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On the Magneto-Heat Coupling Model for Large Power Transformers

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Abstract. This paper studies the magneto-heat coupling model which describes iron loss of conductors and energy exchange between magnetic field and Ohmic heat. The temperature influences Maxwell's equations through the variation of electric conductivity, while electric eddy current density provides the heat equation with Ohmic heat source. It is in this way that Maxwell's equations and the heat equation are coupled together. The system also incorporates the heat exchange between conductors and cooling oil which is poured into and out of the transformer. We propose a weak formulation for the coupling model and establish the well-posedness of the problem. The model is more realistic than the traditional eddy current model in numerical simulations for large power transformers. The theoretical analysis of this paper paves a way for us to design efficient numerical computation of the transformer in the future.

AMS subject classifications: 35K55, 35Q60, 78A25

Key words: Magneto-heat coupling model, eddy current problem, Maxwell's equations, radiative boundary condition.

1 Introduction

Large power transformers are broadly and increasingly used in countries undergoing rapid industrialization. Numerical simulation plays an important role in both computing iron loss and optimal design of large power transformers. It attracts great attention in both societies of computational electromagnetism and computational mathematics. The International Compumag Society has announced 34 benchmark problems or families of benchmark problems, called TEAM Workshop problems, to test electromagnetic analysis methods. Among them, the 21st family, Problem 21, was proposed to test numerical

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methods for computing iron loss in large power transformers [6]. It consists of 13 different models which represent various component accessories of a power transformer. Problem 21 is one of the most active TEAM Workshop problems and has promoted the development of numerical simulations of power transformer.

Power transformer consists of magnetic and nonmagnetic conductors, exciting coils, and cooling oils etc. The Ohmic heat and magnetic hysteresis lead to energy loss and can damage the devices of a power transformer. The oil is poured into and out of the power transformer to cool down the devices. Therefore, the system turns out to be the coupling between Maxwell's equations and the heat equation, or even more accurately, with fluid dynamics of the oil [17]. Traditional studies only consider quasi-static Maxwell's equations for TEAM Workshop Problem 21 and neglect the variation of electric conductivity caused by the temperature (see e.g. [7,8,12,13,16] and the references therein). In numerical simulation of power transforms, eddy current loss accounts for the major part of the total energy loss. The nonlinear dependency of the conductivity on the temperature is no longer negligible. The magneto-heat coupling model is more realistic and can yield better results (see [7, Chapter 3] and [14]). To the authors' best knowledge, there still lacks rigorous analysis in the literature on the well-posedness of the magneto-heat coupling model for power transformers. This paper is to fulfill the task.

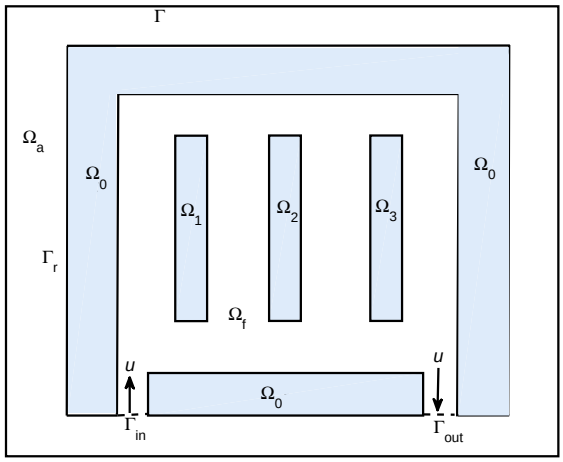


Figure 1: A 2D illustration of the problem geometry, $\Omega=\Omega_a\cup\Omega_f\cup\Omega_c$, $\Omega_c=\cup_{i=0}^{M}\Omega_i$.

The transformer system consists of the oil tank Ω_0, the cooling oil Ω_f contained by the tank, the conductors $\Omega_1,\cdots,\Omega_M$ which are immersed in the oil, and the air region Ω_a outside of the tank. The problem geometry is illustrated in Fig. 1. We assume that the oil tank is electrically conducting and both the air and the cooling oil are insulating. Then the conducting region and the non-conducting region are given respectively by

$$\Omega_c=\Omega_0\cup\cdots\cup\Omega_M, \qquad \Omega_{nc}=\Omega_f\cup\Omega_a.$$

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We assume that Ω_c and Ω_f are heat-conductive and define

$$\Omega_h = \text{interior}\left(\bar{\Omega}_c \cup \bar{\Omega}_f\right).$$

The domain for the magneto-heat problem is then defined by the nonintersecting combination

$$\Omega = \Omega_{nc} \cup \bar{\Omega}_c.$$

For convenience we introduce some notations for boundaries or parts of boundaries. Let $\Gamma = \partial\Omega$ be the truncation boundary of the unbounded air region and $\Gamma_h = \partial\Omega_h$ be the boundary of heat-conductive materials. For a transformer, $\Gamma_h = \Gamma_r \cup \Gamma_{\text{in}} \cup \Gamma_{\text{out}}$ where $\Gamma_r = \partial\Omega_a \cap \partial\Omega_0$ stands for the outer boundary of the oil tank where the heat is radiated into the air, Γ_{in} stands for the inlet of oil pouring into the tank, and Γ_{out} stands for the outlet of oil pouring out. We shall focus on simply connected conductors, namely, $\Omega_0, \cdots, \Omega_M$ are simply-connected Lipschitz domains. It is clear that Ω_a, Ω_f, Ω_c are also simply-connected and have Lipschitz-continuous boundaries.

