamiltonian* fashion, with moment map given by

$$\mathscr{A}_E^c \to \mathrm{Lie}\mathscr{G}_E = \Omega^0(Y, \mathfrak{g}_E), \quad (A, \Phi) \mapsto d_A^*\Phi.$$

Indeed, the vector field X_ξ given by the infinitesimal action of $\xi \in \Omega^0(Y, \mathfrak{g}_E)$ is $X_\xi = (-d_A\xi, [\xi, \Phi])$ and we have

$$\omega_J(X_\xi, (\dot{A}, \dot{\Phi})) = -\langle \dot{\Phi}, d_A\xi\rangle_{L^2} - \langle \dot{A}, [\xi, \Phi]\rangle_{L^2} = \langle *[\dot{A}, *\Phi] - d_A^*\dot{\Phi}, \xi\rangle_{L^2},$$

where $*[\dot{A}, *\Phi] - d_A^*\dot{\Phi}$ is the linearization of $-d_A^*\Phi$ at (A, Φ). We may thus attempt to carry out the *Kähler quotient* construction, albeit in infinite dimensions, of the Hilbert Kähler manifold $(\mathscr{A}_E, \omega_J, J)$ by the $\mathscr{G}_E$-action.

The zeros of the moment map (i.e., $d_A^*\Phi = 0$) have the following interpretation.

PROPOSITION 2.1. *Suppose $\mathbb{A} = A + i\Phi \in \mathscr{A}_E^c$ satisfies $d_A^*\Phi = 0$. Then:*

(1) *every $\mathbb{A}$-invariant complex line subbundle $L \subset E$ admits an $\mathbb{A}$-invariant complement;*
(2) *if the $\mathscr{G}_E$-stabilizer of $\mathbb{A}$ is $\{\pm I\}$, then E admits no $\mathbb{A}$-invariant complex line subbundles.*

Proof. Assertion (1) is proved in [Cor88, Proposition 3.2]. See also [WZ23, § 2.3] where the following result is established: if $L \subset E$ is an $\mathbb{A}$-invariant line subbundle and $L^\perp$ is denotes its orthogonal complement with respect to the unitary structure on E, then the vanishing of $d_A^*\Phi = 0$ implies that $L^\perp$ is also $\mathbb{A}$-invariant.

In this case, the $U(2)$ bundle with connection $(E, \mathbb{A})$ splits as a sum $(L, a) \oplus (L^\perp, a^\perp)$ of $U(1)$ bundles with connection (such that $(\Lambda, \lambda) = (L, a) \otimes (L^\perp, a^\perp)$), and Φ acts diagonally with respect to this decomposition. Hence, the $\mathscr{G}_E$-stabilizer of $\mathbb{A} = A + i\Phi$ contains a $U(1)$ subgroup. Assertion (2) follows. □

A complex connection $\mathbb{A}$ satisfying the conclusion of Proposition 2.1(1) is said to be *polystable* (or completely reducible, or reductive), whereas a complex connection $\mathbb{A}$ satisfying the conclusion of Proposition 2.1(2) is said to be *stable* (or irreducible)). By a recent result by Wu and Zhang [WZ23], the converse to Proposition 2.1 is known (but we do not need this result): $\mathbb{A}$ is polystable if and only if there exists a complex gauge transformation $u \in \mathscr{G}_E^c$ such that $u \cdot \mathbb{A} = A_u + i\Phi_u$ solves $d_{A_u}^*\Phi_u = 0$ (in the case when $\mathbb{A}$ is *flat*, this is a fundamental result by Donaldson [Don87b] and Corlette [Cor88]); and $\mathbb{A}$ and is stable if and only if, in addition, the $\mathscr{G}_E$-stabilizer of $u \cdot \mathbb{A}$ is $\{\pm I\}$. In light of this, and in an abuse of terminology, we make the following definition.

DEFINITION 2.2. The space of *polystable* configurations $\mathscr{C}_E \subset \mathscr{A}_E^c = \mathscr{A}_E \times \Omega^1(Y, \mathfrak{g}_E)$ is the subspace of pairs (A, Φ) satisfying $d_A^*\Phi = 0$. The unitary gauge group $\mathscr{G}_E$ acts on $\mathscr{C}_E$, and the space of *stable* configurations is defined as the open subset $\mathscr{C}_E^* \subset \mathscr{C}_E$ consisting of configurations with smallest possible stabilizer under the action of $\mathscr{G}_E$, namely $\{\pm I\} = Z(SU(2))$. When reference to the Sobolev structure is needed, we use the notation $\mathscr{C}_{E,k}^*, \mathscr{C}_{E,k}$.

The first step towards carrying out the infinite-dimensional Kähler quotient construction is the following result.

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Proposition 2.3. *The space of stable configurations $\mathscr{C}^*_{E,k} \subset \mathscr{A}^c_{E,k}$ is a Hilbert submanifold. Its tangent space is*

$$T_{(A,\Phi)}\mathscr{C}^*_{E,k} = \{(\dot{A},\dot{\Phi}) \in (\Omega^1 \oplus \Omega^1)(Y,\mathfrak{g}_E)_k \mid *[\dot{A},*\Phi] - d^*_A\dot{\Phi} = 0\}.$$

(Here, and likewise subsequently, the subscript k stands for the Sobolev L^2_k completion of the space of sections shown.)

Before proving this result we make some general observations. Consider, for each $i = 1, 2$ and $0 \leq j \leq k$, the pair of bounded linear operators

$$\begin{aligned}\delta^i_{(A,\Phi)} &: \Omega^0(Y,\mathfrak{g}_E)_{j+1} \to (\Omega^1 \oplus \Omega^1)(Y,\mathfrak{g}_E)_j,\\ (\delta^i_{(A,\Phi)})^* &: (\Omega^1 \oplus \Omega^1)(Y,\mathfrak{g}_E)_j \to \Omega^0(Y,\mathfrak{g}_E)_{j-1},\end{aligned}$$

given by

$$\begin{aligned}\delta^1_{(A,\Phi)}(\dot{\xi}) &= (-d_A\dot{\xi}, [\dot{\xi},\Phi]),\\ (\delta^1_{(A,\Phi)})^*(\dot{A},\dot{\Phi}) &= -d^*_A\dot{A} - *[\dot{\Phi},*\Phi],\\ \delta^2_{(A,\Phi)}(\dot{\xi}) &= (-[\dot{\xi},\Phi], -d_A\dot{\xi}),\\ (\delta^2_{(A,\Phi)})^*(\dot{A},\dot{\Phi}) &= *[\dot{A},*\Phi] - d^*_A\dot{\Phi}.\end{aligned}$$

It is easily seen that $(\delta^i_{(A,\Phi)})^*$ is the formal L^2 adjoint of $\delta^i_{(A,\Phi)}$. Let us also introduce the following notation:

$$\mathscr{T}^i_{(A,\Phi),j} := \mathrm{Im}\delta^i_{(A,\Phi)} \subset (\Omega^1 \oplus \Omega^1)(\mathfrak{g}_E)_j, \tag{2.3}$$

$$\mathscr{K}^i_{(A,\Phi),j} := \mathrm{Ker}(\delta^i_{(A,\Phi)})^* \subset (\Omega^1 \oplus \Omega^1)(\mathfrak{g}_E)_j. \tag{2.4}$$

The significance of these operators is as follows. Identifying $(\Omega^0 \oplus \Omega^0)(Y,\mathfrak{g}_E)_{k+1} \cong \mathrm{Lie}\mathscr{G}^c_{E,k+1}$, the map sending

$$(\dot{\xi}_1,\dot{\xi}_2) \in (\Omega^0 \oplus \Omega^0)(Y,\mathfrak{g}_E)_{k+1} \mapsto \delta^1_{(A,\Phi)}(\dot{\xi}_1) + \delta^2_{(A,\Phi)}(\dot{\xi}_2) = \delta^1_{(A,\Phi)}(\dot{\xi}_1) + J\delta^1_{(A,\Phi)}(\dot{\xi}_2) \tag{2.5}$$

can be identified with the infinitesimal action of the complex gauge group $\mathscr{G}^c_{E,k+1}$ on the space of complex connections.

The bounded linear operator $\delta^i_{(A,\Phi)}$ is associated to a first-order differential operator with injective symbol, and as a matter of Hodge theory we thus have that for $0 \leq j \leq k$ the subspace $\mathrm{Ker}\delta^i_{(A,\Phi)} \subset \Omega^0(Y,\mathfrak{g}_E)_{j+1}$ is finite-dimensional (and independent of j), and there are L^2-orthogonal decompositions

$$\Omega^0(Y,\mathfrak{g}_E)_{j+1} = \mathrm{Ker}\delta^i_{(A,\Phi)} \oplus \mathrm{Im}(\delta^i_{(A,\Phi)})^*, \tag{2.6}$$

$$(\Omega^1 \oplus \Omega^1)(Y,\mathfrak{g}_E)_j = \mathrm{Im}\delta^i_{(A,\Phi)} \oplus \mathrm{Ker}(\delta^i_{(A,\Phi)})^* = \mathscr{T}^i_{(A,\Phi),j} \oplus \mathscr{K}^i_{(A,\Phi),j}. \tag{2.7}$$

In particular, Proposition 2.3 asserts that, at a stable configuration,

$$T_{(A,\Phi)}\mathscr{C}^*_{E,k} = \mathscr{K}^2_{(A,\Phi),k}.$$

Observe also that the composition $(\delta^1_{(A,\Phi)})^* \circ \delta^2_{(A,\Phi)}$ vanishes at a polystable configuration (A,Φ):

$$(\delta^1_{(A,\Phi)})^* \circ \delta^2_{(A,\Phi)}(\dot{\xi}) = - * d_A[\dot{\xi},*\Phi] + *[d_A\dot{\xi},\Phi] = [\dot{\xi}, d^*_A\Phi] = 0,$$

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where the last equality used $d_A^*\Phi = 0$. In particular, it follows from this that the decomposition (2.7) restricts to L^2-orthogonal decompositions for $0 \le j \le k$:

$$\mathscr{K}^2_{(A,\Phi),j} = \mathscr{T}^1_{(A,\Phi),j} \oplus (\mathscr{K}^1_{(A,\Phi),j} \cap \mathscr{K}^2_{(A,\Phi),j}), \tag{2.8}$$

$$\mathscr{K}^1_{(A,\Phi),j} = \mathscr{T}^2_{(A,\Phi),j} \oplus (\mathscr{K}^1_{(A,\Phi),j} \cap \mathscr{K}^2_{(A,\Phi),j}). \tag{2.9}$$

Proof of Proposition 2.3. As noted above, the linearization of the equation $d_A^*\Phi = 0$ is $-\delta_2^*(\dot{A}, \dot{\Phi}) = 0$. By the implicit function theorem for Hilbert spaces, it suffices to establish the surjectivity of $\delta_2^* : (\Omega^1 \oplus \Omega^1)(Y, \mathfrak{g}_E)_k \to \Omega^0(Y, \mathfrak{g}_E)_{k-1}$ whenever $(A, \Phi) \in \mathscr{C}^*_{E,k}$. By the decomposition (2.6) with $i = 2$ on the L^2_{k-1} spaces, this is equivalent to $\mathrm{Ker}\delta^2_{(A,\Phi)} = 0$. Indeed, $\mathrm{Ker}\delta^2_{(A,\Phi)}$ is identified as the (rotation by J) of the tangent space to the $\mathscr{G}_{E,k}$-stabilizer of (A, Φ), which vanishes when (A, Φ) is stable.

The latter assertion can be verified directly as follows. Suppose for a contradiction that $[\dot{\xi}, \Phi]$ and $d_A\dot{\xi}$ vanish identically, where $0 \neq \dot{\xi} \in \Omega^0(Y, \mathfrak{g}_E)_{k-1}$. Elliptic regularity ensures that $\dot{\xi}$ has regularity L^2_{k+1} (and, in particular, $\dot{\xi}$ is C^0). Since $\dot{\xi}$ is A-parallel (i.e., $d_A\dot{\xi} = 0$) and non-trivial, the eigenspaces of $\dot{\xi}$ induce a bundle decomposition $E = L \oplus L^{-1}\Lambda$ which is preserved by the connection A, for some $U(1)$ bundle L. The equation $[\dot{\xi}, \Phi] = 0$ says that Φ is diagonal with respect to this decomposition. Altogether, it follows that the $\mathscr{G}_{E,k+1}$-stabilizer of (A, Φ) contains a $U(1)$ subgroup, which is a contradiction. □

2.2.3 *Local structure of the stable orbit space.* Formally, the Kähler quotient construction yields the space of *stable orbits* $\mathscr{B}^*_{E,k} = \mathscr{C}^*_{E,k}/\mathscr{G}_{E,k+1}$. For the moment, we regard it as a topological space with the quotient topology. We now describe how to equip $\mathscr{B}^*_{E,k}$ with a Hilbert manifold structure by constructing slices for the action of $\mathscr{G}_{E,k+1}$ on $\mathscr{C}^*_{E,k}$.

DEFINITION 2.4. Let $(A, \Phi) \in \mathscr{C}^*_{E,k}$ be a stable configuration. The *stable Coulomb slice at* (A, Φ) is the subspace $\mathcal{S}_{(A,\Phi),k} \subset \mathscr{C}^*_{E,k}$ given by

$$\mathcal{S}_{(A,\Phi),k} = ((A, \Phi) + \mathscr{K}^1_{(A,\Phi),k}) \cap \mathscr{C}^*_{E,k}.$$

PROPOSITION 2.5 (Local slices). *Let $(A, \Phi) \in \mathscr{C}^*_{E,k}$ be a stable configuration. Then the following assertions hold:*

(1) *$\mathcal{S}_{(A,\Phi),k}$ is a Hilbert submanifold of $\mathscr{C}^*_{E,k}$ in a neighborhood of (A, Φ);*
(2) *$\mathcal{S}_{(A,\Phi),k}$ meets the $\mathscr{G}_{E,k+1}$-orbit of (A, Φ) transversely and L^2-orthogonally at (A, Φ);*
(3) *there exists a neighborhood $\mathscr{U} \subset \mathscr{C}^*_{E,k}$ of (A, Φ) such for any $(A', \Phi') \in \mathscr{U}$ there exists a unitary gauge transformation $u \in \mathscr{G}_{E,k+1}$ such that $u \cdot (A', \Phi') \in \mathcal{S}_{(A,\Phi),k}$ (the gauge transformation u is necessarily unique up to multiplication by $\pm I$, by stability).*

Proof. To prove assertion (1) it suffices, by the implicit function theorem for Hilbert spaces, to show that the operator

$$T_{(A,\Phi)}\mathscr{C}^*_{E,k} = \mathscr{K}^2_{(A,\Phi),k} \xrightarrow{(\delta^1_{(A,\Phi)})^*} \Omega^0(Y, \mathfrak{g}_E)_{k-1}$$

is surjective. But the operator $(\delta^1_{(A,\Phi)})^* : (\Omega^1 \oplus \Omega^1)(Y, \mathfrak{g}_E)_k \to \Omega^0(Y, \mathfrak{g}_E)_{k-1}$ is surjective (this follows by the same argument as for $(\delta^2_{(A,\Phi)})^*$ from the proof of Proposition 2.3, which uses the fact that (A, Φ) is stable). Thus, the assertion follows from this, the vanishing of $(\delta^1_{(A,\Phi)})^*\delta^2_{(A,\Phi)}$ (using $d_A^*\Phi = 0$) and the decomposition (2.7) with $i = 2$.

Assertion (2) follows now from the decomposition (2.8), noting that the tangent spaces at (A, Φ) to $\mathscr{C}^*_{E,k}$, to the $\mathscr{G}_{E,k+1}$-orbit of (A, Φ) and to $\mathcal{S}_{(A,\Phi),k}$ are given by $\mathscr{K}^2_{(A,\Phi),k}$, $\mathscr{T}^1_{(A,\Phi),k}$ and $\mathscr{K}^1_{(A,\Phi),k} \cap \mathscr{K}^2_{(A,\Phi),k}$, respectively.

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We now show assertion (3). In fact, we show that for an arbitrary pair $(A', \Phi') \in \mathscr{A}^c_{E,k}$, not necessarily in $\mathscr{C}_{E,k}$, there exists a unitary gauge transformation u such that

$$(\delta^1_{(A,\Phi)})^*(u \cdot (A', \Phi') - (A, \Phi)) = 0. \tag{2.10}$$

Writing $(A', \Phi') = (A, \Phi) + (a, \phi)$, we seek, for sufficiently small (a, ϕ), a unitary gauge transformation of the form $u = \exp \xi$ where $\xi \in \Omega^0(Y, \mathfrak{g}_E)_{k+1}$, such that (2.10) holds. Note that

$$u \cdot (A', \Phi') - (A, \Phi) = (-d_A\xi + (\exp \xi)a(\exp \xi)^{-1}, (\exp \xi)\Phi(\exp \xi)^{-1} + (\exp \xi)\phi(\exp \xi)^{-1}).$$

By the implicit function theorem for Hilbert spaces, it suffices to show that the linearization of (2.10) in the ξ direction is surjective at $(\xi, a, b) = (0, 0, 0)$, and this is the operator given by

$$\dot{\xi} \in \Omega^0(Y, \mathfrak{g}_E)_{k+1} \mapsto (\delta^1_{(A,\Phi)})^*\delta^1_{(A,\Phi)}\dot{\xi} \in \Omega^0(Y, \mathfrak{g}_E)_{k-1}. \tag{2.11}$$

By (2.6) and (2.7) the operator (2.11) surjects onto the complement of $\mathrm{Ker}\delta^1_{(A,\Phi)}$. But $\mathrm{Ker}\delta^1_{(A,\Phi)} = 0$ by stability of (A, Φ) (this follows by the same argument we used for $\delta^2_{(A,\Phi)}$ in the proof of Proposition 2.3), and hence the assertion follows. □

By Proposition 2.5, the map $\mathcal{S}_{(A,\Phi)} \to \mathscr{B}^*_{E,k} = \mathscr{C}^*_{E,k}/\mathscr{G}_{E,k+1}$ is a local homeomorphism at (A, Φ), and the Hilbert manifold structure on the neighborhood of $(A, \Phi) \in \mathcal{S}_{(A,\Phi),k}$ provides Hilbert manifold charts for the stable orbit space $\mathscr{B}^*_{E,k}$. The tangent spaces of $\mathscr{B}^*_{E,k}$ are thus given by

$$\begin{aligned} T_{[(A,\Phi)]}\mathscr{B}^*_{E,k} &= T_{(A,\Phi)}\mathcal{S}_{(A,\Phi),k} = \mathcal{K}^1_{(A,\Phi),k} \cap \mathcal{K}^2_{(A,\Phi),k} \\ &= \{(\dot{A}, \dot{\Phi}) \in (\Omega^1 \oplus \Omega^1)(Y, \mathfrak{g}_E)_k \mid -d^*_A\dot{A} - *[\dot{\Phi}, *\Phi] = *[\dot{A}, *\Phi] - d^*_A\dot{\Phi} = 0\}. \end{aligned} \tag{2.12}$$

The projection $\mathscr{C}^*_{E,k} \to \mathscr{B}^*_{E,k}$ then becomes a submersion of Hilbert manifolds. Noting that the $\mathscr{G}_{E,k+1}$-action on $\mathscr{A}^c_{E,k}$ is by isometries of the flat L^2 metric, there is an induced L^2 *metric* on $\mathscr{B}^*_{E,k}$ making this projection a Riemannian submersion. By Proposition 2.5(2), in the description (2.12) of $T_{(A,\Phi)}\mathscr{B}^*_{E,k}$ the L^2 metric coincides with that induced from the standard L^2 inner product.

Furthermore, recall that $\mathscr{G}_{E,k+1}$ acts on the Hilbert Kähler manifold $(\mathscr{A}^c_{E,k}, \omega_J, J)$ preserving the Kähler structure and in a Hamiltonian fashion. By the infinite-dimensional version of the Kähler quotient theorem, the complex structure J and the symplectic form ω_J from (2.1) to (2.2) thus descend to give an (integrable) complex structure and symplectic form on the stable orbit space $\mathscr{B}^*_{E,k}$, compatible with the L^2 metric on $\mathscr{B}^*_{E,k}$. In the description (2.12) of $T_{[(A,\Phi)]}\mathscr{B}^*_{E,k}$, these two structures are given by the expressions as equations (2.1) and (2.2). Summing up, we have the following result.

Corollary 2.6. *The stable orbit space $\mathscr{B}^*_{E,k}$ is a Hilbert Kähler manifold when given the L^2 Riemannian metric, the complex structure J and the symplectic form ω_J.*

2.2.4 *The variational viewpoint.* We have obtained the stable orbit space $\mathscr{B}^*_E$ as the Kähler quotient of $(\mathscr{A}^c_E, \omega_J, J)$ by the $\mathscr{G}_E$-action. There is an alternative viewpoint in which the 'polystability' condition $d^*_A\Phi = 0$ arises from a minimization problem. We only mention this briefly, referring to [Cor88] for the details.

Namely, consider a complex orbit $\mathscr{G}^c_E \cdot (A_0, \Phi_0) \subset \mathscr{A}^c_E$ and the restriction of the *squared L^2-norm functional* $\|\Phi\|^2_{L^2}$ onto the orbit, that is, the $\mathscr{G}_E$-invariant functional

$$\mathscr{G}^c_E \cdot (A_0, \Phi_0) \ni (A, \Phi) \mapsto \|\Phi\|^2_{L^2}. \tag{2.13}$$

Then $(A, \Phi) \in \mathscr{G}^c_E \cdot (A_0, \Phi_0)$ satisfies $d^*_A\Phi = 0$ if and only if (A, Φ) is a critical point of (2.13). Furthermore, when the $\mathscr{G}_E$-stabilizer of (A, Φ) is smallest possible, the function (2.13) induces a

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convex function on $\mathscr{G}^c_E/\mathscr{G}_E$, and hence if a critical point on the orbit exists then it is unique and attains the minimum of $\|\Phi\|^2_{L^2}$ on that orbit.

2.3 Fredholm theory of complex flat connections

We now use our framework from § 2.1 to discuss deformation-theoretic aspects of complex flat connections on 3-manifolds.

2.3.1 *The complex Chern–Simons functional.* Throughout, we make a choice of a 'basepoint' in $\mathscr{A}^c_{E,k}$, that is, we fix a $GL(2,\mathbb{C})$ connection on E with fixed determinant (Λ,λ), which we denote by $\mathbb{A}_0$.

DEFINITION 2.7. The *(complex) Chern–Simons functional* is the function $CS:\mathscr{A}^c_{E,k}\to\mathbb{C}$ defined at a connection $\mathbb{A}=\mathbb{A}_0+\alpha$ by

$$CS(\mathbb{A})=\int_Y \mathrm{Tr}(\alpha\wedge((F_{\mathbb{A}}+F_{\mathbb{A}_0})-\tfrac{1}{6}[\alpha,\alpha])).$$

The Chern–Simons functional has the following property. Suppose M is an oriented and compact 4-manifold with boundary Y equipped with an extension of E to M (as a bundle with fixed determinant) and extensions of the connection $\mathbb{A}$ and the basepoint connection $\mathbb{A}_0$ (as connections with fixed determinant). Then, using Stokes' theorem, one can show that

$$CS(\mathbb{A})=\int_M \mathrm{Tr}(F_{\mathbb{A}}\wedge F_{\mathbb{A}})-\mathrm{Tr}(F_{\mathbb{A}_0}\wedge F_{\mathbb{A}_0}). \tag{2.14}$$

We can use (2.14) to understand the periods of $CS(\mathbb{A})$ under the action of complex gauge transformations with fixed determinant. Namely, for a given $u\in\mathscr{G}^c_{E,k+1}$ choose any path of connections $\mathbb{A}(t)\in\mathscr{A}^c_{E,k}$ from $\mathbb{A}(0)=\mathbb{A}$ to $\mathbb{A}(1)=u\cdot\mathbb{A}$. This path can be regarded as a connection $\underline{\mathbb{A}}$ on the bundle over $[0,1]\times Y$ obtained as the trivial extension of $E\to Y$. We extend the basepoint connection $\mathbb{A}_0$ trivially to $[0,1]\times Y$. Denote by $E_u\to S^1\times Y$ the bundle obtained by clutching with u. By (2.14) and Chern–Weil theory we have

$$\begin{aligned}CS(u\cdot\mathbb{A})-CS(\mathbb{A})&=\int_{[0,1]\times Y}\mathrm{Tr}(F^2_{\underline{\mathbb{A}}})=8\pi^2(c_2(E_u)-\tfrac{1}{2}c_1(E_u)^2)\cdot[S^1\times Y]\\&=8\pi^2c_2(E_u)\cdot[S^1\times Y]\in 8\pi^2\mathbb{Z},\end{aligned} \tag{2.15}$$

where we note that $c_1(E_u)^2=0$ because u acts as the identity on $\Lambda=\Lambda^2E$.

In particular, $\mathrm{Re}CS$ has periods in the lattice $8\pi^2\mathbb{Z}\subset\mathbb{R}$. In turn, $\mathrm{Im}CS$ has no periods; furthermore, it is independent of the chosen basepoint connection $\mathbb{A}_0$ and is given by the following formula: in terms of the decomposition $\mathbb{A}=A+i\Phi$,

$$\mathrm{Im}CS(A+i\Phi)=2\int_Y\mathrm{Tr}(\Phi\wedge(F_A-\tfrac{1}{6}[\Phi,\Phi])). \tag{2.16}$$

2.3.2 *Stable flat connections.* The variation of the Chern–Simons functional $CS:\mathscr{A}^c_{E,k}\to\mathbb{C}$ is given by

$$CS(\mathbb{A}+\dot{\mathbb{A}})=CS(\mathbb{A})=2\int_Y\mathrm{Tr}(\dot{\mathbb{A}}\wedge F_{\mathbb{A}})+\int_Y\mathrm{Tr}(\dot{\mathbb{A}}\wedge d_{\mathbb{A}}\dot{\mathbb{A}})+\frac{1}{3}\int_Y\mathrm{Tr}(\dot{\mathbb{A}}\wedge[\dot{\mathbb{A}},\dot{\mathbb{A}}]). \tag{2.17}$$

From (2.17) one sees that CS is a smooth (in fact, holomorphic) function on the L^2_k connections (as long as $k\geq 1$), and its critical points are the connections $\mathbb{A}$ whose curvature has vanishing traceless part (since the 'Killing form' $(X,Y)\mapsto-\mathrm{Tr}(XY)$ on the Lie algebra $\mathfrak{gl}(2,\mathbb{C})$ has kernel

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$Z(\mathfrak{gl}(2,\mathbb{C})) = \mathbb{C}\cdot I$):

$$F_{\mathbb{A}} - \frac{I}{2}F_\lambda = 0.$$

Said differently, these are the $GL(2,\mathbb{C})$ connections $\mathbb{A}$ on E with fixed determinant (Λ,λ) which are *projectively flat*, in the sense that the connection that $\mathbb{A}$ induces on the $PSL(2,\mathbb{C})$ bundle $P^c = E\times_{U(2)} PSL(2,\mathbb{C})$ is flat. Informally, we might often refer to such connections plainly as 'flat'. In terms of the decomposition $\mathbb{A} = A + i\Phi$, this equation reads

$$F_A - \frac{I}{2}F_\lambda - \frac{1}{2}[\Phi,\Phi] = 0, \tag{2.18}$$

$$d_A\Phi = 0. \tag{2.19}$$

Definition 2.8. A *polystable* (projectively) flat $GL(2,\mathbb{C})$ connection with fixed determinant is a pair $(A,\Phi)\in\mathscr{A}^c_{E,k}$ which solves (2.18) and (2.19) and

$$d_A^*\Phi = 0. \tag{2.20}$$

That is, $A+i\Phi$ is flat and (A,Φ) lies in the polystable locus $\mathscr{C}_{E,k}$. The group $\mathscr{G}_{E,k+1}$ of unitary gauge transformations fixing the determinant preserves the solutions to (2.18)–(2.20). If, in addition, (A,Φ) has stabilizer $\pm I$ under the $\mathscr{G}_E$-action then (A,Φ) is *stable*.

We denote by $\mathscr{M}_{E,k}\subset\mathscr{B}_{E,k}$ (respectively, $\mathscr{M}^*_{E,k}\subset\mathscr{B}^*_{E,k}$) the topological spaces of $\mathscr{G}_{E,k+1}$-orbits of solutions of (2.18)–(2.20) (respectively, those which are stable). It follows from Remark 2.9 that every such $\mathscr{G}_{E,k+1}$-orbit contains a *smooth* solution and the topological spaces $\mathscr{M}_{E,k}, \mathscr{M}^*_{E,k}$ do not depend on the chosen k. Thus, we subsequently drop the subscript k from our notation.

Remark 2.9 (Ellipticity). Equations (2.18)–(2.20) do not form an elliptic system, even modulo $\mathscr{G}_E$. However, this can be achieved if we 'augment' the equations by introducing two additional fields $\Psi_1,\Psi_2\in\Omega^0(Y,\mathfrak{g}_E)_k$ and considering the equations

$$\begin{aligned} F_A - \frac{I}{2}F_\lambda - \frac{1}{2}[\Phi,\Phi] &= *d_A\Psi_1 + *[\Psi_2,\Phi],\\ d_A\Phi &= \pm *([\Phi,\Psi_1] + d_A\Psi_2),\\ d_A^*\Phi &= [\Psi_1,\Psi_2], \end{aligned}$$

on whose space of solutions $\mathscr{G}_{E,k+1}$ acts (by $u\Psi_i u^{-1}$ on the additional fields Ψ_i). Above, we mean that there are two possible systems of equations according to the choice of sign $\pm$. (Both have been studied in [NO20]; for the $-$ sign and $\Psi_2 = 0$ the equations are known as the *extended Bogomolny equations* [HM19, HM20].) Both of these augmented systems are elliptic modulo $\mathscr{G}_E$, in the usual sense.

Remark 2.10 (Invariance under tensoring by line bundles; see also § 2.1.2). For a given $U(1)$ bundle with connection (L,A_L), we can consider the $U(2)$ bundle as $E\otimes L$ and give its determinant $\Lambda\otimes L^2$ the connection $\lambda\otimes A_L^2$. This naturally induces biholomorphisms $\mathscr{A}^c_{E,k}\cong\mathscr{A}^c_{E\otimes L,k}$, $\mathscr{C}^*_{E,k}\cong\mathscr{C}_{E\otimes L,k}$, $\mathscr{B}^*_{E,k}\cong\mathscr{B}^*_{E\otimes L,k}$ which restrict to homeomorphisms $\mathscr{M}_E\cong\mathscr{M}_{E\otimes L}$ and $\mathscr{M}^*_E\cong\mathscr{M}^*_{E\otimes L}$. In particular, it follows that, up to homeomorphism, the spaces $\mathscr{M}_E,\mathscr{M}^*_E$ are independent of the fixed connection λ on $\Lambda = \Lambda^2 E$; and furthermore, they only depend on the $SO(3)$ bundle P underlying E.

2.3.3 *Hessians.* From now on we fix an angle $\theta\in[0,2\pi)$, and consider the functional $S = \mathrm{Re}(e^{-i\theta}CS):\mathscr{A}^c_{E,k}\to\mathbb{R}$ with CS as in (2.14). For simplicity, when performing calculations

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and displaying formulas we shall assume $\theta = \pi/2$ (i.e., use the imaginary part of the Chern–Simons functional (2.16), which has the advantage of having no periods under the action of $\mathscr{G}^c_{E,k+1}$). However, the discussion that follows goes through for all θ with little modification.

