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Lamplighters, Diestel–Leader Graphs, Random Walks, and Harmonic Functions

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The lamplighter group over $\mathbb{Z}$ is the wreath product $\mathbb{Z}_q \wr \mathbb{Z}$. With respect to a natural generating set, its Cayley graph is the Diestel–Leader graph $DL(q,q)$. We study harmonic functions for the 'simple' Laplacian on this graph and, more generally, for a class of random walks on $DL(q,r)$, where $q,r \geqslant 2$. The DL-graphs are horocyclic products of two trees, and we give a full description of all positive harmonic functions in terms of the boundaries of these two trees. In particular, we determine the minimal Martin boundary, that is, the set of minimal positive harmonic functions.

1. Introduction

Think of a (typically infinite) connected graph X where in each vertex there is a lamp that may be switched off (state 0), or switched on with $q-1$ different intensities (states $1,\dots,q-1$). Initially, all lamps are turned off, and a lamplighter starts at some vertex of X and walks around. When he visits a vertex, he may switch the lamp sitting there into one of its q different states (including 'off'). Our information consists of the position $x \in X$ of the lamplighter and of the finitely supported configuration $\eta : X \to \mathbb{Z}_q = \{0,\dots,q-1\}$ of the lamps that are switched on, including their respective intensities. The set $\mathbb{Z}_q \wr X$ of all such pairs (η, x) can be equipped in several ways with a natural connected graph structure, giving rise to a *lamplighter graph*.

When X is a Cayley graph of a group Γ, then underlying this construction there is the *wreath product* $\mathbb{Z}_q \wr \Gamma$, which is the semidirect product of Γ with the group of all finitely supported functions $\eta : \Gamma \to \mathbb{Z}_q$ (*i.e.*, a direct sum), on which Γ acts by $g\eta(h) = \eta(g^{-1}h)$. Instead of $\mathbb{Z}_q = \mathbb{Z}/(q\mathbb{Z})$, one may of course take any other group L of 'lamps', leading to the wreath product $L \wr \Gamma$.

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Various aspects of random walks on lamplighter groups have received considerable attention recently: the Poisson boundary (Kaimanovich and Vershik [11] and Kaimanovich [10]), the rate of escape (Lyons, Pemantle and Peres [13], Erschler [8], Revelle [18]), spectral theory (Grigorchuk and Żuk [9]), and the asymptotic behaviour of transition probabilities (Saloff-Coste and Pittet [16, 17], Revelle [19]). Here we shall consider *harmonic functions*.

A harmonic function on a locally finite graph is a real-valued function whose value at each vertex coincides with the arithmetic average of its values in the neighbour vertices. More generally, we can consider the transition matrix P of a random walk on the graph, suitably adapted to the graph's geometry; a harmonic function h is then one that satisfies $Ph = h$.

In the present paper, we shall determine all positive harmonic functions on certain Cayley graphs of the simplest lamplighter group, $\Gamma = \mathbb{Z}_q \wr \mathbb{Z}$. Namely, we first explain that the Diestel–Leader graph $DL(q,q)$ is a Cayley graph of Γ. More generally, if $q, r \geqslant 2$ then $DL(q,r)$ is obtained as a 'horocyclic product' of two homogeneous trees $\mathbb{T}_q$ and $\mathbb{T}_r$ with degrees $q+1$ and $r+1$, respectively. We remark that this does not mean that $DL(q,r)$ is 'almost' a tree in any sense; indeed, it is a one-ended, vertex-transitive graph which is a Cayley graph only when $r = q$. When $r \neq q$, it is believed to be an example of a transitive graph that is not quasi-isometric with any Cayley graph of some finitely generated group – see Diestel and Leader [6].

Nevertheless, we can use the boundary of each of the two trees that compose $DL(q,r)$ for giving an integral representation of all positive harmonic functions: in that representation, we start with the projections of the random walk on $DL(q,r)$ to each of the two trees and the corresponding Martin kernels. Our main result is that every positive harmonic function on $DL(q,r)$ is of the form $h = h_1 + h_2$, where h_1 is obtained by lifting a harmonic function from $\mathbb{T}_q$ to $DL(q,r)$, and h_2 is obtained analogously from $\mathbb{T}_r$. Thereby, we also determine all minimal positive harmonic functions.

We now give an outline of the contents of this paper.

Section 2, although it does not contain *proofs*, is crucial, since it *explains* the geometry of the structures that we are working with, and in particular, the correspondence between lamplighter groups and Diestel–Leader graphs. As a matter of fact, it is precisely this geometric realization that allows us to determine all positive harmonic functions on $\mathbb{Z}_q \wr \mathbb{Z}$. At the end of Section 2, we state the first main result, regarding the decomposition of positive harmonic functions over the two trees (Theorem 2.3).

In Section 3, we recall basic results on positive harmonic functions for irreducible Markov chains. In particular, we consider finite sets with boundaries, the Martin boundary at infinity and its minimal part, and the Martin compactification for nearest neighbour random walks on trees.

In Section 4, we use all the preceding ingredients to prove the Decomposition Theorem (Theorem 2.3). It is then quite simple to determine all minimal positive harmonic functions (Theorem 4.1); they are the Martin kernels of the two projected random walks, up to one, resp. two exceptions. We then retranslate these results to the lamplighter group $\mathbb{Z}_q \wr \mathbb{Z}$ (Example 4.2).

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In Section 5, we adapt the preceding results to the 'switch-walk-switch' random walk, which is in some sense more natural from the point of view of the lamplighter than the simple random walk on $DL(q,q)$.

Section 6 is devoted to some additional remarks and speculations.

2. Diestel–Leader graphs and lamplighters

Let $\mathbb{T} = \mathbb{T}_q$ be the homogeneous tree with degree $q+1$, $q \geqslant 2$. A *geodesic path*, resp. *geodesic ray*, resp. *infinite geodesic* in $\mathbb{T}$ is a finite, resp. one-sided infinite, resp. doubly infinite sequence (x_n) of vertices of $\mathbb{T}$ such that $d(x_i, x_j) = |i-j|$ for all i, j, where $d(\cdot,\cdot)$ denotes the graph distance.

Two rays are *equivalent* if their symmetric difference is finite. An *end* of $\mathbb{T}$ is an equivalence class of rays. The space of ends is denoted by $\partial\mathbb{T}$, and we write $\widehat{\mathbb{T}} = \mathbb{T} \cup \partial\mathbb{T}$. For all $w, z \in \widehat{\mathbb{T}}$ there is a unique geodesic $\overline{w\,z}$ that connects the two. In particular, if $x \in \mathbb{T}$ and $\xi \in \partial\mathbb{T}$ then $\overline{x\,\xi}$ is the ray that starts at x and represents ξ. Furthermore, if $\xi, \zeta \in \partial\mathbb{T}$ ($\xi \neq \zeta$) then $\overline{\zeta\,\xi}$ is the infinite geodesic whose two halves (split at any vertex) are rays that represent ζ and ξ, respectively.

For $x, y \in \mathbb{T}$, $x \neq y$, we define the cone $\widehat{\mathbb{T}}(x,y) = \{w \in \widehat{\mathbb{T}} : y \in \overline{x\,w}\}$. The collection of all cones is the basis of a topology which makes $\widehat{\mathbb{T}}$ a compact, totally disconnected Hausdorff space with $\mathbb{T}$ as a dense, discrete subset. We denote $\mathbb{T}(x,y) = \mathbb{T} \cap \widehat{\mathbb{T}}(x,y)$ and $\partial\mathbb{T}(x,y) = \partial\mathbb{T} \cap \mathbb{T}(x,y)$.

We fix a root $o \in \mathbb{T}$. If $w, z \in \widehat{\mathbb{T}}$, then their *confluent* $c = w \wedge z$ with respect to the root vertex o is defined by $\overline{o\,w} \cap \overline{o\,z} = \overline{o\,c}$. Similarly, we choose and fix a reference end $\omega \in \partial\mathbb{T}$. For $z, v \in \widehat{\mathbb{T}}\backslash\{\omega\}$, their confluent $b = v \curlywedge z$ with respect to ω is defined by $\overline{v\,\omega} \cap \overline{z\,\omega} = \overline{b\,\omega}$. The *Busemann function* $\mathfrak{h} : \mathbb{T} \to \mathbb{Z}$ and the *horocycles* H_k with respect to ω are defined as

$$\mathfrak{h}(x) = d(x, x \curlywedge o) - d(o, x \curlywedge o) \quad \text{and} \quad H_k = \{x \in \mathbb{T} : \mathfrak{h}(x) = k\}.$$

Every horocycle is infinite. Every vertex x in H_k has one neighbour x^- (its predecessor) in H_{k-1} and q neighbours (its successors) in H_{k+1}. We set $\partial^*\mathbb{T} = \partial\mathbb{T}\backslash\{\omega\}$.