Let T be the final time before which the physical fields are interested in. The electromagnetic fields of the power transformer are governed by the quasi-static Maxwell's equations

$$\frac{\partial \boldsymbol{B}}{\partial t} + \mathbf{curl}\,\boldsymbol{E} = 0 \quad \text{in } \Omega \times (0,T), \tag{1.1a}$$

$$\mathbf{curl}\,\boldsymbol{H} - \sigma \boldsymbol{E} = \boldsymbol{f} \quad \text{in } \Omega \times (0,T), \tag{1.1b}$$

$$\boldsymbol{H}(0) = 0 \quad \text{in } \Omega, \tag{1.1c}$$

$$\boldsymbol{E} \times \boldsymbol{n} = 0 \quad \text{on } \Gamma \times (0,T). \tag{1.1d}$$

where $\boldsymbol{E}$ is the electric field, $\boldsymbol{B}$ is the magnetic flux density, $\boldsymbol{H}$ is the magnetic field, and $\sigma = \sigma(\theta)$ is the conductivity depending on the temperature θ. Throughout the paper, we assume that σ is Lipschitz continuous with respect to θ and satisfies

$$0 < \sigma_{\min} \le \sigma \le \sigma_{\max} \quad \text{in } \Omega_c, \qquad \sigma = 0 \quad \text{in } \Omega_{nc},$$

where $\sigma_{\min}$ and $\sigma_{\max}$ are constants. We remark that the above assumption on the conductivity is rather mild in electrical engineering. The source current density $\boldsymbol{f}$ is carried in some coils and satisfies

$$\text{div}\,\boldsymbol{f} = 0, \qquad \boldsymbol{f}(0) = 0.$$

For engineering applications, it is reasonable to assume that $\boldsymbol{f}$ is Lipschitz continuous in time through this paper. For simplicity, we only consider nonmagnetic materials, namely, $\boldsymbol{B} = \mu \boldsymbol{H}$, where $\mu = \mu_0$ stands for the magnetic permeability in empty space. The quasi-static Maxwell's equations are complemented with the initial and boundary conditions in (1.1c)-(1.1d).

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The temperature θ is governed by the heat convection-diffusion equation with initial and boundary conditions

$$C_\rho\left(\frac{\partial\theta}{\partial t}+\boldsymbol{u}\cdot\nabla\theta\right)-\nabla\cdot(\kappa\nabla\theta)=\sigma(\theta)E^2 \quad \text{in } \Omega_h\times(0,T), \tag{1.2a}$$

$$\theta(0)=\theta_0 \quad \text{in } \Omega_h, \tag{1.2b}$$

$$\theta=\theta_0 \quad \text{on } \Gamma_{\text{in}}\times(0,T), \tag{1.2c}$$

$$-\kappa\frac{\partial\theta}{\partial\boldsymbol{n}}=C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})(\theta-\theta_0) \quad \text{on } \Gamma_{\text{out}}\times(0,T), \tag{1.2d}$$

$$-\kappa\frac{\partial\theta}{\partial\boldsymbol{n}}=h(\theta^4-\theta_0^4)+\lambda(\theta-\theta_0) \quad \text{on } \Gamma_r\times(0,T), \tag{1.2e}$$

where $\theta_0\in L^\infty(\bar{\Omega}_h)$ is the background temperature, C_ρ equals to the constant-volume specific heat times the material density, λ is the heat convection coefficient, h is the radiation coefficient, κ is the thermal conductivity, $\boldsymbol{n}$ stands for the unit outer normal to Ω_h, and $\boldsymbol{u}\in\boldsymbol{L}^\infty(0,T;\boldsymbol{H}^1(\Omega_h))$ stands for the fluid velocity satisfying

$$\begin{cases}\operatorname{div}\boldsymbol{u}=0 \quad \text{in } \Omega_h, & \operatorname{supp}(\boldsymbol{u})=\bar{\Omega}_f,\\ \boldsymbol{u}\cdot\boldsymbol{n}\le 0 \quad \text{on } \Gamma_{\text{in}}, & \boldsymbol{u}\cdot\boldsymbol{n}\ge 0 \quad \text{on } \Gamma_{\text{out}}.\end{cases} \tag{1.3}$$

On Γ_{in}, the cold oil at temperature θ_0 pours into the power transformer and gives the Dirichlet boundary condition for θ. While on Γ_{out}, the hot oil pours out of the power transformer and takes away the heat. It is reasonable in practice to assume that

$$\theta_0\ge\theta_{\min}>0, \quad \kappa\ge\kappa_{\min}>0, \quad C_\rho\ge C_{\min}>0, \tag{1.4}$$

where $\theta_{\min}$, $\kappa_{\min}$, and $C_{\min}$ are positive constants. We remark that (1.2e) stands for the nonlinear radiation condition on the outer boundary of Ω_0. For simplicity, we do not consider the fluid dynamics and assume that ρ, $\boldsymbol{u}$ are given functions. Moreover, the conductivity $\sigma(\theta)$ depends on the temperature through (1.1b), while the electric current density provides the Ohmic heat source $\sigma E^2=\sigma\boldsymbol{E}\cdot\boldsymbol{E}$ to (1.2a). It is in this way that problem (1.1) and problem (1.2) are coupled.

The eddy current model (1.1) is the quasi-static approximation of Maxwell's equations at very low frequency by neglecting the displacement currents in Ampere's law (see [1]). For linear eddy current problems, there are many interesting works in the literature on numerical methods (cf. e.g. [4, 5, 11, 15, 22]) and on the regularity of the solution (cf. e.g. [9]). In [3], Bachinger et al. studied the numerical analysis of nonlinear multi-harmonic eddy current problems in isotropic materials. In [12, 16], Zheng and his coauthors studied nonlinear eddy current model in grain-oriented silicon steel laminations. The homogenization of eddy current problem in silicon steel laminations was also studied by Nédélec and Wolf for linear problem [18] and by Jiang and Zheng for nonlinear problem [13]. In [20], Yin studied the weak solution of Maxwell's equations with

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a thermal effect. In [10], P.-É. Duret proved the existence of weak solutions to the time-dependent magnetohydrodynamic equations coupled to the heat equation with nonlocal radiation boundary conditions. The purpose of this paper is to prove the existence of the solution for the coupled system of (1.1) and (1.2) for power transformers.

