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Essentially unbounded chains in compact sets

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We discuss the possible cofinalities of $\subseteq^*$-chains in $\mathcal{P}\mathbb{N}$ which are $\subseteq^*$-unbounded in their topological closures.

1. *The problem.* Throughout this paper, the power set $\mathcal{P}\mathbb{N}$ of the set $\mathbb{N}$ of natural numbers will be given its usual compact metrizable topology corresponding to the product topology of $\{0,1\}^{\mathbb{N}}$. We say that $a \subseteq^* b$ if $a \setminus b$ is finite.

We are interested in questions of the following type. Suppose that we are given a closed (= compact) set $K \subseteq \mathcal{P}\mathbb{N}$ and a family $\langle a_\xi \rangle_{\xi<\kappa}$ in K, where κ is a cardinal, such that $a_\xi \subseteq^* a_\eta$ whenever $\xi \leqslant \eta < \kappa$. Under what circumstances can we conclude that there is an $a \in K$ such that $a_\xi \subseteq^* a$ for every $\xi < \kappa$?

There are easy examples to show that we cannot always find such an a (Theorem 4(a) below). Our aim here is to present some less obvious examples, and to describe some circumstances in which (contrary to the general trend that will appear) such an upper bound will exist.

2. *Notation.* (a) Let κ be an ordinal. We say that a family $\langle a_\xi \rangle_{\xi<\kappa}$ in $\mathcal{P}\mathbb{N}$ is $\subseteq^*$-*increasing* if $a_\xi \subseteq^* a_\eta$ whenever $\xi \leqslant \eta < \kappa$, and *strictly* $\subseteq^*$-*increasing* if moreover $a_\eta \not\subseteq^* a_\xi$ whenever $\xi < \eta < \kappa$. Nearly always, κ will be an infinite regular cardinal; in this case, if $\langle a_\xi \rangle_{\xi<\kappa}$ is $\subseteq^*$-increasing, either there is a $\zeta < \kappa$ such that $a_\xi \subseteq^* a_\zeta$ for every $\xi < \kappa$, or there is a strictly increasing family $\langle \gamma(\xi) \rangle_{\xi<\kappa}$ in κ such that $\langle a_{\gamma(\xi)} \rangle_{\xi<\kappa}$ is strictly $\subseteq^*$-increasing.

We shall occasionally wish to mention the corresponding relation $\leqslant^*$ on $\mathbb{N}^{\mathbb{N}}$, defined by saying that $f \leqslant^* g$ if $\{n : f(n) > g(n)\}$ is finite; and we speak of $\leqslant^*$-increasing and strictly $\leqslant^*$-increasing families in $\mathbb{N}^{\mathbb{N}}$.

(b) It will be helpful to have names for some cardinals. Recall that $\mathfrak{b}$ is the least cardinal of any subset of $\mathbb{N}^{\mathbb{N}}$ which is not $\leqslant^*$-bounded above in $\mathbb{N}^{\mathbb{N}}$. (See [3].) We shall write $\mathfrak{b}^*$ for the supremum of those cardinals κ for which there is a strictly $\subseteq^*$-increasing family $\langle a_\xi \rangle_{\xi<\kappa}$ in $\mathcal{P}\mathbb{N}$; by the arguments of Proposition 15 below, $\mathfrak{b}^*$ is also the supremum of those cardinals κ for which there is a strictly $\leqslant^*$-increasing family of length κ in $\mathbb{N}^{\mathbb{N}}$. Accordingly we have $\mathfrak{b} \leqslant \mathfrak{b}^* \leqslant \mathfrak{c}$. We note that (if there can be an inaccessible cardinal at all) it is possible that there should be no strictly $\subseteq^*$-increasing family in $\mathcal{P}\mathbb{N}$ of length $\mathfrak{b}^*$, but that in this case $\mathfrak{b}^*$ must be weakly inaccessible.

(c) Now we come to the main definitions of this paper. Let $K \subseteq \mathcal{P}\mathbb{N}$ be a closed set.

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Write S_K for the set of those infinite regular cardinals κ for which there is a $\subseteq^*$-increasing family $\langle a_\xi \rangle_{\xi<\kappa}$ in K such that there is no $a \in K$ with $a_\xi \subseteq^* a$ for every $\xi < \kappa$. Note that we get the same set S_K if we consider only strictly $\subseteq^*$-increasing families. Write S for $\bigcup\{S_K : K \subseteq \mathcal{P}\mathbb{N} \text{ is closed}\}$. Most of our work will be concerned with the problem of determining S. We see at once that $\kappa \leqslant \mathfrak{b}^*$ for every $\kappa \in S$. Also it is clear that for many K (e.g. $K = \mathcal{P}\mathbb{N}$ itself) $S_K = \varnothing$. It is worth noting straight away that we need consider only sets K which are 'closed under subsets', i.e., such that if $a \subseteq b \in K$ then $a \in K$. For if K is any closed, therefore compact, subset of $\mathcal{P}\mathbb{N}$, then

$$\tilde{K} = \{a : \exists\, b \in K \text{ such that } a \subseteq b\} = \{b \cap c : b \in K, c \in \mathcal{P}\mathbb{N}\}$$

is compact, therefore closed, and closed under subsets, and $S_K \subseteq S_{\tilde{K}}$, because any family in K with no $\subseteq^*$-upper bound in K has no $\subseteq^*$-upper bound in $\tilde{K}$. Thus S is just

$$\bigcup\{S_K : K \subseteq \mathcal{P}\mathbb{N} \text{ is topologically closed and closed under subsets}\}.$$

(*d*) The next cardinal for which we shall need a name is $\operatorname{add}(\mathcal{N})$, defined as follows. Let us write W for the set of those $w \subseteq \mathbb{N} \times \mathbb{N}$ such that the vertical section

$$w[\{i\}] = \{j : (i,j) \in w\}$$

has at most i members for every $i \in \mathbb{N}$. Write

$$\operatorname{add}(\mathcal{N}) = \min\{\#(F) : F \subseteq \mathbb{N}^{\mathbb{N}}, \forall\, w \in W\, \exists f \in F \text{ such that } f \nsubseteq^* w\},$$

where we identify a function f with its graph, so that $f \subseteq^* w$ if

$$\{i : i \in \operatorname{dom}(f), (i, f(i)) \notin w\}$$

is finite. (We use the name 'add $(\mathcal{N})$' because this is also the additivity of the Lebesgue null ideal; see [1], [5] or [6]; but we shall not need this fact essentially here.) It is easy to see from the definition above that $\omega_1 \leqslant \operatorname{add}(\mathcal{N}) \leqslant \mathfrak{b}$, and not hard to check that $\operatorname{add}(\mathcal{N})$ is regular (the necessary idea is in Theorem 4(*c*)).