The formal gradient of $S : \mathscr{A}^c_{E,k} \to \mathbb{R}$ with respect to the flat L^2 metric makes sense as a smooth section, denoted by grad$S$, of the L^2_{k-1} completion of the tangent bundle $T\mathscr{A}^c_{E,k-1} \to \mathscr{A}^c_{E,k}$. It is given by

$$\mathrm{grad}S = 2 \cdot \left(* \, d_A \Phi, * \left(F_A - \frac{I}{2} F_\lambda - \frac{1}{2}[\Phi, \Phi] \right) \right). \tag{2.21}$$

Since S is the imaginary part of the holomorphic function CS, the zeros of grad$S$ agree with the critical points of CS (i.e., the flat connections). By the following proposition, the critical points of the restriction $S|_{\mathscr{C}^*_{E,k}}$ of S to the stable locus are the stable flat connections.

PROPOSITION 2.11. *The section* grad$S$ *of* $T\mathscr{A}^c_{E,k-1} \to \mathscr{A}^c_{E,k}$ *takes values in* $T\mathscr{C}^*_{E,k-1} \subset T\mathscr{A}^c_{E,k-1}$ *along* $\mathscr{C}^*_{E,k} \subset \mathscr{A}^c_{E,k}$. *(That is,* grad$S$ *is 'tangent' to the stable locus.)*

Proof. Using the Bianchi identity $d_A F_A = 0$, we compute that $(\delta^2_{(A,\Phi)})^*(\mathrm{grad}S)$ is given by

$$*[*d_A\Phi, *\Phi] - d^*_A \left(* \left(F_A - \frac{I}{2} F_\lambda - \frac{1}{2}[\Phi, \Phi] \right) \right) = *[d_A\Phi, \Phi] - *[d_A\Phi, \Phi] = 0. \qquad \square$$

Remark. More generally, the conclusion of Proposition 2.11 follows by the invariance of S under the complex gauge-group action $\mathscr{G}^c_E$. (More precisely, all that is needed is that the derivative $(dS)_{(A,\Phi)}$ vanishes in the directions tangent to the complex orbit $\mathscr{G}^c_E \cdot (A, \Phi)$.)

Thus, grad$S$ gives a section of $T\mathscr{C}^*_{E,k-1} \to \mathscr{C}^*_{E,k}$. This section is $\mathscr{G}_{E,k+1}$-equivariant; hence, it descends to a well-defined 'vector field' on the stable orbit space $\mathscr{B}^*_{E,k}$. More precisely, we obtain the following smooth section $[\mathrm{grad}S]$ of the Hilbert vector bundle.

$$\begin{array}{ccc} & & T\mathscr{B}^*_{E,k-1} \\ & \overset{[\mathrm{grad}S]}{\curvearrowleft} & \downarrow \\ & & \mathscr{B}^*_{E,k} \end{array} \tag{2.22}$$

Equivalently, the section [grad] can be regarded as the formal gradient of $S : \mathscr{B}^*_{E,k} \to \mathbb{R}$ with respect to the L^2 metric. In what follows we show that the section (2.22) is Fredholm.

We first introduce the notation $\mathscr{K}_{(A,\Phi),j} = \mathscr{K}^1_{(A,\Phi),j} \cap \mathscr{K}^2_{(A,\Phi),j}$ for $0 \le j \le k$ (cf. Definitions (2.3) and (2.4) and the L^2-orthogonal decompositions (2.6)–(2.9)). As (A, Φ) varies in the stable configuration space, these subspaces form $\mathscr{G}$-equivariant Hilbert bundles $\mathscr{K}_j \to \mathscr{C}^*_{E,k}$ (and similarly for $\mathscr{K}^1_j$ and $\mathscr{K}^2_j$). In particular, $\mathscr{K}_j$ is the pullback of the tangent bundle of $\mathscr{B}^*_{E,j}$ along the projection $\mathscr{C}^*_{E,j} \to \mathscr{B}^*_{E,j}$ and $\mathscr{K}^2_j$ is the tangent bundle of $\mathscr{C}^*_{E,j}$.

Now, the L^2 Riemannian metric on $\mathscr{B}^*_{E,k}$ provides the Levi-Civita connection on the Hilbert vector bundle (2.22), and we use this connection to define the vertical derivative of the section $[\mathrm{grad}S]$. This is a vector bundle map

$$\mathscr{D} := \mathcal{D}[\mathrm{grad}S] : \mathscr{K}_k \to \mathscr{K}_{k-1}.$$

Formally, $\mathscr{D}_{(A,\Phi)}$ is the covariant Hessian of S at $[(A, \Phi)]$ with respect to the L^2 metric. The following proposition gives a more explicit description.

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Proposition 2.12. *We have*

$$\mathscr{D} = (\Pi_{\mathscr{K}_{k-1}} \circ \mathrm{Hess} S)|_{\mathscr{K}_k} : \mathscr{K}_k \to \mathscr{K}_{k-1}, \tag{2.23}$$

where:

- $\Pi_{\mathscr{K}_j} : (\Omega^1 \oplus \Omega^1)(Y, \mathfrak{g}_E)_j \to \mathscr{K}_j$ *is the* L^2*-orthogonal projection (for* $0 \leq j \leq k$*); here we are using the following* L^2*-orthogonal decomposition obtained from (2.7) to (2.8) at a stable configuration* $(A, \Phi) \in \mathscr{C}^*_{(A,\Phi),k}$*,*

$$(\Omega^1 \oplus \Omega^1)(Y, \mathfrak{g}_E)_j = \mathscr{T}^1_{(A,\Phi),j} \oplus \mathscr{T}^2_{(A,\Phi),j} \oplus \mathscr{K}_{(A,\Phi),j}; \tag{2.24}$$

- $\mathrm{Hess} S : (\Omega^1 \oplus \Omega^1)(Y, \mathfrak{g}_E)_k \to (\Omega^1 \oplus \Omega^1)(Y, \mathfrak{g}_E)_{k-1}$ *is the Hessian of* $S : \mathscr{A}^c_{E,k} \to \mathbb{R}$ *for the flat* L^2 *metric; this is given by*

$$\mathrm{Hess} S = 2 \cdot \begin{pmatrix} *[\Phi, \cdot] & *d_A \\ *d_A & -*[\Phi, \cdot] \end{pmatrix}. \tag{2.25}$$

Proof. By definition, the projection $\mathscr{C}^*_{E,k} \to \mathscr{B}^*_{E,k}$ is a Riemannian submersion with respect to the L^2 metric on $\mathscr{B}^*_{E,k}$ and the restriction to $\mathscr{C}^*_{E,k} \subset \mathscr{A}^c_{E,k}$ of the flat L^2 metric. As a matter of general principle, the covariant Hessian $\mathscr{D}$ of S on $\mathscr{B}^*_E$ is then given by

$$\mathscr{D} = (\Pi^{\mathscr{K}^2_{k-1}}_{\mathscr{K}_{k-1}} \circ \mathscr{D}')|_{\mathscr{K}_k} : \mathscr{K}_k \to \mathscr{K}_{k-1}, \tag{2.26}$$

where:

- $\mathscr{D}' : \mathscr{K}^2_k \to \mathscr{K}^2_{k-1}$ denotes the covariant Hessian of $S|_{\mathscr{C}^*_{E,k}}$ with respect to the restriction of the flat L^2 metric; and
- $\Pi^{\mathscr{K}^2_{k-1}}_{\mathscr{K}_{k-1}} : \mathscr{K}^2_{k-1} \to \mathscr{K}_{k-1}$ is the L^2-orthogonal projection in (2.8)).

Again by general principle, because $\mathrm{grad} S$ is tangent to the stable locus $\mathscr{C}^*_{E,k} \subset \mathscr{A}^c_{E,k}$ (Proposition 2.11), the operator $\mathscr{D}'$ is related to the flat Hessian by

$$\mathscr{D}' = (\Pi_{\mathscr{K}^2_{k-1}} \circ \mathrm{Hess} S)|_{\mathscr{K}^2_k} : \mathscr{K}^2_k \to \mathscr{K}^2_{k-1}, \tag{2.27}$$

where $\Pi_{\mathscr{K}^2_{k-1}} : (\Omega^1 \oplus \Omega^1)(Y, \mathfrak{g}_E)_{k-1} \to \mathscr{K}^2_{k-1}$ is the L^2-orthogonal projection in (2.7). Putting together (2.26) and (2.27) gives the required result. (Recall that $(\delta^1)^* \circ \delta^2 = 0$ at stable flat connections, which implies that $\Pi_{\mathscr{K}_{k-1}}$ agrees with $\Pi^{\mathscr{K}^2_{k-1}}_{\mathscr{K}_{k-1}} \circ \Pi_{\mathscr{K}^2_{k-1}}$.) □

Theorem 2.13. *The smooth section* $[\mathrm{grad} S]$ *from (2.22) is Fredholm, that is, at a stable connection* (A, Φ) *the Hessian* $\mathscr{D}_{(A,\Phi)} : \mathscr{K}_{(A,\Phi),k} \to \mathscr{K}_{(A,\Phi),k-1}$ *is Fredholm. In addition, the following assertions hold.*

(1) *We have that* $\mathscr{D}_{(A,\Phi)}$ *is formally* L^2*-self-adjoint and, hence, of index zero. Furthermore, there exists a complete* L^2*-orthonormal system of eigenvectors* $\mathscr{D}_{(A,\Phi)}$ *contained in* L^2_{k+1}*, and its eigenvalues form a discrete and doubly infinite subset of* $\mathbb{R}$*.*

(2) *At a stable (projectively) flat connection* (A, Φ)*,* $\mathscr{D}_{(A,\Phi)}$ *is given plainly by (2.25).*

Proof. We use an argument similar to [KM07, Proposition 12.3.1] to establish the Fredholm property of $\mathscr{D}$. For a configuration (A, Φ) we consider the *extended Hessian* operator

$$(\widehat{\mathrm{Hess}} S)_{(A,\Phi),k} : (\Omega^1 \oplus \Omega^1 \oplus \Omega^0 \oplus \Omega^0)(Y, \mathfrak{g}_E)_k \to (\Omega^1 \oplus \Omega^1 \oplus \Omega^0 \oplus \Omega^0)(Y, \mathfrak{g}_E)_{k-1},$$

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given in block form by

$$(\widehat{\mathrm{Hess}S})_{(A,\Phi),k} = \begin{pmatrix} 2*[\Phi,\cdot] & 2*d_A & -d_A & -[\cdot,\Phi] \\ 2*d_A & -2*[\Phi,\cdot] & [\cdot,\Phi] & -d_A \\ -d_A^* & -*[\cdot,*\Phi] & 0 & 0 \\ *[\cdot,*\Phi] & -d_A^* & 0 & 0 \end{pmatrix}. \tag{2.28}$$

The extended Hessian is a Fredholm operator of index zero, since it is a first-order self-adjoint elliptic operator. The L^2-orthogonal decomposition (2.24) induces a further decomposition when $(A,\Phi) \in \mathscr{C}^*_{E,k}$ is stable: for $0 \le j \le k$,

$$(\Omega^1 \oplus \Omega^1 \oplus \Omega^0 \oplus \Omega^0)(Y,\mathfrak{g}_E)_j = \mathscr{T}^1_j \oplus \mathscr{T}^2_j \oplus \mathscr{K}_j \oplus \Omega^0(Y,\mathfrak{g}_E)_j \oplus \Omega^0(Y,\mathfrak{g}_E)_j. \tag{2.29}$$

With respect to the latter decomposition the extended Hessian has the block form

$$(\widehat{\mathrm{Hess}S})_{(A,\Phi),k} = \begin{pmatrix} 0 & 0 & x_1 & \delta^1 & 0 \\ 0 & 0 & x_2 & 0 & \delta^2 \\ x_1^* & x_2^* & \mathscr{D} & 0 & 0 \\ (\delta^1)^* & 0 & 0 & 0 & 0 \\ 0 & (\delta^2)^* & 0 & 0 & 0 \end{pmatrix},$$

where

$$x_i = ((1 - \Pi_{\mathscr{K}^i_{k-1}}) \circ \mathrm{Hess}S)|_{\mathscr{K}_k}.$$

We claim that x_i is a compact operator (and hence so is its adjoint x_i^*). Indeed, the operator $(1 - \Pi_{\mathscr{K}^i_{k-1}})$ (equivalently, the projection to $\mathscr{T}^i_{k-1}$ in (2.7)) is given by

$$(1 - \Pi_{\mathscr{K}^i_{k-1}}) = \delta^i \circ ((\delta^i)^*\delta^i)^{-1} \circ (\delta^i)^*,$$

and we then observe that:

- $\delta^i \circ ((\delta^i)^*\delta^i)^{-1}$ is a pseudo-differential operator of order -1 (i.e., a smoothing operator of order 1);
- $\delta_i^* \circ \mathrm{Hess}S$ is a *first*-order differential operator; in particular, it defines a bounded linear operator $L^2_k \to L^2_{k-1}$.

The second item is verified by direct calculation; for example, as a differential operator $(\Omega^1 \oplus \Omega^1)(Y,\mathfrak{g}_E) \to \Omega^0(Y,\mathfrak{g}_E)$ we have

$$(\delta^2)^*\mathrm{Hess}S = 2 \cdot \left(\left[* \left(F_A - \frac{I}{2} F_\lambda - \frac{1}{2}[\Phi,\Phi] \right), \cdot \right], - * [d_A\Phi, \cdot] \right).$$

At this point, we have that the operator obtained from $\widehat{\mathrm{Hess}S}$ by dropping the entries x_i, x_i^* is also Fredholm. Furthermore, this operator contains $\mathscr{D}$ as a summand, hence the Fredholm property of $\mathscr{D}$ follows.

Assertion (1) holds for the extended Hessian, which is a self-adjoint elliptic operator of order 1. Hence, it also holds for its compact perturbation obtained by dropping the entries x_i, x_i^*. Since the latter operator contains $\mathscr{D}$ as a summand, assertion (1) follows. Assertion (2) follows from the fact that x_i vanishes at a critical point. □

The fact that the complex Chern–Simons functional CS is holomorphic manifests itself in the symmetries of the Hessian of $S = \mathrm{Im}CS$. To give some context, we recall that on a finite-dimensional complex manifold (M,J) the real part $f = \mathrm{Re}F$ of a holomorphic function $F : M \to \mathbb{C}$ is *pluriharmonic*: $dd^c f = 0$. If (M,J) admits a Kähler metric g, then we obtain from g the covariant Hessian $\mathrm{Hess}f : TM \to TM$, and pluriharmonicity of f becomes equivalent to the vanishing

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of the 'complex average' of its Hessian: for all $x \in M$, $v, w \in T_x M$,

$$\tfrac{1}{2}(\langle (\mathrm{Hess} f)_x v, w\rangle + \langle (\mathrm{Hess} f)_x Jc, Jw\rangle) = 0.$$

Proposition 2.14. *For any stable connection (A, Φ) (not necessarily flat), the Hessian $\mathscr{D}_{(A,\Phi)} : \mathscr{K}_{(A,\Phi),k} \to \mathscr{K}_{(A,\Phi),k-1}$ satisfies the following identity: for all $v \in \mathscr{K}_{(A,\Phi),k}$ and $w \in \mathscr{K}_{(A,\Phi),k-1}$,*

$$\langle \mathscr{D} v, w\rangle_{L^2} + \langle \mathscr{D}(Jv), Jw\rangle_{L^2} = 0.$$

Proof. The required identity holds for the flat Hessian defined in (2.25):

$$\langle (\mathrm{Hess} S)_{(A,\Phi)} v, w\rangle_{L^2} + \langle (\mathrm{Hess} S)_{(A,\Phi)} Jv, Jw\rangle_{L^2} = 0.$$

The claim now follows since $\mathscr{K}_k$ is a J-linear subspace of $(\Omega^1 \oplus \Omega^1)(Y, \mathfrak{g}_E)$. □

In particular, we have that the kernel of $\mathscr{D}_{(A,\Phi)}$ is a complex-linear subspace of $\mathscr{K}_{(A,\Phi),k}$.

2.3.4 *Kuranishi models.* An important point that is used later is the existence of finite-dimensional models for S, which readily follows from Theorem 2.13.

Definition 2.15. At a stable flat connection $[(A, \Phi)] \in \mathscr{M}^*_E$, the *Zariski tangent space* to $\mathscr{M}^*_E$ is the finite-dimensional vector space $\mathscr{H}^1_{(A,\Phi)} = \mathrm{Ker} \mathscr{D}_{(A,\Phi)}$.

Let (A, Φ) be a stable flat connection. Recall that the Hilbert space $\mathscr{K}_{(A,\Phi),k}$ is identified as the tangent space to the stable configuration space $\mathscr{B}^*_{E,k}$. Let $U \subset \mathscr{K}_{(A,\Phi),k}$ (respectively, $U' \subset \mathscr{K}_{(A,\Phi),k-1}$) be the closed subspace of $\mathscr{K}_{(A,\Phi),k}$ (respectively, $\mathscr{K}_{(A,\Phi),k-1}$) given as the L^2-orthogonal complement of $\mathscr{H}^1_{(A,\Phi)}$. Then the Hessian $\mathscr{D}_{(A,\Phi)}$ restricts to a bounded linear isomorphism from U to U', since $\mathscr{D}_{(A,\Phi)}$ is a self-adjoint operator.

Corollary 2.16. *Let (A, Φ) be a stable flat connection. Then there exist:*

- *a diffeomorphism G from an open neighborhood of $[(A, \Phi)] \in \mathscr{B}^*_{E,k}$ to an open neighborhood of $(0, 0)$ in $U \oplus \mathscr{H}^1_{A,\Phi}$, such that $G([(A, \Phi)]) = (0, 0)$; and*
- *a smooth function S' on a neighborhood of 0 in $\mathscr{H}^1_{(A,\Phi)}$ with $(dS')_0 = 0$ and $(\mathrm{Hess} S')_0 = 0$;*

such that on the codomain of G we have

$$S(G^{-1}(u, h)) = \tfrac{1}{2}\langle \mathscr{D}_{(A,\Phi)} u, u\rangle_{L^2} + S'(h).$$

In particular, a neighborhood of $[(A, \Phi)]$ in the moduli space of stable flat connections $\mathscr{M}^*_E$ is homeomorphic to a neighborhood of 0 in the critical locus of $S' : \mathscr{H}^1_{(A,\Phi)} \to \mathbb{R}$.

Proof. The argument closely follows [Don02, Proposition 2.5]. At a connection $(A + a, \Phi + \phi)$ the gradient $\mathrm{grad} S$ vector field is tangent to the subspace $\mathscr{K}_{(A+a,\Phi+\phi),k-1}$. The L^2-orthogonal projection $\Pi_{\mathscr{K}_{(A,\Phi),k-1}}$ from $(\Omega^1 \oplus \Omega^1)(Y, \mathfrak{g}_E)_{k-1}$ onto $\mathscr{K}_{(A,\Phi),k-1}$ thus restricts to an isomorphism from $\mathscr{K}_{(A+a,\Phi+\phi),k-1}$ to $\mathscr{K}_{(A,\Phi),k-1}$ for (a, ϕ) in a small neighborhood of $(0, 0)$. Consider the following smooth map defined along the stable Coulomb slice (cf. Proposition 2.5):

$$f : (A + a, \Phi + \phi) \in \mathcal{S}_{(A,\Phi),k} \mapsto \Pi_{\mathscr{K}_{(A,\Phi),k-1}}((\mathrm{grad} S)_{(A+a,\Phi+\phi)}) \in \mathscr{K}_{(A,\Phi),k-1}.$$

Now, the function $\Pi_{U'} \circ f : \mathcal{S}_{(A,\Phi),k} \to U'$ has the property that its derivative at $(a, \phi) = (0, 0)$ is surjective when restricted to directions in $U \subset \mathscr{K}_{(A,\Phi),k} = T_{(A,\Phi)} \mathcal{S}_{(A,\Phi)}$. Thus, a change of coordinates linearizes $\Pi_U \circ f$. Namely, using the inverse function theorem for Hilbert spaces, one obtains a diffeomorphism F from an open neighborhood of (A, Φ) in $\mathcal{S}_{(A,\Phi),k}$ onto an open neighborhood of $(0, 0)$ in $U \oplus \mathscr{H}^1_{(A,\Phi)}$, sending (A, Φ) to $(0, 0)$, together with a smooth function

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f' from an open neighborhood of $(0,0)$ in $U \oplus \mathscr{H}^1_{(A,\Phi)}$ to an open neighborhood of 0 in $\mathscr{H}^1_{(A,\Phi)}$, $f'(0,0) = 0$, and such that

$$f(F^{-1}(u,h)) = (\mathscr{D}_{(A,\Phi)}u, f'(u,h)).$$

Consider now the family of smooth functions $S_h : U \to \mathbb{R}$ obtained by restricting $S \circ F^{-1}$ onto $U \times \{h\}$. By the above, it follows that each of these functions has a critical point at $0 \in U$ and this is a non-degenerate critical point for $h = 0$. Thus, for all h sufficiently close to 0, the function f_h will have a non-degenerate critical point at 0. Then by the parametric version of the Morse–Palais lemma [Lan85], a further coordinate change ensures that each f_h is given by a quadratic function on U plus a constant. A further change of coordinates by a family of linear isomorphisms of U ensures that the quadratic functions are independent of h, and the constants then give the required function $S'(h)$. □

For future reference, we include here the following definition.

DEFINITION 2.17. The moduli space $\mathscr{M}^*_E$ is *Morse–Bott* if it is a smooth submanifold of $\mathscr{B}^*_E$ (hence, a complex submanifold under J) and its dimension at a point $[(A,\Phi)] \in \mathscr{M}^*_E$ is given by $\dim \operatorname{Ker} \mathscr{D}_{(A,\Phi)}$. By Corollary 2.16, this is equivalent to being able to choose the function $S' : \mathscr{H}^1_{(A,\Phi)} \to \mathbb{R}$ constant.

2.4 Other structure groups

Throughout the discussion we have stuck to the structure group $G = SU(2)$ and its complexification $G^c = SL(2,\mathbb{C})$ for the sake of concreteness, and because this is the main case of interest. However, the discussion from this section can be easily adapted to the case when G^c is a (real or complex) semi-simple Lie group.

For this we fix a global Cartan involution Θ of G^c, whose fixed subgroup gives a maximal compact subgroup $G \subset G^c$. Let $\mathfrak{g}^c = \mathfrak{g} \oplus \mathfrak{m}$ be the induced 'Cartan decomposition' at the Lie algebra level (in terms of the eigenspace decomposition of the involution θ of $\mathfrak{g}^c$ induced from Θ). We obtain a G-invariant inner product on $\mathfrak{g}_c$ by using minus the Killing form on $\mathfrak{g}_c$ with θ inserted in one entry (for $\mathfrak{g}^c = \mathfrak{sl}(2,\mathbb{C})$ this gave $\operatorname{Tr}(AB^*)$).

Our basic setup is then modified as follows. We fix a principal G bundle E. Similarly to the case we discussed, a G^c connection on $E \times_G G^c$ is decomposed uniquely as $A + \Phi$ where A is a G connection and $\Phi \in \Omega^1(Y, E \times_G \mathfrak{m})$.

When G^c is real, the Kähler quotient viewpoint on $\mathscr{B}^*_E = \{(A,\Phi) \mid d^*_A \Phi = 0\}/\mathscr{G}_E$ no longer makes sense; however, the same definition and the variational viewpoint still do. Much as in the case we presented, one can show that the stable configuration space thus obtained will again be a Hilbert Riemannian manifold equipped with the natural L^2 metric (albeit not Kähler, when G^c is real).

Of course, the Chern–Simons functional (which uses the Killing form of $\mathfrak{g}^c$ in place of Tr) is just real-valued when G^c is real. All the results discussed above, such as the Fredholm property of the Hessian, carry through in the more general setup.

3. Seifert-fibered 3-manifolds and dimensional reduction

In this section we study the moduli spaces of stable (projectively) flat $GL(2,\mathbb{C})$ connections when (Y,g) is a closed, oriented, *Seifert-fibered* 3-manifold equipped with a Riemannian metric g belonging in a suitable class, by establishing dimensional reduction results. Dimensional

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reduction for Seifert-fibered 3-manifolds is a familiar theme in gauge theory: for flat $SU(2)$ connections, this was studied by Furuta and Steer [FS92] and for the $U(1)$ Seiberg–Witten equations by Mrowka, Ozsváth and Yu [MOY97] (see also [MST96, MW05, Doa19]).

We recall that a closed, oriented 3-manifold Y is *Seifert-fibered* if it admits a *Seifert fibration* over a closed, oriented, orbifold surface C: there exist an orbifold complex line bundle $N \to C$, an (orbifold) hermitian metric on N such that the unit sphere $S(N)$ is a smooth 3-manifold (i.e., a 3-orbifold with empty orbifold locus), and a diffeomorphism of Y with $S(N)$. The projection $\pi : Y \to C$ makes Y the total space of an orbifold principal $U(1)$ bundle. For a treatment of differential and complex geometry on orbifolds we refer the reader to [BG08].

Throughout this section Y denotes a Seifert-fibered 3-manifold equipped with a fixed Seifert fibration $\pi : Y \to C$.

3.1 Seifert metrics and adiabatic connection

We now introduce the class of Riemannian metrics on Y that we consider. Let g_C be an orbifold Riemannian metric on the base orbifold C, with associated volume form denoted by ω_C. In particular, C becomes an orbifold Riemann surface. Let μ be a unitary connection on $N \to C$ with *harmonic curvature*: that is, the curvature of μ is a constant multiple of the volume form,

$$F_\mu = 2i\xi \cdot \omega_C, \quad \xi := -\frac{\pi \deg N}{\mathrm{Vol}(C, g_C)}. \tag{3.1}$$

The connection μ can be regarded as an imaginary 1-form on Y, and when using this viewpoint we denote it by $\mu = i\eta$, where $\eta \in \Omega^1(Y, \mathbb{R})$. (The harmonic curvature condition then says that $d\eta = 2\xi \cdot \pi^*\omega_C$.) We obtain a splitting of TY,

$$TY = \underline{\mathbb{R}} \oplus \pi^*(TC), \tag{3.2}$$

where $\underline{\mathbb{R}} = \mathrm{Ker}(d\pi)$ is the vertical subbundle, and $\pi^*(TC) = \mathrm{Ker}(\eta)$ is the horizontal subbundles associated to the connection μ.

Definition 3.1. A *Seifert metric* on Y is a Riemannian metric obtained from the data $(g_C, \mu = i\eta)$ by the formula

$$g = \eta \otimes \eta + \pi^*(g_C).$$

Throughout this section we fix such a Seifert metric g on Y. Denote by ζ the vector field on Y that generates the Seifert S^1-action on Y. Then ζ has unit length and is metric dual to the 1-form η.

A Seifert metric g makes the splitting (3.2) orthogonal. However, the Levi-Civita connection of g, which we denote by $\widehat{\nabla}$, is *not* compatible with this splitting in general. Instead, we consider a different orthogonal connection on TY, namely the *adiabatic connection* ∇ defined in terms of the splitting (3.2) and the Levi-Civita connection ∇_C of C by

$$\nabla = d \oplus \pi^*(\nabla_C).$$

This is the natural connection to consider for the purpose dimensional reduction. (For example, a 1-form α on Y is the pullback of a 1-form on C precisely when $\nabla_\zeta \alpha = 0$.) We need to compare the Levi-Civita and adiabatic connections: the following calculation can be found in [Nic98, § 1].

Lemma 3.2. *Let $\zeta_0, \zeta_1, \zeta_2$ be a local orthonormal frame of (TY, g) with $\zeta_0 = \zeta$, and ζ_2, ζ_3 given by the horizontal lifts of a local orthonormal frame of (TC, g_C). Denote the dual local orthonormal frame of T^*Y by $\eta^0 (= \eta), \eta^1, \eta^2$. Then the difference between the Levi-Civita and the adiabatic*

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connections on T^*Y, regarded as an endomorphism-valued 1-form $\widehat{\nabla} - \nabla \in \Omega^1(Y, \mathfrak{so}(T^*Y))$, is given by

$$\widehat{\nabla} - \nabla = \begin{pmatrix} 0 & \xi\eta^2 & -\xi\eta^1 \\ -\xi\eta^2 & 0 & -\xi\eta \\ \xi\eta^1 & \xi\eta & 0 \end{pmatrix}.$$

Lemma 3.2 shows, in particular, that the adiabatic connection ∇ is obtained as the C^∞ limit as $t \to \infty$ of the Levi-Civita connection associated to the metric $g_t = \eta \otimes \eta + t\pi^* g_C$.

3.2 Vanishing results

We now discuss the vanishing results that will enable the dimensional reduction. We fix throughout a $U(2)$ bundle $E \to Y$ with fixed determinant $(\Lambda = \Lambda^2 E, \lambda)$. We also fix throughout a $GL(2,\mathbb{C})$ connection $\mathbb{A} = A + i\Phi$ on E with fixed determinant.

3.2.1 *Covariant derivatives.* We first describe how the exterior covariant derivatives $d^A : \Omega^n(Y, \mathfrak{g}_E) \to \Omega^{n+1}(Y, \mathfrak{g}_E)$, $n \geq 0$, interact with the Seifert fibration $\pi : Y \to C$.

By pullback, we have the subbundle $\pi^*(T^*C) \subset T^*Y$. (This is the annihilator of the vertical subbundle $\mathbb{R} = \mathrm{Ker}(d\pi) \subset TY$.) Denote its nth exterior power by $\Lambda^n_H := \Lambda^n(\pi^*(T^*C))$, and the space of sections of Λ^n_H by $\Omega^n_H(Y, \mathfrak{g}_E)$. Using the connection $\mu = i\eta$ on N, we obtain the splitting $T^*Y = \mathbb{R} \oplus \pi^*(T^*C)$ dual to (3.2), which induces a splitting for all $n \geq 0$:

$$\begin{aligned} \Omega^n(Y, \mathfrak{g}_E) &= \Omega^{n-1}_H(Y, \mathfrak{g}_E) \oplus \Omega^n_H(Y, \mathfrak{g}_E) \\ \eta \cdot \alpha + \beta &\leftrightarrow (\alpha, \beta). \end{aligned} \tag{3.3}$$

We consider the following differential operators:

- the adiabatic covariant derivative $\nabla^A_\zeta : \Omega^n(Y, \mathfrak{g}_E) \to \Omega^n(Y, \mathfrak{g}_E)$ along the *vertical* vector field ζ that generates the Seifert S^1-action;
- the *horizontal* exterior covariant derivative $d^H_A : \Omega^n_H(Y, \mathfrak{g}_E) \to \Omega^{n+1}_H(Y, \mathfrak{g}_E)$ defined by $(d^H_A \gamma)(V_1, \ldots, V_{n+1}) = (d_A \gamma)(V_1^H, \ldots, V_{n+1}^H)$, where V^H stands for the projection of a vector field V on Y onto the horizontal subspace of (3.2);
- the formal L^2-adjoint of d^H_A, given by $(d^H_A)^* = - *_H d^H_A *_H : \Omega^n(Y, \mathfrak{g}_E) \to \Omega^{n-1}(Y, \mathfrak{g}_E)$, where $*_H : \Lambda^*_H \xrightarrow{\cong} \Lambda^{2-*}_H$ is the bundle isomorphism induced by the Hodge star operator $*_C$ associated to the orbifold surface (C, g_C).

We need the following result.