The layout of the paper is organized as follows. In Section 2, we introduce some notations and Sobolev spaces used in this paper. In Section 3, we propose a weak formulation for the magneto-heat coupling problem and prove some useful lemmas for the nonlinear problem. Due to the low regularity of the Ohmic heat, it is difficult to analyze the nonlinear heat equation directly. In Section 4, we introduce a regularized model to deal with the Ohmic heat and study the well-posedness of the regularized problem. In Section 5, we prove the existence of the original problem by passage to limit for the regularized solutions. Throughout the paper we denote vector-valued quantities by boldface notation, such as $\boldsymbol{L}^2(\Omega):=(L^2(\Omega))^3$. For convenience, we also write tensor products for time and spatial domains as follows

$$\begin{aligned} &Q=\Omega\times(0,T), \quad Q_c=\Omega_c\times(0,T), \quad Q_h=\Omega_h\times(0,T), \quad Q_f=\Omega_f\times(0,T),\\ &S=\Gamma\times(0,T), \quad S_r=\Gamma_r\times(0,T), \quad S_h=\Omega_h\times(0,T), \quad S_{\text{out}}=\Gamma_{\text{out}}\times(0,T). \end{aligned}$$

2 Preliminaries

For any $p\geq 1$, let $L^p(\Omega)$ be the Sobolev space of p-power integrable functions with the norm

$$\|v\|_{L^p(\Omega)}=\left(\int_\Omega |v(\boldsymbol{x})|^p \mathrm{d}\boldsymbol{x}\right)^{1/p}.$$

For $p=\infty$, $L^\infty(\Omega)$ denotes the space of essentially bounded functions with the norm

$$\|v\|_{L^\infty(\Omega)}=\operatorname{esssup}|v(\boldsymbol{x})|.$$

For $p=2$, $L^2(\Omega)$ denotes the Hilbert space equipped with the inner product and norm

$$(u,v):=\int_\Omega u(\boldsymbol{x})v(\boldsymbol{x})\mathrm{d}\boldsymbol{x}, \qquad \|u\|_{0,\Omega}:=\|u\|_{L^2(\Omega)}=(u,u)^{1/2}.$$

Define $H^m(\Omega):=\{v\in L^2(\Omega): D^\xi v\in L^2(\Omega), |\xi|\leq m\}$ which is equipped with the following norm and semi-norm

$$\|u\|_{m,\Omega}:=\Big(\sum_{|\xi|\leq m}\|D^\xi u\|_{0,\Omega}^2\Big)^{1/2}, \qquad |u|_{m,\Omega}:=\Big(\sum_{|\xi|=m}\|D^\xi u\|_{0,\Omega}^2\Big)^{1/2},$$

where ξ represents non-negative triple index. As usual, $H_0^1(\Omega)$ is the subspace of $H^1(\Omega)$ whose functions have zero traces on $\partial\Omega$.

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Define

$$\begin{aligned} \boldsymbol{H}(\mathbf{curl},\Omega) &:= \{v\in \boldsymbol{L}^2(\Omega): \mathbf{curl}\, v\in \boldsymbol{L}^2(\Omega)\},\\ \boldsymbol{H}_0(\mathbf{curl},\Omega) &:= \{v\in \boldsymbol{H}(\mathbf{curl},\Omega): \boldsymbol{n}\times v=0 \text{ on}\partial\Omega\}, \end{aligned}$$

where $\boldsymbol{n}$ is the unit outer normal to Ω and the spaces are equipped with the inner product

$$(v,w)_{\boldsymbol{H}(\mathbf{curl},\Omega)}:=(v,w)+(\mathbf{curl}\, v,\mathbf{curl}\, w)$$

and the norm

$$\|v\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}:=\sqrt{(v,v)_{\boldsymbol{H}(\mathbf{curl},\Omega)}}.$$

Define the spaces of functions with square integrable divergence by

$$\begin{aligned} \boldsymbol{H}(\mathrm{div},\Omega) &:= \{v\in \boldsymbol{L}^2(\Omega): \mathrm{div}\, v\in L^2(\Omega)\},\\ \boldsymbol{H}_0(\mathrm{div},\Omega) &:= \{v\in \boldsymbol{H}(\mathrm{div},\Omega): \boldsymbol{n}\cdot v=0 \text{ on}\partial\Omega\}, \end{aligned}$$

which are equipped with the inner product

$$(v,w)_{\boldsymbol{H}(\mathrm{div},\Omega)}:=(v,w)+(\mathrm{div}\, v,\mathrm{div}\, w)$$

and the norm

$$\|v\|_{\boldsymbol{H}(\mathrm{div},\Omega)}:=\sqrt{(v,v)_{\boldsymbol{H}(\mathrm{div},\Omega)}}.$$

To study the weak solution of (1.1) and (1.2), we shall use the subspaces of $\boldsymbol{H}(\mathbf{curl},\Omega)$:

$$\boldsymbol{X} := \nabla H^1(\Omega)+\boldsymbol{H}_0(\mathbf{curl},\Omega_c), \tag{2.1}$$

$$\boldsymbol{X}_c := \{v\in \boldsymbol{H}_0(\mathbf{curl},\Omega_c): \mathrm{div}\, v=0 \text{ in}\Omega_c\}. \tag{2.2}$$

It is easy to see that

$$\|v\|^2_{\boldsymbol{H}(\mathbf{curl},\Omega)}=\|v\|^2_{\boldsymbol{L}^2(\Omega)}+\|\mathbf{curl}\, v\|^2_{\boldsymbol{L}^2(\Omega_c)} \qquad \text{for all } v\in\boldsymbol{X}.$$

We shall use the convention that all functions in $\boldsymbol{H}_0(\mathbf{curl},D)$ and $H^1_0(D)$ are extended by zero to the exterior of D for any $D\subset\Omega$. The following decomposition can be found in [16].