(*e*) Recall that if κ and λ are regular infinite cardinals, then an *unfilled* (κ, λ^*)*-gap* in $\mathcal{P}\mathbb{N}$ (modulo the ideal $[\mathbb{N}]^{<\omega}$ of finite sets) is a pair $(\langle a_\xi \rangle_{\xi<\kappa}, \langle b_\eta \rangle_{\eta<\lambda})$ such that $\langle a_\xi \rangle_{\xi<\kappa}$ is $\subseteq^*$-increasing, $\langle b_\eta \rangle_{\eta<\lambda}$ is $\subseteq^*$-decreasing, and $a_\xi \subseteq^* b_\eta$ whenever $\xi < \kappa$ and $\eta < \lambda$, but there is no $c \subseteq \mathbb{N}$ for which $a_\xi \subseteq^* c$ for every $\xi < \kappa$ and $c \subseteq^* b_\eta$ for every $\eta < \lambda$. (Some authors require that the families $\langle a_\xi \rangle_{\xi<\kappa}, \langle b_\eta \rangle_{\eta<\lambda}$ should be strictly $\subseteq^*$-increasing and -decreasing; but as we are concerned only with regular κ and λ, it makes no difference here.) Observe that if there is an unfilled (κ, λ^*)-gap then $\max(\kappa, \lambda) \leqslant \mathfrak{b}^*$.

3. PROPOSITION. *Let L be the closed set*

$$\{a : a \subseteq \mathbb{N}, \forall i \in \mathbb{N} \quad \text{either} \quad 2i \notin a \quad \text{or} \quad 2i+1 \notin a\}$$

in $\mathcal{P}\mathbb{N}$. Then S_L is just the set of those infinite regular cardinals κ such that there is an unfilled (κ, κ^*)-gap in $\mathcal{P}\mathbb{N}$ modulo the finite sets.

Proof. (*a*) Suppose that $\kappa \in S_L$. Let $\langle a_\xi \rangle_{\xi<\kappa}$ be a $\subseteq^*$-increasing family in L with no $\subseteq^*$-upper bound in L. Set

$$b_\xi = \{i : 2i \in a_\xi\}, \quad c_\xi = \{i : 2i+1 \notin a_\xi\}$$

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for $\xi < \kappa$. Then $b_\xi \subseteq {}^* b_\eta \subseteq c_\eta \subseteq {}^* c_\xi$ whenever $\xi \leqslant \eta < \kappa$. If $d \subseteq \mathbb{N}$ set

$$a = \{2i : i \in d\} \cup \{2i+1 : i \in \mathbb{N} \setminus d\} \in L.$$

Then there is a $\xi < \kappa$ such that $a_\xi \not\subseteq {}^* a$. Now

$$\{i : 2i \in a_\xi \setminus a\} = b_\xi \setminus d, \quad \{i : 2i+1 \in a_\xi \setminus a\} = d \setminus c_\xi,$$

so one of $b_\xi \setminus d$, $d \setminus c_\xi$ is infinite. Thus $(\langle b_\xi \rangle_{\xi<\kappa}, \langle c_\xi \rangle_{\xi<\kappa})$ is an unfilled (κ, κ^*)-gap.

(*b*) Suppose that $(\langle b_\xi \rangle_{\xi<\kappa}, \langle c_\xi \rangle_{\xi<\kappa})$ is an unfilled (κ, κ^*)-gap. Set

$$a_\xi = \{2i : i \in b_\xi\} \cup \{2i+1 : i \in \mathbb{N} \setminus (b_\xi \cup c_\xi)\} \in L$$

for each $\xi < \kappa$. Then $a_\xi \subseteq {}^* a_\eta$ whenever $\xi \leqslant \eta < \kappa$. Take any $a \in L$. Set $d = \{i : 2i \in a\}$. Then there is a $\xi < \kappa$ such that $(b_\xi \setminus d) \cup (d \setminus c_\xi)$ is infinite, in which case $a_\xi \not\subseteq {}^* a$. Thus $\langle a_\xi \rangle_{\xi<\kappa}$ witnesses that $\kappa \in S_L$.

4. The following is our principal result in ZFC.

THEOREM. *ω, ω_1 and $\mathrm{add}(\mathcal{N})$ all belong to S.*

Proof. (*a*) Set $a_n = \mathbb{N} \setminus (\{2^n j : j \in \mathbb{N}\} \cup n)$ for each $n \in \mathbb{N}$. (We identify a natural number n with the set of its predecessors.) Set $K = \{a_n : n \in \mathbb{N}\} \cup \{\varnothing\}$. Then $\langle a_n \rangle_{n \in \mathbb{N}}$ witnesses that $\omega \in S_K \subseteq S$.

(*b*) A famous theorem of Hausdorff ([**7**]; see [**4**], 21L, or [**3**], 4·3) shows that there is an unfilled (ω_1, ω_1^*)-gap; so Proposition 3 above shows that $\omega_1 \in S$.

(*c*) Observe that W, as defined in 2(*d*), is actually a closed subset of $\mathcal{P}(\mathbb{N} \times \mathbb{N})$; since $\mathcal{P}(\mathbb{N} \times \mathbb{N})$ can obviously be transferred to $\mathcal{P}\mathbb{N}$ in a way preserving both its topological and $\subseteq$ *-structures, we may speak of S_W and we have $S_W \subseteq S$. We therefore propose to show that $\mathrm{add}(\mathcal{N}) \in S_W$. Let $\langle f_\xi \rangle_{\xi < \mathrm{add}(\mathcal{N})}$ be a family in $\mathbb{N}^\mathbb{N}$ such that for every $w \in W$ there is a $\xi < \mathrm{add}(\mathcal{N})$ such that $f_\xi \not\subseteq {}^* w$. Set

$$Z = \left\{ w : w \subseteq \mathbb{N} \times \mathbb{N}, \lim_{i \to \infty} \frac{1}{i} \# (w[\{i\}]) = 0 \right\}.$$

We seek to construct a $\subseteq$ *-increasing family $\langle w_\xi \rangle_{\xi < \mathrm{add}(\mathcal{N})}$ in $Z \cap W$ such that $f_\xi \subseteq {}^* w_\xi$ for every $\xi < \mathrm{add}(\mathcal{N})$. This family will evidently witness that $\mathrm{add}(\mathcal{N}) \in S_W \subseteq S$.