LEMMA 3.3. *With respect to the splitting (3.3), the exterior covariant derivative $d_A : \Omega^n(Y, \mathfrak{g}_E) \to \Omega^{n+1}(Y, \mathfrak{g}_E)$ is given in block form by*

$$d_A = \begin{pmatrix} -d^H_A & \nabla^A_\zeta \\ 2\xi(\pi^*\omega_C) \wedge \cdot & d^H_A \end{pmatrix}$$

and its formal L^2-adjoint $d^*_A : \Omega^n(Y, \mathfrak{g}_E) \to \Omega^{n-1}(Y, \mathfrak{g}_E)$ by

$$d^*_A = \begin{pmatrix} -(d^H_A)^* & 2\xi(\pi^*\omega_C \wedge \cdot)^* \\ -\nabla^A_\zeta & (d^H_A)^* \end{pmatrix}.$$

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Proof. Fix a local orthonormal frame and coframe as in Lemma 3.2. We recall the standard formula expressing the exterior derivative d_A in terms of the Levi-Civita connection

$$d_A = \eta \wedge \widehat{\nabla}^A_\zeta + \eta^1 \wedge \widehat{\nabla}^A_{\zeta_1} + \eta^2 \wedge \widehat{\nabla}^A_{\zeta_2}.$$

Let $\Psi \in \Omega^n_H(Y, \mathfrak{g}_E)$. Then the horizontal exterior covariant derivative is given by

$$d^H_A \Psi = \eta^1 \wedge \nabla^A_{\zeta_1} \Psi + \eta^2 \wedge \nabla^A_{\zeta_2} \Psi. \tag{3.4}$$

By Lemma 3.2 when $n = 1$, and hence by the Leibniz rule for general n, we have

$$\eta \wedge \widehat{\nabla}^B_\zeta \beta + \eta^1 \wedge \widehat{\nabla}^B_{\zeta_1} \Psi + \eta^2 \wedge \widehat{\nabla}^B_{\zeta_2} \Psi = \eta \wedge \nabla^B_\zeta \Psi + \eta^1 \wedge \nabla^B_{\zeta_1} \Psi + \eta^2 \wedge \nabla^B_{\zeta_2} \Psi.$$

Putting everything together, we have $d_A \Psi = \eta \wedge \nabla^A_\zeta \Psi + d^H_A \Psi$, which gives the second column in the block matrix for d_A. By this formula and the Leibniz rule we also have

$$d_A(\eta \wedge \Psi) = d\eta \wedge \Psi - \eta \wedge d_A \Psi = 2\xi \pi^*(\omega_C) \wedge \Psi - \eta \wedge d^H_A \Psi,$$

which gives the second column of the block matrix for d_A. The formula for d^*_A follows by taking the adjoint of the block matrix for d_A. □

3.2.2 *A Bochner–Weitzenböck formula.* We decompose the imaginary part $\Phi \in \Omega^1(Y, \mathfrak{g}_E)$ of a $GL(2, \mathbb{C})$ connection $A + i\Phi$ according to (3.3) as

$$\Phi = \eta \cdot \alpha + \Psi, \tag{3.5}$$

where $\alpha \in \Omega^0(Y, \mathfrak{g}_E)$ and $\Psi \in \Omega^1_H(Y, \mathfrak{g}_E)$.

Proposition 3.4. *Let $A + i\Phi$ be a $GL(2, \mathbb{C})$ connection on E with fixed determinant, and let α, Ψ be as in (3.5). Then the following pointwise identity holds:*

$$\begin{aligned}\langle (d_A d^*_A + d^*_A d_A)\Phi, \Phi\rangle = &\langle -(\nabla^A_\zeta)^2 \alpha, \alpha\rangle + \langle (*_H \nabla^B_\zeta)^2 \Psi, \Psi\rangle \\ &+ \langle (d^H_A)^* d^H_A \alpha, \alpha\rangle + \langle d^H_A (d^H_A)^* \Psi, \Psi\rangle + \langle [\iota_\zeta(F_A), \alpha], \Psi\rangle \\ &+ \langle (d^H_A)^* d^H_A \Psi, \Psi\rangle + 4\xi \langle *_H d^H_A \Psi, \alpha\rangle + 4\xi^2 \cdot |\alpha|^2.\end{aligned}$$

(Here $\iota_\zeta F_A = (F_A)(\zeta, \cdot)$ denotes contraction of F_A with the vertical vector field ζ.)

Proof. Consider the self-adjoint differential operator $D_A : (\Omega^0 \oplus \Omega^1)(Y, \mathfrak{g}_E) \to (\Omega^0 \oplus \Omega^1)(Y, \mathfrak{g}_E)$ given by the *odd signature operator* coupled with the connection A on $\mathfrak{g}_E$:

$$D_A = \begin{pmatrix} 0 & -d^*_A \\ -d_A & *d_A \end{pmatrix}.$$

This operator squares to give

$$(D_A)^2 = \begin{pmatrix} d_A d^*_A + d^*_A d_A & *[F_A, \cdot] \\ -*[F_A, \cdot] & d_A d^*_A + d^*_A d_A \end{pmatrix}. \tag{3.6}$$

We further decompose $(\Omega^0 \oplus \Omega^1)(Y, \mathfrak{g}_E) = (\Omega^0 \oplus \Omega^0 \oplus \Omega^1_H)(Y, \mathfrak{g}_E)$ using the splitting (3.3) for $n = 1$. By Lemma 3.3 the operator D_A is given by $D_A = D^A_\zeta + D^A_H$, where

$$D^A_\zeta = \begin{pmatrix} 0 & \nabla^A_\zeta & 0 \\ -\nabla^A_\zeta & 2\xi & 0 \\ 0 & 0 & *_H \nabla^A_\zeta \end{pmatrix}, \quad D^A_H = \begin{pmatrix} 0 & 0 & -(d^H_A)^* \\ 0 & 0 & *_H d^H_A \\ -d^H_A & -*_H d^H_A & 0 \end{pmatrix}.$$

We have the identity

$$(D_A)^2 = (D^A_\zeta)^2 + (D^A_H)^2 + (D^A_\zeta D^A_H + D^A_H D^A_\zeta) \tag{3.7}$$

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and in what follows now we compute the three operators on the right-hand side of (3.7) one by one:

$$(D_\zeta^A)^2 = \begin{pmatrix} -(\nabla_\zeta^A)^2 & 2\xi\nabla_\zeta^A & 0 \\ -2\xi\nabla_\zeta^A & -(\nabla_\zeta^A)^2 + 4\xi^2 & 0 \\ 0 & 0 & (*_H\nabla_\zeta^A)^2 \end{pmatrix}, \tag{3.8}$$

$$(D_H^A)^2 = \begin{pmatrix} (d_A^H)^* d_A^H & *_H (d_A^H)^2 & 0 \\ -*_H (d_A^H)^2 & (d_A^H)^* d_A^H & 0 \\ 0 & 0 & d_A^H (d_A^H)^* + (d_A^H)^* d_A^H \end{pmatrix}, \tag{3.9}$$

and $(D_\zeta^B D_H^B + D_H^B D_\zeta^B)$ is given by

$$\begin{pmatrix} 0 & 0 & \nabla_\zeta^A *_H d_A^H - *_H d_A^H \nabla_\zeta^A \\ 0 & 0 & \nabla_\zeta^A (d_A^H)^* - (d_A^H)^* \nabla_\zeta^A + 2\xi *_H d_A^H \\ -*_H \nabla_\zeta^A d_A^H + *_H d_A^H \nabla_\zeta^A - *_H \nabla_\zeta^A *_H d_A^H - d_A^H \nabla_\zeta^A - 2\xi *_H d_A^H & 0 \end{pmatrix}.$$

The entries in the last matrix can be described more succinctly. For this, we fix a local orthonormal frame $\zeta_0(=\zeta), \zeta_2, \zeta_2$ and dual coframe $\eta^0(=\eta), \eta^1, \eta^2$ as in Lemma 3.2. By definition of the adiabatic connection, we have $\nabla_\zeta \eta^1 = \nabla_\zeta \eta^2 = 0$. We then compute as follows, for an auxiliary $s \in \Omega^0(Y, \mathfrak{g}_E)$:

$$\begin{aligned}
(-*_H \nabla_\zeta^A *_H d_A^H - d_A^H \nabla_\zeta^A)s &= -*_H \nabla_\zeta^A (\eta^2 \wedge \nabla_{\zeta_1}^A s - \eta^1 \wedge \nabla_{\zeta_2}^A s) - (\eta^1 \wedge \nabla_{\zeta_1}^A \nabla_\zeta^A s + \eta^2 \wedge \nabla_{\zeta_2}^A \nabla_\zeta^A s) \\
&= -*_H (\eta^2 \wedge \nabla_\zeta^A \nabla_{\zeta_1}^A s - \eta^1 \wedge \nabla_\zeta^A \nabla_{\zeta_2}^A s) \\
&\quad - (\eta^1 \wedge \nabla_{\zeta_1}^A \nabla_\zeta^A s + \eta^2 \wedge \nabla_{\zeta_2}^A \nabla_\zeta^A s) \\
&= e_1[\nabla_\zeta^A, \nabla_{e_1}^A] + e_2[\nabla_\zeta^A, \nabla_{e_2}^A])s = \tfrac{1}{2}[\iota_\zeta F_B, s], \\
(-*_H \nabla_\zeta^A d_A^H + *_H d_A^H \nabla_\zeta^A)s &= -*_H \nabla_\zeta^A (\eta^1 \wedge \nabla_{\zeta_1}^A s + \eta^2 \wedge \nabla_{\zeta_2}^A s) + *_H d_A^H \nabla_\zeta^A s \\
&= -*_H (\eta^1 \wedge \nabla_\zeta^A \nabla_{\zeta_1}^A s + \eta_2 \wedge \nabla_\zeta^A \nabla_{\zeta_2}^A s) \\
&\quad + *_H (\eta_1 \wedge \nabla_{\zeta_1}^A \nabla_\zeta^A s + \eta_2 \wedge \nabla_{\zeta_2}^A \nabla_\zeta^A s) \\
&= -*_H (\eta_1 \wedge [\nabla_\zeta^B, \nabla_{\zeta_1}^B] + \eta_2 \wedge [\nabla_\zeta^B, \nabla_{\zeta_2}^B])s = -*_H \tfrac{1}{2}[\iota_\zeta F_B, s].
\end{aligned}$$

By the self-adjointness of $(D_\zeta^A D_H^A + D_H^A D_\zeta^A)$, the remaining entries in the matrix can be determined from the previous two. Putting everything together we obtain

$$(D_\zeta^A D_H^A + D_H^A D_\zeta^A) = \begin{pmatrix} 0 & 0 & -*_H \tfrac{1}{2}[\iota_\zeta F_A, \cdot] \\ 0 & 0 & *_H \tfrac{1}{2}[\iota_\zeta F_A, *_H \cdot] + 2\xi *_H d_A^H \\ -*_H \tfrac{1}{2}[\iota_\zeta F_A, \cdot] & \tfrac{1}{2}[\iota_\zeta F_A, \cdot] - 2\xi *_H d_A^H & 0 \end{pmatrix}. \tag{3.10}$$

Finally, the required identity follows by applying both sides of (3.7) to $(0, \alpha, \Psi)$, inserting the expressions (3.6), (3.8)–(3.10), and then taking the pointwise inner product against $(0, \alpha, \Psi)$. □

We have now arrived at the crucial vanishing result.

COROLLARY 3.5. *Let $A + i\Phi$ be a $GL(2,\mathbb{C})$ connection on E with fixed determinant, and let α, Ψ be as in (3.5). Suppose that $A + i\Phi$ is projectively flat (cf. (2.18) and (2.19)) and polystable (cf. (2.20)). Then the following quantities vanish identically:*

$$\nabla_\zeta^A \alpha = 0, \quad \nabla_\zeta^A \Psi = 0, \quad d_A^H \alpha = 0, \quad [\alpha, \Psi] = 0,$$

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and we have

$$F_A - \frac{I}{2}F_\lambda - \frac{1}{2}[\Psi,\Psi] = 0,$$
$$d_A^H \Psi = 0,$$
$$(d_A^H)^* \Psi = 0.$$

Furthermore, if either the orbifold degree $\deg N$ *is non-zero or* $A + i\Phi$ *is stable, then* $\alpha = 0$ *also.*

Proof. By (2.18) we have $\iota_\zeta F_A = [\alpha,\Psi] + \frac{I}{2}\iota_\zeta F_\lambda$, and thus

$$\langle [\iota_\zeta F_A, \alpha], \Psi\rangle = \langle [[\alpha,\Psi],\alpha],\Psi\rangle = |[\alpha,\Psi]|^2.$$

By (2.19) and (2.20) the left-hand side of the identity in Proposition 3.4 vanishes identically. Integrating that identity over Y and integrating by parts yields the following identity:

$$\begin{aligned} 0 = &\|\nabla_\zeta^A \alpha\|^2_{L^2(Y)} + \|\nabla_\zeta^A \Psi\|^2_{L^2(Y)} + \|d_A^H \alpha\|^2_{L^2(Y)} + \|(d_A^H)^*\Psi\|^2_{L^2(Y)} + \|[\alpha,\Psi]\|^2_{L^2(Y)} \\ &+ \| *_H d_A^H \Psi + 2\xi\alpha\|^2_{L^2(Y)}. \end{aligned}$$

Thus,

$$\nabla_\zeta^A \alpha = 0, \quad \nabla_\zeta^A \Psi = 0, \quad d_A^H \alpha = 0,$$
$$(d_A^H)^*\Psi = 0, \quad [\alpha,\Psi] = 0, \quad *_H d_A^H \Psi + 2\xi\alpha = 0.$$

Applying d_A^H to the last identity above and using $d_A^H\alpha = 0$ gives $d_A^H *_H d_A^H \Psi = 0$. Integration by parts then gives $d_A^H \Psi = 0$ and hence also $\xi\alpha = 0$.

Thus, if $\xi \neq 0$ (equivalently, if $\deg N \neq 0$) then $\alpha = 0$. Finally, suppose that $\xi = 0$ but $A + i\Phi$ is stable. Then from $\nabla_\zeta^A \alpha = 0$, $d_A^H \alpha = 0$ and Lemma 3.3 we have $d_A \alpha = 0$. This, combined with $[\alpha,\Phi] = [\alpha,\Psi] = 0$, implies that α vanishes by stability. With this, the proof is complete. □

We also need an 'infinitesimal' analog of Corollary 3.5. Let $A + i\Phi$ be a stable projectively flat $GL(2,\mathbb{C})$ connection with fixed determinant. Then the Zariski tangent space $\mathscr{H}^1_{(A,\Phi)} = \operatorname{Ker}\mathscr{D}_{(A,\Phi)}$ to the moduli space $\mathscr{M}_E^*$ at $[(A,\Phi)]$ is given by the space of $(\dot A, \dot\Phi) \in (\Omega^1 \oplus \Omega^1)(Y, \mathfrak{g}_E)$ such that

$$\begin{aligned} -d_A^* \dot A - *[\dot\Phi, *\Phi] &= 0, \\ *[\dot A, *\Phi] - d_A^* \Phi &= 0, \\ *[\dot A, \Phi] + *d_A \dot\Phi &= 0, \\ *d_A \dot A - *[\dot\Phi, \Phi] &= 0. \end{aligned} \tag{3.11}$$

Decompose $\dot A = \eta\cdot\dot a + \dot B$ and $\dot\Phi = \eta\cdot\dot\alpha + \dot\Psi$ according to the splitting (3.3).

Proposition 3.6. *Suppose that* $A + i\Phi$ *is a polystable projectively flat* $GL(2,\mathbb{C})$ *connection on* E *with fixed determinant, and let* α, Ψ *be as in (3.5). Suppose that* $(\dot A, \dot\Phi) \in (\Omega^1 \oplus \Omega^1)(Y, \mathfrak{g}_E)$ *solves equations (3.11). Then the following quantities vanish identically:*

$$\nabla_\zeta \dot a = 0, \nabla_\zeta^A \dot B = 0, [\dot a,\alpha] = 0, [\dot a,\Psi] = 0, [\dot B,\alpha] = 0,$$
$$\nabla_\zeta \dot\alpha = 0, \quad \nabla_\zeta^A \dot\Psi = 0, [\dot\alpha,\alpha] = 0, [\dot\alpha,\Psi] = 0, [\dot\Psi,\alpha] = 0,$$

and we have

$$\begin{aligned} -(d_A^H)^* \dot B - *_H[\dot\Psi, *_H\Psi] &= 0, \\ *_H[\dot B, *_H\Psi] - (d_A^H)^*\dot\Psi &= 0, \\ *_H[\dot B,\Psi] + *_H d_A^H \dot\Psi &= 0, \\ *_H d_A^H \dot B - *_H[\dot\Psi,\Psi] &= 0. \end{aligned}$$

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Furthermore, if either the orbifold degree $\deg N$ *is non-zero or* $A+i\Phi$ *is stable, then* $\dot{a}=0$ *and* $\dot{\alpha}=0$ *also.*

Proof. Equations (3.11) can be decomposed into vertical and horizontal parts using Lemma 3.3:

$$\begin{aligned}
-(d_A^H)^*\dot{B} - *_H[\dot{\Psi}, *_H\Psi] &= -\nabla_\zeta^A\dot{a} + [\dot{\alpha},\alpha],\\
*_H[\dot{B}, *_H\Psi] - (d_A^H)^*\dot{\Psi} &= -\nabla_\zeta^A\dot{\alpha} - [\dot{a},\alpha],\\
*_H[\dot{B},\Psi] + *_H d_A^H\dot{\Psi} &= -2\xi\dot{\alpha},\\
- *_H d_A^H\dot{\alpha} + *_H\nabla_\zeta^A\dot{\Psi} + *_H[\dot{a},\Psi] - *_H[\dot{B},\alpha] &= 0,\\
*_H d_A^H\dot{B} - *_H[\dot{\Psi},\Psi] &= -2\xi\dot{a},\\
- *_H d_A^H\dot{a} + *_H\nabla_\zeta^A\dot{B} - *_H[\dot{\alpha},\Psi] + *_H[\dot{\Psi},\alpha] &= 0.
\end{aligned} \tag{3.12}$$

We first establish the vanishing of the quantities $\nabla_\zeta\dot{\alpha}, d_A^H\dot{\alpha}, [\dot{\alpha},\alpha], [\dot{\alpha},\Psi]$. For this, we express the second and third equations in (3.11) jointly in terms of the operator D_A introduced in the proof of Proposition 3.4, as

$$D_A\begin{pmatrix}0\\ \dot{\Phi}\end{pmatrix} + \begin{pmatrix}*[\dot{A}, *\Phi]\\ *[\dot{A},\Phi]\end{pmatrix} = 0.$$

We then apply D_A to this identity and take the inner product against $(0,\eta\dot{\alpha})$:

$$\left\langle D_A^2\begin{pmatrix}0\\ \dot{\Phi}\end{pmatrix}, \begin{pmatrix}0\\ \eta\dot{\alpha}\end{pmatrix}\right\rangle + \left\langle D_A\begin{pmatrix}*[\dot{A}, *\Phi]\\ *[\dot{A},\Phi]\end{pmatrix}, \begin{pmatrix}0\\ \eta\dot{\alpha}\end{pmatrix}\right\rangle = 0. \tag{3.13}$$

We now compute each term in (3.13). For the first term, by (3.6), (3.8), (3.9), and the vanishing of the terms involving $\iota_\zeta F_A$ in (3.10) (since $[\alpha,\Psi]=0$ by Corollary 3.5), we have

$$\begin{aligned}
\left\langle D_A^2\begin{pmatrix}0\\ \dot{\Phi}\end{pmatrix}, \begin{pmatrix}0\\ \eta\dot{\alpha}\end{pmatrix}\right\rangle &= \langle -\nabla_\zeta^A\dot{\alpha},\dot{\alpha}\rangle + 4\xi^2|\dot{\alpha}|^2 + \langle (d_A^H)^*d_A^H\dot{\alpha},\dot{\alpha}\rangle + 2\xi * \langle *_H d_A^H\dot{\Psi},\dot{\alpha}\rangle\\
&= \langle -\nabla_\zeta^A\dot{\alpha},\dot{\alpha}\rangle + \langle (d_A^H)^*d_A^H\dot{\alpha},\dot{\alpha}\rangle - 2\xi\langle *_H[\dot{B},\Psi],\dot{\alpha}\rangle,
\end{aligned} \tag{3.14}$$

where in the last line we have used the third identity in (3.12). For the second term in (3.13), we use Lemma 3.3 and the vanishing of $\nabla_\zeta^A\alpha, \nabla_\zeta^A\Psi, d_A^H\alpha$ and $(d_A^H)^*\Psi$ to obtain

$$\begin{aligned}
&\left\langle D_A\begin{pmatrix}*[\dot{A}, *\Phi]\\ *[\dot{A},\Phi]\end{pmatrix}, \begin{pmatrix}0\\ \eta\dot{\alpha}\end{pmatrix}\right\rangle\\
&\quad= \langle\, -[\nabla_\zeta^A\dot{a},\alpha] - *_H[\nabla_\zeta^A\dot{B}, *_H\Psi] + *_H[d_A^H\dot{a}, *_H\Psi] + [(d_A^H)^*\dot{B},\alpha] + 2\xi *_H[\dot{B},\Psi],\dot{\alpha}\,\rangle\\
&\quad= \langle\, -[*_H[\dot{\Psi}, *_H\Psi],\alpha] - [[\dot{\alpha},\alpha],\alpha] - *_H[[\dot{\alpha},\Psi], *_H\Psi] + *_H[[\dot{\Psi},\alpha], *_H\Psi] + 2\xi *_H[\dot{B},\Psi],\dot{\alpha}\,\rangle\\
&\quad= |[\dot{\alpha},\alpha]|^2 + |[\dot{\alpha},\Psi]|^2 + 2\xi\langle *_H[\dot{B},\Psi],\dot{\alpha}\rangle,
\end{aligned} \tag{3.15}$$

where to get the penultimate line we used the first and last identities in (3.12), and to get the last line we took adjoints and used the vanishing of the terms

$$-[*_H[\dot{\Psi}, *_H\Psi],\alpha] + *_H[[\dot{\Psi},\alpha], *_H\Psi] = *_H[[\Psi,\alpha], *_H\dot{\Psi}] = 0,$$

due to $[\Psi,\alpha]=0$ (by Corollary 3.5).

Altogether, combining (3.13)–(3.15), canceling terms, and integrating by parts gives the identity

$$0 = \|\nabla_\zeta^A\dot{\alpha}\|_{L^2(Y)}^2 + \|d_A^H\dot{\alpha}\|_{L^2(Y)}^2 + \|[\dot{\alpha},\alpha]\|_{L^2(Y)}^2 + \|[\dot{\alpha},\Psi]\|_{L^2(Y)}^2,$$

and, hence, the vanishing of $\nabla_\zeta\dot{\alpha}, d_A^H\dot{\alpha}, [\dot{\alpha},\alpha], [\dot{\alpha},\Psi]$.

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An analogous argument gives the identity

$$0 = \|\nabla^A_\zeta \dot{a}\|^2_{L^2(Y)} + \|d^H_A \dot{a}\|^2_{L^2(Y)} + \|[\dot{a}, \alpha]\|^2_{L^2(Y)} + \|[\dot{a}, \Psi]\|^2_{L^2(Y)},$$

and, hence, the vanishing of $\nabla_\zeta \dot{a}, d^H_A \dot{a}, [\dot{a}, \alpha], [\dot{a}, \Psi]$.

By the vanishing of all these quantities, (3.11) now read

$$\begin{aligned} -(d^H_A)^* \dot{B} - *_H[\dot{\Psi}, *_H \Psi] &= 0, \\ *_H[\dot{B}, *_H \Psi] - (d^H_A)^* \dot{\Psi} &= 0, \\ *_H[\dot{B}, \Psi] + *_H d^H_A \dot{\Psi} &= -2\xi\dot{\alpha}, \\ *_H \nabla^A_\zeta \dot{\Psi} - *_H[\dot{B}, \alpha] &= 0, \\ *_H d^H_A \dot{B} - *_H[\dot{\Psi}, \Psi] &= -2\xi\dot{a}, \\ *_H \nabla^A_\zeta \dot{B} + *_H[\dot{\Psi}, \alpha] &= 0. \end{aligned} \tag{3.16}$$

Taking ∇^A_ζ of the fourth equation of (3.16), using $\nabla^A_\zeta \alpha = 0$ (cf. Corollary 3.5), then inserting the last equation of (3.16), and finally integrating by parts yields

$$\|\nabla^A_\zeta \dot{\Psi}\|^2_{L^2(Y)} + \|[\dot{\Psi}, \alpha]\|^2_{L^2(Y)} = 0.$$

This says the quantities $\nabla^A_\zeta \dot{\Psi}, [\dot{\Psi}, \alpha]$ vanish. An analogous argument gives the vanishing of $\nabla^A_\zeta \dot{B}, [\dot{B}, \alpha]$. Finally, observe that $\xi\dot{\alpha} = 0$, which follows easily by taking the inner product of the third equation in (3.16) with $\dot{\alpha}$, integrating by parts and using the vanishing of $[\Psi, \dot{\alpha}], d^H_A \dot{\alpha}$. Likewise, $\xi\dot{a} = 0$.

At this point, we have established all but the final assertion in Proposition 3.6. The required conclusion for $\dot{\alpha}$ readily follows from the vanishing of $\xi\dot{\alpha}, [\Psi, \dot{\alpha}], [\alpha, \dot{\alpha}], d_A\dot{\alpha}$, and similarly for $\dot{a}$. This concludes the proof. □

3.3 Dimensional reduction

We now use the vanishing results from the previous section (Corollary 3.5 and Proposition 3.6) to describe the moduli spaces of stable flat connections on Y in terms of stable Higgs pairs on the orbifold base C.

3.3.1 *Stable flat connections and Higgs pairs on C.* Let $E_0 \to C$ be an orbifold $U(2)$ bundle with fixed determinant $(\Lambda_0 = \Lambda^2 E_0, \lambda_0)$ over the closed oriented Riemannian 2-orbifold (C, g_C), where λ_0 is a fixed $U(1)$ connection on Λ_0. We now gather some basic results about the moduli space of stable projectively flat connections on the bundle $E_0 \to C$, which are discussed in further detail in [Hit87, Don87b, FS92, NS95].

The discussion from § 2.2 applies also to the Riemannian 2-orbifold (C, g_C), and we obtain the spaces $\mathscr{B}_{E_0,k}(C), \mathscr{B}^*_{E_0,k}(C)$ of polystable or stable orbits of $GL(2,\mathbb{C})$ connections $B + i\Psi$ with fixed determinant λ_0 and regularity L^2_k under the action of the group $\mathscr{G}_{E_0,k+1}(C)$ of L^2_{k+1} gauge transformations. As before, 'polystable' here means $d^*_B \Psi = 0$, and 'stable' means that, in addition, the stabilizer of $B + i\Psi$ under the $\mathscr{G}_{E_0,k+1}$-action is $\pm I$. Choosing any $k \geq 2$ will suffice for our purposes, and $\mathscr{B}_{E_0,k}(C)$ then becomes a Hilbert Kähler manifold when equipped with the complex structure J and symplectic form ω_J given in (2.1) and (2.2).

The moduli space $\mathscr{M}^*_{E_0}(C, g_C)$ of polystable projectively flat $GL(2,\mathbb{C})$ orbifold connections on $E_0 \to C$ with fixed determinant (Λ_0, λ_0) is given by the $\mathscr{G}_{E_0}$-orbits of configurations $\mathbb{B} = B + i\Phi$

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solving

$$F_B - \frac{I}{2}F_{\lambda_0} - \frac{1}{2}[\Psi,\Psi] = 0, \tag{3.17}$$

$$d_B\Psi = 0, \tag{3.18}$$

$$d_B^*\Psi = 0. \tag{3.19}$$

The moduli space $\mathscr{M}^*_{E_0}(C) \subset \mathscr{B}^*_{E_0}(C)$ arises as the zero locus of a holomorphic Fredholm section, denoted by s, of a holomorphic Hilbert vector bundle, denoted by $\mathscr{V}_{k-1} \to \mathscr{B}^*_{E_0}(C)$. Namely, $\mathscr{V}_{k-1}$ is the bundle induced from the $\mathscr{G}_{E_0,k+1}$-equivariant vector bundle over the stable configuration space $\mathscr{C}^*_{E_0,k}(C)$ with fiber $\Omega^2(C,\mathfrak{g}_E)\oplus\Omega^2(C,\mathfrak{g}_E)$ at every point, and s is induced from the equivariant section given the left-hand side of equations (3.17) and (3.18).

As a matter of general principle, the moduli space $\mathscr{M}^*_{E_0}$ is naturally a complex-analytic space with complex structure J, with local charts provided by finite-dimensional Kuranishi models. In fact, more is true in this case.

PROPOSITION 3.7. *The moduli space $\mathscr{M}^*_{E_0}(C) = s^{-1}(0)$ of stable projectively flat $GL(2,\mathbb{C})$ connections on a closed oriented Riemannian 2-orbifold (C, g_C) is transversely cut-out, hence is naturally a complex manifold with the complex structure J.*

For the formula for the dimension of $\mathscr{M}^*_{E_0}(C)$ in terms of the isotropy data of E_0 (which we shall not need) we refer to [NS95, Theorem 3.5(1)]. The proof of Proposition 3.7 appears in [Hit87, § 5] in a slightly different guise (and in the case when C has no orbifold points). We now briefly outline a version of this argument, for completeness.

Proof of Proposition 3.7. At a stable flat connection (B,Ψ) the vertical derivative of s is the operator

$$(\mathcal{D}s)_{(B,\Psi)} : \mathscr{K}_{(B,\Psi)} = \mathrm{Ker}(\delta^1_{(B,\Psi)})^* \cap \mathrm{Ker}(\delta^2_{(B,\Psi)})^* \subset (\Omega^1\oplus\Omega^1)(C,\mathfrak{g}_E) \to (\Omega^2\oplus\Omega^2)(C,\mathfrak{g}_E),$$
$$(\mathcal{D}s)_{(B,\Psi)}(\dot{B},\dot{\Psi}) = (d_B\dot{B} - [\dot{\Psi},\Psi], [\dot{B},\Psi] + d_B\dot{\Psi}),$$

where δ^1 and δ^2 are the corresponding versions on (C, g_C) of the operators defined in § 2.2.2. Thus, an element $(\dot{F},\dot{G})$ which is L^2-orthogonal to the image of $\mathcal{D}s$ satisfies the following identity coming from taking adjoints: for any $(\cdot B, \cdot\Psi) \in \mathscr{K}$,

$$0 = \langle (d_B^*\dot{F} - *[\Psi, *\dot{G}], *[\Psi, *\dot{F}] + d_B^*\dot{G}), (\dot{B},\dot{\Psi})\rangle_{L^2(C)}. \tag{3.20}$$

One can then show that the element $(d_B^*\dot{F} - *[\Psi,*\dot{G}], *[\Psi,*\dot{F}] + d_B^*\dot{G})$ lies in the kernel of $(\delta^1)^*$ and $(\delta^2)^*$ when (B,Ψ) is flat and, hence, (2.7) and (3.20) imply that

$$d_B^*\dot{F} - *[\Psi,*\dot{G}] = 0, \quad *[\Psi,*\dot{F}] + d_B^*\dot{G} = 0.$$

Taking d_B on both of the two identities, then integrating the first against $\dot{F}$ and the second against $\dot{G}$, and finally adding everything up leads to $0 = \|d_B^*\dot{F}\|_{L^2(C)} + \|d_B^*\dot{G}\|_{L^2(C)}$ and, hence, $\dot{F} = \dot{G} = 0$ as required. □

It is well known that if $\dim C = 2$, then $\mathscr{M}^*_{E_0}(C, g_C)$ is naturally a *hyperkähler* manifold and admits a holomorphic description as a moduli space of *stable Higgs pairs*. We briefly recall this now (see [Hit87, NS95] for details).