2.1. Tree and sequences

Now consider the set Σ_q of all sequences $(\sigma(n))_{n\leqslant 0}$ over $\mathbb{Z}_q$ with finite support $\{n : \sigma(n) \neq 0\}$. We denote by τ the (negative) shift, $\tau\sigma(n) = \sigma(n-1)$. Then the set $\Sigma_q \times \mathbb{Z}$ carries the structure of $\mathbb{T}_q$ in horocylic layers as above: the kth horocycle is $H_k = \Sigma_q \times \{k\}$, and the predecessor of vertex $x = (\sigma, k)$ is $x^- = (\tau\sigma, k-1)$. This corresponds to labelling the edges of $\mathbb{T}_q$ by elements of $\mathbb{Z}_q$ such that all edges on the ray from ω to o have label 0, and for every vertex x and every $\ell \in \mathbb{Z}_q$ there is a successor y among the q successors of x such that the edge $[x, y]$ carries label ℓ. See Figure 1: the origin is the leftmost point on the horocycle H_0, we have indicated the labels on the edges that lead to the point marked with a '•', and that point has coordinates (σ, k) with $\sigma = (\ldots, 0, \ldots, 0, 1, 0, 1)$ and $k = -1$.

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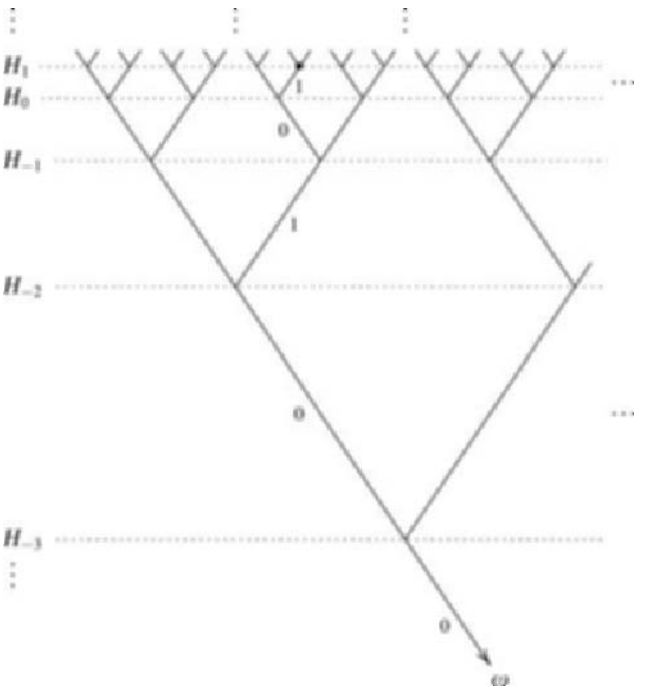


Figure 1.

Now consider two trees $\mathbb{T}_q$ and $\mathbb{T}_r$ with roots o_1 and o_2 and reference ends ω_1 and ω_2, respectively.

Definition 2.1. The Diestel–Leader graph $DL(q,r)$ is

$$DL(q,r) = \{x_1x_2 \in \mathbb{T}_q \times \mathbb{T}_r : \mathfrak{h}(x_1) + \mathfrak{h}(x_2) = 0\},$$

and the neighbourhood is given by

$$x_1x_2 \sim y_1y_2 \iff x_1 \sim y_1 \quad \text{and} \quad x_2 \sim y_2.$$

To visualize $DL(q,r)$, draw $\mathbb{T}_q$ in horocyclic layers with ω_1 at the top and $\partial^*\mathbb{T}_q$ at the bottom, and right to it $\mathbb{T}_r$ in the same way, but upside down, with the respective horocycles $H_k(\mathbb{T}_q)$ and $H_{-k}(\mathbb{T}_r)$ on the same level. Connect the two origins o_1, o_2 by an elastic spring. It is allowed to move along each of the two trees, may expand infinitely, but must always remain in a horizontal position. The vertex set of $DL_{q,r}$ consists of all admissible positions of the spring. From a position x_1x_2 with $\mathfrak{h}(x_1) + \mathfrak{h}(x_2) = 0$ the spring may move downwards to one of the r successors of x_2 in $\mathbb{T}_r$, and at the same time to the predecessor of x_1 in $\mathbb{T}_q$, or it may move upwards in the analogous way. Such a move corresponds to going to a neighbour of x_1x_2. Figure 2 depicts $DL(2,2)$.

As the reference point in $DL(q,r)$, we choose $o = o_1o_2$. We shall keep in mind that $\mathbb{T}_q$ is the first and $\mathbb{T}_r$ the second tree; when $r = q$, it will be sometimes convenient to write $\mathbb{T}^1$ and $\mathbb{T}^2$ for the first and second trees, both copies of $\mathbb{T}_q$.

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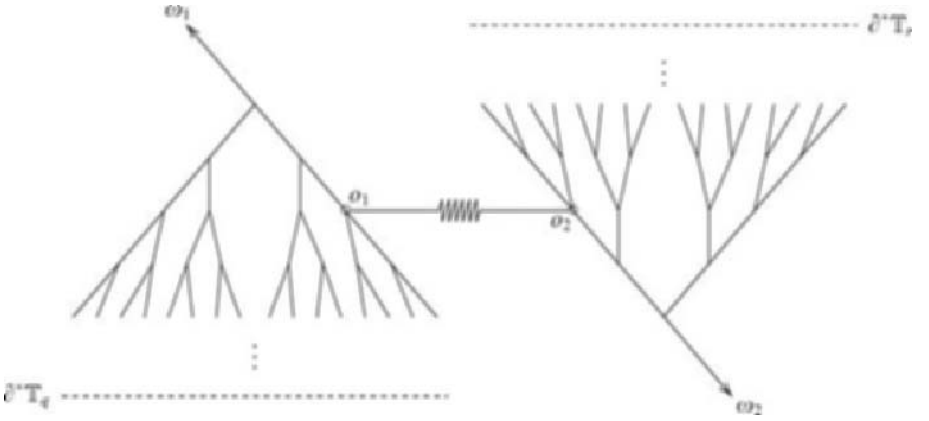


Figure 2.

Next, we explain what the lamplighter group $\mathbb{Z}_q \wr \mathbb{Z}$ has to do with $DL(q,q)$. Let $(\eta,k) \in \mathbb{Z}_q \wr \mathbb{Z}$, and recall that $\eta : \mathbb{Z} \to \mathbb{Z}_q$ is a finitely supported configuration. We identify (η,k) with the vertex $x_1x_2 \in DL(q,q)$, where vertices x_i are given by

$$\begin{aligned} x_1 = (\eta_k^-, k) \quad \text{and} \quad x_2 = (\eta_k^+, -k), \quad \text{where} \\ \eta_k^- = (\eta(k+n))_{n\leqslant 0} \quad \text{and} \quad \eta_k^+ = (\eta(k+1-n))_{n\leqslant 0}, \end{aligned} \tag{2.1}$$

that is, we split η at k, with $\eta_k^- = \eta|_{(-\infty,k]}$ and $\eta_k^+ = \eta|_{[k+1,\infty)}$, both written as sequences over the nonpositive integers.

This is clearly a one-to-one correspondence between $DL(q,q)$ and $\mathbb{Z}_q \wr \mathbb{Z}$, and it is also straightforward that this group acts transitively and fixed-point-freely on the graph: the action of $m \in \mathbb{Z}$ is given by $x_1x_2 = (\sigma_1,k)(\sigma_2,k) \mapsto y_1y_2 = (\sigma_1,k+m)(\sigma_2,-k-m)$, and the action of the group of configurations is pointwise addition modulo q in the obvious way; the reader is invited to work out the simple details. We have to determine the symmetric set of generators of our group with respect to which $DL(q,q)$ is its Cayley graph. (Here, we mean the *right* Cayley graph, where an edge corresponds to multiplying with a generator on the right.)