To determine w_ξ, where $\xi < \mathrm{add}(\mathcal{N})$, we suppose that $\langle w_\zeta \rangle_{\zeta<\xi}$ has already been constructed. For each $\zeta < \xi$, let $h_\zeta \in \mathbb{N}^\mathbb{N}$ be such that $\# (w_\zeta[\{i\}]) \leqslant 2^{-n} i$ whenever $n \in \mathbb{N}$ and $i \geqslant h_\zeta(n)$. Because

$$\# (\xi) < \mathrm{add}(\mathcal{N}) \leqslant \mathfrak{b},$$

there is an $h \in \mathbb{N}^\mathbb{N}$ such that $h_\zeta \leqslant {}^* h$ for every $\zeta < \xi$. We may take h to be strictly increasing. For $\zeta < \xi$, let $m_\zeta \in \mathbb{N}$ be such that $h_\zeta(i) \leqslant h(i)$ for $i \geqslant m_\zeta$, and define $F_\zeta : \mathbb{N} \to [\mathbb{N} \times \mathbb{N}]^{<\omega}$ by setting

$$F_\zeta(n) = \begin{cases} w_\zeta \cap ((h(n+1) \setminus h(n)) \times \mathbb{N}) & \text{if} \quad n \geqslant m_\zeta, \\ \varnothing & \text{if} \quad n < m_\zeta. \end{cases}$$

Because $[\mathbb{N} \times \mathbb{N}]^{<\omega}$ is countable and $\# (\xi) < \mathrm{add}(\mathcal{N})$, there is a set $\Phi \subseteq \mathbb{N} \times [\mathbb{N} \times \mathbb{N}]^{<\omega}$ such that $F_\zeta \subseteq {}^* \Phi$ for every $\zeta < \xi$ and $\# (\{I : (n, I) \in \Phi\}) \leqslant n$ for

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every $n \in \mathbb{N}$. We may of course suppose that whenever $(n, I) \in \Phi$ there is some $\zeta < \xi$ such that $F_\zeta(n) = I$, so that

$$I \subseteq (h(n+1) \setminus h(n)) \times \mathbb{N} \quad \text{and} \quad \#(I[\{i\}]) \leqslant 2^{-n} i$$

for each $$i \in h(n+1) \setminus h(n).$$

In this case, if we set

$$w = \bigcup \{I : (n, I) \in \Phi\} \subseteq \mathbb{N} \times \mathbb{N},$$

then $\#(w[\{i\}]) \leqslant 2^{-n} ni$ whenever $h(n) \leqslant i < h(n+1)$, so that $w \in Z$. Also, if $\zeta < \xi$, there is an $m \geqslant m_\zeta$ such that $(n, F_\zeta(n)) \in \Phi$ for every $n \geqslant m$; in which case $w_\zeta \subseteq w \cup (h(m) \times \mathbb{N})$, so that $w_\zeta \subseteq {}^* w$. Now consider $w' = w \cup f_\xi \in Z$. There is a $k \in \mathbb{N}$ such that $w' \setminus (k \times \mathbb{N}) \in W$; take $w_\xi = w' \setminus (k \times \mathbb{N})$, and see that this serves to continue the induction.

Thus our construction is successful, and $\operatorname{add}(\mathcal{N}) \in S$.

5. Theorem 4 exhausts the named cardinals which we know always belong to S. Under the continuum hypothesis, of course, there is no more to be said, and $4(c)$ is already redundant. So most of the rest of this paper will concern various possibilities consistent with ZFC.

THEOREM. *If Martin's Axiom is true,* $S = \{\omega, \omega_1, \mathfrak{c}\}$.

Proof. Note first that MA implies that $\operatorname{add}(\mathcal{N}) = \mathfrak{c}$. (Using the characterization of $\operatorname{add}(\mathcal{N})$ as the additivity of Lebesgue measure, this is a consequence of the theorem of Martin and Solovay ([4], 32F); but there is also a straightforward direct proof from the definition in $2(d)$ above.) We therefore have only to show that if MA is true and $\omega_1 < \kappa < \mathfrak{c}$ then $\kappa \notin S$.

We may assume that κ is regular, by the definition of S. Suppose that $K \subseteq \mathcal{P}\mathbb{N}$ is topologically closed and closed under subsets and that $\langle a_\xi \rangle_{\xi < \kappa}$ is a $\subseteq^*$-increasing family in K. For $b \subseteq \mathbb{N}$ set

$$A(b) = \{\xi : \xi < \kappa, b \subseteq a_\xi\} \subseteq \kappa.$$

Set $$P = \{p : p \subseteq \mathbb{N}, \sup A(p) = \kappa\}.$$

We claim that the partially-ordered set $(P, \subseteq)$ is upwards-ccc. For let $\langle p_\xi \rangle_{\xi < \omega_1}$ be any family in P. For each $\xi < \omega_1$ there is a $\zeta(\xi) < \kappa$ such that $p_\xi \subseteq a_{\zeta(\xi)}$. Because $\operatorname{cf}(\kappa) = \kappa > \omega_1$, $\theta = \sup_{\xi < \omega_1} \zeta(\xi) < \kappa$, and $p_\xi \subseteq {}^* a_\theta$ for every $\xi < \omega_1$. For $\xi < \omega_1$, $n \in \mathbb{N}$ set

$$B_{\xi n} = \{\eta : \eta \in A(p_\xi), \quad a_\theta \setminus n \subseteq a_\eta\}.$$

Then $\bigcup_{n \in \mathbb{N}} B_{\xi n} \supseteq A(p_\xi) \setminus \theta$ is cofinal with κ; as $\operatorname{cf}(\kappa) > \omega$, $\sup B_{\xi n} = \kappa$ for all n large enough. Let $m_\xi \in \mathbb{N}$ be such that

$$p_\xi \subseteq a_\theta \cup m_\xi, \quad \sup B_{\xi, m_\xi} = \kappa;$$

set $I_\xi = p_\xi \cap m_\xi \in [\mathbb{N}]^{<\omega}$.

Now there must be distinct $\xi, \xi' < \omega_1$ such that $m_\xi = m_{\xi'} = m$, $I_\xi = I_{\xi'} = I$ say. Consider $p = p_\xi \cup p_{\xi'}$. If $\eta \in B_{\xi m}$ then of course $p_\xi \subseteq a_\eta$. But also

$$p_{\xi'} = (p_{\xi'} \cap m) \cup (p_{\xi'} \setminus m) \subseteq I \cup (a_\theta \setminus m) \subseteq a_\eta.$$

So $B_{\xi m} \subseteq A(p)$ and $\sup A(p) = \kappa$ and $p \in P$. Thus $\langle p_\xi \rangle_{\xi < \omega_1}$ is not an up-antichain in P and P is upwards-ccc.

