A MEASURELESS ONE-DIMENSIONAL SET

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It has been known for some time that there are sets in Euclidean space which are of infinite measure in a certain Hausdorff dimension, say α, and yet which contain no subsets that are of finite positive α measure*. The properties of such a set, say X, could be developed in much the same way as those of sets which are of positive finite α measure, if there existed a Carathéodory outer measure Γ, defined over the subsets of X, which was such that $\infty > \Gamma(X) > 0$, and for any subset Y of X, $\Lambda_\alpha(Y) = 0$ implied $\Gamma(Y) = 0$†. The object of this note is to show that there are sets for which no such outer measure exists. It is shown that a set defined by Sierpinski (7) is one-dimensional and is such that any Carathéodory outer measure defined over its subsets either takes infinite values or is identically zero or is not zero for all subsets that consist of a single point. It has been remarked by Prof. Besicovitch that these properties imply that the set is measurable with respect to any Carathéodory outer measure defined over subsets of the plane, and in fact any subset is measurable with respect to any such outer measure.

Only a few properties of α-dimensional sets, which do not depend upon the set being of finite positive α measure, are known‡. The techniques available for dealing with such sets are those introduced by Besicovitch in (3). The example given here emphasizes the importance of these methods.

By a transfinite induction argument Sierpinski (7) established the existence of a subset of the x-axis, say E_x, such that

(i) E_x intersects any non-dense perfect subset of the x-axis in an at most countable subset§,

(ii) if t is any real number there exist points of E_x, p, q with abscissae x_p, x_q such that $x_p - x_q = t$.

(i) implies that E is a set with property C, i.e. given any infinite sequence of positive numbers $\{l_k\}$ there exists a sequence of intervals $\{I_k\}$ such that I_k is of length l_k and

$$\bigcup_{k=1}^{\infty} I_k \supset E_x.$$

Let $F = E_x \times E_y$ be the plane set which is the Cartesian product of E_x with a similarly defined subset E_y of the y-axis. Let S be the square $\{(x, y) \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$, and let L be the segment $\{(0, y) \mid 0 \leqslant y \leqslant 1\}$. F is a set with the required properties.

We show, first, that F is one-dimensional. By (ii) the projection of F on the line $y = -x$ contains the whole of that line; thus F is of infinite one-dimensional measure. We have now to show that the α measure of F is zero for any α greater than one.

* See (1); for related results see (3) and (4). 'Measure' means Hausdorff measure and α is a positive real number.

† See (5); Λ_α is α-dimensional Hausdorff measure.

‡ See (3, 4, 6).

§ See also (8).

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Actually we show that the h measure of $S \cap F$ is zero for any measure function $h(x)$ that satisfies

$$\lim_{x \to 0+} h(x)/x = 0. \tag{1}$$

It will be seen that the argument given applies for any square whose sides are of unit length in place of S, and thus we shall have h.m. $F = 0$. Now from (1), to any positive number ϵ and any pair of positive integers k, l, we can find an integer $n_{k,l}$ such that

$$n_{k,l} h(\sqrt{2}/n_{k,l}) < \epsilon/2^{k+l} \quad (n_{k,l} > l). \tag{2}$$

For each integer $l\,(l = 1, 2, 3, \ldots)$, let $\{I_{k,l}\}\,(k = 1, 2, 3, \ldots)$ be a sequence of intervals such that the length of $I_{k,l}$ is $1/n_{k,l}$ and all of E_x is contained in $\bigcup_{k=1}^{\infty} I_{k,l}$. The set $S \cap (E_x \times E_y)$ is covered by $\left(\bigcup_{k=1}^{\infty} I_{k,l}\right) \times L$ for each $l = 1, 2, 3, \ldots$. Divide each rectangle $I_{k,l} \times L$ into $n_{k,l}$ closed squares of side-length $1/n_{k,l}\,(k = 1, 2, 3, \ldots)$, and denote the totality of squares so obtained by U_l. Then for each l, U_l covers $S \cap (E_x \times E_y)$ and from (2)

$$\sum_{U_l} h(d) = \sum_{k=1}^{\infty} n_{k,l} h(\sqrt{2}/n_{k,l}) < \epsilon/2^l. \tag{3}$$

Further, by the second inequality of (2), the diameter of each of the squares of U_l is less than $\sqrt{2}/l$. Thus

$$\text{h.m. } S \cap (E_x \times E_y) = 0, \tag{4}$$

and this implies that F is one-dimensional.