The Hodge star operator of the metric g_C takes 1-forms to 1-forms, hence gives an additional complex structure on $\mathscr{A}^c_{E_0,k}$ denoted by $I(\dot{B},\dot{\Psi}) = (*_C\dot{B}, *_C\dot{\Psi})$. The complex structures I, J anticommute, and hence we have a third complex structure $K = IJ$ (altogether, I, J, K satisfy the quaternion relations). All three complex structures are compatible with the flat Riemannian

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metric G on $\mathscr{A}^c_{E_0,k}$ obtained by the L^2 inner product, and we obtain three corresponding Kähler (symplectic) forms $\omega_I, \omega_J, \omega_K$. For example, ω_J was given in (2.2) and ω_I is given by

$$\omega_I((\dot{B}_1, \dot{\Psi}_1), (\dot{B}_2, \dot{\Psi}_2)) = G(I(\dot{B}_1, \dot{\Psi}_1), (\dot{B}_2, \dot{\Psi}_2)) = -\int_C \mathrm{Tr}(\dot{B}_1 \wedge \dot{B}_2 + \dot{\Psi}_1 \wedge \dot{\Psi}_2). \tag{3.21}$$

Now, the $\mathscr{G}_{E_0,k+1}$-action is Hamiltonian for each of the three symplectic forms. With respect to ω_J, the moment map is given by the left-hand side of (3.19), as we saw also in the three-dimensional case (this holds in any dimension). The moment maps for ω_I and ω_K can be identified with the left-hand sides in (3.17) and (3.18), respectively. Thus, $\mathscr{M}^*_{E_0}(C)$ arises as an infinite-dimensional *hyperkähler reduction* of $(\mathscr{A}^c_{E_0,k}, G, I, J, K)$ by the $\mathscr{G}_{E_0,k+1}$-action.

In turn, the complex gauge group $\mathscr{G}^c_{E_0,k+1}$ preserves the structures I and J. By infinite-dimensional versions of the Kempf–Ness theorem due to Hitchin [Hit87] and Donaldson [Don87b] (see [NS95] for the orbifold case) the moduli space $\mathscr{M}^*_{E_0}(C, g_C)$ then admits a holomorphic interpretation in the complex structures I and J.

– With the complex structure I, $\mathscr{M}^*_{E_0}(C, g_C)$ is identified with the moduli space of *stable Higgs pairs* on the orbifold bundle $E_0 \to C$ with fixed determinant ([Hit87, NS95]):

$$(\mathscr{M}^*_{E_0}(C, g_C), I) \cong \mathscr{M}^{\mathrm{Higgs},*}_{E_0}(C).$$

We recall the definition of $\mathscr{M}^{\mathrm{Higgs},*}_{E_0}(C)$. For this, we use the orbifold complex structure on C defined by g_C, and the orbifold holomorphic structure on Λ_0 induced by λ_0. Then $\mathscr{M}^{\mathrm{Higgs},*}_{E_0}(C)$ is given by the isomorphism classes of pairs $(\mathscr{E}, \theta)$ where $\mathscr{E}$ is an orbifold holomorphic structure $\mathscr{E}$ on E_0 with fixed determinant and $\theta \in H^0(C, T^*C \otimes \mathfrak{sl}(\mathscr{E}))$ (i.e., θ is a holomorphic section of the orbifold holomorphic bundle $T^*C \otimes \mathfrak{sl}(\mathscr{E})$ of T^*C-valued trace-free endomorphisms of $\mathscr{E}$), subject to the requirement that $(\mathscr{E}, \theta)$ is *stable*: if $\mathscr{L} \subset \mathscr{E}$ is a θ-invariant proper non-trivial holomorphic subbundle, then

$$\deg \mathscr{L} < \tfrac{1}{2} \deg \mathscr{E}.$$

The correspondence between stable (projectively) flat connections and stable Higgs pairs is given by

$$\mathbb{B} = B + i\Psi \mapsto (\mathscr{E}, \theta) = (\overline{\partial}_B, -i\Psi^{1,0}).$$

– Even though we do not need it, we mention for context that with the complex structure J the moduli space $\mathscr{M}^*_{E_0}(C, g_C)$ is biholomorphic to the smooth quasi-affine variety consisting of conjugacy classes of irreducible representations of the orbifold fundamental group of C in $SL(2, \mathbb{C})$, suitably compatible with the isotropy data of E_0 ([Don87b, NS95]):

$$(\mathscr{M}^*_{E_0}(C, g_C), J) \cong \mathrm{Hom}^*_{E_0}(\pi_1^{\mathrm{orb}}(C), SL(2, \mathbb{C}))/SL(2, \mathbb{C}).$$

Remark 3.8. The moduli space $\mathscr{M}^{\mathrm{Higgs},*}_{E_0}(C)$ of stable Higgs pairs on the rank-2 orbifold bundle $E_0 \to C$ can be equivalently described as the moduli space of *stable parabolic Higgs bundles* on a certain rank-2 bundle $|E_0| \to |C|$ over the underlying coarse space $|C|$ of C (which is an ordinary Riemann surface) with appropriate parabolic weights given by the isotropy data of E_0. See [NS95, § 5A] (also [FS92, § 5]) for a description of this correspondence, and [BY96] for generalizations of the aforementioned facts to the general setting of parabolic Higgs bundles. However, we shall not make use of the framework of parabolic Higgs bundles in what follows.

3.3.2 *From stable flat connections on Y to stable flat connections on C.* Consider a $U(2)$ bundle E over a Seifert-fibered 3-manifold Y with fixed determinant Λ. We now describe the moduli space $\mathscr{M}^*_E(Y)$ in terms of the moduli spaces on C.

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If E satisfies any of the three equivalent conditions from the next lemma, we say that E 'pulls back from C up to tensoring'.

LEMMA 3.9. *Let $E \to Y$ be a $U(2)$ bundle over a Seifert-fibered 3-manifold $Y = S(N) \to C$, with determinant $\Lambda = \Lambda^2 E$. The following assertions are equivalent:*

(1) *E is the pullback of an orbifold $U(2)$ bundle on C, up to tensoring by some $U(1)$ bundle L on Y;*
(2) *Λ is the pullback of an orbifold $U(1)$ bundle on C, up to tensoring by some $U(1)$ bundle squared $L^{\otimes 2}$ on Y;*
(3) *the associated $SO(3)$ bundle $P = E \times_{U(2)} SO(3)$ is the pullback of an orbifold $SO(3)$ bundle on C.*

Proof. Since Y is three-dimensional, $E \cong \mathbb{C} \oplus \Lambda$. From this the equivalence of assertions (1) and (2) follows. Clearly, assertion (1) implies assertion (3). To show that assertion (3) implies assertion (1), suppose P is the pullback of an orbifold $SO(3)$ bundle P_0. One can then find an orbifold $U(2)$ bundle E_0 on C whose associated orbifold $SO(3)$ bundle is $P_0 \cong E_0 \times_{U(2)} SO(3)$ (see [FS92, § 1]). It follows that $P = \pi^* P_0$ is the associated bundle of both E and $\pi^* E_0$, so they differ by tensoring by a line bundle. □

We refer to [FS92, Corollary 1.6 and Proposition 1.8] for the classification of orbifold $U(1)$ and $U(2)$ bundles over C, and to [FS92, Theorem 2.3] for the classification of the $U(1)$ and $U(2)$ bundles on Y which pull back from C. Given an orbifold $U(2)$ bundle $E_0 \to C$ with fixed determinant, and setting $E = \pi^* E_0$, we have a well-defined holomorphic map on stable orbit spaces given by pulling back configurations from C onto Y:

$$\pi^* : \mathscr{B}^*_{E_0,k}(C) \to \mathscr{B}^*_{E,k}(Y).$$

Furthermore, one can easily see that

$$\pi^*(\mathscr{M}^*_{E_0}(C)) \subset \mathscr{M}^*_E(Y).$$

THEOREM 3.10 (Dimensional reduction for Seifert-fibered spaces). *Let $Y \to C$ be a closed, oriented Seifert-fibered 3-manifold equipped with a Seifert metric $g = \eta^2 + g_C$ (Definition 3.1), and let E be a $U(2)$ bundle $E \to Y$ with fixed determinant $\Lambda = \Lambda^2 E$. Equip C with the metric g_C.*

If E does not pull back from C up to tensoring, then the moduli space $\mathscr{M}^*_E(Y, g)$ is empty. If it does, then after tensoring E by a $U(1)$ bundle (which leaves $\mathscr{M}^*_E(Y, g)$ unchanged; cf. Remark 2.10) we may fix an isomorphism $\Lambda \cong \pi^* \Lambda_0$ for some orbifold line bundle Λ_0 on C. Then $\mathscr{M}^*_E(Y, g)$ is Morse–Bott (cf. Definition 2.17), and the pullback map π^* induces a biholomorphism

$$\left(\bigsqcup_{E_0, \Lambda^2 E_0 \cong \Lambda_0} \mathscr{M}^*_{E_0}(C, g_C) \right) \sqcup \left(\bigsqcup_{E_0, \Lambda^2 E_0 \cong \Lambda_0 \otimes N} \mathscr{M}^*_{E_0}(C, g_C) \right) \cong \mathscr{M}^*_E(Y, g),$$

where E_0 ranges over all isomorphism classes of orbifold $U(2)$ bundles on C with fixed determinant Λ_0 and $\Lambda_0 \otimes N$, respectively.

To establish this result, we combine the vanishing results established above together with the following standard fact, which essentially appears in [FS92, Proof of Theorem 4.1].

LEMMA 3.11. *Let $P \to Y$ be a principal G bundle with connection A, where G is a Lie group with Lie algebra $\mathfrak{g}$. Let X denote the vector field on the total space P given by the horizontal lift with respect to A of the vertical vector field ζ on Y that generates the Seifert S^1-action on Y.*

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Then the pair (P, A) is the pullback of a corresponding pair (P_0, B) on C if and only if the holonomy of A is trivial along the fibers of $Y \to C$ and the contraction with X of the curvature 2-form $F_A \in \Omega^2(P, \mathfrak{g})$ vanishes: $\iota_X F_A = 0$.

We recall the proof, which is quite simple.

Proof. Only the 'if' statement is non-trivial. The vector field X on P generates an S^1-action on P because the holonomy of A is trivial on the fibers. This action makes P into the total space of an orbifold principal S^1 bundle $P \to P_0 := P/S^1$. Since the S^1-action on P commutes with the G-action then $P_0 = P/S^1$ inherits the structure of an orbifold principal G bundle over C, and the projection $P \to P_0$ is G-equivariant.

To argue that A descends to a connection on P_0, regard A as a 1-form (i.e., as an element in $\Omega^1(P, \mathfrak{g})$). Then $\iota_X A = 0$ since X is horizontal. Since $\iota_X F_A = 0$, we have that

$$\mathscr{L}_X A = d(\iota_X A) + \iota_X(dA) = \iota_X(F_B) - \iota_X(\tfrac{1}{2}[A, A]) = 0.$$

The conditions $\iota_X A = 0$ and $\mathscr{L}_X A = 0$ combined say that A descends to a connection B on the orbifold principal G bundle $P_0 \to C$. □

Proof of Theorem 3.10. Let (A, Φ) be a projectively flat stable $GL(2, \mathbb{C})$ connection on $E \to Y$ with fixed determinant, and consider the induced flat stable $GL(2, \mathbb{C})$ connection (A', Φ') on the associated $SO(3)$ bundle $P \to Y$. By the vanishing of the α component of Φ (Corollary 3.5), it follows that the holonomy of A' along a fiber of $\pi : Y \to C$ is the same as the holonomy of the flat connection $A' + i\Phi'$. Since the fibers represent central elements in the fundamental group of Y (see [FS92, Theorem 2.1]) and $A' + i\Phi'$ is flat, it follows that the holonomy of A' lies in the center of $SO(3)$, that is, it must be trivial. In addition, because $[\alpha, \Psi] = 0$ by Corollary 3.5, then $\iota_\zeta F_{A'} = 0$. Thus, Lemma 3.11 implies that the pair (P, A') pulls back from a corresponding pair (P_0, B').

By Lemma 3.9 it follows that Λ pulls back from C, up to tensoring E by a line bundle. Without loss of generality, we may assume there is an orbifold line bundle Λ_0 on C, which is fixed once and for all, together with an identification $\Lambda = \pi^*\Lambda_0$. We also fix a connection λ on $\Lambda = \pi^*\lambda_0$ of the form $\lambda = \pi^*\lambda_0$, for a connection λ_0 on Λ_0. Now, by again running the argument from the first paragraph but applied to (A, Φ) rather than (A', Φ'), we obtain that the holonomy of A along the non-singular fibers lies in $\pm I$, the center of $SL(2, \mathbb{C})$. Let $k \in \{0, 1\}$ be such that the holonomy of a non-singular fiber is $(-1)^k I$. Consider the $U(2)$ bundle with connection given by $(E, A_k) = (E, A) \otimes (\mathbb{C}, A_{\text{trivial}} + (1/2)i\eta)^{\otimes k}$ (this is identified with E as a bundle). Then (E, A_k) has trivial holonomy along the non-singular fibers and its induced connection on $\Lambda = \Lambda^2 E$ is now $\lambda + i\eta$ instead. For the same reason as in the first paragraph, we also have $\iota_\zeta F_{A_k} = 0$. Thus, by Lemma 3.11 we have that (E, A_k) descends to a pair (E_0, B) on C whose determinant is $(\Lambda_0, \lambda_0) \otimes (N, i\eta)^{\otimes k}$. It follows from Corollary 3.5 that then Φ descends to a section Ψ of $\Omega^1(C, \mathfrak{su}(E_0))$ and that (B, Ψ) is a polystable projectively flat $GL(2, \mathbb{C})$ connection on $E_0 \to C$, with fixed determinant $(\Lambda_0, \lambda_0) \otimes (N, i\eta)^{\otimes k}$. Furthermore, since (A, Φ) is stable, it follows that so is (B, Ψ).

Altogether, the construction thus described provides a well-defined map

$$\mathscr{M}^*_E(Y, g) \to \left(\bigsqcup_{E_0, \Lambda^2 E_0 \cong \Lambda_0} \mathscr{M}^*_{E_0}(C, g_C) \right) \sqcup \left(\bigsqcup_{E_0, \Lambda^2 E_0 \cong \Lambda_0 \otimes N} \mathscr{M}^*_{E_0}(C, g_C) \right) =: \mathscr{M}_E(C, g_C),$$

which gives a topological inverse of the pullback map π^*.

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Now, the map

$$\pi^* : \mathscr{B}^*_{E,k}(C, g_C) := \left(\bigsqcup_{E_0, \Lambda^2 E_0 \cong \Lambda_0} \mathscr{B}^*_{E_0,k}(C, g_C) \right) \sqcup \left(\bigsqcup_{E_0, \Lambda^2 E_0 \cong \Lambda_0 \otimes N} \mathscr{B}^*_{E_0,k}(C, g_C) \right) \to \mathscr{B}^*_{E,k}(Y, g) \tag{3.22}$$

is a holomorphic map of complex Hilbert manifolds, mapping the complex submanifold $\mathscr{M}^*_E(C, g_C)$ bijectively onto the subspace $\mathscr{M}_E(Y, g) \subset \mathscr{B}^*_{E,k}(Y, g)$. By Proposition 3.6, π^* also induces an isomorphism from the tangent space of $\mathscr{M}^*_E(C, g_C)$ onto the Zariski tangent space of $\mathscr{M}^*_E(Y, g)$. This implies that the finite-dimensional Kuranishi models of S from Corollary 2.16 are given by constant functions; hence, $\mathscr{M}^*_E(Y, g)$ is a smooth (in fact, complex) submanifold of $\mathscr{B}^*_E(Y, g)$, and π^* gives a complex-analytic diffeomorphism from $\mathscr{M}^*_E(C, g_C)$ onto $\mathscr{M}^*_E(Y, g)$. □

Theorem C follows immediately from Theorem 3.10 and the holomorphic description of the moduli spaces $\mathscr{M}^*_{E_0}(C)$ in terms of stable Higgs pairs.

Example 3.12. The simplest case is to take $Y = S^1 \times C$ for a (smooth) genus g closed surface C, and $E = \pi^* E_0$ given by the pullback of a rank-2 bundle E_0 of *odd* degree, where Y is equipped with a product metric $g = dt^2 + g_C$. Then polystable projectively flat connections on E and E_0 are automatically stable (which follows from equations (2.18), (3.17) and the Chern–Weil formula), and in order to find solutions we need $g \geq 2$. The moduli space of polystable connections on E_0 is a complex manifold of complex dimension $6(g-1)$ (see [Hit87]), and Theorem 3.10 gives

$$\mathscr{M}_E(Y, g) \cong \mathscr{M}_{E_0}(C, g_C) \sqcup \mathscr{M}_{E_0}(C, g_C).$$

4. Perturbation theory

In this section we study the critical locus of the perturbation S_ε given in Definition 1.1 (the angle θ will be omitted from notation, for convenience). We first discuss a finite-dimensional version of the argument that we employ in the infinite-dimensional case, and establish a finite-dimensional analog of Theorem A. We then discuss the $SL(2,\mathbb{C})$ holonomy perturbations that we use to achieve transversality of our perturbation. Finally, after verifying that the linear analysis results from § 2 and basic compactness properties carry through in the perturbed case, we establish Theorem A.

We show that when (Y, g) is a Seifert-fibered 3-manifold equipped with a Seifert metric, the effect of the perturbation S_ε in this manner is roughly speaking as follows.

- The term $\varepsilon \|\Phi\|^2_{L^2}$ plays the following role. As $\varepsilon \neq 0$ becomes small, the irreducible critical points of $S(A, \Phi) + \varepsilon \|\Phi\|^2_{L^2}$ either will tend to *localize* at a certain *compact* submanifold $\mathscr{Z} \subset \mathscr{M}^*_E = \mathrm{Crit} S$, or else the function $\|\Phi\|_{L^\infty}$ becomes unbounded on them.
- The term $\varepsilon^2 f(\underline{\sigma}(A, \Phi))$ is then added in to ensure the *non-degeneracy* of the critical points that localize at $\mathscr{Z}$.

4.1 Perturbations of Morse–Bott functions

We consider the following setup. Let $S : B \times \mathbb{R} \to \mathbb{R}$ be a smooth function, which we regard as a one-parameter family of functions $S_\varepsilon = S(\cdot, \varepsilon) : B \to \mathbb{R}$, and consider the sequence of smooth functions $S_j : B \to \mathbb{R}$ $(j = 0, 1, 2, \ldots)$ given by the (rescaled) derivatives of S with respect to ε at $\varepsilon = 0$:

$$\left(\frac{\partial}{\partial \varepsilon} \right)^j S|_{\varepsilon = 0} = \frac{1}{j!} S_j.$$

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When S and the sequence S_j have the above relationship we write

$$S_\varepsilon := S_0 + \varepsilon S_1 + \varepsilon^2 S_2 + \varepsilon^3 S_3 + \cdots . \tag{4.1}$$

(Expression (4.1) should be understood only 'formally': without any convergence assumption.)

We regard $S_0 : B \to \mathbb{R}$ as our 'original' function, and $S_\varepsilon : B \to \mathbb{R}$ as a 'perturbation' of S_0. Our goal is to describe the *change in the critical locus* $\mathrm{Crit}(S_\varepsilon)$ as ε goes from zero to non-zero. More precisely, for each critical point $x \in \mathrm{Crit} S_0$ we obtain an open neighborhood $\mathscr{U}_x$ of x in B, a constant $\varepsilon_x > 0$ and a parametrization of the critical locus of S_ε on the neighborhood $\mathscr{U}_0$ for all $0 < |\varepsilon| < \varepsilon_x$. For this, we consider the following restricted assumptions, which suffice for our purposes:

(1) S_0 is a Morse–Bott function; we denote its critical locus by $\mathscr{Z}_0 := \mathrm{Crit}(S_0)$;
(2) the restriction of S_1 to $\mathscr{Z}_0$ is a Morse–Bott function; we denote its critical locus by $\mathscr{Z}_1 := \mathrm{Crit}(S_1|_{\mathscr{Z}_0})$.

4.1.1 *Local theory.* We begin by noting that the critical points of S_ε are constrained in the following way.

Lemma 4.1. *Let $\varepsilon_n \in \mathbb{R}$ be a sequence converging to zero, with $\varepsilon_n \neq 0$ for all n. Suppose that $x_n \in B$ is a sequence converging to $x \in B$ such that $b_n \in \mathrm{Crit}(S_{\varepsilon_n})$ for all n. (Clearly, then $x \in \mathscr{Z}_0$.) Then x is contained in $\mathscr{Z}_1$.*

Proof. Since S_0 is Morse–Bott, there is a neighborhood of b in B modeled on an open neighborhood of $(0,0)$ in the product $U \times H$ of two inner product vector spaces U and H, such that in the coordinates $(u,h) \in U \times H$ we have

$$S_0(u,h) = \tfrac{1}{2}\langle Lu, u\rangle + \text{constant}$$

for a self-adjoint linear isomorphism $L : U \xrightarrow{\cong} U$. (Thus, in these local coordinates the critical locus $\mathscr{Z}_0$ is the linear subspace $u = 0$.)

Writing $x_n = (u_n, h_n)$, the equation that x_n is a critical point of S_{ε_n} says

$$0 = Lu_n + \varepsilon_n \frac{\partial S_1}{\partial u}(u_n, h_n) + \varepsilon_n^2 \frac{\partial S_2}{\partial u}(u_n, h_n) + \cdots \in U,$$
$$0 = \varepsilon_n \frac{\partial S_1}{\partial h}(u_n, h_n) + \varepsilon_n^2 \frac{\partial S_2}{\partial h}(u_n, h_n) + \cdots \in H.$$

Dividing out ε_n from the second equation and taking the limit $n \to \infty$ shows that $\partial S_1/\partial h$ vanishes at $(0,0)$. This says that $x \in B$ is contained not only in $\mathscr{Z}_0$, but also in $\mathscr{Z}_1 \subset \mathscr{Z}_0$. □

(We note that the proof of Lemma 4.1 only used the Morse–Bott hypothesis on S_0.)

We must thus describe the behavior of the critical locus of our perturbation near a point $x \in \mathscr{Z}_1$: in particular, determine that x appear as limits of a sequence as in Lemma 4.1. Here we make use of the Morse–Bott hypothesis on both S_0 and $S_1|_{\mathscr{Z}_0}$. This ensures that for any point $x \in \mathscr{Z}_1 = \mathrm{Crit}(S_1|_{\mathscr{Z}_0})$, there is a neighborhood $\mathscr{U}_x$ of x in B diffeomorhic to a neighborhood $\mathscr{U}_0$ of $(0,0,0)$ in the product $U \times V \times W$ of inner product vector spaces, such that in the $U \times V \times W$ coordinates we have:

– $S_0(u,v,w) = (1/2)\langle L_0 u, u\rangle + \text{constant}$, for a self-adjoint linear isomorphism $L_0 : U \xrightarrow{\cong} U$; in particular, the critical locus $\mathscr{Z}_0 = \mathrm{Crit}(S_0)$ is given by $0 \times V \times W \subset U \times V \times W$;
– $S_1(0,v,w) = (1/2)\langle L_1 v, v\rangle + \text{constant}$, for a self-adjoint linear isomorphism $L_1 : V \xrightarrow{\cong} V$; in particular, the critical locus $\mathscr{Z}_1 = \mathrm{Crit}(S_1|_{\mathscr{Z}_0})$ is given by $0 \times 0 \times W \subset U \times V \times W$.

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Consider the equations for a point (u,v,w) to be a critical point of S_ε:

$$0 = L_0 u + \varepsilon \frac{\partial S_1}{\partial u}(u,v,w) + \varepsilon^2 \frac{\partial S_2}{\partial u}(u,v,w) + \cdots \in U, \tag{4.2}$$

$$0 = \varepsilon \frac{\partial S_1}{\partial v}(u,v,w) + \varepsilon^2 \frac{\partial S_2}{\partial v}(u,v,w) + \cdots \in V, \tag{4.3}$$

$$0 = \varepsilon \frac{\partial S_1}{\partial w}(u,v,w) + \varepsilon^2 \frac{\partial S_2}{\partial w}(u,v,w) + \cdots \in W. \tag{4.4}$$

Putting together the right-hand side of (4.2) with the right-hand side of (4.3) divided by ε, we obtain a smooth map $F : U \times V \times W \times \mathbb{R} \to U \times V$:

$$F(u,v,w,\varepsilon) = \left(L_0 u + \varepsilon \frac{\partial S_1}{\partial u}(u,v,w) + \varepsilon^2 \frac{\partial S_2}{\partial u}(u,v,w) + \cdots, \frac{\partial S_1}{\partial v}(u,v,w) + \varepsilon \frac{\partial S_2}{\partial v}(u,v,w) + \cdots \right),$$

with the property that its derivative at $(u,v,w,\varepsilon) = (0,0,0,0)$ induces an isomorphism of $U \oplus V \oplus 0 \oplus 0 \subset T_{(0,0,0,0)}(U \times V \times W \times \mathbb{R})$ with $U \oplus V = T_{(0,0)}(U \times V)$. Indeed, the derivative of F at $(0,0,0,0)$ has the shape

$$(dF)_{(0,0,0,0)} = \begin{pmatrix} L_0 & 0 & 0 & * \\ * & L_1 & * & * \end{pmatrix}.$$

It follows, by the implicit function theorem, that the equation $F(u,v,w,\varepsilon) = 0$ uniquely determines u, v, w as functions of variables $(t,\varepsilon) \in W \times \mathbb{R}$. More precisely, we find a neighborhood $\mathscr{U}_0'$ of 0 in W and $\varepsilon_0 > 0$ and a unique smooth function

$$(t,\varepsilon) \in \mathscr{U}_0' \times (-\varepsilon_0, \varepsilon_0) \subset W \times \mathbb{R} \mapsto (u(t,\varepsilon), v(t,\varepsilon), t) \in \mathscr{U}_0 \subset U \times V \times W$$

subject to $F(u(t,\varepsilon), v(t,\varepsilon), t) = 0$ and $u(0,0) = 0$, $v(0,0) = 0$. Furthermore, it also follows that for every t we have $u(t,0) = 0$ and $v(t,0) = 0$ (since $\partial S_1/\partial v$ vanishes when $u = 0$ and $v = 0$).

Thus, for $0 < |\varepsilon| < \varepsilon_0$ the critical locus of S_ε on the neighborhood $\mathscr{U}_0$ of $(0,0,0)$ in $U \times V \times W$ is parametrized by the set of $t \in \mathscr{U}_0' \subset W$ such that the triple $(u,v,w) = (u(t,\varepsilon), v(t,\varepsilon), t)$ solves the remaining equation (4.4). We now describe this parametrization more precisely as a critical locus. For this, introduce the smooth function on $\mathscr{U}_0' \times (-\varepsilon_0, \varepsilon_0) \setminus 0 \subset W \times \mathbb{R}$ given by

$$\begin{aligned} S'(t,\varepsilon) &:= \sum_{j=0}^{\infty} S_j(u(t,\varepsilon), v(t,\varepsilon), t)\varepsilon^{j-2} - \varepsilon^{-2} S_0(0,0,0) - \varepsilon^{-1} S_1(0,0,0) \\ &= \varepsilon^{-2} S_\varepsilon(u(t,\varepsilon), v(t,\varepsilon), t) - \varepsilon^{-2} S_0(0,0,0) - \varepsilon^{-1} S_1(0,0,0), \end{aligned}$$

and note that S' extends smoothly to $\mathscr{U}_0' \times (-\varepsilon_0, \varepsilon_0)$. Indeed, regarding S_0 and S_1 as functions of (t,ε) and recalling that $u(t,0) = 0$, $v(t,0) = 0$, we have

$$\begin{aligned} S_0(t,0) &= S_0(u(t,0), v(t,0), t) = \text{constant}, \\ S_1(t,0) &= S_1(u(t,0), v(t,0), t) = \text{constant}, \\ \frac{\partial S_0}{\partial \varepsilon}(t,0) &= \frac{\partial S_0}{\partial u}(u(t,0), v(t,0), t) \cdot \frac{\partial u}{\partial \varepsilon} + \frac{\partial S_0}{\partial v}(u(t,0), v(t,0), t) \cdot \frac{\partial v}{\partial \varepsilon} = 0. \end{aligned}$$

Denote $S_\varepsilon'(t) = S'(t,\varepsilon)$. The next lemma provides the promised local parametrization of the critical locus of S_ε:

LEMMA 4.2. *For $0 < |\varepsilon| \le \varepsilon_0$, the smooth embedding*

$$t \in \mathscr{U}_0' \subset W \mapsto (u(t,\varepsilon), v(t,\varepsilon), t) \in \mathscr{U}_0 \subset U \times V \times W$$

takes the critical locus $\mathrm{Crit}(S_\varepsilon')$ *on the neighborhood* $\mathscr{U}_0' \subset W$ *of* $t = 0$ *to the critical locus* $\mathrm{Crit}(S_\varepsilon)$ *on the neighborhood* $\mathscr{U}_0 \subset U \times V \times W$ *of* $(u,v,w) = (0,0,0)$.

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Proof. The remaining equation (4.4), after setting $u = u(t,\varepsilon), v = v(t,\varepsilon), w = t$ and dividing by $\epsilon^2 \neq 0$, is

$$\varepsilon^{-2}\frac{\partial S_\varepsilon}{\partial w}(u(t,\varepsilon), v(t,\varepsilon), t) = 0.$$

But since $\partial S_\varepsilon/\partial u$ and $\partial S_\varepsilon/\partial v$ vanish at $(u,v,w) = (u(t,\varepsilon), v(t,\varepsilon), t)$, the left-hand side can be written as

$$\varepsilon^{-2}\left(\frac{\partial S_\varepsilon}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial S_\varepsilon}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial S_\varepsilon}{\partial w}\right) = \frac{\partial S'_\varepsilon}{\partial t}.$$

□

Consider the sequence of smooth functions on $\mathscr{U}'_0 \subset W$ given by

$$S'_j(t) := \frac{1}{j!}\frac{\partial^{(j)} S'}{\partial \varepsilon^{(j)}}(t,0), \quad j \geq 0.$$

We note that the smooth function $S'(t,\varepsilon)$ need not be analytic in ε anymore: we may only 'formally' write

$$S'_\varepsilon(t) = S'_0(t) + \varepsilon S'_1(t) + \varepsilon^2 S'_2(t) + \cdots .$$

At this point, it follows that the points $\mathscr{Z}_1 = \mathrm{Crit}(S_1|_{\mathscr{Z}_0})$ which arise as limits of a sequence as in Lemma 4.1 are further constrained to be critical points of the *leading term* S'_0 in this formal expansion. The next lemma describes this leading term.