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Now consider, for $\xi < \kappa$,

$$Q_\xi = \{p : p \in P, a_\xi \subseteq {}^* p\}.$$

This is cofinal with P. For if $p \in P$, then

$$A(p) \backslash \xi \subseteq \bigcup_{n \in \mathbb{N}} \{\eta : p \subseteq a_\eta, \quad a_\xi \subseteq a_\eta \cup n\},$$

so there is an $n \in \mathbb{N}$ such that

$$A(p \cup (a_\xi \backslash n)) = \{\eta : p \subseteq a_\eta, \quad a_\xi \subseteq a_\eta \cup n\}$$

is cofinal with κ and $p \subseteq p \cup (a_\xi \backslash n) \in Q_\xi$.

Accordingly (because we are assuming that $\kappa < \mathfrak{c}$ and that MA is true) there is an upwards-directed $R \subseteq P$ meeting every Q_ξ. Now $R \subseteq K$ (because K is closed under subsets) so $a = \bigcup R \in K$ (because K is compact) and $a_\xi \subseteq {}^* a$ (because $R \cap Q_\xi \neq \emptyset$) for every $\xi < \kappa$. As K, $\langle a_\xi \rangle_{\xi < \kappa}$ are arbitrary, $\kappa \notin S$, as claimed.

Remark. Note that the partially ordered set P of the proof above is much better than just 'ccc'. In fact it has the property that every subset of P of cardinal less than κ can be expressed as the union of a sequence of centred sets. In particular, ω_1 is a precaliber of P. So, in the notation of [**4**], §B1, our argument shows that if $\omega_1 < \kappa < \mathfrak{m}_{\mathrm{pc}(\omega_1)}$, then $\kappa \notin S$.

6. Apart from add $(\mathcal{N})$, a great many other cardinals have been described which lie between ω_1 and $\mathfrak{c}$ and are therefore candidates for membership of S. Most of them are rejected in the context of Proposition 8 below. Our argument follows a well-travelled path, but as it has not been fully described in print we give details of the principal turning points. We begin with a well-known lemma which may be proved with the techniques of [**9**], §VIII·5. (We shall use only the case $\kappa = \omega_1$.)

LEMMA. *Let* $(\langle \mathbb{P}_\xi \rangle_{\xi \leqslant \alpha}, \langle \pi_\xi \rangle_{\xi < \alpha})$ *be a finite-support iteration of* p.o. *sets as described in* [**9**], §*VIII*·5. *Suppose that* κ *is a regular uncountable cardinal and that*

$$\Vdash_{\mathbb{P}_\xi} \kappa \text{ is a precaliber of } \pi_\xi$$

for each $\xi < \alpha$. Then κ *is a precaliber of* $\mathbb{P}_\alpha$.

7. LEMMA. *Let* κ *be a regular uncountable cardinal and* $\langle a_\xi \rangle_{\xi < \kappa}$ *a family in* $\mathcal{P}\mathbb{N}$. *Let* $C \subseteq \mathcal{P}\mathbb{N}$ *be any set, and suppose that there is a* p.o. *set* $\mathbb{P}$ *such that* κ *is a precaliber of* $\mathbb{P}$ *and*

$$\Vdash_{\mathbb{P}} \exists a \in \bar{C} \quad \text{such that} \quad a_\xi \subseteq {}^* a \; \forall \xi < \kappa.$$

Then there is really an $a \in \bar{C}$ *such that* $a_\xi \subseteq {}^* a$ *for cofinally many* $\xi < \kappa$.

Proof. Let τ be a $\mathbb{P}$-name for a subset of $\mathbb{N}$ such that

$$\Vdash_{\mathbb{P}} \tau \in \bar{C}$$

and, for every $\xi < \kappa$,

$$\Vdash_{\mathbb{P}} \exists n \in \mathbb{N}, \quad a_\xi \backslash n \subseteq \tau.$$

Then there is a family $\langle (p_\xi, n_\xi) \rangle_{\xi < \kappa}$ in $\mathbb{P} \times \mathbb{N}$ such that

$$p_\xi \Vdash_{\mathbb{P}} a_\xi \backslash n_\xi \subseteq \tau$$

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for every $\xi < \kappa$. Because κ is a precaliber of $\mathbb{P}$ there is a set $A \subseteq \kappa$, cofinal with κ, such that $\{p_\xi : \xi \in A\}$ is centred in $\mathbb{P}$. Now if $I \in [A]^{<\omega}$, $m \in \mathbb{N}$ there is a $p \in \mathbb{P}$ such that $p \leqslant p_\xi$ for every $\xi \in I$, so that

$$p \Vdash_{\mathbb{P}} a_\xi \backslash n_\xi \subseteq \tau \in \bar{C} \forall \xi \in I;$$

consequently

$$p \Vdash_{\mathbb{P}} \exists c \in C, \quad (a_\xi \backslash n_\xi) \cap m \subseteq c \forall \xi \in I.$$

But this last statement is absolute, so is absolutely true, and we have, for every finite set $I \subseteq A$ and for every $m \in \mathbb{N}$, a $c \in C$ such that $\bigcup_{\xi \in I} (a_\xi \backslash n_\xi) \cap m \subseteq c$ for every $\xi \in I$. A compactness argument now shows that there is an $a \in \bar{C}$ such that

$$(a_\xi \backslash n_\xi) \cap m \subseteq a \forall \xi \in A, m \in \mathbb{N},$$

so that $a_\xi \subseteq {}^* a$ for every $\xi \in A$, as required.

8. PROPOSITION. *It is relatively consistent with* ZFC *to suppose that* $\mathfrak{p} = \mathfrak{c} = \omega_2$ *and that* $S = \{\omega, \omega_1\}$.