Stepping from a vertex x_1x_2 to $y_1x_2^-$, where y_1 is one of the successors of $x_1 \in H_k(\mathbb{T}^1)$ (horocycle in the first tree) means that the lamplighter walks from position k to $k+1$ and then switches the lamp at the new position to some state in $\mathbb{Z}_q$. Thus, the 'downward' edges of this type correspond to multiplying on the right with the group elements $(\delta_1^\ell, 1)$, $\ell \in \mathbb{Z}_q$, where δ_k^ℓ is the configuration with value ℓ at k and 0 elsewhere. On the other hand, we have the 'upward' edges from x_1x_2 to $x_1^-y_2$, where y_2 is one of the successors of $x_2 \in H_{-k}(\mathbb{T}^2)$ (horocycle in the second tree). They correspond to multiplying on the right with the inverses of the above generators, *i.e.*, the elements $(\delta_0^\ell, -1)$, where $\ell \in \mathbb{Z}_q$.

Thus the simple random walk on $DL(q,q)$ is the following lamplighter walk: its *law*, the probability measure on $\mathbb{Z}_q \wr \mathbb{Z}$ that describes the one step transition probabilities, is equidistribution on

$$\left\{ \left(\delta_1^\ell, 1\right),\ \left(\delta_0^\ell, -1\right) : \ell \in \mathbb{Z}_q \right\}. \tag{2.2}$$

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If at some step the lamplighter stands at $k \in \mathbb{Z}$, (s)he chooses with equal probability either to step to $k+1$ and then to switch the lamp at $k+1$ to a random state, or (s)he chooses to switch the lamp at k to a random state (before leaving k) and then to step to $k-1$. While this is a symmetric random walk on $\mathbb{Z}_q \wr \mathbb{Z}$, resp. $DL(q,q)$, this type of action does not appear 'symmetric' from the point of view of the lamplighter. For this reason, other types of 'simple' random walks have been considered in the past: the one whose law is equidistribution on

$$\{(\mathbf{0}, \pm 1),\ (\delta_0^\ell, 0)\ : 0 \neq \ell \in \mathbb{Z}_q\} \tag{2.3}$$

('walk or switch'), and the one where the lamplighter standing at k first switches the lamp where he stands to a random state, then walks to $k \pm 1$, and then switches the lamp at the arrival point to a random state ('switch-walk-switch'). The corresponding law is equidistribution on

$$\left\{\left(\delta_0^\ell + \delta_{\pm 1}^m, \pm 1\right)\ : \ell, m \in \mathbb{Z}_q\right\}. \tag{2.4}$$

Harmonic functions for the 'walk or switch' model cannot be determined by the methods that we elaborate here, since it is not very well adapted to the structure of $DL(q,q)$; see the comments at the end.

On the other hand, the 'switch-walk-switch' model (2.4) corresponds to simple random walk on the following modification of $DL(q,q)$: in the first of the two trees, we add edges between every vertex and the siblings of its predecessor (*i.e.*, its 'uncles'), and the resulting neighbourhood relation on the horocyclic product is as in Definition 2.1. It will be easy to adapt our results to this random walk.

In the first place, we shall study the following slight generalization $P = P_\alpha$ of simple random walk on $DL(q,r)$, where $0 < \alpha < 1$. For $x_1x_2 \in DL(q,r)$,

$$p(x_1x_2, y_1y_2) = \begin{cases} \alpha/q & \text{if } y_1^- = x_1 \text{ and } y_2 = x_2^-, \\ (1-\alpha)/r & \text{if } y_1 = x_1^- \text{ and } y_2^- = x_2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.5}$$

P acts on functions $h : DL(q,r) \to \mathbb{R}$ by

$$Ph(x_1x_2) = \sum_{y_1y_2} p(x_1x_2, y_1y_2) h(x_1x_2).$$

A *harmonic*, or more precisely, *P_α-harmonic function*, is one that satisfies $Ph = h$.

We can consider the projections $P_1 = P_{1,\alpha}$ and $P_2 = P_{2,1-\alpha}$ of P_α on $\mathbb{T}_q$ and $\mathbb{T}_r$, respectively:

$$p_1(x_1, y_1) = \begin{cases} \alpha/q & \text{if } y_1^- = x_1, \\ (1-\alpha) & \text{if } y_1 = x_1^-, \\ 0 & \text{otherwise,} \end{cases} \qquad p_2(x_1, y_2) = \begin{cases} \alpha & \text{if } y_2 = x_2^-, \\ (1-\alpha)/r & \text{if } y_2^- = x_2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

The following is straightforward.

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Lemma 2.2.

(a) *If h_1 is a P_1-harmonic function on $\mathbb{T}_q$, then $h(x_1x_2) = h_1(x_1)$, $x_1x_2 \in DL(q,r)$, defines a P-harmonic function on $DL(q,r)$.*

(b) *If h_2 is a P_2-harmonic function on $\mathbb{T}_q$, then $h(x_1x_2) = h_2(x_2)$, $x_1x_2 \in DL(q,r)$, defines a P-harmonic function on $DL(q,r)$.*

Our first main result is the following.

Theorem 2.3. *If h is a nonnegative P-harmonic function on $DL(q,r)$, then there are nonnegative P_i-harmonic functions h_i, $i = 1, 2$, on $\mathbb{T}_q$ and $\mathbb{T}_r$, respectively, such that*

$$h(x_1x_2) = h_1(x_1) + h_2(x_2) \quad \text{for all } x_1x_2 \in DL(q,r).$$

Conversely, it is of course clear that every sum of the latter form defines a P-harmonic function. Recall that in this type of decomposition, x_2 cannot vary independently of x_1, since one must have $\mathfrak{h}(x_1) + \mathfrak{h}(x_2) = 0$ for the horocycle indices.

The next short section contains some basic preparatory material for the proof of Theorem 2.3.

3. Basic results about harmonic functions

Let X be a denumerable set and $P = (p(x,y))_{x,y\in X}$ the stochastic transition matrix of a Markov chain $(Z_n)_{n\geqslant 0}$ on X. We write $\Pr_x$ for probability conditioned to the starting point $Z_0 = x$. The n-step transition probability $p^{(n)}(x,y) = \Pr_x[Z_n = y]$ is the (x,y)-entry of the matrix power P^n. We assume that P is *irreducible*: $\forall\ x, y\ \exists\ n : p^{(n)}(x,y) > 0$.

As above, a function h on X is called *P-harmonic* or just *harmonic at x*, if $Ph(x) = h(x)$, where $Ph(x) = \sum_y p(x,y)h(y)$. It is called harmonic when it is harmonic at each x.

For a subset $A \subset X$, we define the stopping time

$$s^A = \inf\{n \geqslant 0 : Z_n \in A\}.$$

For $y \in X$, we write $s^y = s^{\{y\}}$. Given $x, y \in X$, let

$$F(x,y) = \Pr_x[s^y < \infty] \quad \text{and} \quad F^A(x,y) = \Pr_x[s^y \leqslant s^A,\ s^y < \infty]. \tag{3.1}$$

Thus, $F(x,y)$ is the probability of ever reaching y, starting from x. The function $F(\cdot, y)$ is harmonic in $X \setminus \{y\}$. Furthermore, if $y \in A$, then $F^A(\cdot, y)$ is harmonic in $X \setminus A$.

3.1. Harmonic functions on finite sets

Let S be a finite subset of X. Define its *boundary* and *interior* by

$$\partial S = \{y \in S : p(y, X \setminus S) > 0\} \quad \text{and} \quad S^o = S \setminus \partial S.$$

For the sake of simplicity, we assume that the restriction of P to S^o is irreducible. We define

$$\mathscr{H}(P,S) = \{h : S \to \mathbb{R} \mid h \text{ is harmonic in } S^o\}.$$

The following is very well known.

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Proposition 3.1. *Under the above assumptions, the functions $F^{\partial S}(\cdot, y)$, $y \in \partial S$, constitute a basis of the linear space $\mathscr{H}(P, S)$. Every $h \in \mathscr{H}(P, S)$ is uniquely represented as*

$$h = \sum_{y \in \partial S} F^{\partial S}(\cdot, y)\, h(y).$$

Proof. The functions $F^{\partial S}(\cdot, y)$, $y \in \partial S$, are linearly independent, since $F^{\partial S}(x, y) = \delta_x(y)$ for $x, y \in \partial S$. Given $h \in \mathscr{H}(P, S)$, let $g = \sum_{y \in \partial S} F^{\partial S}(\cdot, y)\, h(y)$. Then $g \in \mathscr{H}(P, S)$, and $(g - h)|_{\partial S} \equiv 0$. By the *Minimum Principle*, every function in $\mathscr{H}(P, S)$ attains its minimum (and its maximum) on the boundary. Therefore $g = h$ on S. □

3.2. Positive and minimal harmonic functions

Regarding the following material, see Woess [20, Section 24] for a more detailed outline and many references.