Lemma 2.1. *Let the conducting region Ω_c be simply connected. Then*

$$\boldsymbol{X}=\{v\in\boldsymbol{H}(\mathbf{curl},\Omega):\mathbf{curl}\, v=0 \text{ in } \Omega_a\cup\Omega_f\}.$$

Furthermore, for any $v\in\boldsymbol{X}$, there exist a unique $v_c\in\boldsymbol{X}_c$ and a unique $\phi\in H^1(\Omega)/\mathbb{R}$ such that

$$v=v_c+\nabla\phi, \qquad \|v_c\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}+\|\phi\|_{H^1(\Omega)}\le C\|v\|_{\boldsymbol{H}(\mathbf{curl},\Omega)},$$

where $C>0$ is a constant only depending on Ω_{nc}.

To end up this section, we introduce the function space for the radiative and conductive heat equation

$$Y=\Big\{v\in H^1(\Omega_h)\cap L^5(\Gamma_r):\ v=0 \text{ on}\,\Gamma_{\text{in}}\Big\}, \qquad \|v\|_Y:=\|v\|_{H^1(\Omega_h)}+\|v\|_{L^5(\Gamma_r)}.$$

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3 A weak formulation

The purpose is to derive a weak formulation for the magneto-heat coupling system (1.1) and (1.2). Since the source current density satisfies div $\boldsymbol{f}=0$, there exists a source magnetic field $\boldsymbol{H}_f$ such that

$$\boldsymbol{f}=\mathbf{curl}\,\boldsymbol{H}_f \quad \text{in } \mathbb{R}^3. \tag{3.1}$$

The field $\boldsymbol{H}_f$ can be written explicitly by the Biot-Savart Law for general coils

$$\boldsymbol{H}_f:=\mathbf{curl}\,\boldsymbol{A}_f \quad \text{where} \quad \boldsymbol{A}_f(\boldsymbol{x}):=\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{\boldsymbol{f}(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|}\mathrm{d}\boldsymbol{y}, \tag{3.2}$$

or be computed numerically in practical simulations. In the following we are going to solve the residual $\boldsymbol{R}:=\boldsymbol{H}-\boldsymbol{H}_f$ which is called the reaction field in [16].

Lemma 3.1. *The reaction field $\boldsymbol{R}\in\boldsymbol{H}(\mathbf{curl},\Omega)$ admits a unique decomposition*

$$\boldsymbol{R}=\boldsymbol{R}_c+\nabla\psi, \qquad \boldsymbol{R}_c\in\boldsymbol{X}_c,\ \psi\in H^1(\Omega)/\mathbb{R}. \tag{3.3}$$

Proof. By (1.1b) and (3.1), clearly we have $\mathbf{curl}\,\boldsymbol{R}=0$ in $\Omega_a\cup\Omega_f$. Then (3.3) is a direct consequence of Lemma 2.1. □

Now we are going to deduce a weak formulation of (1.1) and (1.2). For any $\boldsymbol{v}\in\boldsymbol{X}$, (1.1a) and (1.1d) imply that

$$\begin{aligned}\int_{\Omega_c}\frac{\partial\boldsymbol{B}}{\partial t}\cdot\boldsymbol{v}&=-\int_{\Omega_c}\mathbf{curl}\,\boldsymbol{E}\cdot\boldsymbol{v}=-\int_{\Omega_c}\boldsymbol{E}\cdot\mathbf{curl}\,\boldsymbol{v}+\int_{\partial\Omega_c}(\boldsymbol{E}\times\boldsymbol{n}_c)\cdot\boldsymbol{v}\\&=\int_{\Omega_c}\sigma^{-1}(\boldsymbol{f}-\mathbf{curl}\,\boldsymbol{H})\cdot\mathbf{curl}\,\boldsymbol{v}+\int_{\partial\Omega_c}(\boldsymbol{E}\times\boldsymbol{n}_c)\cdot\boldsymbol{v},\end{aligned} \tag{3.4}$$

$$\int_{\Omega_{nc}}\frac{\partial\boldsymbol{B}}{\partial t}\cdot\boldsymbol{v}=-\int_{\Omega_{nc}}\mathbf{curl}\,\boldsymbol{E}\cdot\boldsymbol{v}=-\int_{\partial\Omega_c}(\boldsymbol{E}\times\boldsymbol{n}_c)\cdot\boldsymbol{v}, \tag{3.5}$$

where $\boldsymbol{n}_c$ is the unit outer normal to Ω_c. Since $\boldsymbol{E}$ and $\boldsymbol{v}$ are continuous tangentially across $\partial\Omega_c$, adding up (3.4) and (3.5) yields

$$\mu\int_\Omega\frac{\partial}{\partial t}(\boldsymbol{R}+\boldsymbol{H}_f)\cdot\boldsymbol{v}+\int_{\Omega_c}\sigma^{-1}(\theta)\,\mathbf{curl}\,\boldsymbol{R}\cdot\mathbf{curl}\,\boldsymbol{v}=0 \qquad \forall\boldsymbol{v}\in\boldsymbol{X},$$

where we deduce from (1.1b) that

$$\sigma^{-1}(\boldsymbol{f}-\mathbf{curl}\,\boldsymbol{H})=\sigma^{-1}(\mathbf{curl}\,\boldsymbol{H}_f-\mathbf{curl}\,\boldsymbol{H})=-\sigma^{-1}\mathbf{curl}\,\boldsymbol{R} \qquad \text{in } \Omega_c.$$