Proof. Recall that $\mathfrak{p} = \mathfrak{m}_{\sigma\text{-centred}}$, so that '$\mathfrak{p} = \mathfrak{c}$' is $P(\mathfrak{c})$ or MAS or MA (σ-centred); see [**4**, **3**, **14**]. We use a standard construction. Start by assuming the continuum hypothesis. Let $(\langle \mathbb{P}_\xi \rangle_{\xi \leqslant \omega_2}, \langle \pi_\xi \rangle_{\xi < \omega_2})$ be a finite-support iteration of p.o. sets such that, writing $\mathbb{P} = \mathbb{P}_{\omega_2}$,

$$\Vdash_{\mathbb{P}} \mathfrak{p} = \mathfrak{c} = \omega_2,$$

exactly as in the proof of the relative consistency of $\mathfrak{m} = \mathfrak{c} = \omega_2$ in [**9**], §VIII·6, *except* that we now require

$$\Vdash_{\mathbb{P}_\xi} \pi_\xi \text{ is } \sigma\text{-centred} \quad \text{and} \quad \# (\pi_\xi) = \omega_1$$

in place of

$$\Vdash_{\mathbb{P}_\xi} \pi_\xi \quad \text{is ccc.}$$

(Use the same book-keeping devices as in [**9**] to ensure that every $\mathbb{P}$-name for a σ-centred partially ordered set of cardinal ω_1 is dealt with adequately in the process of the iteration.) This time, however, we see from Lemma 6 that ω_1 will be a precaliber of $\mathbb{P}$; indeed, if we take any $\zeta < \omega_2$, we can think of $\mathbb{P}$ as a simple iteration $\mathbb{P}_\zeta * \pi'_\zeta$ where

$$\Vdash_{\mathbb{P}_\zeta} \omega_1 \quad \text{is a precaliber of} \quad \pi'_\zeta.$$

Note that

$$\Vdash_{\mathbb{P}_\zeta} \mathfrak{c} = \omega_1$$

for every $\zeta < \omega_2$.

(*b*) We seek now to show that

$$\Vdash_{\mathbb{P}} S = \{\omega, \omega_1\};$$

in view of Theorem 4, this amounts to saying that

$$\Vdash_{\mathbb{P}} \omega_2 \notin S.$$

So suppose we have $\mathbb{P}$-names for a closed set $K \subseteq \mathscr{P}\mathbb{N}$ and a $\subseteq {}^*$-increasing family $\langle a_\xi \rangle_{\xi < \omega_2}$ in K. In $V^{\mathbb{P}}$, take a countable dense subset C of K. Because $\mathbb{P}$ is ccc, there is a $\zeta_0 < \omega_2$ such that there is a $\mathbb{P}_{\zeta_0}$-name for C. Also, for each $\xi < \omega_2$, there is a $\gamma(\xi) < \omega_2$ such that there is a $\mathbb{P}_{\gamma(\xi)}$-name for a_ξ. Now the set

$$A = \{\zeta : \zeta_0 \leqslant \zeta < \omega_2, \quad \gamma(\xi) < \zeta \forall \xi < \zeta, \operatorname{cf}(\zeta) = \omega_1\}$$

is a stationary set in ω_2.

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(*c*) We claim that if $\zeta \in A$ there is a $\mathbb{P}_\zeta$-name for an $a'_\zeta \subseteq \mathbb{N}$ such that

$$\Vdash_{\mathbb{P}_\zeta} a'_\zeta \in \bar{C} \;\&\; a_\xi \subseteq^* a'_\zeta \forall\, \xi < \zeta.$$

For express $\mathbb{P}$ as $\mathbb{P}_\zeta * \pi'_\zeta$, as remarked above. Working in $V^{\mathbb{P}_\zeta}$, we know that there is a π'_ζ-name for a_ζ and that

$$\Vdash_{\pi'_\zeta} a_\zeta \in \bar{C} \;\&\; a_\xi \subseteq^* a_\zeta \forall\, \xi < \zeta.$$

Applying Lemma 7 within $V^{\mathbb{P}_\zeta}$ to a cofinal subfamily of $\langle a_\xi \rangle_{\xi<\zeta}$ of order type ω_1, we see that already in $V^{\mathbb{P}_\zeta}$ there is a $\subseteq^*$-bound in $\bar{C}$ for a cofinal subfamily of $\langle a_\xi \rangle_{\xi<\zeta}$; that is to say, there is a $\mathbb{P}_\zeta$-name for an $a'_\zeta \subseteq \mathbb{N}$ such that

$$\Vdash_{\mathbb{P}_\zeta} a'_\zeta \in \bar{C} \;\&\; \{\xi : a_\xi \subseteq^* a'_\zeta\} \quad \text{is cofinal with} \quad \zeta.$$

But because (still in $V^{\mathbb{P}_\zeta}$) the family $\langle a_\xi \rangle_{\xi<\zeta}$ is $\subseteq^*$-increasing, this a'_ζ will do.

(*d*) Now we insisted that $\mathrm{cf}(\zeta) = \omega_1$ for every $\zeta \in A$, so in fact there is for each $\zeta \in A$ a $\theta(\zeta) < \zeta$ such that a'_ζ has a $\mathbb{P}_{\theta(\zeta)}$-name. By the pressing-down lemma, there is a $\delta < \omega_2$ such that

$$A' = \{\zeta : \zeta \in A, \quad \theta(\zeta) = \delta\}$$

is cofinal with ω_2. Because

$$\Vdash_{\mathbb{P}_\delta} \mathfrak{c} = \omega_1,$$

there is a $\mathbb{P}_\delta$-name for a set $a \subseteq \mathbb{N}$ such that

$$A'' = \{\zeta : \zeta \in A', \Vdash_{\mathbb{P}_\delta} a'_\zeta = a\}$$

is cofinal with ω_2. Now of course

$$\Vdash_{\mathbb{P}} a \in K.$$

Also, if $\xi < \omega_2$, there is a $\zeta \in A''$ such that $\xi < \zeta$, in which case

$$\Vdash_{\mathbb{P}_\zeta} a_\xi \subseteq^* a'_\zeta = a,$$

so that

$$\Vdash_{\mathbb{P}} a_\xi \subseteq^* a.$$

As ξ is arbitrary, we have found, in $V^{\mathbb{P}}$, a $\subseteq^*$-upper bound for $\langle a_\xi \rangle_{\xi<\omega_2}$ in K.

(*e*) Because K and $\langle a_\xi \rangle_{\xi<\omega_2}$ are arbitrary,

$$\Vdash_{\mathbb{P}} \omega_2 \notin S,$$

as required.

9. *Remark.* It is clear that the arguments above can be widely generalized into propositions of the form 'forcing with $\mathbb{P}$ cannot add κ to S'.