Lemma 4.3. *Let $\lambda(t) \in U$ be the function defined by the requirement*

$$L_0(\lambda(t)) = -\frac{\partial S_1}{\partial u}(0,0,t) \text{ as elements of } U.$$

Then we have

$$\begin{aligned}
S'_0(t) &= \frac{1}{2}\frac{\partial S_1}{\partial u}(0,0,t)\cdot\lambda(t) + S_2(0,0,t),\\
\frac{\partial S'_0}{\partial t} &= \frac{\partial^2 S_1}{\partial u \partial w}(0,0,t)\cdot\lambda(t) + \frac{\partial S_2}{\partial w}(0,0,t),\\
\frac{\partial^2 S'_0}{\partial t^2} &= \frac{\partial^3 S_1}{\partial u \partial w^2}(0,0,t)\cdot\lambda(t) + \frac{\partial^2 S_1}{\partial u \partial w}(0,0,t)\cdot\frac{\partial \lambda}{\partial t} + \frac{\partial^2 S_2}{\partial w^2}(0,0,t).
\end{aligned}$$

Proof. We have

$$S'_0(t) = S'(t,0) = \frac{1}{2}\frac{\partial^2 S_0}{\partial \varepsilon^2}(t,0) + \frac{\partial S_1}{\partial \varepsilon}(t,0) + S_2(t,0),$$

and we now compute each of the first two terms on the right-hand side separately (in the following calculations we always evaluate at $\varepsilon = 0$):

$$\begin{aligned}
\frac{1}{2}\frac{\partial^2 S_0}{\partial \varepsilon^2} &= \frac{1}{2}\frac{\partial}{\partial \varepsilon}\left(\frac{\partial S_0}{\partial u}\frac{\partial u}{\partial \varepsilon} + \frac{\partial S_0}{\partial v}\frac{\partial v}{\partial \varepsilon}\right)\\
&= \frac{1}{2}\frac{\partial^2 S_0}{\partial u^2}\left(\frac{\partial u}{\partial \varepsilon}\right)^2 + \frac{\partial^2 S_0}{\partial u \partial v}\left(\frac{\partial u}{\partial \varepsilon}\right)\left(\frac{\partial v}{\partial \varepsilon}\right) + \frac{1}{2}\frac{\partial^2 S_0}{\partial v^2}\left(\frac{\partial v}{\partial \varepsilon}\right)^2\\
&\quad + \frac{1}{2}\frac{\partial S_0}{\partial u}\frac{\partial^2 u}{\partial \varepsilon^2} + \frac{1}{2}\frac{\partial S_0}{\partial v}\frac{\partial^2 v}{\partial \varepsilon^2}\\
&= \frac{1}{2}\frac{\partial^2 S_0}{\partial u^2}\left(\frac{\partial u}{\partial \varepsilon}\right)^2,
\end{aligned}$$

$$\frac{\partial S_1}{\partial \varepsilon} = \frac{\partial S_1}{\partial u}\frac{\partial u}{\partial \varepsilon} + \frac{\partial S_1}{\partial v}\frac{\partial v}{\partial \varepsilon} = \frac{\partial S_1}{\partial u}\frac{\partial u}{\partial \varepsilon}.$$

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Now, by differentiating (4.2) with respect to ε and then setting $\varepsilon = 0$, we obtain the following relation in U^*:

$$\frac{\partial^2 S_0}{\partial u^2}\frac{\partial u}{\partial \varepsilon} = -\frac{\partial S_1}{\partial u},$$

that is, $\partial u/\partial \varepsilon(t,0) = \lambda(t)$. Inserting this into the above, we obtain the required formula for S_0':

$$\begin{aligned} S_0'(t) &= \frac{1}{2}\frac{\partial^2 S_0}{\partial u^2}\left(\frac{\partial u}{\partial \varepsilon}\right)^2 + \frac{\partial S_1}{\partial u}\frac{\partial u}{\partial \varepsilon} + S_2 \\ &= -\frac{1}{2}\frac{\partial S_1}{\partial u}\cdot\lambda + \frac{\partial S_1}{\partial u}\cdot\lambda + S_2 \\ &= \frac{1}{2}\frac{\partial S_1}{\partial u}\cdot\lambda + S_2. \end{aligned}$$

For $\partial S_0'/\partial t$, recall that

$$\frac{\partial S_\varepsilon'}{\partial t} = \varepsilon^{-2}\frac{\partial S_\varepsilon}{\partial w} = \varepsilon^{-1}\frac{\partial S_1}{\partial w} + \frac{\partial S_2}{\partial w} + \mathcal{O}(\varepsilon).$$

Now, we have

$$\frac{\partial}{\partial \varepsilon}\frac{\partial S_1}{\partial w} = \frac{\partial^2 S_1}{\partial u \partial w}\frac{\partial u}{\partial \varepsilon} + \frac{\partial^2 S_1}{\partial v \partial w}\frac{\partial v}{\partial \varepsilon} = \frac{\partial^2 S_1}{\partial u \partial w}(0,0,t)\cdot\lambda + \mathcal{O}(\varepsilon).$$

Inserting this into the previous formula yields

$$\frac{\partial S_\varepsilon'}{\partial t} = \frac{\partial^2 S_1}{\partial u \partial w}(0,0,t)\cdot\lambda + \frac{\partial S_2}{\partial w}(0,0,t) + \mathcal{O}(\varepsilon), \tag{4.5}$$

from which the formula for $\partial S_0'/\partial t$ follows.

Finally,

$$\begin{aligned} \frac{\partial^2 S_0'}{\partial t^2} =& \frac{\partial^3 S_1}{\partial u^2 \partial w}(0,0,t)\cdot\frac{\partial u}{\partial t}(0,t)\cdot\lambda(t) + \frac{\partial^3 S_1}{\partial u \partial v \partial w}(0,0,t)\cdot\frac{\partial v}{\partial t}(0,t)\cdot\lambda(t) \\ &+ \frac{\partial^3 S_1}{\partial u \partial w^2}(0,0,t)\cdot\lambda(t) + \frac{\partial^2 S_1}{\partial u \partial w}(0,0,t)\cdot\frac{\partial \lambda}{\partial t} \\ &+ \frac{\partial^2 S_2}{\partial u \partial w}(0,0,t)\cdot\frac{\partial u}{\partial t}(t,0) + \frac{\partial^2 S_2}{\partial v \partial w}(0,0,t)\cdot\frac{\partial v}{\partial t}(t,0) \\ &+ \frac{\partial^2 S_2}{\partial w^2}(0,0,t) + \mathcal{O}(\varepsilon). \end{aligned}$$

The required formula then follows by noting that $\partial u/\partial t$ and $\partial v/\partial t$ vanish identically at $(u,v) = (t,0)$ since $u(t,0)$ and $v(t,0)$ vanish identically. □

Of course, there may be further obstructions for a critical point of the leading term S_0' to appear as a limit of a sequence as in Lemma 4.1. However, in the situation where $t=0$ is a *non-degenerate* (i.e., Morse) critical point of the leading term S_0', we obtain the following conclusion.

PROPOSITION 4.4. *If $t=0$ is a non-degenerate critical point of the leading term $S_0' : \mathscr{U}_0' \subset W \to \mathbb{R}$, then there exist a neighborhood $\mathscr{U}_0 \subset U \times V \times W$ of $x_0 = (0,0,0)$ and $\epsilon_0 > 0$ such that S_ε has a unique critical point x_ε in $\mathscr{U}_0$ for each $0 < |\varepsilon| \le \varepsilon_0$, and the function $\varepsilon \mapsto x_\varepsilon$ extends to a*

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smooth function on $[-\varepsilon_0, \varepsilon_0]$. *Furthermore, this critical point is non-degenerate and when* $\pm\varepsilon > 0$ *its index is*

$$\operatorname{ind}(S_\varepsilon, x_\varepsilon) = \operatorname{ind}(S_0, x_0) + \operatorname{ind}(\pm S_1|_{\mathscr{Z}_0}, x_0) + \operatorname{ind}(S_0', 0).$$

(On the other hand, if $t = 0$ *is not a critical point of* $S_0' : \mathscr{U}_0' \subset W \to \mathbb{R}$, *then there exist a neighborhood* $\mathscr{U}_0 \subset U \times V \times W$ *of* $(0,0,0)$ *and* $\epsilon_0 > 0$ *such that* S_ε *has no critical points in* $\mathscr{U}_0$ *for each* $0 < |\varepsilon| < \varepsilon_0$.*)*

We recall that the index $\operatorname{ind}(f, x) \in \mathbb{Z}_{\geq 0}$ of a critical point x of a Morse or Morse–Bott function f is defined as the dimension of the negative eigenspace of the Hessian of f along the normal bundle to the critical locus of f at x. This only depends on the connected component C of the critical locus of f that contains x, so we may also use the notation $\operatorname{ind}(f, C)$ instead.

Proof. Since $t = 0$ is a non-degenerate critical point of S_0', it persists in the perturbation S_ε' to give a unique non-degenerate critical point $t(\varepsilon)$ of S_ε' for ε sufficiently close to $\varepsilon = 0$. By Lemma 4.2 we obtain a unique critical point x_ε of the original perturbation S_ε in a neighborhood of $(0,0,0)$ for sufficiently small $\varepsilon \neq 0$, given in the $U \times V \times W$ coordinates by $x_\varepsilon = (u_\varepsilon, v_\varepsilon, w_\varepsilon) := (u(t(\varepsilon), \varepsilon), v(t(\varepsilon), \varepsilon), t(\varepsilon))$. (In addition, clearly the path x_ε extends to $\varepsilon = 0$ to give a smooth path.)

Consider smooth paths $\dot{x}_\varepsilon = (\dot{u}_\varepsilon, \dot{v}_\varepsilon, \dot{w}_\varepsilon) \in U \oplus V \oplus W$ and $\mu_\varepsilon \in \mathbb{R}$, defined for ε sufficiently close to $\varepsilon = 0$, such that $\dot{x}_\varepsilon \neq 0$ and

$$(\operatorname{Hess} S_\varepsilon)_{x_\varepsilon}(\dot{x}_\varepsilon) = \mu_\epsilon \cdot \dot{x}_\varepsilon. \tag{4.6}$$

Formally, we write $\mu_\varepsilon = \mu_0 + \varepsilon\mu_1 + \varepsilon^2\mu_2 + \cdots$ (i.e., μ_1 is the first derivative of μ_ε at $\varepsilon = 0$, and so forth, but with no assumption of convergence), and likewise for other quantities (e.g., $\dot{u}_\varepsilon = \dot{u}_0 + \varepsilon\dot{u}_1 + \varepsilon^2\dot{u}_2 + \cdots$). We consider the following scenarios.

(I) Suppose that the path μ_ε vanishes at $\varepsilon = 0$, that is, $\mu_0 = 0$. Thus, $(\operatorname{Hess} S_0)_{x_0}\dot{x}_0 = 0$, and hence $\dot{x}_0 \in 0 \oplus V \oplus W$ (i.e., $\dot{u}_0 = 0$). Differentiating (4.6) in ε yields the identity

$$\left(\operatorname{Hess}\frac{\partial S_\varepsilon}{\partial \varepsilon}\right)_{x_\varepsilon}(\dot{x}_\varepsilon) + (\nabla^3 S_\varepsilon)_{x_\varepsilon}\left(\frac{\partial x_\varepsilon}{\partial \varepsilon}, \dot{x}_\varepsilon\right) + (\operatorname{Hess} S_\varepsilon)_{x_\varepsilon}\left(\frac{\partial \dot{x}_\varepsilon}{\partial \varepsilon}\right) = \frac{\partial \mu_\varepsilon}{\partial \varepsilon}\dot{x}_\varepsilon + \mu_\varepsilon\frac{\dot{x}_\varepsilon}{\partial \varepsilon}, \tag{4.7}$$

and evaluation at $\varepsilon = 0$ gives the relation

$$(\operatorname{Hess} S_1)_{x_0}(\dot{x}_0) + (\nabla^3 S_0)_{x_0}(x_1, \dot{x}_0) + (\operatorname{Hess} S_0)_{x_0}(\dot{x}_1) = \dot{\mu}_1\dot{x}_0.$$

Noting that in our $U \times V \times W$ coordinates the function S_0 is $(1/2)\langle L_0 u, u\rangle + \text{constant}$ and, hence, $\nabla^3 S_0$ vanishes identically, the previous relation says

$$\operatorname{Hess}(S_1|_{0\times V\times W})_{x_0}(\dot{x}_0) = \mu_1 \cdot \dot{x}_0, \tag{4.8}$$

$$\left(\frac{\partial^2 S_1}{\partial u \partial v}\right)_{x_0}(\dot{v}_0) + \left(\frac{\partial^2 S_1}{\partial u \partial w}\right)_{x_0}(\dot{w}_0) + L_0\dot{u}_1 = 0. \tag{4.9}$$

(II) Suppose further that $\mu_0 = \mu_1 = 0$. By (4.8) we now have that $\dot{x}_0 \in 0 \times 0 \times W$ (i.e., $\dot{u}_0 = \dot{v}_0 = 0$). Differentiating (4.7) at $\varepsilon = 0$ (and noting the vanishing of $\nabla^3 S_0$ and $\nabla^4 S_0$ in the $U \times V \times W$ coordinates) yields the identity

$$(\operatorname{Hess} S_2)_{x_0}(\dot{x}_0) + (\nabla^3 S_1)_{x_0}(x_1, \dot{x}_0) + (\operatorname{Hess} S_1)_{x_0}(\dot{x}_1) + (\operatorname{Hess} S_0)_{x_0}(\dot{x}_2) = \mu_2 \cdot \dot{x}_0.$$

Projecting this identity onto the W factor (and using also that $S_1(0, v, w) = (1/2)\langle L_1 v, v\rangle + \text{constant}$ in our coordinates, and $\lambda(0) = u_1$) yields

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$$\begin{aligned}
\mu_2 \cdot \dot{w}_0 &= \left(\frac{\partial^2 S_2}{\partial w^2}\right)_{x_0}(\dot{w}_0) + \left(\frac{\partial^3 S_1}{\partial u \partial w^2}\right)_{x_0}(\lambda(0), \dot{w}_0) + \left(\frac{\partial^2 S_1}{\partial u \partial w}\right)_{x_0}(\dot{u}_1) \\
&= \left(\frac{\partial^2 S_0'}{\partial t^2}\right)_{x_0}(\dot{w}_0) - \left(\frac{\partial^2 S_1}{\partial u \partial w}\right)_{x_0} \cdot \left(\left(\frac{\partial \lambda}{\partial t}\right)(0), \dot{w}_0\right) + \left(\frac{\partial^2 S_1}{\partial u \partial w}\right)_{x_0}(\dot{u}_1) \quad \text{by Lemma (4.3)} \\
&= \left(\frac{\partial^2 S_0'}{\partial t^2}\right)_{x_0}(\dot{w}_0) + \left\langle L_0 \dot{u}_1, \frac{\partial \lambda}{\partial t}(0)\right\rangle + \left(\frac{\partial^2 S_1}{\partial u \partial w}\right)_{x_0}(\dot{u}_1) \quad \text{by (4.9)} \\
&= \left(\frac{\partial^2 S_0'}{\partial t^2}\right)_{x_0}(\dot{w}_0) + \left\langle \dot{u}_1, L_0 \frac{\partial \lambda}{\partial t}(0)\right\rangle + \left(\frac{\partial^2 S_1}{\partial u \partial w}\right)_{x_0}(\dot{u}_1) \quad \text{by (4.9).}
\end{aligned}$$

Differentiating in t the identity defining $\lambda(t)$ in Lemma 4.3 gives $L_0(\partial\lambda/\partial t) = -(\partial^2 S_1/\partial u \partial w)_{(0,0,t)}$, and inserting this in the previous identity gives the relation

$$(\mathrm{Hess} S_0')_{x_0}(\dot{w}_0) = \mu_2 \cdot \dot{w}_0. \tag{4.10}$$

Furthermore, we must have $\mu_2 \neq 0$ because otherwise (4.10) implies that $\dot{w}_0 = 0$ and hence the contradiction $\dot{x}_0 = (0,0,0)$.

From these calculations all the required results follow at once. □

4.1.2 *Invariant formulation.* Our previous construction of parametrizations of the critical locus of S_ε made use of local coordinates adapted to the functions S_0 and S_1. We now address the 'intrinsic' aspects of the previous discussion. Namely, we show that the leading term S_0' of the function S_ε' is, in fact, intrinsically defined as a smooth function on the whole of $\mathscr{Z}_1 = \mathrm{Crit}(S_1|_{\mathscr{Z}_0})$.

For this, we consider the 'constrained optimization problem' of extremizing the function S_1 restricted to the critical locus of S_0. The associated *Lagrange multipliers functional* is given by the smooth function on the total space of the tangent bundle of B given by

$$\mathscr{L} : TB \to \mathbb{R}, \quad \mathscr{L}(x, \lambda) = S_1(x) + (dS_0)_x(\lambda),$$

where $x \in B$ and $\lambda \in T_x B$ (the 'Lagrange multiplier'). The critical points of $\mathscr{L}$ consist of pairs (x, λ) such that

$$x \in \mathscr{Z}_0 = \mathrm{Crit} S_0 \quad \text{and} \quad (\mathrm{Hess} S_0)_x(\lambda) = -(dS_1)_x \quad (\text{in } T_x^* B). \tag{4.11}$$

Here $(\mathrm{Hess} S_0)_x : T_x B \otimes T_x B \to \mathbb{R}$ is the quadratic form given by the Hessian of S_0 at x (which is intrinsically defined since b is a critical point of S_0). Note that, since we are assuming S_0 is Morse–Bott, $(\mathrm{Hess} S_0)_x$ induces a non-degenerate quadratic form $(T_x B/T_x \mathscr{Z}_0) \otimes (T_x B/T_x \mathscr{Z}_0) \to \mathbb{R}$. Thus, the condition that a point $x \in B$ admits a $\lambda \in T_x B$ such that (4.11) holds says, precisely, that x is a critical point simultaneously both of S_0 and of the restriction of S_1 to $\mathscr{Z}_0$. Given such $x \in B$, the collection of all λ such that (4.11) holds forms an affine space over $T_x \mathscr{Z}_0 = \mathrm{Ker}(\mathrm{Hess} S_0)_x$.

DEFINITION 4.5. The *bundle of Lagrange multipliers* associated to $\mathscr{L}$ is the affine bundle $\Lambda \to \mathscr{Z}_1 = \mathrm{Crit}(S_1|_{\mathscr{Z}_0})$ with fibers

$$\Lambda_x = \{\lambda \in T_x B \mid (\mathrm{Hess} S_0)_x(\lambda) = -(dS_1)_x\} \subset T_x B,$$

whose associated vector bundle is $(T\mathscr{Z}_0)|_{\mathscr{Z}_1} \to \mathscr{Z}_1$.

Observe that the leading term function $S_0' : \mathscr{U}_0' \subset W \to \mathbb{R}$ from our previous discussion, which we calculated in Lemma 4.3, agrees with the intrinsic and globally defined smooth function on $\mathscr{Z}_1$ given by

$$\mathfrak{f} := \tfrac{1}{2}(dS_1)(\lambda) + (S_2)|_{\mathscr{Z}_1}, \tag{4.12}$$

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where $\lambda \in \Gamma(\mathscr{Z}^1, \Lambda)$ is any smooth section of the bundle of Lagrange multipliers. (Note that expression (4.12) is indeed independent of the choice of section λ, since Λ is an affine bundle over $(T\mathscr{Z}_0)|_{\mathscr{Z}_1}$ and dS_1 vanishes on $(T\mathscr{Z}_0)|_{\mathscr{Z}_1}$.)

This observation allows us to formulate the results of our coordinate-dependent discussion in an intrinsic manner.

Theorem 4.6 (Perturbation theorem, finite-dimensional version). *Let $S_\varepsilon = S_0 + \varepsilon S_1 + \varepsilon^2 S_2 + \cdots : B \to \mathbb{R}$ be a perturbation such that the following assumptions hold:*

- *S_0 is Morse–Bott; denote its critical locus by $\mathscr{Z}_0 = \mathrm{Crit}(S_0)$;*
- *the restriction of S_1 to $\mathscr{Z}_0$ is Morse–Bott; denote its critical locus by $\mathscr{Z}_1 = \mathrm{Crit}(S_1|_{\mathscr{Z}_0})$;*
- *$\mathscr{Z}_1$ is compact.*

Then there exist a canonical smooth function $\mathfrak{f} : \mathscr{Z}_1 \to \mathbb{R}$ (namely (4.12)), an open neighborhood $\mathscr{U}_0$ of $\mathscr{Z}_1 \subset B$ and a constant $\varepsilon_0 > 0$, such that the following assertion holds when $0 < |\varepsilon| \leq \varepsilon_0$: if $\mathfrak{f}$ is a Morse function and $\pm\varepsilon > 0$, then S_ε is a Morse function in the neighborhood $\mathscr{U}_0$ and there is a bijection

$$F_\varepsilon : \mathrm{Crit}\mathfrak{f} \cong \mathrm{Crit}S_\varepsilon \cap \mathscr{U}_0$$

such that for each critical point $x \in \mathscr{Z}_1$,

$$\mathrm{ind}(S_\varepsilon, F(x)) = \mathrm{ind}(S_0, x) + \mathrm{ind}(\pm S_1|_{\mathscr{Z}_0}, x) + \mathrm{ind}(\mathfrak{f}, x).$$

(Above, $\mathrm{ind}(S_0, x)$ denotes the Morse–Bott index of the critical manifold of S_0 passing through x, and likewise for $\mathrm{ind}(\pm S_1|_{\mathscr{Z}_0}, x)$.)

The following theorem is the finite-dimensional prototype of the localization theorem that we establish for the Chern–Simons functional on a Seifert-fibered space.

Theorem 4.7 (Localization theorem, finite-dimensional version). *Let $S_\varepsilon = S_0 + \varepsilon S_1 + \varepsilon^2 S_2 + \cdots : B \to \mathbb{R}$ be a perturbation such that the following assumptions hold:*

- *S_0 is Morse–Bott; denote its critical locus by $\mathscr{Z}_0 = \mathrm{Crit}(S_0)$;*
- *the restriction of S_1 to $\mathscr{Z}_0$ is Morse–Bott; denote its critical locus by $\mathscr{Z}_1 = \mathrm{Crit}(S_1|_{\mathscr{Z}_0})$;*
- *$\mathscr{Z}_1$ is compact;*
- *the function on $\mathscr{Z}_1$ given by $(1/2)(dS_1)(\lambda) + (S_2)|_{\mathscr{Z}_1}$ is Morse; here λ stands for any section of the bundle $\Lambda \to \mathscr{Z}_1$ of Lagrange multipliers associated to $\mathscr{L} := S_1 + (dS_0)(\lambda) : TB \to \mathbb{R}$;*
- *the restriction of S_1 to $\mathscr{Z}_0$ is a proper and bounded-below function.*

Let $\psi : B \to \mathbb{R}_{\geq 0}$ be a given proper continuous function on B. Then, for any given open set $\mathscr{U} \subset B$ and a constant $C > 0$ such that

$$\mathscr{Z}_1 \subset \mathscr{U} \subset \{\psi < C\},$$

there exists a constant $\varepsilon_0 = \varepsilon_0(B, S_i, \psi) > 0$ such that the following assertions hold for $0 < |\varepsilon| \leq \varepsilon_0$.

(1) *For any critical point $x \in \mathrm{Crit}S_\varepsilon$ either $x \in \mathscr{U}$ or $\psi(x) > C$. Furthermore, the critical points of S_ε contained in $\mathscr{U}$ are non-degenerate and comprise a finite set.*

(2) *Let $\mathscr{Z}_0 = \cup \mathscr{C}_i$ denote the decomposition of $\mathscr{Z}_0$ into connected components. For $\pm\varepsilon > 0$ the count of critical points of S_ε in $\mathscr{U}$ with signs given by their indices agrees with a signed count of the Euler characteristics (respectively, compactly supported Euler characteristics) of the components $\mathscr{C}_i$:*

$$\sum_{x \in \mathrm{Crit}(S_\varepsilon) \cap \mathscr{U}} (-1)^{\mathrm{ind}(S_\varepsilon, x)} = \sum_i (-1)^{\mathrm{ind}(S_0, \mathscr{C}_i)} \cdot \chi(\mathscr{C}_i) \quad \text{(respectively, } \sum_i (-1)^{\mathrm{ind}(S_0, \mathscr{C}_i)} \cdot \chi_c(\mathscr{C}_i)\text{)}.$$

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Proof. If the first part of assertion (1) were false, then we can find a sequence $\varepsilon_n \neq 0$ converging to zero, and a sequence $x_n \in B$ such that for all n,

$$\begin{aligned} x_n &\in \mathrm{Crit}(S_{\varepsilon_n}), \\ \psi(x_n) &\leq C, \\ x_n &\notin \mathscr{U}. \end{aligned}$$

Since ψ is proper, we may assume by the second condition that x_n converges to a point $x \in B$ after passing to a subsequence. By the first condition and Lemma 4.1 we have $x \in \mathscr{Z}_1$ and, hence, x_n eventually lies in $\mathscr{U}$, which contradicts the third condition.

Thus, there is $\varepsilon_0 > 0$ such that the first part of assertion (1) holds. For the second part of assertion (1) we may shrink $\varepsilon_0 > 0$ further so as to ensure that for $0 < |\varepsilon| \leq \varepsilon_0$ the set $\mathrm{Crit}_\varepsilon \cap \mathscr{U}$ is contained in a neighborhood $\mathscr{U}_0$ of $\mathscr{Z}_1 \subset B$ such that Theorem 4.6 holds for $\mathscr{U}_0$ and ε_0. In particular, all critical points of S_ε in $\mathscr{U}$ are non-degenerate. Thus, assertion (1) follows.

For assertion (2), let $\mathscr{Z}_1 \cap \mathscr{C}_i = \cup C_{ij}$ denote the decomposition into connected components. By Theorem 4.6, for $\pm\varepsilon > 0$ the count of critical points of S_ε in $\mathscr{U}$ with signs given by their index is

$$\sum_{x \in \mathrm{Crit}(S_\varepsilon) \cap \mathscr{U}} (-1)^{\mathrm{ind}(S_\varepsilon, x)} = \sum_{i,j} (-1)^{\mathrm{ind}(S_0, \mathscr{C}_i) + \mathrm{ind}(\pm S_1|_{\mathscr{C}_i}, C_{ij})} \cdot \chi(C_{ij}).$$

But S_1 is a proper and bounded-below Morse–Bott function on $\mathscr{C}_i$ and, hence, for each i,

$$\sum_j (-1)^{\mathrm{ind}(\pm S_1|_{\mathscr{C}_i}, C_{ij})} \chi(C_{ij})$$

agrees with the Euler characteristic $\chi(\mathscr{C}_i)$ in the $+$ case, and the compactly supported Euler characteristic $\chi_c(\mathscr{C}_i)$ in the $-$ case. The result now follows. □

4.2 Holonomy perturbations

We now discuss the properties of a suitable class of functions on the stable orbit space $\mathscr{B}^*_E$ given by an $SL(2,\mathbb{C})$ analog of the *holonomy perturbations* which have been employed in several gauge-theoretic contexts to obtain transversality results [Don87a, Flo88, Tau90, BD95, Don02, KM07, KM11].

4.2.1 *Holonomy functions.* We fix a smooth embedding of a solid torus in Y, denoted by $q : S^1 \times D^2 \hookrightarrow Y$, which we regard as a family of parametrized embedded loops $q_z = q(-, z) : S^1 \hookrightarrow Y$ parametrized by $z \in D^2$. For the $U(1)$ connection λ on the determinant line bundle $\Lambda = \Lambda^2 E$, taking its holonomy along the loop q_z defines an element $\mathrm{Hol}_\lambda(q_z) \in U(1)$ for each z. We equip q with a choice of square root for the holonomy around the central loop, that is, a fixed element $\mathrm{Hol}^{1/2}_\lambda(q_0) \in U(1)$ squaring to $\mathrm{Hol}_\lambda(0)$. In turn, this determines a unique smooth map $\mathrm{Hol}^{1/2}_\lambda : D^2 \to U(1)$ such that $\mathrm{Hol}^{1/2}_\lambda(q_z)$ squares to $\mathrm{Hol}_\lambda(q_z)$ for every $z \in D^2$. (From now on, we drop the square-root data from the notation.)

Now, given a $GL(2,\mathbb{C})$ connection $\mathbb{A}$ on E with fixed determinant (Λ, λ), we can assign to each loop q_z a well-defined 'holonomy' in $SL(2,\mathbb{C})$ (rather than just in $GL(2,\mathbb{C})$):

$$\widetilde{\mathrm{Hol}}_{\mathbb{A}}(q_z) := \mathrm{Hol}_{\mathbb{A}}(q_z) \cdot ((\mathrm{Hol}^{1/2}_\lambda(q_z))^{-1} \in SL(E_{q(0,z)}). \tag{4.13}$$

More precisely, (4.13) can be regarded as a smooth section, denoted by $\widetilde{\mathrm{Hol}}_{\mathbb{A}}(q)$, of the pullback bundle $q^* SL(E)$ which is $\mathbb{A}$-covariant constant along the loops. (Here $SL(E) = E \times_{\mathrm{Ad}} SL(2,\mathbb{C})$ is the bundle of Lie groups.)

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Finally, choose a 2-form $\mu \in \Omega^2(D^2)$ on the unit disk D^2 which is non-negative, vanishes on a neighborhood of the boundary of D^2, and integrates to 1.

Definition 4.8. The $SL(2,\mathbb{C})$ *holonomy function* associated to the data q, μ is the $\mathscr{G}_E^c$-invariant complex-valued function on the set $\mathscr{A}_E^c$ of complex connections given by

$$\sigma_{q,\mu}(\mathbb{A}) = \int_{D^2} \mathrm{Tr}(\widetilde{\mathrm{Hol}}_{\mathbb{A}}(q_z))\mu(z) \in \mathbb{C}. \tag{4.14}$$

Let $\pi_E : SL(E) \to \mathfrak{sl}(E) = \mathfrak{g}_E \otimes \mathbb{C}$ be the bundle projection induced by the conjugation-invariant map $\pi : SL(2,\mathbb{C}) \to \mathfrak{sl}(2,\mathbb{C})$ which sends a matrix M to its traceless part $M - (I/2)\mathrm{Tr}(M)$. We then define the function

$$T_q(\mathbb{A}) = \pi_E \circ \widetilde{\mathrm{Hol}}_{\mathbb{A}}(q) \in \Omega^0(S^1 \times D^2, q^*\mathfrak{sl}(E)),$$

which again is $\mathbb{A}$-covariant constant over the loops. A short calculation (see, for example, [BD95, Don02]) shows that the directional derivatives of σ_q are given by

$$(d\sigma_{q,\mu})_{\mathbb{A}}(\dot{\mathbb{A}}) = -\int_Y \mathrm{Tr}(\dot{\mathbb{A}} \wedge T_q(\mathbb{A})\mu). \tag{4.15}$$

We set

$$V_{q,\mu}(\mathbb{A}) = T_q(\mathbb{A}) \cdot (*\mu) \in \Omega^1(Y, \mathfrak{sl}(E)). \tag{4.16}$$

4.2.2 *Bounds on holonomy functions.* We now discuss the differentiability of the holonomy functions $\sigma_{q,\mu}$ in the L_k^2 Hilbert manifold structure of the stable orbit spaces, together with various bounds needed ensure that the basic linear analysis and compactness properties of (2.18)–(2.20) remain valid after perturbation. The main result we need is the following proposition.

Proposition 4.9. *Fix q, μ as above, and let $k \geq 2$. For each $n \geq 0$ there is a continuous increasing function $h_n : \mathbb{R} \to \mathbb{R}_{>0}$ such that the following assertions hold.*

(1) *The function $\sigma_{q,\mu}$ extends to a smooth (in fact, holomorphic) function on the L_k^2 connections,*

$$\sigma_{q,\mu} : \mathscr{A}_{E,k}^c \to \mathbb{C},$$

satisfying the following bounds: for every $n \geq 0$ and $\dot{\mathbb{A}}_i \in \Omega^1(Y, \mathfrak{sl}(E))_k$,

$$|(\nabla^{(n)}\sigma_{q,\mu})_{\mathbb{A}}(\dot{\mathbb{A}}_1, \ldots, \dot{\mathbb{A}}_n)| \leq h_n(\|\Phi\|_{L^\infty(Y)}) \cdot \prod_{i=1}^n \|\dot{\mathbb{A}}_i\|_{L_k^2(Y)}.$$

(2) *The function $V_{q,\mu}$ extends to a smooth function*

$$V_{q,\mu} : \mathscr{A}_{E,k}^c \to \Omega^1(Y, \mathfrak{sl}(E))_k$$

(i.e., a smooth section of the tangent bundle of $\mathscr{A}_{E,k}^c$), satisfying similar bounds for $n \geq 0$:

$$\|(\nabla^{(n)}T_{\mathbb{A}})(\dot{\mathbb{A}}_1, \ldots, \dot{\mathbb{A}}_n)\|_{L_k^2(Y)} \leq h_n(\|\Phi\|_{L^\infty(Y)}) \cdot \prod_{i=1}^n \|\dot{\mathbb{A}}_i\|_{L_k^2(Y)}.$$

Remark 4.10. Since $k \geq 2$, the L_k^2 space continuously embeds in C^0, and thus the quantity $\|\Phi\|_{L^\infty}$ is defined for $\Phi \in L_k^2$. Unlike in the $SU(2)$ case (see [Tau90]), when $k = 1$ we are no longer able to ensure the differentiability of the functions $\sigma_{q,\mu}$ and $V_{q,\mu}$ on the L_1^2 connections.