We return to the infinite set X with irreducible transition matrix P. For the sake of simplicity, we assume that P has *finite range*, that is, $\{y : p(x, y) > 0\}$ is finite for all $x \in X$. The set $\mathscr{H}^+ = \mathscr{H}^+(P, X)$ of nonnegative P-harmonic functions constitutes a convex cone that is closed in the topology of pointwise convergence. We choose a reference point $o \in X$. Then the set $\mathscr{B} = \{h \in \mathscr{H}^+ : h(o) = 1\}$ is a compact, convex base of the cone $\mathscr{H}^+$. Its extremal elements are called *minimal harmonic functions*. Thus, $h \in \mathscr{H}^+$ is minimal if

$$h(o) = 1 \quad \text{and} \quad h \geqslant h_1 \in \mathscr{H}^+ \Longrightarrow h_1/h \equiv \text{constant}.$$

The set $\mathscr{B}_{\min}$ of minimal harmonic functions is a Borel subset of $\mathscr{B}$, and every $h \in \mathscr{H}^+$ is an integral of minimal ones with respect to a Borel measure on $\mathscr{B}_{\min}$. This can be made more precise by the following construction. Define the *Martin kernel*

$$K(x, y) = F(x, y)/F(o, y).$$

The *Martin compactification* is the smallest metrizable compactification of X containing X as a discrete, dense subset, and to which all functions $K(x, \cdot)$, $x \in X$, extend continuously. The *Martin boundary* $\mathscr{M} = \mathscr{M}(P)$ is the ideal boundary added to X in this compactification. Then every minimal harmonic function is of the form $K(\cdot, \xi)$ for some $\xi \in \mathscr{M}$, and the set

$$\mathscr{M}_{\min} = \{\xi \in \mathscr{M} : K(\cdot, \xi) \text{ is minimal harmonic}\}$$

is a Borel set. The *Poisson–Martin Representation Theorem* says that for every $h \in \mathscr{H}^+$ there is a unique Borel measure ν^h on $\mathscr{M}$ with $\nu^h(\mathscr{M} \setminus \mathscr{M}_{\min}) = 0$ such that

$$h(x) = \int_{\mathscr{M}} K(x, \cdot)\, d\nu^h \quad \forall\, x \in X.$$

Furthermore, considering the constant harmonic function $\mathbf{1}$, we set $\nu = \nu^{\mathbf{1}}$. Then every *bounded* harmonic function h has a unique representation as above, where $d\nu^h(\xi) = \varphi(\xi)\, d\nu(\xi)$ with $\varphi \in L^\infty(\mathscr{M}, \nu)$. The probability space $(\mathscr{M}, \nu)$ is a model of the *Poisson boundary* of the random walk. While the Martin boundary is a topological object, the Poisson boundary is a measure-theoretical one, and finding it means its determination up to isomorphisms between measure spaces.

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See Kaimanovich and Vershik [11] for a profound introduction and impressive results regarding Poisson boundaries of random walks on groups, Kaimanovich [10] for lamplighter groups over $\mathbb{Z}^d$ and other semidirect products, and Kaimanovich and Woess [12] for Poisson boundaries of random walks on homogeneous graphs, including the Diestel–Leader graphs.

3.3. Harmonic functions on trees

Next, let us suppose that $X = T$ carries the structure of an infinite, locally finite tree. We assume that P is of *nearest neighbour* type, that is,

$$p(x, y) > 0 \iff x \sim y \text{ in } T \tag{3.2}$$

($\sim$ denotes neighbourhood). We also assume that the random walk (Markov chain) with transition matrix P is *transient*, that is, $\sum_n p^{(n)}(x, y) < \infty$ for some ($\iff$ all) $x, y \in T$. Geodesics and boundary of T are defined as in Section 2, with the general tree T in the place of $\mathbb{T}_q$. The results regarding the Martin compactification in this setting are contained in the seminal paper by Cartier [4].

The basic link between tree structure and random walk is the following well-known lemma: see, *e.g.*, Cartier [4], or Woess [20, Lemmas 1.23 and 1.13(d)].

Lemma 3.2. *For a nearest neighbour random walk on a tree T,*

$$F(x, y) = F(x, w)F(w, z) \quad \text{for all } x, y \in T \text{ and } w \in \overline{x\,y}.$$

Furthermore, if $x \sim y$, then

$$F(y, x) = p(y, x) + \sum_{w \neq x} p(y, w)F(w, y)F(y, x).$$

For $x, y \in T$, let $c = x \wedge y$ be their confluent with respect to o. Then the lemma implies that $K(x, y) = K(x, c)$. From here, the following is almost immediate.

Proposition 3.3. *Suppose that P defines a transient nearest neighbour random walk on the locally finite tree T with root o. Then the Martin compactification is the end compactification $\widehat{T}$, and for $\xi \in \partial T$, the Martin kernel is given by*

$$K(x, \xi) = K(x, c) = \frac{F(x, c)}{F(o, c)}, \quad \text{where} \quad c = x \wedge \xi.$$

Furthermore, each function $K(\cdot, \xi)$, $\xi \in \partial T$, is minimal harmonic.

For various different proofs, see Cartier [4], Picardello, Taibleson and Woess [15], or Woess [20, Section 26], or the one that is implicit in the proof of Theorem 2.3 below.

Example 3.4. Consider the random walk on $\mathbb{T}_q$ with transition matrix $P_1 = P_{1,\alpha}$, defined in (2.6). It is clear that for this random walk, the probabilities

$$F_1^- = F_1(x, x^-) \quad \text{and} \quad F_1^+ = F_1(x^-, x)$$

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Figure 3.

are independent of $x \in \mathbb{T}_q$ (x^- is the predecessor with respect to ω). Using Lemma 3.2, we find the two quadratic equations

$$F_1^- = (1-\alpha) + \alpha (F_1^-)^2 \quad \text{and} \quad F_1^+ = \frac{\alpha}{q} + (q-1)\frac{\alpha}{q} F_1^- F_1^+ + (1-\alpha)(F_1^+)^2.$$

Among the two solutions of each equation, the smaller one is the right one (compare, *e.g.*, with the generating functions argument in the proof of Lemma 1.24 in Woess [20]). Thus

$$F_1^- = \begin{cases} \dfrac{1-\alpha}{\alpha} & \text{if } \alpha \geqslant \dfrac{1}{2}, \\ 1 & \text{if } \alpha \leqslant \dfrac{1}{2}, \end{cases} \qquad F_1^+ = \begin{cases} \dfrac{1}{q} & \text{if } \alpha \geqslant \dfrac{1}{2}, \\ \dfrac{\alpha}{(1-\alpha)q} & \text{if } \alpha \leqslant \dfrac{1}{2}. \end{cases} \tag{3.3}$$

We can now compute the associated Martin kernels $K_1(\cdot, \xi)$, $\xi \in \partial \mathbb{T}_q$. First, since $x \wedge \omega = x \curlywedge o$ (where $\wedge$ and $\curlywedge$ denote confluents with respect to o and ω), it is immediate that

$$K_1(x, \omega) = (F_1^-)^{\mathfrak{h}(x)} = \begin{cases} \left(\dfrac{1-\alpha}{\alpha}\right)^{\mathfrak{h}(x)} & \text{if } \alpha \geqslant \dfrac{1}{2}, \\ 1 & \text{if } \alpha \leqslant \dfrac{1}{2}. \end{cases} \tag{3.4}$$

Next, if $\xi \in \partial^* \mathbb{T}_q$, we set $c = x \wedge \xi$ and write $k = d(o, o \curlywedge x)$, $l = d(x, o \curlywedge x)$, so that $\mathfrak{h}(x) = l - k$. We distinguish two cases (see Figure 3).