Of course there are simpler models of '$\mathfrak{c} \notin S$' than those formed by the method above. The point of the above construction is that we get $\mathfrak{b}^* = \omega_2 \notin S$. We observe also that $\mathrm{add}(\mathscr{M})$, the least cardinal of any family of meagre subsets of $\mathbb{R}$ with non-meagre union, need not belong to S (because $\mathfrak{p} \leqslant \mathrm{add}(\mathscr{M}) \leqslant \mathfrak{c}$).

We now turn to a type of cardinal which cannot by its nature belong to S. The following argument is derived from [**8**].

10. Proposition. *Let λ be a real-valued-measurable cardinal. Then $\mathfrak{b}^* < \lambda$.*

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Proof. Let ν be a λ-additive probability, zero on singletons, with domain $\mathscr{P}\lambda$. We proceed in two stages.

(*a*) First let us suppose, if possible, that there is a strictly $\subseteq^*$-increasing family $\langle a_\xi \rangle_{\xi<\lambda}$ in $\mathscr{P}\mathbb{N}$. Consider the sets

$$A_n = \{\xi : \xi < \lambda, n \in a_\xi\} \subseteq \lambda,$$
$$C = \{(\xi,\eta) : \xi \leqslant \eta < \lambda\} \subseteq \lambda \times \lambda.$$

Then $$C = \{(\xi,\eta) : a_\xi \subseteq^* a_\eta\} = \bigcup_{n\in\mathbb{N}} \bigcap_{i\geqslant n} (\lambda \times A_i) \cup ((\lambda \setminus A_i) \times \lambda)$$

is measurable for the product measure $\nu \times \nu$ on $\lambda \times \lambda$. So, by Fubini's theorem,

$$\int \nu C[\{\xi\}]\, \nu(d\xi) = \int \nu C^{-1}[\{\eta\}]\, \nu(d\eta).$$

But $$\nu C[\{\xi\}] = \nu(\lambda \setminus \xi) = 1, \quad \nu C^{-1}[\{\eta\}] = \nu(\eta+1) = 0$$

for all $\xi, \eta < \lambda$, so this is impossible.

(*b*) Thus there can be no strictly $\subseteq^*$-increasing family in $\mathscr{P}\mathbb{N}$ of length λ, and $\mathfrak{b}^* \leqslant \lambda$. Now suppose, if possible, that $\mathfrak{b}^* = \lambda$. In this case we can find, for each ordinal $\theta < \lambda$, a strictly $\subseteq^*$-increasing family $\langle a_{\theta\xi} \rangle_{\xi<\theta}$ in $\mathscr{P}\mathbb{N}$. As before, let ν be a λ-additive probability with domain $\mathscr{P}\lambda$ which is zero on singletons. Let $\mathbb{P}$ be the p.o. set $\{p : p \subseteq \lambda, \nu p > 0\}$, ordered by $\subseteq$. Then $\mathbb{P}$ is ccc. We have a $\mathbb{P}$-name for a family $\langle b_\xi \rangle_{\xi<\lambda}$ in $\mathscr{P}\mathbb{N}$, writing

$$\{\theta : \xi < \theta, \quad n \in a_{\theta\xi}\} \Vdash_{\mathbb{P}} n \in b_\xi,$$
$$\{\theta : \xi < \theta, \quad n \notin a_{\theta\xi}\} \Vdash_{\mathbb{P}} n \notin b_\xi.$$

If $\xi \leqslant \eta < \lambda$ then

$$\nu\{\theta : \eta < \theta, \exists n \in \mathbb{N}, \quad a_{\theta\xi} \subseteq a_{\theta\eta} \cup n\} = 1,$$

so $$\{p : \exists n \in \mathbb{N}, \quad p \Vdash_{\mathbb{P}} b_\xi \subseteq b_\eta \cup n\}$$

is dense in $\mathbb{P}$, and $$\Vdash_{\mathbb{P}} b_\xi \subseteq^* b_\eta.$$

Also, by Solovay's theorem ([**13**], theorem 7),

$$\Vdash_{\mathbb{P}} \lambda \text{ is real-valued-measurable.}$$

So by (*a*) above, argued in $V^{\mathbb{P}}$,

$$\Vdash_{\mathbb{P}} \exists \xi < \lambda, b_{\xi+1} \subseteq^* b_\xi.$$

Accordingly there must be $p \in \mathbb{P}$, $\xi < \lambda$ and $n \in \mathbb{N}$ such that

$$p \Vdash_{\mathbb{P}} b_{\xi+1} \subseteq b_\xi \cup n.$$

But since $a_{\theta,\xi+1} \not\subseteq^* a_{\theta\xi}$ for every $\theta > \xi+1$,

$$\nu\{\theta : \theta > \xi+1, \exists i \in a_{\theta,\xi+1} \setminus (a_{\theta\xi} \cup n)\} = 1,$$

and there is an $i \in \mathbb{N} \setminus n$ such that $\nu q > 0$, where

$$q = \{\theta : \theta \in p, i \in a_{\theta,\xi+1} \setminus a_{\theta\xi}\}.$$

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But now $q \leqslant p$ and

$$q \Vdash_{\mathbb{P}} i \in b_{\xi+1} \backslash (b_\xi \cup n),$$

which is impossible.

Thus $\mathfrak{b}^* < \lambda$, as claimed.

Remark. We have another proof of this proposition, not involving forcing, which is illuminating in different ways.

11. COROLLARY. *If λ is any real-valued-measurable cardinal, then $\lambda \notin S$, and in fact* $\sup S < \lambda$.

12. The results of Sections 5–11 all tend to show that S is small. It is easy to show also that S can be large.

PROPOSITION. *Let θ be any finite cardinal such that $\theta^\omega = \theta$. Then there is a* ccc p.o. set $\mathbb{P}$ *such that*

$\Vdash_{\mathbb{P}} \mathfrak{c} = \theta$ *and whenever κ, λ are infinite regular cardinals not greater than θ and not both equal to ω, there is an unfilled (κ, λ^*)-gap in $\mathscr{P}\mathbb{N}$.*

Consequently, using Proposition 3 *and Theorem* 4(*a*),

$$\Vdash_{\mathbb{P}} \kappa \in S \text{ for every infinite regular cardinal } \kappa \geqslant \mathfrak{c}.$$

Proof. (*a*) The first step is to define, for each pair κ, λ of infinite regular cardinals with $\kappa \geqslant \max(\omega_1, \lambda)$, a p.o. set $\mathbb{P}_{\kappa\lambda}$ such that

$$\Vdash_{\mathbb{P}_{\kappa\lambda}} \text{there is an unfilled } (\kappa, \lambda^*)\text{-gap.}$$