We establish Proposition 4.9 in what follows. First, recall the Frobenius norm on $\mathfrak{gl}(2,\mathbb{C})$ given by $|M| := \sqrt{\mathrm{Tr}(MM^*)}$. We shall use the following properties of this norm:

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– with the chosen normalization, we have $|M \cdot M'| \leq (1/\sqrt{2})|M| \cdot |M'|$ for any two $M, M' \in \mathfrak{gl}(2,\mathbb{C})$;
– $|\cdot|$ is invariant under right or left multiplication by $U(2)$; in particular, $|\cdot|$ is well defined on the associated bundles $SL(E)$ and $\mathfrak{sl}(E)$;
– the projection $\pi : \mathfrak{gl}(2,\mathbb{C}) \to \mathfrak{sl}(2,\mathbb{C})$ to the traceless part is an orthogonal projection with respect to the Frobenius inner product, so we have

$$|M|^2 = |\pi(M)|^2 + \tfrac{1}{2}|\mathrm{Tr}M|^2. \tag{4.17}$$

We have the following estimate.

LEMMA 4.11. *Let $E \to S^1 = \mathbb{R}/\mathbb{Z}$ be a an $SU(2)$ bundle with a smooth $SL(2,\mathbb{C})$ connection $\mathbb{A} = A + i\Phi$. Then the Frobenius norm of the holonomy of $\mathbb{A}$ along S^1 satisfies the following bound:*

$$|\mathrm{Hol}(\mathbb{A})|^2 \leq 2 \cdot \exp \int_0^1 \sqrt{2}|\Phi(t)|\, dt.$$

Furthermore, for every $n \geq 0$ there is an increasing continuous function $h_n : \mathbb{R} \to \mathbb{R}_{>0}$ such that for any collection $\dot{\mathbb{A}}_i$ $(i = 1, \ldots, n)$ of smooth sections of $\mathfrak{sl}(E) \to S^1$ we have the following bounds on directional derivatives:

$$|(\nabla^{(n)}\mathrm{Hol})_{\mathbb{A}}(\dot{\mathbb{A}}_1, \ldots, \dot{\mathbb{A}}_n)| \leq h_n(\|\Phi\|_{L^2(S^1)}) \cdot \prod_{i=1}^{n} \|\dot{\mathbb{A}}_i\|_{L^2(S^1)}.$$

In particular, it follows that the holonomy along S^1 extends to a smooth function on the space $\mathscr{A}_0(S^1)$ of L^2-regular $SL(2,\mathbb{C})$ connections on $E \to S^1$:

$$\mathrm{Hol} : \mathscr{A}_0(S^1) \to SL(2,\mathbb{C}).$$

Proof. We fix once and for all a trivialization of the pullback of E along $[0,1] \to S^1$ which is A-covariant constant. Thus, the holonomy $\mathrm{Hol}(\mathbb{A}) \in SL(2,\mathbb{C})$ is given by

$$\mathrm{Hol}(\mathbb{A}) = \xi(1) \cdot \mathrm{Hol}(A),$$

where $\xi : [0,1] \to SL(2,\mathbb{C}) \subset \mathfrak{gl}(2,\mathbb{C})$ is the unique solution to the initial value problem

$$\left(\frac{\partial}{\partial t} + i\Phi(t)\right) \cdot \xi(t) = 0, \quad \xi(0) = I. \tag{4.18}$$

Now, observe that

$$\begin{aligned}\frac{\partial}{\partial t}|\xi(t)|^2 &= \mathrm{Tr}\left(\frac{\partial \xi}{\partial t}\xi^* + \xi\left(\frac{\partial \xi}{\partial t}\right)^*\right) \\ &= -2\mathrm{Tr}(i\Phi(t) \cdot \xi\xi^*) \quad \text{using (4.18) and } \Phi^* = -\Phi \\ &\leq 2|\Phi(t)| \cdot |\xi\xi^*| \leq \sqrt{2}|\Phi(t)| \cdot |\xi(t)|^2.\end{aligned}$$

Integrating, we obtain, for all $t \in [0,1]$,

$$|\xi(t)|^2 \leq 2 \cdot \exp \int_0^t \sqrt{2}|\Phi(x)|\, dx. \tag{4.19}$$

The required bounds on $\mathrm{Hol}(\mathbb{A})$ now follow from (4.19) with $t = 1$. For the bounds on the higher directional derivatives we proceed as follows. Similarly as above, the holonomy of the connection $\mathbb{A} + \sum_{i=1}^{n} s_i \dot{\mathbb{A}}_i$ is given by $\xi_s(1) \cdot \mathrm{Hol}(A)$, where, for each $s \in \mathbb{R}^n$, $\xi_s : [0,1] \to SL(2,\mathbb{C})$

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is the unique solution of the initial value problem

$$\left(\frac{\partial}{\partial t} + i\Phi(t) + \sum_{i=1}^{n} s_i \dot{\mathbb{A}}(t)\right) \cdot \xi_s(t) = 0, \quad \xi_s(0) = I. \tag{4.20}$$

Define the following $\mathfrak{gl}(2,\mathbb{C})$-valued functions on $[0,1]$:

$$X(t) := \left(\frac{\partial}{\partial s_1} \cdots \frac{\partial}{\partial s_n} \xi_s(t)\right)|_{s=0}, \quad Y_i(t) := \left(\frac{\partial}{\partial s_1} \cdots \widehat{\frac{\partial}{\partial s_i}} \cdots \frac{\partial}{\partial s_n} \xi_s(t)\right)|_{s=0},$$

and note that

$$(\nabla^{(n)}\mathrm{Hol})_{\mathbb{A}}(\dot{\mathbb{A}}_1, \ldots, \dot{\mathbb{A}}_n) = X(1) \cdot \mathrm{Hol}(A), \quad (\nabla^{(n-1)}\mathrm{Hol})_{\mathbb{A}}(\dot{\mathbb{A}}_1, \ldots, \widehat{i}, \ldots, \dot{\mathbb{A}}_n) = Y_i(1) \cdot \mathrm{Hol}(A).$$

Differentiating (4.20) in all the s_i variables and setting $s_i = 0$ yields the identity

$$\left(\frac{\partial}{\partial t} + i\Phi(t)\right) \cdot X(t) + \sum_{i=1}^{n} \dot{\mathbb{A}}(t) \cdot Y_i(t) = 0. \tag{4.21}$$

Here $X(t)$ is the unique solution of the initial value problem given by (4.21) and $X(0) = 0$. A short calculation gives the closed formula

$$X(t) = -\xi(t) \cdot \sum_{i=1}^{n} \int_0^t \xi(x)^{-1} \cdot \dot{\mathbb{A}}_i(x) \cdot Y_i(x)\, dx. \tag{4.22}$$

Now, it is easy to see that for an invertible matrix $M \in GL(2,\mathbb{C})$ the Frobenius norm of its inverse satisfies the bound

$$|M^{-1}| \leq \sqrt{2} \frac{|M|}{|\det M|}. \tag{4.23}$$

Combining (4.19), (4.23) and (4.22) yields the following estimate:

$$|X(t)| \leq \left(\exp \int_0^t \sqrt{2}|\Phi(x)|\, dx\right) \cdot \sum_{i=1}^{n} \int_0^t |\dot{\mathbb{A}}_i(x)| \cdot |Y_i(x)|\, dx.$$

The required bounds on the directional derivatives of the holonomy map now follow inductively from this estimate. □

Proof of Proposition 4.9. When $\mathbb{A}$ and the $\dot{\mathbb{A}}_i$ are smooth, the bounds from assertion (1) follow from Lemma 4.11, noting the fact that the L^∞ and L^2_k norms on Y with $k \geq 2$ control the L^2 norm on the restriction to any loop $q_z : S^1 \hookrightarrow Y$. In turn, such bounds imply the smoothness of $\sigma_{q,\mu}$ as a function on the L^2_k connections.

For the smoothness and bounds on $V_{q,\mu}$ we consider a variant of the argument in [Kro05, Proof of Proposition 7]. Namely, let $\mathscr{H}$ stand for the Hilbert bundle over D^2 with fibers

$$\mathscr{H}_z = L^2(S^1, (q^*\mathfrak{sl}(E))|_{S^1 \times z}), \quad z \in D^2.$$

Fixing a smooth unitary connection A_0 on E identifies $\mathscr{H}_z$ as the space of L^2 connections along the loop over $z \in D^2$. Furthermore, A_0 yields a canonical connection on $\mathscr{H}$, which we use to define Sobolev spaces of sections of $\mathscr{H}$. By Lemma 4.11, we have a (non-linear) smooth bundle map given by taking holonomy:

$$\underline{\mathrm{Hol}} : \mathscr{H} \to (q^*\mathfrak{gl}(E))|_{0 \times D^2}.$$

This induces a smooth map on L^2_k sections

$$\mathrm{Hol} : L^2_k(D^2, \mathscr{H}) \to L^2_k(D^2, (q^*\mathfrak{gl}(E))|_{0 \times D^2}), \tag{4.24}$$

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which by Lemma 4.11 satisfies bounds of the following form: for an L^2_k section s of $\mathscr{H}$, written in components as $s = (s_A, s_\Phi)$, and $k \geq 2$,

$$\|(\nabla^{(n)}\mathrm{Hol})_s(\dot{s}_1, \ldots, \dot{s}_n)\|_{L^2_k(D^2)} \leq h_n(\|s_\Phi\|_{L^\infty(D^2)}) \cdot \prod_{i=1}^{n} \|\dot{s}_i\|_{L^2_k(D^2)}.$$

An L^2_k connection $\mathbb{A}$ on Y gives, by restriction to each loop, an L^2_k section of $\mathscr{H}$, from which we obtain $T_{\mathbb{A}}$ by applying the map Hol, then applying the projection $\pi : \mathfrak{gl}(2,\mathbb{C}) \to \mathfrak{sl}(2,\mathbb{C})$ and then multiplying with $*\mu$. Thus, it follows that $T_{\mathbb{A}}$ lies in L^2_k and defines a smooth function on the L^2_k connections satisfying the required bounds. □

4.2.3 *Separating properties.* The crucial property of the holonomy functions σ_q that we shall use to obtain transversality results is given in the following proposition.

PROPOSITION 4.12. *Let $\mathscr{Z} \subset \mathscr{B}^*_E$ be a compact finite-dimensional submanifold. Then there exists a finite collection of disjoint smooth embeddings $q_1, \ldots, q_N$ (together with corresponding square roots and 2-forms), such that the map*

$$\underline{\sigma} := (\sigma_{q_1}, \ldots, \sigma_{q_N}) : \mathscr{B}^*_E \to \mathbb{C}^N$$

induces a smooth embedding of $\mathscr{Z}$ into $\mathbb{C}^N$, given by the restriction $\underline{\sigma}|_{\mathscr{Z}} : \mathscr{Z} \hookrightarrow \mathbb{C}^N$.

Before establishing this, we recall various facts about character schemes that we shall need (we refer to [CS83, Sik12, HP04] for further details). The $SL(2,\mathbb{C})$ *character scheme* of a finitely presented group Γ is the affine scheme (over $\mathbb{C}$) given by the affine GIT quotient of the $SL(2,\mathbb{C})$ representation scheme $\mathrm{Hom}(\Gamma, SL(2,\mathbb{C})$ (an affine scheme) by the conjugation action of $SL(2,\mathbb{C})$:

$$\mathscr{M}(\Gamma, SL(2,\mathbb{C})) := \mathrm{Hom}(\Gamma, SL(2,\mathbb{C}))//SL(2,\mathbb{C}).$$

The (closed) points of the character scheme $\mathscr{M}(\Gamma, SL(2,\mathbb{C})$ correspond one-to-one with the $SL(2,\mathbb{C})$-orbits of polystable representations. A representation $\rho : \Gamma \to SL(2,\mathbb{C})$ is *polystable* (or completely reducible) if every ρ-invariant line $L \subset \mathbb{C}^2$ admits a ρ-invariant complementary line $L' \subset \mathbb{C}^2$, and it is *stable* (or irreducible) if there exists no ρ-invariant line $L \subset \mathbb{C}^2$.

When $\Gamma = F_M$ is the *free group* on M letters the character variety is well understood. In this case, the non-singular locus in $\mathscr{M}(\Gamma, SL(2,\mathbb{C}))$ coincides with the locus of stable orbits [HP04, § 5]. The set of stable representations of F_M is explicitly described as the open set $(SL(2,\mathbb{C})^{\times M})_* \subset SL(2,\mathbb{C})^{\times M}$ of M-tuples of matrices which leave no line in $\mathbb{C}^2$ simultaneously invariant. Thus,

$$\mathscr{M}(F_M, SL(2,\mathbb{C}))^{\text{non-sing}} = (SL(2,\mathbb{C})^{\times M})_*/SL(2,\mathbb{C}).$$

For a finitely presented group Γ, the character scheme $\mathscr{M}(\Gamma, SL(2,\mathbb{C})$ can be embedded as an affine subscheme of some affine space $\mathbb{C}^N$ via the embedding which sends a conjugacy class of representation $[\rho]$ to its trace on a suitable collection of N elements of Γ (see [CS83, Corollary 1.4.5]). When $\Gamma = F_M$ it suffices to use the collection of words given by

$$X_{i_1} \cdots X_{i_M},$$

where $X_1, \ldots, X_M$ denote the M letters generating F_M, and the indices $i_1, \ldots, i_r$ run over all possible collections of distinct positive integers up to M (see the proof of [CS83, Proposition 1.4.1]).

Proof of Proposition 4.12. Recall that a stable connection $\mathbb{A} \in \mathscr{C}^*_E$ on E leaves no complex line subbundle $L \subset E$ invariant under parallel transport, by Proposition 2.1. In particular, one can

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find a finite collection of disjoint embedded loops $\underline{\gamma} = (\gamma_1, \dots, \gamma_M)$ at a given basepoint $x \in Y$ (equipped with square roots) such that the associated holonomy map

$$\widetilde{\mathrm{Hol}}(\underline{\gamma}) : \mathscr{B}^*_{E,k} \to (SL(2,\mathbb{C})^{\times M})/SL(2,\mathbb{C}) \tag{4.25}$$

maps the compact subset $\mathscr{Z} \subset \mathscr{B}^*_E$ into the manifold locus $(SL(2,\mathbb{C})^{\times N})_*/SL(2,\mathbb{C})$.

As noted in [Don02, § 5.5.1], it is a basic fact that a connection $\mathbb{A}$ is determined up to gauge equivalence by its holonomy. In particular:

- given two connections $\mathbb{A}_0, \mathbb{A}_1 \in \mathscr{A}^c_E$ that are not in the same $\mathscr{G}^c_E$-orbit, there exists $\underline{\gamma}$ such that $\widetilde{\mathrm{Hol}}(\underline{\gamma})$ takes on distinct values at $\mathbb{A}_0$ and $\mathbb{A}_1$; and
- given a connection $\mathbb{A} \in \mathscr{A}^c_E$ and a tangent vector $\dot{\mathbb{A}} \in T_{\mathbb{A}}\mathscr{A}^c_E$ not lying in the tangent space to the $\mathscr{G}^c_E$-orbit at $\mathbb{A}$, there exists $\underline{\gamma}$ such that the derivative of $\widetilde{\mathrm{Hol}}(\underline{\gamma})$ along $\dot{\mathbb{A}}$ is non-trivial.

Thus, because $\mathscr{Z}$ is a compact submanifold of $\mathscr{B}^*_E$, one can choose the loops $\underline{\gamma}$ so that the map (4.25) yields a *smooth embedding* of $\mathscr{Z}$ into the manifold locus $(SL(2,\mathbb{C})^{\times M})_*/SL(2,\mathbb{C})$.

Next, fix an embedding of $\mathscr{M}(F_M, SL(2,\mathbb{C}))$ as an affine subscheme of some $\mathbb{C}^N$, as described above. In particular, this restricts to a smooth embedding of the manifold locus

$$(SL(2,\mathbb{C})^{\times M})_*/SL(2,\mathbb{C}) \hookrightarrow \mathbb{C}^N. \tag{4.26}$$

The composition of the maps (4.25) and (4.26) gives a map $\mathscr{B}^*_E \to \mathbb{C}^N$ inducing a smooth embedding of $\mathscr{Z}$ into $\mathbb{C}^N$. We then modify this map as follows.

- Slightly deform the N loops in Y obtained as the concatenations $\gamma_{i_1} \cdots \gamma_{i_M}$ to obtain N *disjoint* embedded loops.
- Thicken the N disjoint embedded loops from the previous step to obtain disjoint embeddings $q_i : S^1 \times D^2 \hookrightarrow Y$ of solid tori neighborhoods with sufficiently concentrated 2-forms μ_i.

It is clear that the map $\underline{\sigma} : \mathscr{B}^*_E \to \mathbb{C}^N$ resulting from this procedure will still induce a smooth embedding of $\mathscr{Z}$ into $\mathbb{C}^N$, as required. □

4.3 Perturbing the Chern–Simons functional

For a closed oriented 3-manifold, consider the functional $S = \mathrm{Re}(e^{-\theta}CS)$ and the perturbed functional

$$S_\varepsilon(A,\Phi) = S(A,\Phi) + \varepsilon \cdot \|\Phi\|^2_{L^2} + \varepsilon^2 \cdot f(\underline{\sigma}(A,\Phi)). \tag{4.27}$$

Here, the function $\underline{\sigma} = (\sigma_{q_1}, \dots, \sigma_{q_N})$ is constructed by using a collection of disjoint embedded solid tori q_i, square-root data and bump forms, and $f : \mathbb{C}^N \to \mathbb{R}$ is a smooth function.

Acting on the space $\mathscr{A}^c_E$ of complex connections, the function S_ε is still invariant under the unitary gauge group $\mathscr{G}_E$, but no longer under the complex gauge group $\mathscr{G}^c_E$. Thus, when $\theta = \pi/2$ so that there are no periods of S under the $\mathscr{G}_E$-action, the perturbation S_ε descends to a well-defined function on the stable orbit space $\mathscr{B}^*_{E,k} = \mathscr{C}^*_{E,k}/\mathscr{G}_{E,k+1}$ (again, setting $\theta = \pi/2$ is not required; for other θ the 1-form dS_ε descends to $\mathscr{B}^*_E$, which is all that we need).

We now establish an infinite-dimensional analog of Theorem 4.7 for the perturbed functional $S_\varepsilon : \mathscr{B}^*_{E,k} \to \mathbb{R}$. The proof will be formally identical to the finite-dimensional case, once we have verified that the basic linear analysis and compactness results carry over to the perturbed setting.

4.3.1 *Perturbed Hessian.* First, Proposition 2.11 holds with S replaced with the perturbed functional S_ε from (4.27). Indeed, the formal L^2 gradient of the functional $\|\Phi\|^2_{L^2}$ on the affine space of complex connections $\mathscr{A}^c_E$ is $\mathrm{grad}\|\Phi\|^2_{L^2} = (0, 2\Phi)$, which is tangent to the stable locus

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$\mathscr{C}^*_E \subset \mathscr{A}^c_E$. On the other hand, the formal L^2 gradient of $f(\underline{\sigma}(A,\Phi))$ is tangent to $\mathscr{C}^*_E$ because this function is invariant under the *complex* gauge group $\mathscr{G}^c_E$ (see the remark to Proposition 2.11).

Proposition 2.12 and Theorem 2.13 then carry through in the perturbed setting, with S_ε in place of S. Indeed, denote the formal Hessian of $S_\varepsilon : \mathscr{B}^*_{E,k} \to \mathbb{R}$ (taken with respect to the L^2 metric) by

$$\mathscr{D}^\varepsilon_{(A,\Phi)} : \mathscr{K}_{(A,\Phi),k} \to \mathscr{K}_{(A,\Phi),k-1}.$$

Then this is a *compact* perturbation of the unperturbed Hessian $\mathscr{D}_{(A,\Phi)}$. Indeed,

$$\mathscr{D}^\varepsilon_{(A,\Phi)} = \mathscr{D}_{(A,\Phi)} + \varepsilon\mathcal{D}[\mathrm{grad}\|\Phi\|^2_{L^2}] + \varepsilon^2\mathcal{D}[\mathrm{grad} f(\underline{\sigma})]$$

since $[\mathrm{grad}\|\Phi\|^2_{L^2}]$ and $[\mathrm{grad} f(\underline{\sigma})]$ (the formal L^2 gradients of $\|\Phi\|^2_{L^2}$ and $f(\underline{\sigma})$ on $\mathscr{B}^*_{E,k}$ with respect to the L^2 metric) are sections of $\mathscr{K}_k$ (the L^2_k completion of the tangent bundle of $\mathscr{B}^*_{E,k}$) rather than just $\mathscr{K}_{k-1}$ (cf. Proposition 4.9), and hence their vertical derivatives $[\mathrm{grad}\|\Phi\|^2_{L^2}]$ and $\mathcal{D}[\mathrm{grad} f(\underline{\sigma})]$ are compact operators from $\mathscr{K}_k$ to $\mathscr{K}_{k-1}$.

Remark 4.13. Proposition 2.14 no longer holds for S_ε. In fact, the function $\|\Phi\|^2_{L^2}$ on $\mathscr{B}^*_{E,k}$ is *strictly plurisubharmonic*: for the flat Hessian $\mathrm{Hess}\|\Phi\|^2_{L^2}$ and $v, w \in \mathscr{K}_{(A,\Phi),k}$ we now have a *positive* symmetric form

$$\langle(\mathrm{Hess}\|\Phi\|^2_{L^2})Jv, (\mathrm{Hess}\|\Phi\|^2_{L^2})Jw\rangle_{L^2} + \langle(\mathrm{Hess}\|\Phi\|^2_{L^2})v, (\mathrm{Hess}\|\Phi\|^2_{L^2})w\rangle_{L^2} = 2\langle v, w\rangle_{L^2}.$$

The perturbation $f(\underline{\sigma})$ can be taken strictly plurisubharmonic as well, by requiring $f : \mathbb{C}^N \to \mathbb{R}$ to be strictly plurisubharmonic (since $\underline{\sigma}$ is holomorphic), and in this case the whole functional S_ε will be strictly plurisubharmonic on $\mathscr{B}^*_{E,k}$.

We now make some comments about the spectral flow of families of Fredholm operators (for the definition we refer to [APS76, Tau90, KM07]). Consider a continuous path of self-adjoint Fredholm operators on a real Hilbert: $L_t : H \to H$, $t \in [0,1]$. We only need to consider the following situation:

- H is equipped with an isometric linear complex structure J (hence, H is a complex Hilbert space);
- for each $i = 0, 1$, L_i satisfies either $\langle L_i Jv, L_i Jw\rangle + \langle L_i v, L_i w\rangle = 0$ for any $v, w \in H$ (hence, $\mathrm{Ker} L_0 \subset (H, J)$ is a $\mathbb{C}$-linear finite-dimensional subspace); otherwise L_i is invertible.

In this situation, there is associated to the path L_t a well-defined *mod 2 spectral flow*:

$$\mathbf{sf}_2(L_t) \in \mathbb{Z}/2.$$

The mod 2 spectral flow has the following basic properties:

- it only depends on the underlying homotopy class of the path L_t;
- it behaves additively with respect to composition of paths;
- it vanishes if the kernel of L_t is a $\mathbb{C}$-linear subspace of (H, J) for every $t \in [0,1]$.

4.3.2 *L^∞ bounds on Φ and compactness.* It is well known that on the moduli space of polystable $GL(2,\mathbb{C})$ projectively flat $GL(2,\mathbb{C})$ connections $\mathscr{M}_E$ on (Y, g) the function $\|\Phi\|_{L^2}$ is *proper*, thus controls the non-compactness of $\mathscr{M}_E$ (see [Tau13b]). In particular, since the L^∞ norm is stronger than the L^2 norm, the function $\|\Phi\|_{L^\infty}$ is proper on $\mathscr{M}_E$ (note that since $k \geq 2$, we have that Φ is continuous; thus its L^∞ norm is defined).

The corresponding *perturbed* moduli space $\mathscr{M}^\varepsilon_E \subset \mathscr{B}_{E,k}$ is defined as the $\mathscr{G}_{E,k+1}$-orbits of polystable configurations (A,Φ) (i.e., $d^*_A\Phi = 0$) which are critical points of the $\mathscr{G}_{E,k+1}$-invariant

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function S_ε acting on $\mathscr{A}^c_{E,k}$. The stable locus $\mathscr{M}^{*,\varepsilon}_E$ in $\mathscr{M}^\varepsilon_E$ is given by the critical points of S_ε on the Hilbert manifold of stable orbits $\mathscr{B}^*_{E,k} = \mathscr{C}^*_{E,k}/\mathscr{G}_{E,k+1}$.

Proposition 4.14. *For any constant $\varepsilon_0 > 0$, the function $\psi(A,\Phi) := \|\Phi\|_{L^\infty}$ is a proper function on the union of the perturbed moduli spaces $\mathscr{M}^\varepsilon_E \subset \mathscr{B}_{E,k}$ over all ε with $0 \le |\varepsilon| \le \varepsilon_0$.*

Proof. The perturbed equations are:

$$*(F_A - \tfrac{1}{2}[\Phi,\Phi]) = \varepsilon\Phi + \varepsilon^2 \mathfrak{q}_2(A,\Phi), \tag{4.28}$$

$$*d_A\Phi = \varepsilon^2 \mathfrak{q}_1(A,\Phi), \tag{4.29}$$

$$d^*_A\Phi = 0, \tag{4.30}$$

where $(\mathfrak{q}_1, \mathfrak{q}_2)$ are the components of the vector field $\operatorname{grad} f(\underline{\sigma})$ on $\mathscr{A}^c_E$.

For a given configuration (A,Φ) a short calculation with the Bochner–Weitzenböck formula yields the following estimate (see [Tau13b, equation 2.4]), where c_0 is a constant depending on the Ricci tensor of the metric:

$$\|\nabla_A\Phi\|^2_{L^2} + \frac{1}{4}\|[\Phi,\Phi]\|^2_{L^2} + \|F_A\|^2_{L^2} \le c_0\|\Phi\|^2_{L^2} + \int_Y |F_A - \frac{1}{2}[\Phi,\Phi]|^2 + |d_A\Phi|^2 + |d^*_A\Phi|^2.$$

Suppose (A,Φ) solves (4.28)–(4.30). Then we obtain

$$\|\nabla_A\Phi\|^2_{L^2} + \tfrac{1}{4}\|[\Phi,\Phi]\|^2_{L^2} + \|F_A\|^2_{L^2} \le (c_0 + 2\varepsilon^2)\|\Phi\|^2_{L^2} + \varepsilon^4\|\mathfrak{q}_1(A,\Phi)\|^2_{L^2} + 2\varepsilon^4\|\mathfrak{q}_2(A,\Phi)\|^2_{L^2}. \tag{4.31}$$

Suppose now we are given bounds

$$\|\Phi\|_{L^\infty} \le C, \quad |\varepsilon| \le \varepsilon_0. \tag{4.32}$$

Then Proposition 4.9 gives an L^∞ bound on $\mathfrak{q}_1(A,\Phi)$ and $\mathfrak{q}_2(A,\Phi)$. Thus, (4.31) gives an L^2 curvature bound

$$\|F_A\|^2_{L^2} \le C'.$$

Given this, it is a routine application of Uhlenbeck's local gauge-fixing theorem [Uhl82], together with standard elliptic bootstrapping and patching arguments (see, for example, [DK90, §§ 2 and 4]), to show that a sequence (A_n,Φ_n) of L^2_k solutions of (4.28)–(4.30) with uniform bounds (4.32) satisfies uniform L^2_{k+1} bounds after applying a sequence of global L^2_{k+1} unitary gauge transformations. Thus, (A_n,Φ_n) converges in L^2_k after passing to a subsequence. This completes the proof. □

4.3.3 *The perturbation and localization theorems for the complex Chern–Simons functional.* With the above understood, we are now in good shape to discuss the infinite-dimensional version of Theorems 4.6 and 4.7.

The setup for this results is as follows. We let (Y,g) be a closed, oriented, Riemannian 3-manifold, equipped with a $U(2)$ bundle $E \to Y$ with fixed determinant. Let $q_1,\dots,q_N : S^1 \times D^2 \hookrightarrow Y$ be a finite collection of disjoint smooth embeddings (equipped with 'square roots' and 2-forms), and let $\underline{\sigma} : \mathscr{B}^*_{E,k} \to \mathbb{C}^N$ denote the associated holonomy function (and we choose $k \ge 2$). Let $f : \mathbb{C}^N \to \mathbb{R}$ be a smooth function. Consider, for each $\varepsilon \in \mathbb{R} \setminus 0$, the smooth functional $S_\varepsilon : \mathscr{B}^*_{E,k} \to \mathbb{R}$ given by

$$S_\varepsilon(A,\Phi) = S(A,\Phi) + \varepsilon\|\Phi\|^2_{L^2(Y)} + \varepsilon^2 f(\underline{\sigma}(A,\Phi)).$$

We make the following assumptions.

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(A1) The moduli space of stable flat connections $\mathscr{M}^*_E = \mathrm{Crit}(S)$ is Morse–Bott (cf. Definition 2.17).

(A2) The restriction of the function $\|\Phi\|^2_{L^2}$ onto $\mathscr{M}^*_E$ is a Morse–Bott function; denote its critical locus by $\mathscr{Z} = \mathrm{Crit}(\|\Phi\|^2_{L^2}|_{\mathscr{M}^*_E})$.

(A3) The subset $\mathscr{Z}$ is compact.

(A4) The function $\underline{\sigma} : \mathscr{B}^*_{E,k} \to \mathbb{C}^N$ restricts to a smooth embedding of $\mathscr{Z}$ (cf. Proposition 4.12).

(A5) The function f restricts 'generically' onto $\mathscr{Z}$: the smooth function on $\mathscr{Z}$ given by $\mathfrak{f} := (1/2)(d\|\Phi\|^2_{L^2})(\lambda) + (f \circ \underline{\sigma})|_{\mathscr{Z}}$ is Morse; here λ stands for any section of the bundle of Lagrange multipliers $\Lambda \to \mathscr{Z}$ (see below).

(A6) The bundle $E \to Y$ is *admissible*: either Y is a homology 3-sphere or $c_1(E)$ defines an odd class in $H^2(Y,\mathbb{Z})/\mathrm{Torsion}$.

DEFINITION 4.15. The *bundle of Lagrange multipliers* associated to the Lagrange functional $\mathscr{L} : T\mathscr{B}^*_{E,k} \to \mathbb{R}$, $\mathscr{L}([\mathbb{A}],\lambda) = \|\Phi\|^2_{L^2} + (dS)_{\mathbb{A}}(\lambda)$, is the affine bundle $\Lambda \to \mathscr{Z} = \mathrm{Crit}(\|\Phi\|^2_{L^2}|_{\mathscr{M}^*_E})$ with fibers

$$\Lambda_{[\mathbb{A}]} = \{\lambda \in \mathscr{K}_{[\mathbb{A}],k} = T_{[\mathbb{A}]}\mathscr{B}^*_{E,k} \mid \mathscr{D}_{\mathbb{A}}(\lambda) = -2(0,\Phi)\} \subset T_{[\mathbb{A}]}\mathscr{B}^*_{E,k},$$

with associated vector bundle $T\mathscr{M}^*_E|_{\mathscr{Z}} \to \mathscr{Z}$.