Case 1: c lies between o and $o \curlywedge x$.
Let $s = d(o, c)$. Then

$$K_1(x, \xi) = \frac{(F_1^-)^l (F_1^+)^{k-s}}{(F_1^-)^s} = K_1(x, \omega)(F_1^- F_1^+)^{k-s} = K_1(x, \omega)(F_1^- F_1^+)^{\mathfrak{h}(o \curlywedge \xi) - \mathfrak{h}(x \curlywedge \xi)}.$$

Case 2: c lies between $o \curlywedge x$ and x.
Let $r = d(x, c)$. Then

$$K_1(x, \xi) = \frac{(F_1^-)^r}{(F_1^-)^k (F_1^+)^{l-r}} = K_1(x, \omega)(F_1^- F_1^+)^{r-l} = K_1(x, \omega)(F_1^- F_1^+)^{\mathfrak{h}(o \curlywedge \xi) - \mathfrak{h}(x \curlywedge \xi)}.$$

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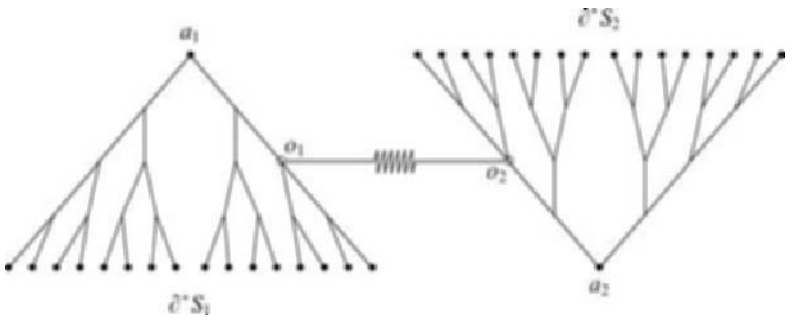


Figure 4.

We write $\mathfrak{h}(x,\xi) = d(x,c) - d(o,c)$, the horocycle number with respect to ξ, while $\mathfrak{h}(x) = \mathfrak{h}(x,\omega)$. Also, we set $\rho = (F_1^- F_1^+)^{1/2}$. Then we find in both cases

$$K_1(x,\xi) = K_1(x,\omega)\,\rho^{\mathfrak{h}(x,\xi)-\mathfrak{h}(x)}, \quad \text{where} \quad \rho = \min\left\{\frac{1-\alpha}{\alpha q}, \frac{\alpha}{(1-\alpha)q}\right\}^{1/2}. \tag{3.5}$$

In particular, if $\alpha = 1/2$ then

$$K_1(x,\omega) = 1 \quad \text{and} \quad K_1(x,\xi) = q^{(\mathfrak{h}(x)-\mathfrak{h}(x,\xi))/2} \quad \text{for} \quad \xi \in \partial^* \mathbb{T}_q. \tag{3.6}$$

4. Minimal harmonic functions on $DL(q,r)$

After all these preliminaries, the proof of Theorem 2.3 depends in the first place on how we look at the underlying structure.

Proof of Theorem 2.3. In $DL(q,r)$, consider the subgraph spanned by all vertices x_1x_2 with $-n \leqslant \mathfrak{h}(x_1) \leqslant n$. It is not connected. We let $S = S^{(n)}$ denote the connected component of the root o_1o_2.

Let $a_1 = a_1^{(n)} \in \mathbb{T}_q$ be the vertex on $\overline{o_1\,\omega_1}$ at distance $d(a_1,o_1) = n$. Then a_1 can be viewed as the root of the q-ary rooted tree $S_1 = S_1^{(n)} = \{x_1 \in \mathbb{T}_q : -n \leqslant \mathfrak{h}(x_1) \leqslant n,\ a_1 \in \overline{x_1\,\omega_1}\}$ of height $2n$, whose set of leaves (elements with $\mathfrak{h}(x_1) = n$) is denoted by $\partial^* S_1$. Analogously, we define $a_2 = a_2^{(n)}$, the r-ary rooted tree $S_2 = S_2^{(n)}$, and its set of leaves $\partial^* S_2$. Then

$$S = \{x_1x_2 \in S_1 \times S_2 : \mathfrak{h}(x_1) + \mathfrak{h}(x_2) = 0\}$$

is the horocyclic product of S_1 and S_2 (see Figure 4).

One may imagine S as a tetrahedron. Two of its faces are copies of S_1 that meet at the common bottom edge $\partial^* S_1 \times \{a_2\}$, and the other two faces are copies of S_2 that meet at the common top edge $\{a_1\} \times \partial^* S_2$. The boundary of S_i is $\{a_i\} \cup \partial^* S_i$.

We now restrict P to S, and also the projections P_1 to S_1 and P_2 to S_2. Then the boundary of S in the sense of Section 3.1 is

$$\partial S = (\partial^* S_1 \times \{a_2\}) \cup (\{a_1\} \times \partial^* S_2).$$

As in Lemma 2.2, if $h_i \in \mathscr{H}(P_i, S_i)$, then it lifts to a function in $\mathscr{H}(P,S)$.

In particular, if $y_1 \in \partial^* S_1$ then $h(x_1x_2) = F_1^{\partial S_1}(x_1,y_1)$ defines a function in $\mathscr{H}(P,S)$ with value one at y_1a_2 and value 0 in $\partial S \backslash \{y_1a_2\}$. But, by Proposition 3.1, these properties

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characterize the function $x_1x_2 \mapsto F^{\partial S}(x_1x_2, y_1a_2)$ on S. Therefore, for all $x_1x_2 \in S$,

$$\begin{aligned} F^{\partial S}(x_1x_2, y_1a_2) &= F_1^{\partial S_1}(x_1, y_1) \quad \forall\ y_1 \in \partial^* S_1 \quad \text{and} \\ F^{\partial S}(x_1x_2, a_1y_2) &= F_2^{\partial S_2}(x_2, y_2) \quad \forall\ y_2 \in \partial^* S_2 \end{aligned} \tag{4.1}$$

Applying Proposition 3.1 once more, we see that every $h \in \mathscr{H}(P, S)$ can be written uniquely as

$$h(x_1x_2) = h_1(x_1) + h_2(x_2) \quad \forall\ x_1x_2 \in S, \quad \text{where}$$

$$\begin{aligned} h_1(x_1) &= \sum_{y_1 \in \partial^* S_1} F_1^{\partial S_1}(x_1, y_1) h(y_1a_2) \quad \text{and} \\ h_2(x_2) &= \sum_{y_2 \in \partial^* S_2} F_2^{\partial S_2}(x_2, y_2) h(a_1y_2). \end{aligned} \tag{4.2}$$

This is true, in particular, if h is P-harmonic on the whole of $DL(q, r)$, since its restriction to $S = S^{(n)}$ is in $\mathscr{H}(P, S)$. Furthermore, if h is nonnegative then so are h_1 and h_2. Note, however, that $h_i = h_i^{(n)}$ $(i = 1, 2)$ *depend on* n, and it is by no means true that the restriction of $h_i^{(n+1)}$ to $S_i^{(n)}$ might coincide with $h_i^{(n)}$. We have to study the behaviour of $h_i^{(n)}$ when $n \to \infty$, and this is the point where the assumption of nonnegativity of h will be used. We define

$$K_i^{(n)}(x_i, y_i) = \frac{F_i^{\partial S_i}(x_i, y_i)}{F_i^{\partial S_i}(o_i, y_i)}, \quad x_i \in S_i, \quad y_i \in \partial S_i, \quad S_i = S_i^{(n)}, \quad i = 1, 2.$$

Then we can rewrite the functions h_i of (4.2) as

$$h_i(x_i) = \sum_{y_i \in \partial S_i} K_i^{(n)}(x_i, y_i) \lambda_i(y_i), \quad \text{where}$$

$$\begin{aligned} &\lambda_1(a_1) = 0, \quad \lambda_1(y_1) = h(y_1a_2)/F_1^{\partial S_1}(o_1, y_1),\ y_1 \in \partial^* S_1, \\ &\lambda_2(a_2) = 0, \quad \lambda_2(y_2) = h(a_1y_2)/F_2^{\partial S_2}(o_2, y_2),\ y_2 \in \partial^* S_2. \end{aligned}$$

Of course, $\lambda_i(y_i) = \lambda_i^{(n)}(y_i)$ also depends on n. For $\xi_1 \in \partial \mathbb{T}_q$, we define

$$K_1^{(n)}(x_1, \xi_1) = K_1^{(n)}(x_1, y_1), \quad \text{if } \xi_1 \in \mathbb{T}_q(o_1, y_1) \text{ with } y_1 \in \partial S_1.$$