This is the obvious one; each condition $p \in \mathbb{P}_{\kappa\lambda}$ describes initial segments of finitely many of the sets a_ξ, b_η which will form the gap. Specifically, p is to be of the form

$$p = (I_p, J_p, n_p, A_p, B_p)$$

where $I_p \in [\kappa]^{<\omega}$, $J_p \in [\lambda]^{<\omega}$, $n_p \in \mathbb{N}$, $A_p \subseteq I_p \times n_p$ and $B_p \subseteq J_p \times n_p$; we shall say that

$$p \Vdash_{\mathbb{P}_{\kappa\lambda}} a_\xi \cap n_p = \{i : (\xi, i) \in A_p\} \forall \xi \in I_p,$$
$$b_\eta \cap n_p = \{i : (\eta, i) \in B_p\} \forall \eta \in J_p.$$

The ordering on $\mathbb{P}_{\kappa\lambda}$ is also the obvious one; say that $p \leqslant q$ if

$$I_q \subseteq I_p, J_q \subseteq J_p, n_q \leqslant n_p,$$
$$A_q = A_p \cap (I_q \times n_q), B_q = B_p \cap (J_q \times n_q),$$

and moreover, for every $i \in n_p \backslash n_q$,

$$\text{if } \xi, \xi' \in I_q \text{ and } \xi \leqslant \xi' \text{ and } (\xi, i) \in A_p \text{ then } (\xi', i) \in A_p;$$
$$\text{if } \eta, \eta' \in J_q \text{ and } \eta' \leqslant \eta \text{ and } (\eta, i) \in B_p \text{ then } (\eta', i) \in B_p;$$
$$\text{if } \xi \in I_q, \eta \in J_q \text{ and } (\xi, i) \in A_p \text{ then } (\eta, i) \in B_p.$$

Evidently this ordering will ensure that

$$\Vdash_{\mathbb{P}_{\kappa\lambda}} \langle a_\xi \rangle_{\xi<\kappa} \text{ is } \subseteq^*\text{-increasing},$$
$$\langle b_\eta \rangle_{\eta<\lambda} \text{ is } \subseteq^*\text{-decreasing},$$
$$a_\xi \subseteq^* b_\eta \forall \xi < \kappa, \eta < \lambda.$$

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We must now check that

$$\Vdash_{\mathbb{P}_{\kappa\lambda}} \text{there is no } c \text{ such that } a_\xi \subseteq^* c \,\forall\, \xi < \kappa,$$
$$c \subseteq^* b_\eta \,\forall\, \eta < \lambda.$$

But suppose, if possible, that there were a $\mathbb{P}_{\kappa\lambda}$-name for such a c. In this case, because $\mathbb{P}_{\kappa\lambda}$ is ccc, there is a limit ordinal $\zeta < \kappa$ such that whenever $p \in \mathbb{P}_{\kappa\lambda}$ and $I_p \cup J_p \subseteq \zeta$ and $\gamma < \zeta$ there are a $q \leqslant p$ and an $m \in \mathbb{N}$ such that $I_q \cup J_q \subseteq \zeta$ and

$$q \Vdash_{\mathbb{P}_{\kappa\lambda}} c \subseteq b_\gamma \cup m.$$

Now fix on $p_0 \in \mathbb{P}_{\kappa\lambda}$ and $n \in \mathbb{N}$ such that

$$p_0 \Vdash_{\mathbb{P}_{\kappa\lambda}} a_\zeta \subseteq c \cup n.$$

Take any $\gamma < \zeta$ such that $J_{p_0} \cap \zeta \subseteq \gamma$, and set

$$p = (I_{p_0} \cap \zeta, J_{p_0} \cap \zeta, n_{p_0}, A_{p_0} \cap (\zeta \times \mathbb{N}), B_{p_0} \cap (\zeta \times \mathbb{N})).$$

Take $q \leqslant p, m \in \mathbb{N}$ such that $I_q \cup J_q \subseteq \zeta$ and

$$q \Vdash_{\mathbb{P}_{\kappa\lambda}} c \subseteq b_\gamma \cup m.$$

Take $k = \max(m, n, n_q)$. Because $I_q \subseteq \zeta$ and $J_{p_0} \cap \zeta \subseteq \gamma$, it is easy to build an r such that $r \leqslant p$, $r \leqslant q$, $n_r > k$, $(\zeta, k) \in A_r$ and $(\gamma, k) \notin B_r$. In this case

$$r \Vdash_{\mathbb{P}_{\kappa\lambda}} a_\zeta \subseteq c \cup n, c \subseteq b_\gamma \cup m \;\&\; k \in a_\zeta \backslash b_\gamma,$$

which is absurd.

(*b*) Thus we have a p.o. set such that

$$\Vdash_{\mathbb{P}_{\kappa\lambda}} \text{there is an unfilled } (\kappa, \lambda^*)\text{-gap}.$$

Now take $\mathbb{P}$ to be the finite support product of

$$\langle \mathbb{P}_{\kappa\lambda} \rangle_{\theta \geqslant \kappa \geqslant \max(\lambda, \omega_1), \lambda \geqslant \omega}.$$

The same Δ-system argument which shows that each $\mathbb{P}_{\kappa\lambda}$ is ccc applies also to $\mathbb{P}$. Also (because all the $\mathbb{P}_{\kappa\lambda}$ are absolute constructions) we can think of $\mathbb{P}$ as an iteration $\mathbb{P}'_{\kappa\lambda} * \mathbb{P}_{\kappa\lambda}$, so that

$$\Vdash_{\mathbb{P}} \text{there is an unfilled } (\kappa, \lambda^*)\text{-gap}$$

for all the relevant κ, λ. The computation of $\mathfrak{c}$ in $V^{\mathbb{P}}$ is easy, because $\#(\mathbb{P}) = \theta$.

13. *Universal sets.*

THEOREM. *There is a closed set $M \subseteq \mathscr{P}\mathbb{N}$ such that $S = S_M$.*

Proof. We shall actually construct a subset M' of $\mathscr{P}(\mathbb{N} \cup T)$ for a suitable countable set T; as already remarked in the proof of 4(*c*) above, we can then speak of $S_{M'}$. Take T to be

$$\{(n, \tau) : n \in \mathbb{N}, \tau \subseteq \mathscr{P}n\};$$

observe that (on the usual coding of ordered pairs as sets) $T \cap \mathbb{N} = \varnothing$, so that T is a countable set disjoint from $\mathbb{N}$. For (m, σ), $(n, \tau) \in T$ say that $(m, \sigma) \leqslant (n, \tau)$ if $m \leqslant n$ and $\sigma = \{I \cap n : I \in \tau\}$. Then T is a tree of height ω. Let B be the set of

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infinite branches of T; then (because each level of T is finite) B is closed in $\mathscr{P}T$. For $b\in B$, $n\in\mathbb{N}$ let $\tau_n(b)$ be that unique subset of $\mathscr{P}n$ such that $(n,\tau_n(b))\in b$. Set

$$M' = \{a \cup b : a\subseteq\mathbb{N}, b\in B, a\cap n\in\tau_n(b)\,\forall\, n\in\mathbb{N}\}.$$

Then M' is a closed subset of $\mathscr{P}(\mathbb{N}\cup T)$.

