A CHARACTERISTICALLY-SIMPLE GROUP

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1. *Introduction.* The object of this note is to give an example of an infinite locally finite p-group which has no proper characteristic subgroup except the unit group. (A group G is a locally finite p-group if every finite set of elements of G generates a subgroup of finite order equal to a power of the prime p.) It is known that an infinite locally finite p-group cannot be simple, for if it were it would satisfy the minimal condition for normal subgroups, and so have a non-trivial centre (see (1)). However our example shows that it can be characteristically-simple. Examples are known of locally finite p-groups with trivial centre ((2), (4)), and of locally finite p-groups coinciding with their commutator groups ((1), (5)). Since the centre and commutator subgroup of a group are characteristic subgroups our example will have both of these properties. We may remark that the direct product of a simple, or even of a characteristically-simple group with itself any number of times is also characteristically-simple, but by Corollary 2·1 our group cannot be so decomposed.

2. *Construction.* Let $\{\alpha, \beta, \ldots\}$ be the set of all rational numbers, and R a division ring $\{a, b, \ldots\}$. Let E be the vector space over R with basis $e_{\alpha\beta}$, for all $\alpha < \beta$, and define a multiplication on E:

$$e_{\alpha\beta}e_{\gamma\delta} = e_{\alpha\delta} \text{ if } \beta = \gamma, \text{ and } = 0 \text{ otherwise},$$

making E an associative algebra. Then the set G of all elements $1+u$, where $u \in E$, forms a multiplicative group (see, for example (1) or (3)).

If R has characteristic p, for example if it is the Galois Field of p elements, then G is a locally finite p-group, as shown in (1); and if R has characteristic zero, G is locally infinite (i.e. every element of G has infinite order). Our results will, of course, hold for both cases.

G is countable provided R is; but the results and proofs of this note are unchanged if we let $\{\alpha, \beta, \ldots\}$ run over the whole set of real numbers, and G is not then countable.

3. *Proofs.* The general element of G is expressible uniquely in the form

$$g = 1 + \sum_{\alpha<\beta} a_{\alpha\beta}e_{\alpha\beta}, \quad a_{\alpha\beta} \in R \tag{1}$$

where only a finite number of the $a_{\alpha\beta}$ are different from zero. We will always impose this condition on sums of the type (1), and will refer to the terms of the sum for which $a_{\alpha\beta} \neq 0$, as the terms *occurring* in (1).

Lemma 1. *G is generated by the set of all $1+ae_{\alpha\beta}$, where $a \in R$ and $\alpha < \beta$.*

We prove that g can be expressed as a product of $1+ae_{\alpha\beta}$ by induction on the number, n, of terms occurring in (1). It is true when there is only one $a_{\alpha\beta} \neq 0$, so we assume that the result holds for all elements of G with less than n terms in the sum (1). Let g have n such. If not all the β are equal, let ζ be the greatest. Then

$$1 + \Sigma a_{\alpha\beta}e_{\alpha\beta} = (1 + \sum_{\beta=\zeta} a_{\alpha\beta}e_{\alpha\beta})(1 + \sum_{\beta\neq\zeta} a_{\alpha\beta}e_{\alpha\beta}).$$

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If all the β are equal, let ξ be the smallest α which occurs. Then

$$1+\Sigma a_{\alpha\beta}e_{\alpha\beta} = (1+\sum_{\alpha\neq\xi} a_{\alpha\beta}e_{\alpha\beta})(1+a_{\xi\beta}e_{\xi\beta}),$$

and in each case the terms on the right-hand side may be generated by the $1+ae_{\alpha\beta}$. Hence g can be expressed as a product of these and the result is proved.

LEMMA 2. *If $N \neq 1$ is a normal subgroup of G, then there exist ξ, ζ such that $1+ae_{\gamma\delta} \in N$, for all $a \in R$ and all γ, δ satisfying $\gamma < \xi < \zeta < \delta$.*

Let $g = 1+\Sigma\alpha_{\alpha\beta}e_{\alpha\beta} \in N$, $g \neq 1$, and let ζ be the largest β occurring in the sum. Put $u = 1+e_{\zeta\delta}$, for any $\delta > \zeta$. Then the commutator $h = gug^{-1}u^{-1} = 1+\sum_{\beta=\zeta} a_{\alpha\beta}e_{\alpha\delta}$ belongs to N. If ξ is the least α occurring in the expression for h, we put $v = 1-aa_{\xi\zeta}^{-1}e_{\gamma\xi}$, for any $a \in R$, and for any $\gamma < \xi$. Then

$$h^{-1}v^{-1}hv = 1+ae_{\gamma\delta} \text{ belongs to } N.$$

COROLLARY 2·1. *If G is the direct product of two groups $H_1 \times H_2$, then one of them $= 1$ and the other $= G$.*

For H_1, H_2 are both normal subgroups of G, and hence, if neither reduces to 1, each will contain the elements $1+e_{\gamma\delta}$ for $\gamma < \text{Min}(\xi_1, \xi_2)$ and $\delta > \text{Max}(\zeta_1, \zeta_2)$. This is impossible, as $H_1 \cap H_2 = 1$.

THEOREM 1. *If $N \neq 1$ is a characteristic subgroup of G, then $N = G$.*

Suppose $N \neq G$. Then there is an element $1+ae_{\alpha\beta}$ not in N, by Lemma 1. Also, as N is normal in G it contains an element $1+ae_{\gamma\delta}$, for some $\gamma < \delta$, by Lemma 2. Now any one-one mapping θ of the rational numbers on to themselves, which preserves order, induces an automorphism Θ of G in the obvious way: if θ maps α on α', we define

$$(1+\Sigma a_{\alpha\beta}e_{\alpha\beta})\Theta = 1+\Sigma a_{\alpha\beta}e_{\alpha'\beta'}.$$

But there exists such a one-one map, which maps γ on α and δ on β. Consequently there is an automorphism of G mapping $1+ae_{\gamma\delta}$ on $1+ae_{\alpha\beta}$. This is impossible, so $N = G$.

We may remark here that, although G has no non-trivial characteristic subgroups, it has an uncountable number of normal subgroups. For if μ,ν are any real numbers (not necessarily rational), then the set of all elements g of G, for which the pairs (α, β) that occur in (1) satisfy $\alpha < \mu < \nu < \beta$, forms an Abelian normal subgroup $G_{\mu\nu}$, and $G_{\mu\nu} = G_{\theta\phi}$ only if $\mu = \theta$ and $\nu = \phi$. Hence by taking a suitable sequence (μ_i, ν_i), $i = 1, 2, 3, \ldots$, we can see that G satisfies neither the ascending nor the descending chain condition for normal subgroups.

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