Multiply both sides of (1.2a) with any $\varphi\in Y$. Using integration by part and the boundary conditions in (1.2d)-(1.2e), we obtain

$$\begin{aligned}&\int_{\Omega_h}C_\rho\left(\frac{\partial\theta}{\partial t}+\boldsymbol{u}\cdot\nabla\theta\right)\varphi+\int_{\Omega_h}\kappa\nabla\theta\cdot\nabla\varphi+\int_{\Gamma_r}\Psi(\theta)\varphi+\int_{\Gamma_{\text{out}}}C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})\theta\varphi\\&=\int_{\Omega_h}\sigma E^2\varphi+\int_{\Gamma_r}\Psi(\theta_0)\varphi+\int_{\Gamma_{\text{out}}}C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})\theta_0\varphi.\end{aligned} \tag{3.6}$$

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Since the temperature is always positive, in the continuous level at least, we rewrite the nonlinear radiating term as follows

$$\Psi(\theta)=h\theta^4+\lambda\theta=\left(h|\theta|^3+\lambda\right)\theta. \tag{3.7}$$

This form is preferable in proving the coercivity of the nonlinear differential operator for the heat equation.

Write $\theta=\eta+\theta_0$ and let $r(\eta)$ be the resistivity defined by

$$r(\eta)=\begin{cases}\sigma^{-1}(\eta+\theta_0) & \text{in } \Omega_c,\\ 0 & \text{elsewhere.}\end{cases}$$

Then the Ohmic heat can be represented as follows

$$\sigma(\theta)E^2=r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2.$$

From (3.4)-(3.6), we obtain a weak formulation of the magneto-heat equations: Find $\boldsymbol{R}\in \boldsymbol{L}^2(0,T;\boldsymbol{X})$ and $\eta\in L^2(0,T;Y)$ such that $\boldsymbol{R}(0)=0$, $\eta(0)=0$, and

$$\mu\int_\Omega \frac{\partial \boldsymbol{R}}{\partial t}\cdot\boldsymbol{v}+\int_{\Omega_c} r(\eta)\,\mathbf{curl}\,\boldsymbol{R}\cdot\mathbf{curl}\,\boldsymbol{v}=-\mu\int_\Omega \frac{\partial \boldsymbol{H}_f}{\partial t}\cdot\boldsymbol{v}, \tag{3.8a}$$

$$\int_{\Omega_h} C_\rho\left(\frac{\partial\eta}{\partial t}+\boldsymbol{u}\cdot\nabla\eta\right)\varphi+a(\eta,\varphi)=\int_{\Omega_h} r(\eta)\,|\mathbf{curl}\,\boldsymbol{R}|^2\varphi+\mathcal{F}(\varphi), \tag{3.8b}$$

for all $\boldsymbol{v}\in\boldsymbol{X}$ and $\varphi\in Y\cap L^\infty(\Omega_h)$, where

$$a(\eta,\varphi):=\int_{\Omega_h}\kappa\nabla\eta\cdot\nabla\varphi+\int_{\Gamma_r}\left[\Psi(\eta+\theta_0)-\Psi(\theta_0)\right]\varphi+\int_{\Gamma_{\text{out}}} C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})\eta\varphi, \tag{3.9}$$

$$\mathcal{F}(\varphi):=-\int_{\Omega_h} C_\rho(\boldsymbol{u}\cdot\nabla\theta_0)\varphi-\int_{\Omega_h}\kappa\nabla\theta_0\cdot\nabla\varphi. \tag{3.10}$$

The rest of this paper will be devoted to the well-posedness of problem (3.8).

To end this section, we present some useful lemmas which play the key role in studying the well-posedness of (3.8). Given a constant $\tau>0$, let $\mathcal{L}$: $Y\to Y'$ be the nonlinear operator defined by

$$\langle\mathcal{L}\phi,w\rangle:=\int_{\Omega_h} C_\rho\left(\frac{1}{\tau}\phi+\boldsymbol{u}\cdot\nabla\phi\right)w+a(\phi,w)\qquad\forall\phi,w\in Y. \tag{3.11}$$

Lemma 3.2. *There exists a constant $C>0$ depending only on $\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(0,T;H^1(\Omega_h))}$, $\|\theta_0\|_{L^\infty(\Omega_h)}$, and material parameters such that*

$$\|\mathcal{L}\phi\|_{Y'}\le C\|\phi\|_{H^1(\Omega_h)}+C\sum_{j=1}^4\|\phi\|^j_{L^5(\Gamma_r)}\qquad\forall\phi\in Y.$$

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Proof. We estimate the nonlinear term first. Since $\theta_0>0$, we have

$$\begin{aligned}|\Psi(\phi+\theta_0)-\Psi(\theta_0)| \leq & h\left|(\phi+\theta_0)^3|\phi+\theta_0|-\theta_0^4\right|+\lambda|\phi| \\ \leq & |\phi|\left(h|\phi+\theta_0|^3+h\phi^2\theta_0+3h\phi\theta_0^2+3h\theta_0^3+\lambda\right).\end{aligned}$$

There exists a constant depending only on λ, h, Γ_r, and $\|\theta_0\|_{L^\infty(\Omega_h)}$ such that

$$\left|\int_{\Gamma_r}[\Psi(\phi+\theta_0)-\Psi(\theta_0)]w\right| \leq C\|w\|_{L^5(\Gamma_r)}\sum_{j=1}^{4}\|\phi\|_{L^5(\Gamma_r)}^j \qquad \forall w\in Y.$$