Then $K_1^{(n)}(x_1, \cdot)$ is locally constant (whence continuous) on $\partial \mathbb{T}_q$. (Recall that $\partial \mathbb{T}_q$ is compact and totally disconnected.) We define a nonnegative Borel measure $\nu_1^{(n)}$ on $\partial \mathbb{T}_q$ by

$$\begin{aligned} &\nu_1^{(n)}(\partial \mathbb{T}_q(o_1, a_1^-)) = 0 \quad \text{and} \\ &\nu_1^{(n)}(\partial \mathbb{T}_q(o_1, w_1)) = \lambda_1^{(n)}(y_1)\, q^{\mathfrak{h}(w_1)-n} \quad \text{if } w_1 \in \mathbb{T}_q(o_1, y_1),\ y_1 \in \partial^* S_1^{(n)}. \end{aligned}$$

This defines a finitely additive, nonnegative measure on the semiring of all sets $\partial \mathbb{T}_q(o_1, x_1)$, $x_1 \in \mathbb{T}_q \backslash \{o_1\}$. Since all these sets are open and compact, the measure is sigma-additive on that semiring and extends to a unique nonnegative Borel measure on $\partial \mathbb{T}_q$. We proceed in precisely the same way on the second tree, and also get a nonnegative Borel measure $\nu_2^{(n)}$ on $\partial \mathbb{T}_r$, such that, for all $x_1x_2 \in S^{(n)}$,

$$h_1^{(n)}(x_1) = \int_{\partial \mathbb{T}_q} K_1^{(n)}(x_1, \cdot)\, d\nu_1^{(n)} \quad \text{and} \quad h_2^{(n)}(x_2) = \int_{\partial \mathbb{T}_r} K_2^{(n)}(x_2, \cdot)\, d\nu_2^{(n)}. \tag{4.3}$$

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Since $K_i^{(n)}(o_i,\cdot)\equiv 1$, we have $\nu_1^{(n)}(\partial\mathbb{T}_q)+\nu_2^{(n)}(\partial\mathbb{T}_r)=h(o_1o_2)$ for all n. Thus, by compactness (Helly's theorem), there are a subsequence (n') and nonnegative measures ν_1 on $\partial\mathbb{T}_q$ and ν_2 on $\partial\mathbb{T}_r$ such that $\nu_i^{(n')}\to\nu_i$ weakly for $i=1,2$.

If $x_1x_2\in DL(q,r)$, then we choose $n_0=n_0(x_1x_2)$ large enough such that, for all $n\geqslant n_0$, the geodesics $\overline{o_i\,x_i}$ are contained in the interior of $S_i^{(n)}$, $i=1,2$.

For every $\xi_i\in\partial\mathbb{T}_q$, resp. $\in\partial\mathbb{T}_r$, the confluent $c_i=x_i\wedge\xi_i$ is one of the finitely many points on $\overline{o_i\,x_i}$. It is clear that $F_i^{\partial S_i^{(n)}}(x_i,c_i)\to F_i(x_i,c_i)$ as $n\to\infty$, and the same is true with o_i in the place of x_i. Therefore, using Lemma 3.2, we find

$$K_i^{(n)}(x_i,\xi_i)=K_i^{(n)}(x_i,c_i)\to K_i(x_i,c_i)=K_i(x_i,\xi_i),$$

as $n\to\infty$, where $K_i(\cdot,\cdot)$ is the Martin kernel of P_i on $\mathbb{T}_q$, resp. $\mathbb{T}_r$, $i=1,2$. Thus, $K_i^{(n)}(x_i,\cdot)\to K_i(x_i,\cdot)$ uniformly, and using (4.3), we get

$$h_1^{(n')}(x_1)\to\int_{\partial\mathbb{T}_q}K_1(x_1,\cdot)\,d\nu_1=:h_1(x_1)\quad\text{and}\quad h_2^{(n')}(x_2)\to\int_{\partial\mathbb{T}_r}K_1(x_2,\cdot)\,d\nu_2=:h_2(x_2).$$

Then h_1 is P_1-harmonic on $\mathbb{T}_q$ and h_2 is P_2-harmonic on $\mathbb{T}_r$, and $h(x_1x_2)=h_1(x_1)+h_2(x_2)$ for all $x_1x_2\in DL(q,r)$. □

We remark that the last part is the key point of the argument, namely a way of recovering a Poisson integration formula from the finite approximations $K_i^{(n)}(\cdot,\cdot)$ of the Martin kernel.

Theorem 4.1.

(a) *Each of the functions*

$$x_1x_2\mapsto K_1(x_1,\xi_1),\ \xi_1\in\partial^*\mathbb{T}_q,\quad\textit{and}\quad x_1x_2\mapsto K_2(x_2,\xi_2),\ \xi_2\in\partial^*\mathbb{T}_r,$$

is minimal P_α-harmonic on $DL(p,q)$.

(b) *If $\alpha\neq 1/2$ then these are all minimal harmonic functions.*

(c) *If $\alpha=1/2$, then these together with the constant function* **1** *are all minimal harmonic functions.*

Proof. **(a)** Let $\xi_1\in\partial^*\mathbb{T}_q$ and suppose that $K_1(x_1,\xi_1)\geqslant h(x_1x_2)$ for all $x_1x_2\in DL(q,r)$, where $h\geqslant 0$ is P_α-harmonic. By Theorem 2.3, $h(x_1x_2)=h_1(x_1)+h_2(x_2)$, where $h_i\geqslant 0$ is P_i-harmonic, $i=1,2$. Then $K_1(\cdot,\xi_1)\geqslant h_1$. By Proposition 3.3, $h_1=c\cdot K_1(\cdot,\xi_1)$, where $0\leqslant c\leqslant 1$. If $c=1$ then we are done. Otherwise, $K_1(x_1,\xi_1)\geqslant c\cdot K_1(\cdot,\xi_1)+h_2(x_2)$, that is,

$$K_1(x_1,\xi_1)\geqslant\frac{1}{1-c}h_2(x_2)=\int_{\partial\mathbb{T}_r}K_2(x_2,\cdot)\,d\nu_2\quad\forall\ x_1x_2\in DL(q,r),$$

where ν_2 is a nonnegative Borel measure on $\partial\mathbb{T}^r$.

Setting $x_2=o_2$, we obtain

$$K_1(x_1,\xi_1)\geqslant\nu_2(\partial\mathbb{T}_r)\quad\forall\ x_1\in H_0(\mathbb{T}_q).$$

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If $x_1 \to \omega_1$ then $\mathfrak{h}(x_1, \xi_1) \to \infty$. Therefore, (3.5) yields

$$\nu_2(\partial \mathbb{T}_r) \leqslant \lim_{x_1 \to \omega_1,\, \mathfrak{h}(x_1)=0} K_1(x_1, \xi_1) = 0,$$

and $\nu_2(\partial \mathbb{T}_r) = 0$. Therefore $h_2 \equiv 0$.

This proves minimality of $K_1(\cdot, \xi_1)$ for all $\xi_1 \in \partial^* \mathbb{T}_q$. Exchanging the roles of the two trees, we get the other 'half' of statement (a).

(b), (c) Conversely, let h be a minimal P_α-harmonic function on $DL(q,r)$. By Theorem 2.3, $h(x_1x_2) = h_1(x_1) + h_2(x_2)$ with h_i nonnegative and P_i-harmonic. One of the h_i, say h_1, must be positive. Minimality yields $h_1(x_1) = c_1 \cdot h(x_1x_2)$ for all $x_1x_2 \in DL(q,r)$, where $c_1 > 0$. Thus $h(x_1x_2)$ depends only on x_1. Without loss of generality, $c_1 = 1$, and $h(x_1x_2) = h_1(x_1)$ for all x_1x_2. Minimality of h with respect to P_α yields minimality of h_1 with respect to P_1. Thus, by Proposition 3.3, $h(x_1x_2) = K_1(x_1, \xi_1)$ for some $\xi_1 \in \partial \mathbb{T}_q$.

The case when $h(x_1x_2)$ depends only on x_2 is analogous.

To complete the proof of statements (b) and (c), we have to study minimality of the functions $x_1x_2 \mapsto K_i(\cdot, \omega_i)$, $i = 1, 2$ with respect to P_α.

If $\alpha = 1/2$ then by (3.4), $K_i(\cdot, \omega_i) \equiv 1$. In this case it is known from Kaimanovich and Woess [12, Section 6.2] that the Poisson boundary is trivial, *i.e.*, all bounded harmonic functions are constant, which is the same as minimality of the constant function **1**. This proves (c).