THEOREM 4.16 (Perturbation theorem). *Assume that assumptions (A1)–(A5) hold. Then there exist an open neighborhood $\mathscr{U}_0$ of $\mathscr{Z} \subset \mathscr{B}^*_{E,k}$ and a constant $\varepsilon_0 > 0$, such that when $0 < |\varepsilon| \le \varepsilon_0$ then the critical points of $S_\varepsilon : \mathscr{B}^*_{E,k} \to \mathbb{R}$ in the neighborhood $\mathscr{U}_0$ are non-degenerate and there is a bijection*

$$F_\varepsilon : \mathrm{Crit}\mathfrak{f} \cong \mathrm{Crit}S_\varepsilon \cap \mathscr{U}_0.$$

Furthermore, if for a point $[\mathbb{A}_0] \in \mathscr{Z}$ we denote $[\mathbb{A}_\varepsilon] := F_\varepsilon([\mathbb{A}_0])$, then the mod 2 spectral flow of the path of operators $\mathscr{D}^t_{\mathbb{A}_\varepsilon}$ from $t = 0$ to $t = \varepsilon$ is, for $\pm\varepsilon > 0$,

$$\mathbf{sf}_2(\mathscr{D}^t_{\mathbb{A}_\varepsilon}) = \mathrm{ind}(\pm\|\Phi\|^2_{L^2}|_{\mathscr{M}^*_E}, [\mathbb{A}_0]) + \mathrm{ind}(\mathfrak{f}, [\mathbb{A}_0]).$$

(Note that the quantity $\mathrm{ind}(\pm\|\Phi\|^2_{L^2}|_{\mathscr{M}^*_E}, [\mathbb{A}_0])$ taken mod 2, as above, is independent of the sign $\pm$ since $\mathscr{M}^*_E$ has even real dimension.)

The proof of Theorem 4.16 is given below. We also have the following result.

THEOREM 4.17 (Localization theorem). *Assume that assumptions (A1)–(A6) hold. Let $\mathscr{U} \subset \mathscr{B}^*_{E,k}$ be an open set, and $C > 0$ a constant such that*

$$\mathscr{Z} \subset \mathscr{U} \subset \{\|\Phi\|_{L^\infty} < C\}.$$

When Y is a homology 3-sphere, let $\mathscr{U}_{\mathrm{triv}}$ be a given open neighborhood in $\mathscr{B}_{E,k}$ of the trivial connection $[\mathbb{A}_{\mathrm{triv}}]$ which is disjoint from $\mathscr{U}$.

Then there exists a constant $\varepsilon_0 = \varepsilon_0(Y, g, E, q_i, f, \mathscr{U}, C) > 0$ such that the following assertions hold for $0 < |\varepsilon| \le \varepsilon_0$.

(1) *If $[\mathbb{A} = (A,\Phi)] \in \mathrm{Crit}S_\varepsilon$ is a critical point of $S_\varepsilon : \mathscr{B}^*_{E,k} \to \mathbb{R}$, then either $[\mathbb{A}] \in \mathscr{U}$ or $[\mathbb{A}] \in \mathscr{U}_{\mathrm{triv}} \setminus \{[\mathbb{A}_{\mathrm{triv}}]\}$ or $\|\Phi\|_{L^\infty} > C$. Furthermore, the critical points of S_ε contained in $\mathscr{U}$ are non-degenerate and comprise a finite set.*

(2) *The count of critical points $[\mathbb{A}]$ of S_ε in $\mathscr{U}$ with signs given by the mod 2 spectral flow of the path of Hessians $\mathscr{D}^t_{\mathbb{A}}$ with t going from $t = 0$ to $t = \varepsilon$ computes the Euler characteristic*

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of the moduli space of stable flat connections:

$$\sum_{[\mathbb{A}]\in \mathrm{Crit}(S_\varepsilon)\cap \mathscr{U}} (-1)^{\mathbf{sf}_2(\mathscr{D}^t_{\mathbb{A}})} = \chi(\mathscr{M}^*_E).$$

Proof of Theorem 4.16. We can follow the argument from the proof of the finite-dimensional result, Theorem 4.6, with the following modifications.

Local perturbation theory. The arguments from § 4.1.1 carry through with minor modifications to the infinite-dimensional case, crucially making use now of the finite-dimensional Kuranishi models for S from Corollary 2.16. That is, we have product coordinates $U \times V \times W$ on a neighborhood of a stable flat connection $[\mathbb{A}_0] \in \mathscr{M}^*_E$ in $\mathscr{B}^*_{E,k}$, with $[\mathbb{A}_0]$ corresponding to $(0,0,0) \in U \times V \times W$. Namely:

- $V \oplus W = \mathscr{H}^1_{\mathbb{A}_0}$ is the (finite-dimensional) tangent space to $\mathscr{M}^*_E$;
- U is the L^2-orthogonal complement of $\mathscr{H}^1_{\mathbb{A}_0}$ in $\mathscr{K}_{\mathbb{A}_0,k} = T_{\mathbb{A}_0}\mathscr{B}^*_{E,k}$;
- W is the tangent space to $\mathscr{Z}$ (taken to be 0 if $[\mathbb{A}_0] \notin \mathscr{Z}$);
- V is the L^2-orthogonal complement of W in $\mathscr{H}^1_{\mathbb{A}_0}$.

We also let U' be the L^2 orthogonal complement of $\mathscr{H}^1_{\mathbb{A}_0}$ in $\mathscr{K}_{\mathbb{A}_0,k-1}$. We thus have L^2 orthogonal decompositions

$$\mathscr{K}_{\mathbb{A}_0,k} = U \oplus V \oplus W, \quad \mathscr{K}_{\mathbb{A}_0,k-1} = U' \oplus V \oplus W.$$

By the finite-dimensional Kuranishi model for S (Corollary 2.16) and assumption (A1) we may assume S is in standard form,

$$S(u,v,w) = \tfrac{1}{2}\langle \mathscr{D}_{(A,\Phi)}u, u\rangle + \text{constant},$$

where now $\mathscr{D}_{\mathbb{A}_0}$ gives an isomorphism from U to U' rather than from U to U (this does not bring in any additional difficulties). By assumption (A2), a further change of coordinates ensures that $\|\Phi\|^2_{L^2}$ is in standard form on the slice $0 \times V \times W$. The proofs of Lemmas 4.1–4.3 now go through with almost identical notation, using the implicit function theorem for Hilbert spaces.

Spectral flow calculation. Proposition 4.4 also goes through using assumptions (A3), (A4) and (A5), except the index assertion no longer makes sense (since the spectrum of the Hessians of S and S_ε is now unbounded in both directions). This is modified as follows. For $0 < |\varepsilon| \le \varepsilon_0$, the configuration $[\mathbb{A}_\varepsilon] = F_\varepsilon([\mathbb{A}_0])$ is the unique critical point of S_ε in a small neighborhood $\mathscr{U}_0$ of $\mathbb{A}_0$, and is non-degenerate. Fix a representative $\mathbb{A}_0$ of $[\mathbb{A}_0]$ and take the unique representative $\mathbb{A}_\varepsilon$ of $[\mathbb{A}_\varepsilon]$ contained in the stable Coulomb slice $\mathcal{S}_{\mathbb{A}_0,k}$ at $\mathbb{A}_0$ (cf. Proposition 2.5). Consider the path of operators $\mathscr{D}^t_{\mathbb{A}_t}$ for t going from $t=0$ to $t=\varepsilon$. The argument from the proof of the index assertion in Proposition 4.4, together with the intrinsic description of the 'leading-order term' as the function $\mathfrak{f}$, shows that the mod 2 spectral flow of the path $\mathscr{D}^t_{\mathbb{A}_t}$ is given by

$$\mathrm{ind}(\pm\|\Phi\|^2_{L^2}|_{\mathscr{M}^*_E}, [\mathbb{A}_0]) + \mathrm{ind}(\mathfrak{f}, [\mathbb{A}_0]).$$

But observe that the path $\mathscr{D}^t_{\mathbb{A}_t}$ is homotopic to the concatenation of the following two paths:

- the path $\mathscr{D}_{\mathbb{A}_t}$ from $\mathscr{D}_{\mathbb{A}_0}$ to $\mathscr{D}_{\mathbb{A}_\varepsilon}$; all operators in this path have $\mathbb{C}$-linear kernel, hence there is no mod 2 spectral flow;
- the path $\mathscr{D}^t_{\mathbb{A}_\varepsilon}$ from $\mathscr{D}_{\mathbb{A}_\varepsilon}$ to $\mathscr{D}^\varepsilon_{\mathbb{A}_\varepsilon}$.

Hence, the mod 2 spectral flow of the path $\mathscr{D}^t_{\mathbb{A}_\varepsilon}$ agrees with the required quantity, and this completes the proof. □

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Proof of Theorem 4.17. Suppose no $\varepsilon_0 > 0$ exists making the first part of assertion (1) hold. Then we can find a sequence $[\mathbb{A}_n = (A_n, \Phi_n)] \in \mathscr{B}^*_{E,k}$ and $\varepsilon_n \neq 0$ with $\varepsilon_n \to 0$, such that

$$\begin{gathered}[\mathbb{A}_n] \in \mathscr{M}^{\varepsilon_n,*}_{E,k}, \\ [\mathbb{A}_n] \notin \mathscr{U}, \\ \|\Phi\|_{L^\infty} \leq C, \\ [\mathbb{A}_n] \notin \mathscr{U}_{\text{triv}}.\end{gathered}$$

By Proposition 4.14, the function $\|\Phi\|_{L^\infty}$ is proper when restricted to the union of the moduli spaces $\mathscr{M}^{\varepsilon}_{E,k}$ with ε taking values in any compact interval. Thus, after applying a sequence of unitary gauge transformations and passing to a subsequence, we may assume that $\mathbb{A}_n$ converges in L^2_k to a configuration $\mathbb{A} = (A, \Phi)$, and necessarily $[(A, \Phi)] \in \mathscr{M}_E$.

If $\mathbb{A}$ is stable, then by the infinite-dimensional analog of Lemma 4.1 (as explained previously, this is established following the same reasoning as in the finite-dimensional case, using now the finite-dimensional Kuranishi models for S from Corollary 2.16) we have that $[\mathbb{A}] \in \mathscr{Z} \subset \mathscr{U}$, which contradicts the fact that $\mathbb{A}_n \notin \mathscr{U}$. If $\mathbb{A}$ is not stable, then there exists a $U(1)$ bundle with connection (L, B) and $\phi \in \Omega^1(Y, i\mathbb{R})$ such that

$$\begin{gathered}E = L \oplus L^{-1}\Lambda, \quad A = B \oplus B^{-1}\lambda, \quad \Phi = \begin{pmatrix} \phi & 0 \\ 0 & -\phi \end{pmatrix}, \\ F_B - \tfrac{1}{2}F_\lambda = 0, \quad d\phi = 0, \quad d^*\phi = 0.\end{gathered} \tag{4.33}$$

From the first equation in (4.33) it follows that $c_1(E)$ is an even class in $H^2(Y, \mathbb{Z})/\text{Torsion}$. Since $E \to Y$ is an admissible bundle, the only possibility is that Y is a homology 3-sphere and both L and Λ are trivial. Then $\varphi = 0$ by the second and third equations in (4.33), and the first equation has a unique solution modulo $U(1)$ gauge transformations: the trivial connection. This contradicts the fact that $\mathbb{A}_n \notin \mathscr{U}_{\text{triv}}$.

Thus, there is $\varepsilon_0 > 0$ such that the first part of assertion (1) holds. For the second part of assertion (1) we only need to shrink $\varepsilon_0 > 0$ further so as to ensure that for $0 < |\varepsilon| \leq \varepsilon_0$ the set $\text{Crit}S_\varepsilon \cap \mathscr{U}$ is contained in a neighborhood $\mathscr{U}_0$ of $\mathscr{Z} \subset \mathscr{B}^*_{E,k}$ such that Theorem 4.16 holds for $\mathscr{U}_0$ and ε_0. For assertion (2), by Theorem 4.16 we then have that if $\mathscr{Z} = \cup \mathscr{Z}_i$ stands for the decomposition into connected components and $\pm\varepsilon > 0$, then

$$\sum_{[\mathbb{A}] \in \text{Crit}S_\varepsilon \cap \mathscr{U}} (-1)^{\mathbf{sf}_2(\mathscr{D}^t_{\mathbb{A}})} = \sum_i (-1)^{\text{ind}(\pm\|\Phi\|^2_{L^2}|_{\mathscr{M}^*_E}, \mathscr{Z}_i)} \cdot \chi(\mathscr{Z}_i).$$

Note that $\|\Phi\|^2_{L^2}$ is a proper and bounded-below function on $\mathscr{M}_E$, and hence also on $\mathscr{M}^*_E$ since E is an admissible bundle. It follows that the right-hand side of the above equation computes the Euler characteristic $\chi(\mathscr{M}^*_E)$ when $\varepsilon > 0$ and the compactly supported Euler characteristic $\chi_c(\mathscr{M}^*_E)$ when $\varepsilon < 0$. But observe that both Euler characteristics are identical, since $\mathscr{M}^*_E$ has *even* real dimension. This completes the proof. □

4.3.4 *The exact case.* So far we have been working with the functional $S = \text{Re}(e^{-i\theta}S)$ for an arbitrary $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. The values $\theta = \pm\pi/2$ modulo 2π (cf. 2.16) are special since then the functional S is exact, that is to say, it has no periods under the action of complex or unitary gauge transformations, and thus defines an honest $\mathbb{R}$-valued function on $\mathscr{B}^*_{E,k}$.

In this case the family of operators $\mathscr{D}_p = \mathscr{D}^{\varepsilon}_{[\mathbb{A}]} + \delta$ parametrized by $p = ([\mathbb{A}], \varepsilon, \delta) \in P := \mathscr{B}^*_{E,k} \times \mathbb{R} \times \mathbb{R}$ has the following property.

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Lemma 4.18. *Suppose $\theta = \pi/2$ or $3\pi/2$. Let $\gamma : [0,1] \to P$ be a path such that the endpoints $\mathscr{D}_{\gamma(0)}$, $\mathscr{D}_{\gamma(1)}$ are invertible. Then the spectral flow $\mathbf{sf}(\mathscr{D}_{\gamma(t)})$ only depends on $\gamma(0)$ and $\gamma(1)$.*

Proof. It suffices to show that if γ is a generic loop in P based at a point $\gamma(0) = \gamma(1) = p$ such that $\mathscr{D}_p$ is invertible, then the spectral flow along γ vanishes:

$$\mathbf{sf}(\mathscr{D}_{\gamma(t)}) = 0.$$

This follows by a standard argument identifying the spectral flow along the loop with the index of a suitable Fredholm operator on $S^1 \times Y$ (see, for example, [KM07, Lemma 14.4.6]). By [MW14, Proposition 4.1], the said operator on $S^1 \times Y$ has vanishing index. □

We can now attempt to attach an *integral* 'Morse index' to the critical points of S_ε using the spectral flow (as opposed to only a $\mathbb{Z}/2$ index). This is explained in the following refinement of Theorem 4.17. In the following, $\mathcal{P}_T(\mathscr{M}_E^*)$ (respectively, $\mathcal{P}_T^c(\mathscr{M}_E^*)$) stands for the Poincaré polynomial (respectively, compactly supported Poincaré polynomial) of $\mathscr{M}_E^*$.

Theorem 4.19. *Suppose that $\theta = \pi/2$ or $3\pi/2$. Assume that assumptions (A1)–(A6) hold. Let $\mathscr{Z} = \cup_i \mathscr{Z}_i$ denote the decomposition into connected components. For each i let $\mathscr{U}_i \subset \mathscr{B}_{E,k}^*$ be an open subset, and $C > 0$ a constant, such that:*

- *$\mathscr{Z}_i \subset \mathscr{U}_i \subset \{\|\Phi\|_{L^\infty} < C\}$;*
- *$\mathscr{U}_i \cap \mathscr{U}_j = \emptyset$ for $i \neq j$.*

Set $\mathscr{U} = \cup_i \mathscr{U}_i$. When Y is a homology 3-sphere, let $\mathscr{U}_{\mathrm{triv}}$ be a given open neighborhood in $\mathscr{B}_{E,k}$ of the trivial connection $[\mathbb{A}_{\mathrm{triv}}]$ which is disjoint from $\mathscr{U}$.

Then there exists a constant $\varepsilon_0 = \varepsilon_0(Y, g, E, q_i, f, \mathscr{U}, C) > 0$ such that the following assertions hold for $0 < |\varepsilon| \leq \varepsilon_0$.

(1) *If $[\mathbb{A} = (A, \Phi)] \in \mathrm{Crit} S_\varepsilon$ is a critical point of $S_\varepsilon : \mathscr{B}_{E,k}^* \to \mathbb{R}$ then either $[\mathbb{A}] \in \mathscr{U}$ or $[\mathbb{A}] \in \mathscr{U}_{\mathrm{triv}} \setminus \{[\mathbb{A}_{\mathrm{triv}}]\}$ or $\|\Phi\|_{L^\infty} > C$. Furthermore, the critical points of S_ε contained in $\mathscr{U}$ are non-degenerate and comprise a finite set.*

(2) *If $[\mathbb{A}]$ is a critical point of S_ε in $\mathscr{U}_i$ then there exists a path $[\mathbb{A}_t]$ defined for t going from $t = 0$ to $t = \varepsilon$ with $[\mathbb{A}_0] \in \mathscr{Z}_i$, $[\mathbb{A}_1] = [\mathbb{A}]$ and $[\mathbb{A}_t] \in \mathscr{U}_i$ for all t; fix one such path for each critical point in $\mathscr{U}$. Let $\delta = \delta(\varepsilon) > 0$ be a constant small enough so that the Hessian $\mathscr{D}$ at every point of $\mathscr{Z}$ has spectral gap greater than δ, and so does the perturbed Hessian $\mathscr{D}^\varepsilon$ at every point in $\mathrm{Crit}(S_\varepsilon) \cap \mathscr{U}$. For each critical point $[\mathbb{A}]$ of S_ε in $\mathscr{U}$ we assign an integral 'Morse index' given by the spectral flow of the path of operators $\mathscr{D}_{\mathbb{A}_t}^t + \delta$ with t going from $t = 0$ to $t = \varepsilon$.*

Define the following Laurent polynomial with non-negative integer coefficients:

$$M_T := \sum_{[\mathbb{A}] \in \mathrm{Crit}(S_\varepsilon) \cap \mathscr{U}} T^{-\mathbf{sf}(\mathscr{D}_{\mathbb{A}_t}^t + \delta)} \in \mathbb{Z}[T, T^{-1}].$$

Then evaluating M_T at $T = 1$ gives $\chi(\mathscr{M}_E^*)$, and we have a 'Morse inequality':

$$M_T \geq \begin{cases} \mathcal{P}_T(\mathscr{M}_E^*) & \text{if } \varepsilon > 0, \\ \mathcal{P}_T^c(\mathscr{M}_E^*) & \text{if } \varepsilon < 0. \end{cases}$$

(Here, $\geq$ is meant as an inequality involving the individual coefficients in the Laurent polynomial).

Proof. We only need to prove assertion (2), as the rest is identical to Theorem 4.17. For a critical point $[\mathbb{A}] \in \mathrm{Crit} S_\varepsilon \cap \mathscr{U}_i$, the index calculation of Proposition 4.4 now gives the following identity

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over $\mathbb{Z}$ (rather than over $\mathbb{Z}/2$; cf. Theorem 4.16): for $\pm\varepsilon > 0$,

$$-\mathbf{sf}(\mathscr{D}^t_{\mathbb{A}_t} + \delta) = \mathrm{ind}(\pm\|\Phi\|^2_{L^2}, [\mathbb{A}_0]) + \mathrm{ind}(\mathfrak{f}, [\mathbb{A}_0]),$$

where $[\mathbb{A}_t]$ is the path provided by Theorem 4.16, with t going from $t=0$ to $t=\varepsilon$, and $[\mathbb{A}_0] \in \mathscr{Z}_i$. By Lemma 4.18, we may replace $[\mathbb{A}_t]$ by any other path with the same endpoints. Furthermore, the family of operators $\mathscr{D}_{[\mathbb{A}_0]} + \delta$ has no spectral flow as we vary $[\mathbb{A}_0]$ in the component $\mathscr{Z}_i$ since $\mathscr{Z}_i$ belongs in the Morse–Bott critical locus $\mathscr{M}^*_E$.

We thus have that if $\pm\varepsilon > 0$, then

$$M_T := \sum_{[\mathbb{A}] \in \mathrm{Crit} S_\varepsilon \cap \mathscr{U}} T^{-\mathbf{sf}(\mathscr{D}^t_{\mathbb{A}_t})} = \sum_i T^{\mathrm{ind}(\pm\|\Phi\|^2_{L^2}|_{\mathscr{M}^*_E}, \mathscr{Z}_i)} \cdot M_T(\mathscr{Z}_i, \mathfrak{f}).$$

Here $M_T(\mathscr{Z}_i, \mathfrak{f})$ is the Morse polynomial of the Morse function $\mathfrak{f}$ on $\mathscr{Z}_i$, and the Morse inequalities say $M_T(\mathscr{Z}_i, \mathfrak{f}) \geq \mathcal{P}_T(\mathscr{Z}_i)$, thus

$$M_T \geq \sum_i T^{\mathrm{ind}(\pm\|\Phi\|^2_{L^2}|_{\mathscr{M}^*_E}, \mathscr{Z}_i)} \cdot \mathcal{P}_T(\mathscr{Z}_i).$$

Now, the Morse–Bott inequalities (see, for example, [AB83, § 1]) for the proper and bounded-below Morse–Bott function $\|\Phi\|^2_{L^2}$ on $\mathscr{M}^*_E$ give

$$\sum_i T^{\mathrm{ind}(\pm\|\Phi\|^2_{L^2}|_{\mathscr{M}^*_E}, \mathscr{Z}_i)} \cdot P_T(\mathscr{Z}_i) \geq \mathcal{P}_T(\mathscr{M}^*_E) \quad \text{or} \quad \mathcal{P}^c_T(\mathscr{M}^*_E)$$

according to whether $\varepsilon > 0$ or $\varepsilon < 0$. Putting everything together concludes the proof. □

4.4 The case of Seifert-fibered 3-manifolds

Let $Y = S(N) \to C$ be a closed, oriented, Seifert-fibered 3-manifold over a closed oriented 2-orbifold C, and equip Y with a Seifert metric $g = \eta^2 + g_C$. We equip C with the holomorphic structure induced from g_C.

4.4.1 *The locus $\mathscr{Z}$.* We now establish the Morse–Bott property of the locus $\mathscr{Z} = \mathscr{Z}_E(g)$ given as the critical locus of the restriction of the functional $\|\Phi\|^2_{L^2}$ to the moduli space $\mathscr{M}^*_E(Y,g)$ of stable projectively flat $GL(2,\mathbb{C})$ connections on E with fixed determinant (Λ, λ).

We use the following notation.

- By Theorem 3.10 and Remark 2.10 we may suppose that (Λ, λ) pulls back from C. Thus, fix an orbifold $U(1)$ bundle with connection (Λ_0, λ_0) on C, with an identification $\Lambda^2 E \cong \pi^*\Lambda_0$.
- We use $\mathscr{N}^*_E(Y) = \mathscr{M}^*_E(Y,g) \cap \{\Phi = 0\}$ to denote the moduli space of irreducible projectively flat $U(2)$ connections on E with fixed determinant (Λ, λ). A similar notation, namely $\mathscr{N}^*_{E_0}(C)$, is used for the corresponding moduli spaces on an orbifold bundle E_0 on C, which can now be identified with the moduli space of stable holomorphic structures on E_0 with fixed determinant.
- For an orbifold complex line bundle L_0 on C, $\mathcal{D}_{L_0}(C)$ denotes the moduli space of effective orbifold divisors on L_0: pairs $(\mathscr{L}_0, s)$ where $\mathscr{L}_0$ is an orbifold holomorphic structure on $L_0 \to C$ and s is a holomorphic section of $\mathscr{L}_0$, modulo the action of the complex automorphisms of L_0. It is a standard fact that this moduli space is smooth (because C has complex dimension 1) and is identified as the symmetric product $\mathcal{D}_{L_0}(C) \cong \mathrm{Sym}^d|C|$ of $|C|$, where d is the degree of the ordinary line bundle given by the desingularization $|L_0| \to |C|$ (see [MOY97, Proposition 2.0.14]).

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Proposition 4.20. *The locus* $\mathscr{Z} = \mathrm{Crit}(\|\Phi\|^2_{L^2}|_{\mathscr{M}^*_E(Y)}) \subset \mathscr{M}^*_E(Y,g)$ *is Morse–Bott and can be described as the disjoint union*

$$\mathscr{Z} = \Bigg(\bigsqcup_{E_0, \Lambda^2 E_0 \cong \Lambda_0} \mathscr{Z}^0_{E_0} \Bigg) \sqcup \Bigg(\bigsqcup_{E_0, \Lambda^2 E_0 \cong \Lambda_0 \otimes N} \mathscr{Z}^1_{E_0} \Bigg) \sqcup \Bigg(\bigsqcup_{L_0, 0 < \deg L_0 - (1/2)\deg \Lambda_0 \leq (1/2)\deg K_C} \mathscr{Z}^0_{L_0} \Bigg) \sqcup \Bigg(\bigsqcup_{0 < \deg L_0 - (1/2)\deg(\Lambda_0 \otimes N) \leq (1/2)\deg K_C} \mathscr{Z}^1_{L_0} \Bigg),$$

where:

(1) $\mathscr{Z}^k_{E_0} = \mathscr{N}^*_{E_0}(C)$ *is the moduli space of stable holomorphic orbifold structures on* E_0 *with fixed determinant* $\Lambda_0 \otimes N^k$ *(these are the minima of* $\|\Phi\|^2_{L^2}$ *on* $\mathscr{M}^*_E(Y)$*);*

(2) $\mathscr{Z}^k_{L_0}$ *is a suitable* 2^{2g}*-fold cover of the moduli space of effective orbifold divisors* $\mathcal{D}_{L_0^{-2}\Lambda_0 N^k K}$ *in the orbifold line bundle* $L_0^{-2}\Lambda_0 N^k K$*, where* g *is the genus of* C*; the Morse–Bott index of* $\mathscr{Z}^k_{L_0}$ *is given by* $2\dim_{\mathbb{C}} H^0(C, L_0^2 \Lambda_0^{-1} N^{-k} K)$.

Proof. We combine the dimensional reduction theorem (Theorem 3.10) with an argument of Hitchin [Hit87, § 7] (see also [NS95, BY96]). Hitchin's argument makes use of the correspondence between moduli spaces of stable Higgs bundles and stable (projectively) flat connections on a Riemann surface (i.e., (3.17)–(3.19). We give a variant of his argument using basic properties of the Kazdan–Warner equation [KW74] instead.

Consider an orbifold $U(2)$ bundle $E_0 \to C$ with fixed determinant given by either (Λ_0, λ_0) or $(\Lambda_1, \lambda_1) = (\Lambda_0 \otimes N, \lambda_0 \otimes i\eta)$. We consider both cases simultaneously by writing (Λ_k, λ_k), and we regard Λ_k as a holomorphic orbifold line bundle with the holomorphic structure induced by the connection λ_k. The moduli space $\mathscr{M}^*_{E_0}(C, g_C)$ of stable (projectively) flat connections has a *natural* S^1*-action* defined as

$$\mathbb{B} = (B, \Psi) \mapsto (B, \cos\theta \cdot \Psi + \sin\theta \cdot *_C \Psi), \quad \lambda = e^{i\theta} \in S^1, \tag{4.34}$$

where $*_C$ denotes the Hodge star operator of the orbifold metric g_C. It is easy to check that (4.34) preserves the set of solutions of (3.17)–(3.19). The S^1-action on the moduli spaces $\mathscr{M}^*_{E_0}(C)$ then induces one on $\mathscr{M}^*_E(Y)$ by passing through the correspondence of Theorem 3.10. The crucial point is that the S^1-action on $\mathscr{M}^*_{E_0}(C, g_C)$ is Hamiltonian in the ω_I symplectic form (cf. (3.21)), with moment map given by $\mu(B, \Psi) = \|\Psi\|^2_{L^2(C)}$ (times $-1/2$), as one can easily verify. From this it follows that μ is a Morse–Bott function on $\mathscr{M}^*_{E_0}(C, g_C)$ whose critical loci agree with the S^1-fixed points. Then, by Theorem 3.10, it follows that the restriction of $\|\Phi\|^2_{L^2(Y)}$ to $\mathscr{M}^*_E(Y, g)$ is also Morse–Bott.

We now describe the S^1-fixed points on $\mathscr{M}^*_{E_0}(C, g_C)$. Clearly, the locus $\Psi = 0$ consists of fixed points, and this accounts for the components of the form $\mathscr{Z}^k_{E_0}$. When $\Psi \neq 0$, the fixed points are given by stable configurations (B, Ψ) solving (3.17)–(3.19) such that for any $\theta \in S^1$ there exists a unitary gauge transformation $u_\theta \in \mathscr{G}_{E_0}$ such that

$$u_\theta \cdot B = B, \quad u_\theta \Psi u_\theta^{-1} = \cos\theta \cdot \Psi + \sin\theta \cdot *_C \Psi.$$

Since $u_\theta \neq \pm I$ for $\theta \neq 0, 2\pi, \ldots$, the first equation shows that B reduces to a $U(1)$ connection: E_0 has a decomposition $E_0 = L_0 \oplus L_0^{-1}\Lambda_k$ and $B = b \oplus b^{-1}\lambda_k$ for a $U(1)$ bundle with connection (L_0, b) on C. The second equation then implies that the diagonal terms in Ψ vanish,

$$\Psi = \begin{pmatrix} 0 & -\sigma^* \\ \sigma & 0 \end{pmatrix},$$

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and $\sigma \in \Omega^1(C, L_0^{-2}\Lambda_k)$ satisfies $*_C\sigma = \pm i\sigma$. After possibly swapping the order of the summands in the decomposition of E_0, we may suppose that $*_C\sigma = -i\sigma$, that is, σ is a section of $L_0^{-2}\Lambda_k K$ where K is the orbifold canonical bundle of the orbifold Riemann surface C. Equations (3.17)–(3.19) for (B, Ψ) are then easily seen to reduce to the following pair of equations on (b, σ):

$$F_b - \tfrac{1}{2}F_{\lambda_k} = -i|\sigma|^2\omega_C, \tag{4.35}$$

$$\overline{\partial}_b\sigma = 0. \tag{4.36}$$

We denote by $\mathscr{Z}^k_{L_0}$ the moduli space of pairs (b, σ) with $\sigma \neq 0$ solving (4.35) and (4.36), modulo the natural action of the $U(1)$ gauge transformations u of L_0 by $(b, \sigma) \mapsto (b - u^{-1}du, u \cdot \sigma)$. Already from the fact that the quadratic term in σ in (4.35) is *non-degenerate*, it follows by a standard argument that the moduli space $\mathscr{Z}^k_{L_0}$ is compact (see, for example, [Mor96, § 5]). If (b, σ) lies in this moduli space, then b induces a holomorphic structure $\mathscr{L}_0$ on L_0 and σ defines a non-trivial holomorphic section of $\mathscr{L}_0^{-2}\Lambda_k K$. Denote the space of such pairs $(\mathscr{L}_0, \sigma)$ modulo the complex gauge transformations of L_0 by $\widetilde{\mathcal{D}}_{L_0^{-2}\Lambda_k K}(C)$. In particular, when this space is non-empty, the orbifold degree of $L_0^{-2}\Lambda_k K$ must be non-negative, which gives

$$\deg L_0 - \tfrac{1}{2}\deg(\Lambda_k) \leq \tfrac{1}{2}\deg K. \tag{4.37}$$

On the other hand, by (4.35), $\sigma \neq 0$ and the Chern–Weil formula, we obtain

$$0 < \deg L_0 - \tfrac{1}{2}\deg(\Lambda_k). \tag{4.38}$$

To describe the topology of $\mathscr{Z}^k_{L_0}$, we show that, provided (4.37) and (4.38) hold, the following map defines a diffeomorphism:

$$\mathscr{Z}^k_{L_0} \xrightarrow{\cong} \widetilde{\mathcal{D}}_{L_0^{-2}\Lambda_k K}(C), \quad (b, \sigma) \mapsto (\mathscr{L}_0 = \overline{\partial}_b, \sigma).$$

This follows by a standard argument, which we briefly outline. Given a pair $(\mathscr{L}_0, \sigma)$ such that $\overline{\partial}_{\mathscr{L}_0}\sigma = 0$, we seek a complex gauge transformation u of L_0 such that if b is the Chern connection of the hermitian holomorphic bundle $u \cdot \mathscr{L}_0$, then the pair (b, σ) solves (4.35). Equivalently, one may seek a new hermitian metric $h_\xi = e^{-\xi} \cdot h_0$, where h_0 is the given metric on L_0 and ξ is a real-valued function, such that if b is the Chern connection of the hermitian holomorphic bundle $(\mathscr{L}_0, h_\xi)$, then (b, σ) solves (4.35) for the new metric h_ξ. Expressing (4.35) as an equation for the function ξ yields the equation

$$\Delta\xi \cdot \omega + 2e^{2\xi}|\sigma|^2 \cdot \omega - 2i(F_{h_0} - \tfrac{1}{2}F_{\lambda_k}) = 0,$$

where $\Delta = d^*d$ is the non-negative Laplace operator of g_C. By the theorem of Kazdan and Warner [KW74] (see also [BW96, Lemma 3.4]) this equation admits a unique smooth solution ξ because

$$\int_C 2i(F_{h_0} - \tfrac{1}{2}F_{\lambda_k}) = 4\pi(\deg L_0 - \tfrac{1}{2}\deg \Lambda_k) > 0.$$

From the diffeomorphism $\mathscr{Z}^k_{L_0} \cong \widetilde{D}_{L_0^{-2}\Lambda_k K}(C)$ the promised description of $\mathscr{Z}^k_{L_0}$ readily follows, as the map $\widetilde{\mathcal{D}}_{L_0^{-2}\Lambda_k K}(C) \to \mathcal{D}_{L_0^{-2}\Lambda_k K}(C)$ sending $(\mathscr{L}_0, \sigma)$ to $(\mathscr{L}_0^{-2}\Lambda_k K, \sigma)$ is easily seen to be a 2^{2g}-fold cover.