Since $H^1(\Omega_h)$ is embedded into $L^6(\Omega_h)$, the linear terms are estimated by using the Cauchy-Schwarz inequality

$$\begin{aligned}|\langle\mathcal{L}\phi,w\rangle| \leq & \frac{C}{\tau}\|\phi\|_{L^2(\Omega_h)}\|w\|_{L^2(\Omega_h)}+C\|\boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega_f)}|\phi|_{H^1(\Omega_f)}\|w\|_{L^6(\Omega_f)}+C|\phi|_{H^1(\Omega_h)}|w|_{H^1(\Omega_h)} \\ & +C\|\boldsymbol{u}\cdot\boldsymbol{n}\|_{L^2(\Gamma_{\rm out})}\|\phi\|_{L^4(\Gamma_{\rm out})}\|w\|_{L^4(\Gamma_{\rm out})}+\left|\int_{\Gamma_r}[\Psi(\phi+\theta_0)-\Psi(\theta_0)]w\right|.\end{aligned}$$

Notice that $\|\boldsymbol{u}\cdot\boldsymbol{n}\|_{L^2(\Gamma_{\rm out})}\leq C\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega_h)}$. Using the trace theorem and the embedding theorem, we also have

$$\|\phi\|_{L^4(\Gamma_{\rm out})}\leq C\|\phi\|_{H^{1/2}(\Gamma_{\rm out})}\leq C\|\phi\|_{H^1(\Omega_h)}, \qquad \|w\|_{L^4(\Gamma_{\rm out})}\leq C\|w\|_{H^1(\Omega_h)}.$$

Combining all estimates above, we complete the proof. □

Lemma 3.3. *There is a constant $C>0$ depending only on λ, h, Γ_r, and $\|\theta_0\|_{L^\infty(\Omega_h)}$ such that*

$$\langle\mathcal{L}\phi,\phi\rangle\geq C\left(\|\phi\|_{H^1(\Omega_h)}^2+\|\phi\|_{L^5(\Gamma_r)}^5\right) \qquad \forall\phi\in Y.$$

Proof. Remember that C_ρ is constant in Ω_f and $\mathrm{supp}(\boldsymbol{u})=\bar{\Omega}_f$. For any $\phi\in Y$, the integration by part yields

$$\int_{\Omega_h}(\boldsymbol{u}\cdot\nabla\phi)\phi=\int_{\Omega_f}(\boldsymbol{u}\cdot\nabla\phi)\phi=-\int_{\Omega_f}(\boldsymbol{u}\cdot\nabla\phi)\phi+\int_{\Gamma_f}(\boldsymbol{u}\cdot\boldsymbol{n})\phi^2-\int_{\Omega_f}\mathrm{div}\,\boldsymbol{u}\phi^2. \tag{3.12}$$

Since $\mathrm{div}\,\boldsymbol{u}=0$, we know that

$$\int_{\Omega_h}(\boldsymbol{u}\cdot\nabla\phi)\phi=\frac{1}{2}\int_{\Gamma_{\rm out}}(\boldsymbol{u}\cdot\boldsymbol{n})\phi^2\geq 0 \qquad \forall\phi\in Y. \tag{3.13}$$

Next, from (3.7), it is easy to see that $\Psi(t)=h|t|^3t+\lambda t$ satisfies

$$\Psi'(t)=4h|t|^3+\lambda>0.$$

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This indicates that Ψ is an increasing function. Careful calculations show that

$$[\Psi(v)-\Psi(w)](v-w)\geq\frac{h}{8}|v-w|^5+\lambda|v-w|^2. \tag{3.14}$$

If follows that

$$[\Psi(\phi+\theta_0)-\Psi(\theta_0)]\phi\geq\frac{h}{8}|\phi|^5+\lambda|\phi|^2 \qquad \forall\phi\in Y.$$

By the definition of a and the assumptions in (1.4), we find that

$$\langle\mathcal{L}\phi,\phi\rangle\geq\frac{1}{\tau}C_\rho\|\phi\|^2_{L^2(\Omega_h)}+\kappa_{\min}\|\nabla\phi\|^2_{L^2(\Omega_h)}+\frac{h}{8}\|\phi\|^5_{L^5(\Gamma_r)}\geq C\left(\|\phi\|^2_{H^1(\Omega_h)}+\|\phi\|^5_{L^5(\Gamma_r)}\right).$$

This completes the proof. □

Lemma 3.4. *The operator $\mathcal{L}: Y\to Y'$ is strictly monotone in the sense that*

$$\langle\mathcal{L}v-\mathcal{L}w,v-w\rangle>\min(\tau^{-1}C_\rho,\kappa_{\min})\|v-w\|^2_{H^1(\Omega_h)}+\frac{h}{8}\|v-w\|^5_{L^5(\Gamma_r)} \qquad \forall v,w\in Y.$$

Proof. From (3.14), the nonlinear term satisfies

$$[\Psi(v+\theta_0)-\Psi(w+\theta_0)](v-w)\geq\frac{h}{8}|v-w|^5+\lambda|v-w|^2.$$

Furthermore, using (3.13) and the fact that $\boldsymbol{u}\cdot\boldsymbol{n}\geq 0$ on Γ_{out}, we have

$$\begin{aligned}\langle\mathcal{L}v-\mathcal{L}w,v-w\rangle=&\frac{1}{\tau}\int_{\Omega_h}C_\rho(v-w)^2+\int_{\Omega_h}\boldsymbol{u}\cdot\nabla(v-w)(v-w)+\int_{\Omega_h}\kappa|\nabla(v-w)|^2\\&+\int_{\Gamma_r}[\Psi(v+\theta_0)-\Psi(w+\theta_0)](v-w)+\int_{\Gamma_{\text{out}}}C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})(v-w)^2\\\geq&\frac{1}{\tau}C_\rho\|v-w\|^2_{L^2(\Omega_h)}+\kappa_{\min}|v-w|^2_{H^1(\Omega_h)}+\frac{h}{8}\|v-w\|^5_{L^5(\Gamma_r)}.\end{aligned}$$

This completes the proof. □

Lemma 3.5. *The operator $\mathcal{L}: Y\to Y'$ is hemi-continuous, namely,*

$$F(t)=\langle\mathcal{L}(u+tv),w\rangle$$

is continuous on $[0,1]$ for any $u,v,w\in Y$.