Suppose $\alpha \neq 1/2$. Then – again by [12] – the Poisson boundary is nontrivial, and the constant function **1** is non-minimal. If $\alpha > 1/2$, then this yields that $x_1x_2 \mapsto K_2(x_2, \omega_2) = 1$ is non-minimal. On the other hand, we know from (3.4) that

$$g(x_1x_2) := K_1(x_1, \omega_1) = \left(\frac{1-\alpha}{\alpha}\right)^{\mathfrak{h}(x_1)}.$$

We can conjugate P_α by g, that is, we set

$$\check{p}(x_1x_2, y_1y_2) = \frac{p(x_1x_2, y_1y_2) g(y_1y_2)}{g(x_1x_2)}.$$

Then g is minimal P_α-harmonic if and only if **1** is minimal $\check{P}_\alpha$-harmonic. However, $\check{P}_\alpha = P_{1-\alpha}$ by a straightforward computation, and the constant function **1** is not minimal $P_{1-\alpha}$-harmonic by nontriviality of the Poisson boundary. Thus, also g is non-minimal for P_α. Again, the case $\alpha < 1/2$ follows by exchanging the roles of the two trees. □

We remark that for our nearest neighbour case, minimality, resp. non-minimality of **1** can be proved in a more elementary (somewhat longer) way than by appealing to the results of Kaimanovich and Woess [12].

Example 4.2. We conclude this section by retranslating the results for simple random walk on $DL(q,q)$ to the setting and notation of the random walk (2.2) on $DL(q,q)$ ('walk forward and switch or switch and walk backward'). We write $\mathbb{T}^1$ and $\mathbb{T}^2$ for the first and the second tree, respectively. (Both are copies of $\mathbb{T}_q$.) We have $\alpha = 1/2$, and the constant harmonic function **1** is minimal.

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In terms of configurations, each $\xi_1 \in \partial^* \mathbb{T}^1$ corresponds to an infinite configuration

$$\xi_1 : \mathbb{Z} \to \mathbb{Z}_q \quad \text{with} \quad \xi_1(n) = 0 \ \forall \ n \leqslant n_0(\xi_1) \in \mathbb{Z}.$$

If we label the edges of $\mathbb{T}^1$ by elements of $\mathbb{Z}_q$, as described in Section 2.1, then $\xi_1(n)$ is the label of the edge between the horocycles $H_{n-1}(\mathbb{T}^1)$ and $H_n(\mathbb{T}^1)$ on the infinite geodesic $\overline{\omega_1\, \xi_1}$. Now let $(\eta, k) \in \mathbb{Z}_q \wr \mathbb{Z}$, and consider the associated pair $x_1 x_2 \in DL(q,q)$. From the computations in Example 3.4, we know that

$$K_1(x_1, \xi_1) = q^{\mathfrak{h}(x_1 \curlywedge \xi_1) - \mathfrak{h}(o_1 \curlywedge \xi_1)}.$$

It is easy to compute in terms of (η, k)

$$\begin{aligned} \mathfrak{h}(x_1 \curlywedge \xi_1) &= \begin{cases} \min\{n \leqslant k : \xi_1(n+1) \neq \eta(n+1)\} & \text{if such } n \text{ exists,} \\ k & \text{otherwise,} \end{cases} \\ \mathfrak{h}(o_1 \curlywedge \xi_1) &= \begin{cases} \min\{m \leqslant 0 : \xi_1(m+1) \neq 0\} & \text{if such } m \text{ exists,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4.4}$$

We shall write $\mathsf{def}^+\big((\eta,k),\xi_1\big) = \mathfrak{h}(x_1 \curlywedge \xi_1) - \mathfrak{h}(o_1 \curlywedge \xi_1)$, the (*positive*) *defect* of (η, k) with respect to ξ_1.

Analogously, each $\xi_2 \in \partial^* \mathbb{T}^2$ corresponds to an infinite configuration

$$\xi_2 : \mathbb{Z} \to \mathbb{Z}_q \quad \text{with} \quad \xi_2(n) = 0 \ \forall \ n \geqslant n_0(\xi_2) \in \mathbb{Z}.$$

Here, we label the edges of $\mathbb{T}^2$ by elements of $\mathbb{Z}_q$, and $\xi_2(n)$ is the label of the edge between the horocycles $H_{-n}(\mathbb{T}^2)$ and $H_{-n+1}(\mathbb{T}^2)$ on the infinite geodesic $\overline{\omega_2\, \xi_2}$. If $(\eta, k) \in \mathbb{Z}_q \wr \mathbb{Z} \leftrightarrow x_1 x_2 \in DL(q,q)$, then

$$K_1(x_2, \xi_2) = q^{\mathfrak{h}(x_2 \curlywedge \xi_2) - \mathfrak{h}(o_2 \curlywedge \xi_2)},$$

and we compute

$$\begin{aligned} -\mathfrak{h}(x_2 \curlywedge \xi_2) &= \begin{cases} \max\{n > k : \xi_2(n) \neq \eta(n)\} & \text{if such } n \text{ exists,} \\ k & \text{otherwise,} \end{cases} \\ -\mathfrak{h}(o_2 \curlywedge \xi_2) &= \begin{cases} \max\{m > 0 : \xi_2(m) \neq 0\} & \text{if such } m \text{ exists,} \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{4.5}$$

We shall write $\mathsf{def}^-\big((\eta,k),\xi_2\big) = \mathfrak{h}(x_2 \curlywedge \xi_2) - \mathfrak{h}(o_2 \curlywedge \xi_2)$, the (*negative*) *defect* of (η, k) with respect to ξ_2.

We conclude: write $\partial^+(\mathbb{Z}_q \wr \mathbb{Z})$ and $\partial^-(\mathbb{Z}_q \wr \mathbb{Z})$ for all infinite configurations ξ_1, resp. ξ_2 as above. (Every *finitely supported* configuration appears once in each of the two boundaries!)

Then all non-constant minimal P-harmonic functions are given by

$$\begin{aligned} K_1(\cdot, \xi_1) &= q^{\ \mathsf{def}^+(\cdot,\xi_1)}, \ \xi_1 \in \partial^+(\mathbb{Z}_q \wr \mathbb{Z}), \quad \text{and} \\ K_2(\cdot, \xi_2) &= q^{\ \mathsf{def}^-(\cdot,\xi_2)}, \ \xi_2 \in \partial^-(\mathbb{Z}_q \wr \mathbb{Z}). \end{aligned} \tag{4.6}$$

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5. Switch-walk-switch

We now turn our attention to the random walk (2.4), where at each step, the lamplighter first switches the lamp at his actual position to a random state, then walks, and then switches the lamp at the arrival point to a random state.

As explained in Section 2, this corresponds to simple random walk on the modification of $DL(q,q)$ where in the first tree, additional edges are drawn between every vertex and the siblings of its predecessor, while the second tree remains as it is.

More generally, consider $DL(q,r)$. For $x_1, u_1 \in \mathbb{T}_q$, we introduce the *sibling relation* $x_1 \overset{s}{\sim} u_1 \iff x_1^- = u_1^-$. We extend this relation to $DL(q,r)$ by setting $x_1x_2 \overset{s}{\sim} u_1x_2 \iff x_1 \overset{s}{\sim} u_1$. The new edge set on the same vertex set $\{x_1x_2 \in \mathbb{T}_q \times \mathbb{T}_r : \mathfrak{h}(x_1) + \mathfrak{h}(x_2) = 0\}$ is now given by

$$\{[x_1x_2, y_1y_2] : y_1^- \overset{s}{\sim} x_1 \text{ and } y_2 = x_2^-\}.$$

We write $DL^s(q,r)$ for the resulting graph. Every vertex x_1x_2 with $\mathfrak{h}(x_1) = k$ has q^2 neighbours y_1y_2 with $\mathfrak{h}(y_1) = k+1$ and qr neighbours with $\mathfrak{h}(y_1) = k-1$. Adapted to this structure, we choose $0 < \alpha < 1$ and consider the random walk on $DL^s(q,r)$ with transition matrix $Q = Q_\alpha$ given by

$$q(x_1x_2, y_1y_2) = \begin{cases} \alpha/q^2 & \text{if } y_1^- \overset{s}{\sim} x_1 \text{ and } y_2 = x_2^-, \\ (1-\alpha)/(qr) & \text{if } y_1 \overset{s}{\sim} x_1^- \text{ and } y_2^- = x_2, \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Now note that when $x_1 \overset{s}{\sim} u_1$ then transitions from x_1x_2 and u_1x_2 go to the same neighbours with the same probabilities. Thus, $Qh(x_1x_2) = Qh(u_1x_2)$ whenever $x_1 \overset{s}{\sim} u_1$, and we have the following.