Suppose $\kappa\in S$. Let $K\subseteq\mathscr{P}\mathbb{N}$ be a closed set such that $\kappa\in S_K$, and $\langle a_\xi\rangle_{\xi<\kappa}$ a $\subseteq^*$-increasing family in K with no $\subseteq^*$-upper bound in K. For $n\in\mathbb{N}$ set $\sigma_n = \{a\cap n : a\in K\}$; then

$$b_K = \{(n,\sigma_n) : n\in\mathbb{N}\}\in B.$$

Write $d_\xi = a_\xi \cup b_K\in M'$ for each $\xi<\kappa$. Then $\langle d_\xi\rangle_{\xi<\kappa}$ is $\subseteq^*$-increasing. We claim that it witnesses that $\kappa\in S_{M'}$. For take any $d\in M'$ and set $a = d\cap\mathbb{N}$, $b = d\cap T$. Then $b\in B$. If $b\neq b_K$ then $b_K\not\subseteq^* b$ (because both b and b_K are infinite branches of T), so that $d_0\not\subseteq^* d$. If $b = b_K$ then $a\cap n\in\tau_n(b) = \tau_n(b_K) = \sigma_n$ for every $n\in\mathbb{N}$, so $a\in\bar{K} = K$. There must therefore be some $\xi<\kappa$ such that $a_\xi\not\subseteq^* a$, in which case $d_\xi\not\subseteq^* d$. As d is arbitrary, we do indeed have $\kappa\in S_{M'}$. As κ is arbitrary, $S = S_{M'}$; copying M' onto a subset M of $\mathscr{P}\mathbb{N}$, our proof is complete.

Remarks. It is easy to see that, in the notation of Sections 3–4 above, $S_L\subseteq S_W$ and that $\omega\notin S_W$. Theorem 5 shows that we can have $\mathfrak{c}\in S_W\backslash S_L$ (because, as is shown in [2], 4·3, it is relatively consistent with ZFC to suppose that MA is true but that there are no $(\mathfrak{c},\mathfrak{c}^*)$-gaps). But we ask: is W nearly universal in the sense that $S = S_W\cup\{\omega\}$?

14. *Transference to* $\mathbb{N}^{\mathbb{N}}$. We were originally led to the questions dealt with in this paper by some questions of K. Alster concerning the behaviour to be expected of $\leqslant^*$-increasing families in $\mathbb{N}^{\mathbb{N}}$. Here an extra refinement can be introduced because we have another ordering $<^*$, writing $f<^*g$ if $\{n : f(n)\geqslant g(n)\}$ is finite, not precisely recoverable from $\leqslant^*$. All the results above transfer to results concerning $\leqslant^*$, $<^*$ and compact (we must now distinguish between 'compact' and 'closed') subsets of $\mathbb{N}^{\mathbb{N}}$, with no surprises; but as the re-coding involved is not absolutely trivial we spell it out.

15. Proposition. *Let κ be a regular infinite cardinal. Then the following are equivalent*:

(i) $\kappa\in S$;

(ii) *there are a compact $K\subseteq\mathbb{N}^{\mathbb{N}}$ and a family $\langle f_\xi\rangle_{\xi<\kappa}$ in K such that $f_\xi\leqslant^* f_\eta$ whenever $\xi\leqslant\eta<\kappa$ but there is no $f\in K$ such that $f_\xi\leqslant^* f$ for every $\xi<\kappa$*;

(iii) *there are a compact $K\subseteq\mathbb{N}^{\mathbb{N}}$ and a family $\langle f_\xi\rangle_{\xi<\kappa}$ in K such that $f_\xi<^* f_\eta$ whenever $\xi<\eta<\kappa$ but there is no $f\in K$ such that $f_\xi<^* f$ for every $\xi<\kappa$.*

Proof. (i)$\Rightarrow$(iii). Suppose that $K\subseteq\mathscr{P}\mathbb{N}$, $\langle a_\xi\rangle_{\xi<\kappa}\in K^\kappa$ witness that $\kappa\in S_K\subseteq S$. We may suppose that $a_\eta\not\subseteq^* a_\xi$ whenever $\xi<\eta<\kappa$. Define $\phi:\mathscr{P}\mathbb{N}\to\mathbb{N}^{\mathbb{N}}$ by writing

$$\phi(a)(n) = \sum_{t\in a\cap n} 2^t\;\forall\, n\in\mathbb{N}, a\subseteq\mathbb{N}.$$

Then ϕ is continuous and, for $a,b\subseteq\mathbb{N}$,

$$a\subseteq^* b\ \&\ b\not\subseteq^* a\Rightarrow\phi(a)<^*\phi(b)\Rightarrow a\subseteq^* b.$$

So $\phi[K]$, $\langle\phi(a_\xi)\rangle_{\xi<\kappa}$ witness (iii).

(iii)$\Rightarrow$(ii). This implication is trivial.

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(ii) $\Rightarrow$ (i). Define $\psi:\mathbb{N}^{\mathbb{N}}\to\mathscr{P}(\mathbb{N}\times\mathbb{N})$ by writing

$$\psi(f)=\{(i,j):i\in\mathbb{N},j\leqslant f(i)\}$$

for $f\in\mathbb{N}^{\mathbb{N}}$. Then ψ is continuous and, for $f,g\in\mathbb{N}^{\mathbb{N}}$,

$$\psi(f)\subseteq{}^*\psi(g)\quad\text{if and only if}\quad f\leqslant{}^*g.$$

So if K, $\langle f_\xi\rangle_{\xi<\kappa}$ witness (ii), $\psi[K]$ and $\langle\psi(f_\xi)\rangle_{\xi<\kappa}$ will witness (i), allowing for a recoding of $\mathscr{P}(\mathbb{N}\times\mathbb{N})$ into $\mathscr{P}\mathbb{N}$.

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