Proof. For convenience we write $u(t)=u+tv$. For any $t_0\in[0,1]$, from (3.7) we have

$$|\Psi(u(t))-\Psi(u(t_0))|\leq h|u(t)v|\left[2u^2(t)+u(t)u(t_0)+u^2(t_0)\right]|t-t_0|\leq 4h(|u|+|v|)^4|t-t_0|.$$

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Then it is easy to see that

$$\begin{aligned}|F(t)-F(t_0)| &= |\langle \mathcal{L}u(t)-\mathcal{L}u(t_0),w\rangle| \\ &\le |t-t_0|\left[\int_{\Omega_h} C_\rho\left|\frac{v}{\tau}+\boldsymbol{u}\cdot\nabla v\right||w|+\int_{\Omega_h}\kappa|\nabla v\cdot\nabla w|+\int_{\Gamma_{\text{out}}} C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})|vw|\right] \\ &\quad+\int_{\Gamma_r}|\Psi(u(t)+\theta_0)-\Psi(u(t_0)+\theta_0)||w| \\ &\le |t-t_0|\left[\int_{\Omega_h} C_\rho\left|\frac{v}{\tau}+\boldsymbol{u}\cdot\nabla v\right||w|+\int_{\Omega_h}\kappa|\nabla v\cdot\nabla w|+\int_{\Gamma_{\text{out}}} C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})|vw|\right] \\ &\quad+|t-t_0|\int_{\Gamma_r} 4h(|u|+|v|)^4|w|.\end{aligned}$$

This shows that $F(t)$ is continuous for any $u,v,w\in Y$. □

4 A regularized problem

Notice that the righthand side of (3.8b) requires $\varphi\in L^\infty(\Omega_h)$ and makes the problem difficult to analyze. To overcome the issue, we first study the regularized problem: Given $0<\epsilon<1$, find $\boldsymbol{R}\in\boldsymbol{L}^2(0,T;\boldsymbol{X})$ and $\eta\in L^2(0,T;Y)$ such that $\boldsymbol{R}(0)=0$, $\eta(0)=0$, and

$$\mu\int_\Omega\frac{\partial\boldsymbol{R}}{\partial t}\cdot\boldsymbol{v}+\int_{\Omega_c} r(\eta)\,\mathbf{curl}\,\boldsymbol{R}\cdot\mathbf{curl}\,\boldsymbol{v}=-\mu\int_\Omega\frac{\partial\boldsymbol{H}_f}{\partial t}\cdot\boldsymbol{v}\qquad\forall\boldsymbol{v}\in\boldsymbol{X},\tag{4.1a}$$

$$\int_{\Omega_h} C_\rho\left(\frac{\partial\eta}{\partial t}+\boldsymbol{u}\cdot\nabla\eta\right)\varphi+a(\eta,\varphi)=\int_{\Omega_h}\left[r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2\right]_\epsilon\varphi+\mathcal{F}(\varphi)\qquad\forall\varphi\in Y,\tag{4.1b}$$

where $[g]_\epsilon$ is the cutoff of g defined by

$$[g]_\epsilon:=\frac{g}{1+\epsilon|g|},\quad\varepsilon>0.$$

It is clear that $[g]_\epsilon\in L^\infty(\Omega)$. If $g\in L^p(\Omega)$, then

$$\lim_{\epsilon\to0}\|[g]_\epsilon-g\|_{L^{p/2}(\Omega)}=0,\qquad 1\le p\le\infty.$$

Without causing confusions, we have used the same notations for the approximate solutions here. The purpose of this section is to prove the existence of the solution to (4.1).

4.1 Semi-discrete approximation

We shall use Rothe's method (cf. [19]) to study the solution of problem (4.1). Let N be a positive integer and let an equidistant partition of $[0,T]$ be given by

$$t_n=n\tau,\quad n=0,\cdots,N,\qquad\tau=T/N.$$

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The semi-discrete approximation to (4.1) reads: Given $\boldsymbol{R}_0=0$ and $\eta_0=0$, find $\boldsymbol{R}_n\in\boldsymbol{X}$ and $\theta_n\in Y$, $1\le n\le N$, such that

$$\mu\int_\Omega \frac{\boldsymbol{R}_n-\boldsymbol{R}_{n-1}}{\tau}\cdot\boldsymbol{v}+\int_\Omega r(\eta_{n-1})\,\mathbf{curl}\,\boldsymbol{R}_n\cdot\mathbf{curl}\,\boldsymbol{v}=-\mu\int_\Omega\frac{\delta\boldsymbol{H}_f}{\delta t}\cdot\boldsymbol{v}, \tag{4.2a}$$

$$\int_{\Omega_h} C_\rho\left(\frac{\eta_n-\eta_{n-1}}{\tau}+\boldsymbol{u}\cdot\nabla\eta_n\right)\varphi+a(\eta_n,\varphi)=\int_{\Omega_c}\left[r(\eta_{n-1})\,|\mathbf{curl}\,\boldsymbol{R}_n|^2\right]_\epsilon\varphi+\mathcal{F}(\varphi), \tag{4.2b}$$

for all $\boldsymbol{v}\in\boldsymbol{X}$ and $\varphi\in Y$, where $\dfrac{\delta}{\delta t}$ denotes the finite difference operator

$$\frac{\delta\boldsymbol{H}_f}{\delta t}=\frac{\boldsymbol{H}_f(t_n)-\boldsymbol{H}_f(t_{n-1})}{\tau}\qquad\text{in } [t_{n-1},t_n).$$

Clearly (4.2) can be solved sequentially, that is, the solution of (4.2a) is independent of (4.2b) for a given η_{n-1}. We shall study (4.2a) first and then study (4.2b) for given $\mathbf{curl}\,\boldsymbol{R}_n$.