Suppose next that there exists a Carathéodory outer measure function Γ, defined over the subsets of F for which $\infty > \Gamma(F) > 0$. Denote the intersection of F with the line $x = t$ by F_t. We shall show that for some t, $\Gamma(F_t) > 0$. Assume that this is not the case, i.e. $\Gamma(F_t) = 0$ for all t. Let X be a subset of E_x and define the set function Θ by

$$\Theta(X) = \Gamma(X \times E_y). \tag{5}$$

$\Theta(X)$ is a Carathéodory outer measure defined over the subsets of E_x that vanishes for sets consisting of a finite or enumerable number of points. Let I_0 be a closed finite interval of the x-axis such that $\Theta(E_x \cap I_0) > 0$. Now, given $\epsilon > 0$, there is a positive number $\delta = \delta(\epsilon)$ such that if I is an interval of length less than or equal to δ, then

$$\Theta(E_x \cap I_0 \cap I) < \epsilon. \tag{6}$$

For if this were not the case we could find a sequence of intervals $I_1, I_2, I_3, \ldots$ whose lengths tend to zero and which converge to a point p and for each of which

$$\Theta(E_x \cap I_0 \cap I_n) \geqslant \epsilon. \tag{7}$$

Let $P(\eta)$ be the interval of the x-axis whose centre is p and whose length is 2η. Then, by the countable subadditivity of Θ,

$$\lim_{\eta \to 0} \Theta(P(\eta) \cap E_x) = 0. \tag{8}$$

Choose η so that $\Theta(P(\eta)) < \epsilon$, and n so that $I_n \subset P(\eta)$, then

$$\epsilon \leqslant \Theta(E_x \cap I_0 \cap I_n) \leqslant \Theta(P(\eta) \cap E_x) < \epsilon, \tag{9}$$

a contradiction which establishes the existence of the positive number $\delta = \delta(\epsilon)$.

Let

$$\delta_k = \delta(\Theta(E_x \cap I_0)/2^{k+1}) \quad (k = 1, 2, \ldots).$$

Let $\{J_k\}$ be a sequence of intervals on the x-axis such that J_k is of length δ_k and $\bigcup_{k=1}^{\infty} J_k \supset E_x$.

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Then

$$\begin{aligned}\Theta(E_x \cap I_0) &\leqslant \sum_{k=1}^{\infty} \Theta(E_x \cap I_0 \cap J_k) \\ &\leqslant \Theta(E_x \cap I_0) \sum_{1}^{\infty} \frac{1}{2^{k+1}} \\ &= \tfrac{1}{2}\Theta(E_x \cap I_0). \end{aligned} \tag{10}$$

This is a contradiction, and thus for some t, $\Gamma(F_t) > 0$. We next argue with Γ defined over the subsets of F_t. Since F_t is a set with property C, the same argument shows that there are points of F_t for which (regarded as subsets of F_t) Γ is positive.

This completes the proof of the statement made earlier, except for the remark about measurability.

To show that $X \subset F$ is measurable with respect to any Carathéodory outer measure Ξ defined over the subsets of the plane, let Z be any planar subset for which $\Xi(Z) < \infty$. There are at most an enumerable sequence of points of Z at which (regarding them as subsets of the plane) Ξ is positive. Let them be $z_1, z_2, z_3, \ldots$, where this sequence may contain none or only a finite number of members. For any subset X of F write

$$\Gamma(X) = \Xi(Z \cap X) - \sum_{z_i \in X} \Xi(z_i). \tag{11}$$

Then $0 \leqslant \Gamma(X) < \infty$ and $\Gamma(X)$ is zero if X is a single point; thus by the above argument $\Gamma(X) = 0$. Denote the set of z_i that belong to X by X_1, then X_1 is an F_σ set and therefore Ξ measurable; thus, by (11),

$$\begin{aligned}\Xi(Z \cap X) + \Xi(Z \cap \mathscr{C}(X)) &= \sum_{z_i \in X} \Xi(z_i) + \Xi(Z \cap \mathscr{C}(X)) \\ &\leqslant \sum_{z_i \in X} \Xi(z_i) + \Xi(Z \cap \mathscr{C}(X_1)) \\ &= \Xi(X_1 \cap Z) + \Xi(\mathscr{C}(X_1) \cap Z) \\ &= \Xi(Z). \end{aligned}$$

Thus X is Ξ measurable, i.e. any subset of F is measurable with respect to any outer function Ξ.

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