Lemma 5.1. *Every Q_α-harmonic function is constant on each equivalence class of the sibling relation.*

We can construct the *factor graph* of $DL^s(q,r)$ with respect to the sibling relation. We write $[x_1]x_2$ for the equivalence class of x_1x_2, since all its elements have the same second 'coordinate', and $[x_1]$ is the sibling class in the first tree. Then the vertex set of the factor graph is $\{[x_1]x_2 : x_1x_2 \in DL^s(q,r)\}$, and two classes $[x_1]x_2$ and $[y_1]y_2$ are connected by an edge of the factor graph if and only if there is an edge between a pair of representatives in $DL^s(q,r)$. Thus, if $\mathfrak{h}(y_2) = \mathfrak{h}(x_2) - 1$, there is an edge from $[x_1]x_2$ to $[y_1]y_2$ precisely when $[y_1^-] = [x_1]$ and $x_2^- = y_2$. We write π for the natural projection. The next lemma is now immediate.

Lemma 5.2. *The factor graph of $DL^s(q,r)$ with respect to the sibling relation is $DL(q,r)$.*

The transition matrix Q_α is compatible with the factorization, and its image under the projection $\pi : DL^s(q,r) \to DL(q,r)$ is P_α, as defined in (2.5).

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By 'compatible' we mean that $q_\alpha(v_1x_2, [y_1]y_2) = \sum_{w_1 \overset{s}{\sim} y_1} q_\alpha(x_1x_2, w_1y_2)$ is the same for each representative $v_1 \in [x_1]$, and this common value is the transition probability from $[x_1]x_2$ to $[y_1]y_2$ of the projection of Q_α.

Corollary 5.3. *Every Q_α-harmonic function is of the form $h \circ \pi$, where h is a P_α-harmonic function on $DL(q,r)$ and $\pi : DL^s(q,r) \to DL(q,r)$ is the factor map with respect to the sibling relation.*

Example 5.4. Our final task is to retranslate once more to the lamplighter group, by giving a direct description of the minimal harmonic functions for the switch-walk-switch model that does not involve the above factor map. We have $Q = Q_{1/2}$ on $DL^s(q,q)$.

Now, it is clear what the factor map does to a pair $(\eta, k) \in \mathbb{Z}_q \wr \mathbb{Z}$: it 'forgets' (cancels) the value $\eta(k)$, and what remains is the pair $(\eta_{\not k}, k)$, where

$$\eta_{\not k}(n) = \begin{cases} \eta(n-1) & \text{if } n \leqslant k, \\ \eta(n) & \text{if } n > k. \end{cases}$$

Thus, with respect to the computations of Example 4.2, (4.5) remains unchanged, while instead of the positive defect we need

$$\mathsf{def}^{\oplus}((\eta,k),\xi_1) = \mathfrak{h}([x_1] \curlywedge \xi_1) - \mathfrak{h}([o_1] \curlywedge \xi_1), \quad \text{where}$$

$$\mathfrak{h}([x_1] \curlywedge \xi_1) = \begin{cases} \min\{n \leqslant k : \xi_1(n) \neq \eta(n)\} & \text{if such } n \text{ exists}, \\ k & \text{otherwise}, \end{cases} \tag{5.2}$$

$$\mathfrak{h}([o_1] \curlywedge \xi_1) = \begin{cases} \min\{m \leqslant 0 : \xi_1(m) \neq 0\} & \text{if such } m \text{ exists}, \\ 0 & \text{otherwise}. \end{cases}$$

Again, the constant function $\mathbf{1}$ is minimal Q-harmonic. All non-constant minimal Q-harmonic functions are given by

$$\begin{aligned} K_1(\cdot,\xi_1) &= q^{\,\mathsf{def}^{\oplus}(\cdot,\xi_1)}, \ \xi_1 \in \partial^+(\mathbb{Z}_q \wr \mathbb{Z}), \quad \text{and} \\ K_2(\cdot,\xi_2) &= q^{\,\mathsf{def}^{-}(\cdot,\xi_2)}, \ \xi_2 \in \partial^-(\mathbb{Z}_q \wr \mathbb{Z}). \end{aligned} \tag{5.3}$$

6. Final observations and speculations

(1) Even when $r \neq q$, one can interpret $DL(q,r)$ as a 'lamplighter graph' over $\mathbb{Z}$: at each point of $\mathbb{Z}$, there are green lamps with q different states, including 'off', and red lamps with r different states, again including 'off'. The lamplighter walking along $\mathbb{Z}$ has to make sure that when his actual position is $k \in \mathbb{Z}$, then the lamps in $(-\infty, k]$ have to be in one of the green states, and those in $[k+1, +\infty)$ in one of the red states.

(2) Several basic properties of random walks on $DL(q,r)$ that are not necessarily of nearest neighbour type, but invariant under a transitive group of automorphisms of $DL(q,r)$, were studied by Bertacchi [1]. For a large class of random walks of this type, the Poisson boundary was determined by Kaimanovich and Woess [12], as mentioned above.

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(3) As a general principle, the three problems of (i) determining the Poisson boundary (≡ all bounded harmonic functions), (ii) determining all minimal harmonic functions (≡ all positive harmonic functions), and (iii) determining the full Martin compactification (≡ finding the directions of convergence of the Martin kernels) have an increasing degree of difficulty. (As a matter of fact, these three problems get sometimes mixed up even by advanced non-experts.) Thus, the reader should not be *a priori* astonished by the fact that in this paper, we were able to solve problem (ii) for a much smaller class of random walks than those for which (i) was solved in [12].

(4) In particular, to the author's knowledge, the present results provide the first example of a complete solution of problem (ii) on a finitely generated, solvable group. In addition, our group is non-polycyclic.

On the other hand, the situation is much better understood for connected solvable Lie groups, because more structure theory is at hand. For the basic example, namely the *affine group* over $\mathbb{R}$, random walks and harmonic functions have been studied in much detail, see Molchanov [14], Elie [7], or Bougerol and Elie [2]. We recall here that the main result of [2] implies existence of non-constant positive harmonic functions for finite range random walks on finitely generated polycyclic groups with exponential growth, but a complete solution of (ii) is not available for those groups.

As for the affine group over $\mathbb{R}$, the Poisson boundary and the Martin compactification are well understood for random walks on the affine group over the p-adic numbers, see Cartwright, Kaimanovich and Woess [5] and Brofferio [3].

(5) As pointed out by Bertacchi [1], there is a natural geometric compactification of $DL(q,r)$. Namely, this graph is a subgraph of the direct product $\mathbb{T}_q \times \mathbb{T}_r$, for which a natural compactification is $\widehat{\mathbb{T}}_q \times \widehat{\mathbb{T}}_r$. Thus, we define $\widehat{DL}(q,r)$ as the closure of $DL(q,r)$ in $\widehat{\mathbb{T}}_q \times \widehat{\mathbb{T}}_r$, and $\partial DL(q,r) = \widehat{DL}(q,r) \setminus DL(q,r)$. Almost sure convergence of random walks to a boundary-valued random variable is studied in [1]. However, at present, we are far from proving that this compactification is in some sense comparable or identical with the Martin compactification even in the case of simple random walk.

(6) The following seems noteworthy regarding the two classes of examples that we have studied: in the case of drift ($\alpha \neq 1/2$), the minimal harmonic functions are parametrized (continuously on each part) by $\partial^* \mathbb{T}_q \cup \partial^* \mathbb{T}_r$. In the drift-free case ($\alpha = 1/2$), the *additional* minimal harmonic function $\mathbf{1}$ enters the stage. Thus, in some sense, the cone of positive harmonic functions is *bigger* in the drift-free case than in the presence of drift. This contrasts with all examples known so far. (Of course, the constant function $\mathbf{1}$ is a Martin kernel even when $\alpha \neq 1/2$, but then it does not belong to $\mathscr{M}_{\min}$.)

(7) Finally, the reason why our method does not adapt to the 'walk or switch' model (2.3) appears to be that in contrast with the cases that we have solved here, this random walk is *not* invariant under a group of automorphisms of $DL(q,r)$ that acts *transitively* both on $\partial^* \mathbb{T}^1$ and $\partial^* \mathbb{T}^2$.

A next step could be to prove the Decomposition Theorem (Theorem 2.3) for all irreducible random walks with the latter transitivity property.

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Added in proof

In the meantime, Sara Brofferio and the author have succeeded in solving problem(s).

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