All that remains is to calculate the Morse–Bott index of the critical manifold $\mathscr{Z}^k_{L_0}$. For this, note that the ω_I-Hamiltonian S^1-action on $\mathscr{M}^*_E$ is also I-holomorphic. The Morse–Bott index of $\mathscr{Z}^k_{L_0}$ then agrees with twice the weight of the S^1-action on the normal bundle to $\mathscr{Z}^k_{L_0} \subset \mathscr{M}^*$. Using an argument by Hitchin [Hit87, Proof of Proposition 7.1] (see also [NS95, BY96]), one can calculate this weight, which gives the Morse–Bott index $2\dim_{\mathbb{C}} H^0(C, L_0^2\Lambda_0^{-1}N^{-k}K)$. □

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Remark 4.21. For future reference, we note here some information we shall use (in the proof of Corollary 5.3) about the dimensions of the connected components of $\mathscr{M}_E^*(Y)$ containing the components of $\mathscr{Z}$ described above, for simplicity when C has genus $g=0$ and $\Lambda_0=\mathbb{C}$. In this case any orbifold line bundle $L_0\to C$ admits a unique holomorphic structure up to isomorphism, so we do not specify it. We have:

(1) $\mathscr{Z}_{E_0}^k=\mathscr{N}_{E_0}^*(C)$ lies in a component of $\mathscr{M}_E^*(Y)$ with twice the dimension;
(2) $\mathscr{Z}_{L_0}^k$ lies in a component of $\mathscr{M}_E^*(Y)$ of complex dimension

$$2(\dim_{\mathbb{C}} H^0(C,L_0^{-2}N^kK)+\dim_{\mathbb{C}} H^0(C,L_0^2N^{-k}K)-1).$$

Both case (1) and case (2) follow from the fact [Hit87, p. 95] that at each point $(\mathscr{E},\theta)\in\mathscr{M}_E^*(Y)$ (thought of as a stable Higgs pair with fixed determinant N^k) the tangent space $T_{(\mathscr{E},\theta)}\mathscr{M}_E^*(Y)$ contains an ω_J-Lagrangian subspace W isomorphic to

$$W\cong H^0(C,\mathrm{End}_0\mathscr{E}\otimes K)/H^0(C,\mathrm{End}_0\mathscr{E})\cdot\theta,$$

where $H^0(C,\mathrm{End}_0\mathscr{E})\cdot\theta$ denotes the subspace of the form $[\psi,\theta]$ in which $\psi\in H^0(C,\mathrm{End}_0\mathscr{E})$. In case (1) we have $W\cong H^1(C,\mathrm{End}_0\mathscr{E})$, which is the tangent space at $\mathscr{E}$ to the moduli space of stable bundles. In case (2) we have a decomposition $\mathscr{E}=L_0\oplus L_0^{-1}N^k$ with respect to which θ is lower triangular, hence $\theta\in H^0(C,L_0^{-2}N^kK)$. Then $\mathrm{End}_0\mathscr{E}=\mathscr{O}_C\oplus L_0^{-2}N^k\oplus L_0^2N^{-k}$. We have $H^0(C,L_0^{-2}N^k)\cdot\theta=0$ since this subspace is given by $[\psi,\theta]$ with ψ lower-triangular; and $H^0(L_0^2N^{-k})\cdot\theta\in H^0(C,K)=0$ since $g=0$. Thus, in this case

$$W\cong(H^0(C,L_0^{-2}N^kK)/\mathbb{C}\theta)\oplus H^0(C,L_0^2N^{-k}K),$$

which has the required dimension.

4.4.2 *Proofs of main theorems.*

Proof of Theorem A. The same result holds under the assumption that $E\to Y$ is an admissible $U(2)$ bundle (which is more general than Y being an integral homology sphere). Then the moduli space $\mathscr{N}_E^*(Y)$ of irreducible projectively flat $U(2)$ connections with fixed determinant is compact. From Theorem 3.10 the moduli space $\mathscr{M}_E^*(Y)$ is Morse–Bott, and by Proposition 4.20 the critical locus $\mathscr{Z}$ is compact and Morse–Bott. The result now follows from Theorem 4.17. □

Proof of Theorem B. It remains to show that we can achieve equality in the inequality of Theorem 4.19(2). Since we are assuming that Y is a homology 3-sphere, the genus of C is zero and hence the components of $\mathscr{Z}\subset\mathscr{M}^*$ are either components of the irreducible $SU(2)$ moduli space $\mathscr{N}^*(Y)$ or diffeomorphic to a complex projective space $\mathbb{C}P^e\cong\mathrm{Sym}^e\mathbb{C}P^1$ of suitable dimension e.

We recall that a Morse–Bott function g on a smooth manifold M is called *perfect* if g achieves equality in the Morse–Bott inequalities [AB83]:

$$\sum_i T^{\mathrm{ind}(g,Z_i)}\cdot\mathcal{P}_T(Z_i)=\mathcal{P}_T(M),$$

where Z_i denotes the connected components of the critical locus of g. In this definition, we also allow M to be non-compact, in which case we assume that g is proper, bounded from below and has compact critical locus.

Of course $\mathbb{C}P^e$ admits a perfect Morse function, and, for a Seifert-fibered integral homology 3-sphere, $\mathscr{N}^*(Y)$ admits a perfect Morse function as well [KK91, BO90, FS92]. Thus, there exists a perfect Morse function g on $\mathscr{Z}$. We may then arrange that the function $\mathfrak{f}\colon\mathscr{Z}\to\mathbb{R}$ agrees

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with g. Indeed, $\mathfrak{f} = 1/2(d\|\Phi\|^2_{L^2})(\lambda) + (f \circ \underline{\sigma})|_{\mathscr{Z}}$, so choosing any smooth function $f : \mathbb{C}^N \to \mathbb{R}$ whose restriction to $\mathscr{Z} \subset \mathbb{C}^N$ (embedded via $\underline{\sigma}$) agrees with $g - 1/2(d\|\Phi\|^2_{L^2})(\lambda)$ will do.

On the other hand, the proper and bounded-below function $\|\Phi\|^2_{L^2}$ on $\mathscr{M}^*$ is a perfect Morse–Bott function. This follows from a result of Frankel [Fra59] (also used in [Hit87, § 7]) since this function is the moment map for a Hamiltonian S^1-action on the Kähler manifold $(\mathscr{M}^*, I, \omega_I)$ (as was explained in the proof of Proposition 4.20).

With this understood, following the argument in the proof of Theorem 4.19 we have that if $\pm\varepsilon > 0$ and $|\varepsilon| \le \varepsilon_0$, then

$$\begin{aligned} M_T &:= \sum_{[\mathbb{A}] \in \mathrm{Crit} S_\varepsilon \cap \mathscr{U}} T^{-\mathbf{sf}(\mathscr{D}^t_{\mathbb{A}_t})} \\ &= \sum_i T^{\mathrm{ind}(\pm\|\Phi\|^2_{L^2}|_{\mathscr{M}^*}, \mathscr{Z}_i)} \cdot M_T(\mathscr{Z}_i, \mathfrak{f}) \\ &= \sum_i T^{\mathrm{ind}(\pm\|\Phi\|^2_{L^2}|_{\mathscr{M}^*}, \mathscr{Z}_i)} \cdot \mathcal{P}_T(\mathscr{Z}_i) \\ &= \mathcal{P}_T(\mathscr{M}^*) \text{ or } \mathcal{P}^c_T(\mathscr{M}^*) \quad \text{(according to whether } \varepsilon > 0 \text{ or } \varepsilon < 0), \end{aligned}$$

where the third line uses the perfection of the Morse function $\mathfrak{f}$ on each $\mathscr{Z}_i$, that is, $M_T(\mathscr{Z}_i, \mathfrak{f}) = P_T(\mathscr{Z}_i)$, and the last line uses the perfection of the Morse–Bott function $\|\Phi\|^2_{L^2}$ on $\mathscr{M}^*$, together with the fact that it is proper and bounded from below. □

5. Calculations

Let $Y = S(N) \to C^2$ be a Seifert-fibered 3-manifold over a 2-orbifold C with exactly n orbifold points with isotropy orders $\alpha_1, \ldots, \alpha_n$. Then (see [FS92, Theorem 2.3]):

- Y is a rational homology 3-sphere if and only if the genus of C is zero (i.e., $|C| \cong S^2$);
- Y is an integral homology 3-sphere if and only if the genus of C is zero, the topological orbifold Picard group of C is infinite cyclic ($\mathrm{Pic}^t(C) \cong \mathbb{Z}$), and N is one of the two generators of $\mathrm{Pic}^t(C)$.

Throughout this section we assume that $Y \to C$ is a Seifert-fibered integral homology 3-sphere over a 2-orbifold $C = S^2(\alpha_1, \ldots, \alpha_n)$. Our goal is to calculate of the Poincaré polynomial of the moduli space of stable $SL(2,\mathbb{C})$ flat connections $\mathscr{M}^*(Y)$ (on the trivial $SU(2)$ bundle).

5.1 The locus $\mathscr{Z}$

For a given complex structure on the 2-orbifold C, let $\mathcal{D}_{<d}(C)$ be the moduli space of orbifold effective divisors on C of orbifold degree less than d.

COROLLARY 5.1. *Let $Y = S(N) \to C$ be a Seifert-fibered integral homology 3-sphere, equipped with a Seifert metric g (and hence also a complex structure on C). Then there is a diffeomorphism*

$$\mathscr{Z}(g) \cong \mathscr{N}^*(Y) \sqcup \mathcal{D}_{<\deg K}(C).$$

Proof. By Proposition 4.20 the locus $\mathscr{Z}(g) \cap \{\Phi \neq 0\}$ is diffeomorphic to

$$\left(\bigsqcup_{L_0, 0 < \deg L_0 \le (1/2)\deg K} \mathcal{D}_{L_0^{-2}K}(C) \right) \sqcup \left(\bigsqcup_{0 < \deg L_0 - (1/2)\deg N \le (1/2)\deg K} \mathcal{D}_{L_0^{-2}NK}(C) \right).$$

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The orbifold line bundles on C with orbifold degree less than $\deg K$ are

$$\mathbb{C} = N^0, N^{-1}, \dots, N^{-d+1}. \tag{5.1}$$

Consider now the following two cases.

Non-spin case: $d = 2m+1$, for some $m \in \mathbb{Z}_{\geq 0}$. The line bundles $L_0^{-2}K$ with $0 < \deg L_0 \leq (1/2)\deg K$ are given by

$$N^{-1}, N^{-3}, \dots, N^{-d+2}, \tag{5.2}$$

and the line bundles $L_0^{-2}NK$ with $0 < \deg L_0 - (1/2)\deg N \leq (1/2)\deg K$ are given by

$$\mathbb{C} = N^0, N^{-2}, \dots, N^{-d+1}. \tag{5.3}$$

The two collections (5.1) and (5.2)–(5.3) consist of the same line bundles. This establishes the required result in the non-spin case.

Spin case: $d = 2m$, for some $m \in \mathbb{Z}_{>0}$. The line bundles $L_0^{-2}K$ with $0 < \deg L_0 \leq (1/2)\deg K$ are given by

$$\mathbb{C} = N^0, N^{-2}, \dots, N^{-d+2}, \tag{5.4}$$

and the line bundles $L_0^{-2}NK$ with $0 < \deg L_0 - (1/2)\deg N \leq (1/2)\deg K$ are given by

$$N^{-1}, N^{-3}, \dots, N^{-d+1}. \tag{5.5}$$

Both collections (5.1) and (5.4)–(5.5) consist of the same line bundles. This establishes the required result in the spin case. □

The space $\mathcal{D}_{<\deg K}(C)$ has one connected component $\mathcal{D}_L$ for each orbifold line bundle L with $0 \leq \deg L < \deg K$ (namely, $\mathcal{D}_L(C)$ is the moduli space of orbifold effective divisors in the line bundle L). Through the isomorphism established in Corollary 5.1, the Morse–Bott index of $\|\Phi\|^2_{L^2}|_{\mathscr{M}^*}$ along the critical submanifold $\mathcal{D}_L(C) \subset \mathscr{Z}$ is given by Proposition 4.20 by

$$\operatorname{ind}(\|\Phi\|^2_{L^2}|_{\mathscr{M}^*}, \mathcal{D}_L) = 2\dim_{\mathbb{C}} H^0(C, L^{-1}K^2),$$

where L is understood to be equipped with its unique holomorphic structure (since C has genus zero). In what follows we denote the complex dimension of the kth sheaf cohomology groups by h^k. Using again the fact that $\|\Phi\|^2_{L^2}$ is a perfect Morse–Bott function on $\mathscr{M}^*(Y)$, we obtain

$$\mathcal{P}_T(\mathscr{M}^*(Y)) = \mathcal{P}_T(\mathscr{N}^*(Y)) + \sum_{0 \leq \deg L < \deg K} (T^2)^{h^0(C, L^{-1}K^2)} \cdot \mathcal{P}_T(\mathcal{D}_L(C)). \tag{5.6}$$

5.2 Formulas for Poincaré polynomials

Concretely, the connected components of $\mathcal{D}_{<d}(C)$, the moduli space of orbifold effective divisors on C (for some orbifold complex structure on C), are indexed by $(n+1)$-tuples of non-negative integers $\underline{e} = (e, \beta_1, \dots, \beta_n) \in (\mathbb{Z}_{\geq 0})^{n+1}$ such that

$$\beta_i < \alpha_i \quad \text{and} \quad \deg(\underline{e}) := e + \sum_{i=1}^{n} \frac{\beta_i}{\alpha_i} < d, \tag{5.7}$$

and the connected component of $\mathcal{D}_{<d}(C)$ corresponding to $\underline{e}$ is diffeomorphic to $\operatorname{Sym}^e S^2 \cong \mathbb{C}P^e$.

Namely, an orbifold effective divisor $(\mathscr{L}, s)$ (i.e., an orbifold holomorphic line bundle with a non-trivial holomorphic section up to $\mathbb{C}^*$-rescaling) vanishes on the ith orbifold point of C with a fractional order, whose fractional part is β_i/α_i with $0 \leq \beta_i < \alpha_i$, and e is the (ordinary) degree of the desingularized holomorphic line bundle $|\mathscr{L}| \to |C|$ (see [MOY97, FS92] for details).

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Proof of Corollary A. Let $\mathscr{Z}(\underline{e}) \subset \mathscr{Z}$ be the connected component of $\mathcal{D}_{<\deg K}(C) \subset \mathscr{Z}$ (cf. Corollary 5.1) parametrized by the data $\underline{e} = (e, \beta_1, \dots, \beta_n) \in (\mathbb{Z}_{\geq 0})^{n+1}$, with

$$\beta_i < \alpha_i, \quad \deg(\underline{e}) < \deg K = -\chi(C) = -2 + n - \sum_{i=1}^{n} \frac{1}{\alpha_i}.$$

Let $L_{\underline{e}}$ denote the unique orbifold line bundle corresponding to the data $\underline{e}$ (equipped with its unique orbifold holomorphic structure up to isomorphism, since C has genus zero). We have $\mathscr{Z}(\underline{e}) = \mathcal{D}_{L_{\underline{e}}}(C) \cong \mathbb{C}P^e$.

By (5.6) we have

$$\mathcal{P}_T(\mathscr{M}^*(Y)) = \mathcal{P}_T(\mathscr{N}^*(Y)) + \sum_{\underline{e}} (T^2)^{h^0(C, L_{\underline{e}}^{-1} K^2)} \cdot \mathcal{P}_T(\mathbb{C}P^e).$$

By the orbifold versions of Serre duality and the Riemann–Roch theorem we have

$$h^0(L_{\underline{e}} K^{-1}) - h^0(L_{\underline{e}}^{-1} K^2) = h^0(L_{\underline{e}} K^{-1}) - h^1(L_{\underline{e}} K^{-1}) = 1 + \deg |L_{\underline{e}} K^{-1}|.$$

But $\deg L_{\underline{e}} K^{-1} < 0$, hence

$$h^0(L_{\underline{e}}^{-1} K^2) = -1 - \deg |L_{\underline{e}} K^{-1}|.$$

To compute $\deg |L_{\underline{e}} K^{-1}|$ (the ordinary degree of the desingularization of $L_{\underline{e}} K^{-1}$), note that the anticanonical bundle K^{-1} has $\deg |K^{-1}| = 2 - n$ and isotropy 1 at each orbifold point of C, and $L_{\underline{e}}$ has $\deg |L_{\underline{e}}| = e$ and isotropy β_i at the ith orbifold point. From this we obtain

$$\deg |L_{\underline{e}} K^{-1}| = e + 2 - n + \sum_{i=1}^{n} \left\lfloor \frac{\beta_i + 1}{\alpha_i} \right\rfloor.$$

Thus,

$$h^0(L_{\underline{e}}^{-1} K^2) = -e - 3 + n - \sum_{i=1}^{n} \left\lfloor \frac{\beta_i + 1}{\alpha_i} \right\rfloor = -\chi(C) - \deg(\underline{e}) - 1 + \sum_{i=1}^{n} \left\{ \frac{\beta_i + 1}{\alpha_i} \right\}.$$

From this calculation the promised formula for $\mathcal{P}_T(\mathscr{M}^*(Y))$ now follows. □

5.3 Relation to the Milnor number

Proof of Corollary B. Let $Y = S(N)$ be the link of a weighted-homogeneous ICIS $(X, 0) \subset (\mathbb{C}^{2+k}, 0)$ of dimension 2, with the induced orientation and Seifert-fibered structure. We denote the Seifert data of Y by

$$(b, \gamma_1, \dots, \gamma_n),$$

where $b, \gamma_i \in \mathbb{Z}$, $0 < \gamma_i < \alpha_i$ and γ_i is coprime to α_i. (We follow the conventions for Seifert data of [MOY97, § 2].) The Seifert data of the canonical bundle K is $(-2, \alpha_1 - 1, \dots, \alpha_n - 1)$. With the given orientation on Y (as the link of the singularity), the orbifold degree $\deg N = b + \sum_{i=1}^{n} \gamma_i/\alpha_i$ is negative.

The geometric genus $p_g(X, 0)$ of the weighted-homogeneous ICIS $(X, 0)$ (more generally, of a weighted-homogeneous normal isolated surface singularity) of dimension 2 can be computed by the following formula due to Pinkham [Pin77, Theorem 5.7] and Dolgachev [Dol75] (also attributed to Demazure). We use a formulation due to Némethi (see [Ném05, Remark 11.14] or [Ném22, § 5.1.26], and beware that Némethi's b_0 is $-b$ in our convention):

$$p_g(X, 0) = \sum_{\ell \geq 0} \max\{0, -N(\ell) - 1\},$$

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where

$$N(\ell) := -\ell b - \sum_{i=1}^{n} \left\lceil \frac{\ell \gamma_i}{\alpha_i} \right\rceil .$$

We observe that, for $\ell \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} \deg |K \otimes N^{\ell}| &= -2 + \ell b + \sum_{i=1}^{n} \left\lfloor \frac{\alpha_i - 1 + \gamma_i}{\alpha_i} \right\rfloor \\ &= -2 + \ell b + \sum_{i=1}^{n} \left\lceil \frac{\ell \gamma_i}{\alpha_i} \right\rceil \\ &= -2 - N(\ell), \end{aligned}$$

and, hence, we have $-N(\ell) - 1 = \deg |K \otimes N^{\ell}| + 1$. By this formula, the integer $-N(\ell) - 1$ is positive if and only if $\deg |K \otimes N^{\ell}| \geq 0$. Since C has genus zero, the condition $\deg |K \otimes N^{\ell}| \geq 0$ implies that $K \otimes N^{\ell}$ admits non-trivial holomorphic sections (since for an orbifold holomorphic line bundle $L \to C$ we have $H^0(C, L) \cong H^0(|C|, |L|)$; see, for example, [MOY97, Proposition 2.0.14]). Hence, we have the following assertions.

- We have $\deg N^{-\ell} \leq \deg K$, that is, $\ell \leq -\deg K / \deg N$.
- We have $h^0(C, K \otimes N^{\ell}) = \deg |K \otimes N^{\ell}| + 1$. One way of seeing this is as follows. Consider for $\mathscr{L} := K \otimes N^{\ell}$ the short exact sequence of (orbifold) sheaves

  $$0 \xrightarrow{s} \mathscr{O}_C \to \mathscr{L} \to \mathscr{L}|_D \to 0$$

  associated to a non-trivial holomorphic section s of $\mathscr{L}$ (where $\mathscr{L}|_D$ is defined as the cokernel sheaf of multiplication with s; here D stands for the 'zero locus' of s). Since $\mathscr{L}|_D$ is supported on a zero-dimensional locus, $H^1(C, \mathscr{L}|_D) = 0$. Since C has genus zero, $H^1(C, \mathscr{O}_C) = 0$. Hence, by the long exact sequence in cohomology, we obtain $H^1(C, \mathscr{L}) = 0$, that is, $h^1(C, \mathscr{L}) = 0$. Now, by this vanishing and the orbifold version of the Riemann–Roch formula we obtain the required identity (here we also use that C has genus zero):

  $$h^0(C, \mathscr{L}) = h^0(C, \mathscr{L}) - h^1(C, \mathscr{L}) = 1 + \deg |\mathscr{L}|.$$

Putting everything together, and using the fact that N generates the topological Picard group of C (since Y is an integral homology sphere), gives

$$p_g(X, 0) = \sum_{0 \leq \ell \leq -(\deg K / \deg N)} h^0(C, K \otimes N^{\ell}) = \sum_{0 \leq \deg L \leq \deg K} h^0(C, L) = \sum_{0 \leq \deg L < \deg K} h^0(C, L),$$

where the last identity follows from the fact that $H^0(C, K) = 0$ when C has genus zero. The right-hand side in this identity is $\chi(\mathcal{D}_{<\deg K}(C))$, thus (5.6) gives

$$\chi(\mathscr{M}^*(Y)) = \chi(\mathscr{N}^*(Y)) + p_g(X, 0).$$

At this point, what remains of the proof of Corollary B was already explained in § 1.6. □

5.4 Comparison with sheaf-theoretic $SL(2, \mathbb{C})$ Floer cohomology

We conclude by discussing the gauge-theoretic interpretation of the sheaf-theoretic $SL(2, \mathbb{C})$ Floer cohomology of Abouzaid and Manolescu.

For a closed oriented 3-manifold Y, Abouzaid and Manolescu [AM20] defined a *sheaf-theoretic* $SL(2, \mathbb{C})$ *Floer cohomology* $HP^*(Y)$ and also cohomology with *compact supports* $HP_c^*(Y)$, both of which are finitely generated $\mathbb{Z}$-graded abelian groups. These invariants arise from a certain

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perverse sheaf $P^\bullet(Y)$ on the stable locus $\mathscr{M}^*(Y)$ in the $SL(2,\mathbb{C})$ character variety, defined using the methods of [Bus14, Joy15], by passing to hypercohomology or hypercohomology with compact supports: $HP^*(Y) = \mathbb{H}^*(\mathscr{M}^*(Y), P^\bullet(Y))$, $HP^*_c(Y) = \mathbb{H}^*_c(\mathscr{M}^*(Y), P^\bullet(Y))$. It remains an open problem to describe these invariants using gauge theory, which would likely help to elucidate their geometric content.

Consider the *sheaf-theoretic* $SL(2,\mathbb{C})$ *Casson invariant* $\lambda^P(Y) \in \mathbb{Z}$, which is defined as the Euler characteristic of $HP^*(Y)$; and, more generally, the *sheaf-theoretic Poincaré polynomial*

$$\mathcal{P}^P_T(Y) := \sum_{n\in\mathbb{Z}} T^n \cdot \mathrm{rank}_{\mathbb{Z}} HP^n(Y) \in \mathbb{Z}[T,T^{-1}].$$

One may also consider their compactly supported analogs, which we denote by $\lambda^P_c(Y)$ and $\mathcal{P}^{P,c}_T(Y)$, but they contain the same information: $\lambda^P_c(Y) = \lambda^P(Y)$ and $\mathcal{P}^{P,c}_T(Y) = \mathcal{P}^P_{T^{-1}}(Y)$ (see [AM20, Proposition 8.2]). We obtain a gauge-theoretic interpretation of these invariants when Y is a Seifert-fibered homology sphere.

COROLLARY 5.2. *Let Y be a Seifert-fibered homology 3-sphere. Then the following assertions hold.*

(1) *In the situation of Theorem A(2) we have*

$$\sum_{[\mathbb{A}]\in\mathscr{M}^{*,\varepsilon}_\theta\cap\mathscr{U}} (-1)^{\mathbf{sf}_2(\mathscr{D}^t_{[\mathbb{A}],\theta})} = \chi(\mathscr{M}^*(Y)) = \lambda^P(Y).$$

(2) *In the situation of Theorem B(2) we have*

$$\sum_{[\mathbb{A}]\in[\mathbb{A}]\in\mathscr{M}^{*,\varepsilon}_\theta\cap\mathscr{U}} T^{\mathbf{sf}(\mathscr{D}^t_{[\mathbb{A}_t],\theta}+\delta)-\dim_{\mathbb{C}}\mathrm{Ker}\mathscr{D}_{[\mathbb{A}_0],\theta}} = \begin{cases} \mathcal{P}^P_T(Y) & \text{if } \varepsilon>0,\\ \mathcal{P}^{P,c}_T(Y) & \text{if } \varepsilon<0.\end{cases}$$

The proof is given below. We also provide formulas for $\lambda^P(Y)$ and $\mathcal{P}^P_T(Y)$. To describe these, for a given smooth compact manifold M with connected components M_j we consider the 'normalized' Poincaré polynomial $\widehat{\mathcal{P}}_T(M) := \sum_j (T^{-1})^{\dim M_j}\cdot \mathcal{P}_T(M_j) \in \mathbb{Z}[T^{-1}]$.

COROLLARY 5.3. *Let Y be a Seifert-fibered homology 3-sphere. Then:*

(1) $\mathcal{P}^P_T(Y) = \widehat{\mathcal{P}}_T(\mathscr{N}^*(Y)) + \sum_{\underline{e}} \widehat{\mathcal{P}}_T(\mathbb{C}P^e)$;

(2) $\lambda^P(Y) = -2\lambda(Y) + \sum_{\underline{e}} \chi(\mathbb{C}P^e)$;

where $\underline{e} = (e,\beta_1,\dots,\beta_n)$ ranges over all vectors of non-negative integers such that $\beta_i < \alpha_i$ and $\deg(\underline{e}) < -\chi(C)$.

Proof of Corollary 5.2. For a complex manifold M, consider the following 'complex normalized' version of the Poincaré polynomial (respectively, compactly supported Poincaré polynomial): decomposing M into connected components M_j we set

$$\widetilde{\mathcal{P}}_T(X) := \sum_j (T^{-1})^{\dim_{\mathbb{C}} M_j}\cdot \mathcal{P}_T(M_j),$$
$$\widetilde{\mathcal{P}}^c_T(X) := \sum_j (T^{-1})^{\dim_{\mathbb{C}} X_j}\cdot \mathcal{P}^c_T(X_j).$$

For a Seifert-fibered integral homology 3-sphere, the Poincaré polynomials of the sheaf-theoretic $SL(2,\mathbb{C})$ Floer cohomologies recover the complex normalized Poincaré polynomials

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of $\mathscr{M}^*(Y)$ (see [AM20, formula 35]):

$$\mathcal{P}_T^P(Y) = \widetilde{\mathcal{P}}_T(\mathscr{M}^*(Y)), \quad \mathcal{P}_T^{P,c}(Y) = \widetilde{\mathcal{P}}_T^c(\mathscr{M}^*(Y)).$$

Since $\mathscr{M}^*(Y)$ is the Morse–Bott critical locus of the functional S_θ by Theorem 3.10, its complex dimension along the connected component containing the point $[\mathbb{A}_0] \in \mathscr{M}^*(Y)$ is $\dim_{\mathbb{C}} \operatorname{Ker} \mathscr{D}_{[\mathbb{A}_0],\theta}$. The argument from the proof of Theorem B readily gives the required formula for $\mathcal{P}_T^P(Y)$ and $\mathcal{P}_T^{P,c}(Y)$ as a gauge-theoretic count of critical points. □

Proof of Corollary 5.3. Recall from the proof of Corollary 5.2 that $\mathcal{P}_T^P(Y) = \widetilde{\mathcal{P}}_T(\mathscr{M}^*(Y))$. Using again the fact that $\|\Phi\|_{L^2}^2$ is a perfect Morse–Bott function, together with Corollary 5.1 and Remark 4.21(2), we obtain

$$\begin{aligned}\widetilde{\mathcal{P}}_T(\mathscr{M}^*(Y)) &= \widehat{\mathcal{P}}_T(\mathscr{N}^*(Y)) + \sum_{0 \le \deg L < \deg K} T^{2h^0(L^{-1}K) - 2(h^0(L) + h^0(L^{-1}K^2) - 1)} \mathcal{P}_T(\mathcal{D}_L(C)) \\ &= \widehat{\mathcal{P}}_T(\mathscr{N}^*(Y)) + \sum_{0 \le \deg L < \deg K} \widehat{\mathcal{P}}_T(\mathcal{D}_L(C)),\end{aligned}$$

from which the required formula now follows. □

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