4.2 Well-posedness of for the discrete eddy current problem

Let $\bar{\boldsymbol{R}}_\tau$ and $\boldsymbol{R}_\tau$ denote the piecewise constant and piecewise linear interpolations using the discrete solutions, namely,

$$\bar{\boldsymbol{R}}_\tau(\cdot,t)=\boldsymbol{R}_n,\qquad \boldsymbol{R}_\tau(\cdot,t)=l_n(t)\boldsymbol{R}_n+(1-l_n(t))\boldsymbol{R}_{n-1}, \tag{4.3}$$

for any $t\in(t_{n-1},t_n]$ and $1\le n\le N$, where $l_n(t):=(t-t_{n-1})/\tau$. Clearly we have

$$\bar{\boldsymbol{R}}_\tau\in\boldsymbol{L}^2(0,T;\boldsymbol{X}),\qquad \boldsymbol{R}_\tau\in\boldsymbol{C}(0,T;\boldsymbol{X}).$$

Lemma 4.1. *For any $1\le n\le N$ and for given $\eta_{n-1}\in Y$, the weak formulation* (4.2a) *has a unique solution $\boldsymbol{R}_n\in\boldsymbol{X}$.*

Proof. First we write (4.2) as: Find $\boldsymbol{R}_n\in\boldsymbol{X}$ such that

$$(\boldsymbol{R}_n,\boldsymbol{v})+\frac{\tau}{\mu}(r(\eta_{n-1})\,\mathbf{curl}\,\boldsymbol{R}_n,\mathbf{curl}\,\boldsymbol{v})=\left(\boldsymbol{R}_{n-1}-\tau\frac{\delta\boldsymbol{H}_f}{\delta t},\boldsymbol{v}\right)\qquad\forall\,\boldsymbol{v}\in\boldsymbol{X}. \tag{4.4}$$

Note that $1/\sigma_{\max}\le r(\eta_{n-1})\le 1/\sigma_{\min}$ in Ω_c. The left-hand side constructs a coercive bilinear form on $\boldsymbol{X}$. Moreover, the righthand side constructs a bounded linear functional on $\boldsymbol{X}$. An application of the Lax-Milgram lemma shows that problem (4.4) attains a unique solution. □

Lemma 4.2. *There is a constant C depending only on Ω_c, T, and material parameters such that*

$$\|\boldsymbol{R}_\tau\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{L}^2(\Omega))}+\|\mathbf{curl}\,\boldsymbol{R}_\tau\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}\le C\|\boldsymbol{H}_f\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}, \tag{4.5}$$

$$\|\bar{\boldsymbol{R}}_\tau\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{L}^2(\Omega))}+\|\mathbf{curl}\,\bar{\boldsymbol{R}}_\tau\|_{\boldsymbol{L}^2(0,T;\boldsymbol{L}^2(\Omega))}\le C\|\boldsymbol{H}_f\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}. \tag{4.6}$$

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Proof. Setting $v=\boldsymbol{R}_n$ in (4.4) shows that

$$(\boldsymbol{R}_n-\boldsymbol{R}_{n-1},\boldsymbol{R}_n)+\frac{\tau}{\mu}(r_{n-1}\mathbf{curl}\,\boldsymbol{R}_n,\mathbf{curl}\,\boldsymbol{R}_n)=-\tau\left(\frac{\delta \boldsymbol{H}_f}{\delta t},\boldsymbol{R}_n\right), \tag{4.7}$$

where $r_{n-1}:=r(\theta_{n-1})$. Using the initial value $\boldsymbol{R}_0=0$ and the inequality

$$2(\boldsymbol{R}_n-\boldsymbol{R}_{n-1},\boldsymbol{R}_n)\geq\|\boldsymbol{R}_n\|^2_{L^2(\Omega)}-\|\boldsymbol{R}_{n-1}\|^2_{L^2(\Omega)},$$

we have

$$\sum_{n=1}^{m}(\boldsymbol{R}_n-\boldsymbol{R}_{n-1},\boldsymbol{R}_n)\geq\frac{1}{2}\|\boldsymbol{R}_m\|^2_{L^2(\Omega)}. \tag{4.8}$$

Remember that $1/\sigma_{\max}\leq r(\theta_{n-1})\leq 1/\sigma_{\min}$. Summing up (4.7) with $1\leq n\leq m$ yields

$$\begin{aligned}\frac{1}{2}\|\boldsymbol{R}_m\|^2_{L^2(\Omega)}+\frac{1}{\mu\sigma_{\max}}\sum_{n=1}^{m}\tau\|\mathbf{curl}\,\boldsymbol{R}_n\|^2_{L^2(\Omega)}&\leq\sum_{n=1}^{m}\tau\left\|\frac{\delta \boldsymbol{H}_f}{\delta t}\right\|_{L^2(\Omega)}\|\boldsymbol{R}_n\|_{L^2(\Omega)}\\&\leq\|\boldsymbol{H}_f\|^2_{H^1(0,T;L^2(\Omega))}+\sum_{n=1}^{m}\tau\|\boldsymbol{R}_n\|^2_{L^2(\Omega)}.\end{aligned}$$

An application of Grönwall's inequality proves (4.5). Furthermore, (4.6) is a direct consequence of (4.5). □

4.3 Well-posedness of for the discrete heat equation

Similarly, we define the piecewise constant and piecewise linear interpolations for the temperature function

$$\bar{\eta}_\tau(\cdot,t)=\eta_n,\qquad \eta_\tau(\cdot,t)=l_n(t)\eta_n+(1-l_n(t))\eta_{n-1}, \tag{4.9}$$

for any $t\in(t_{n-1},t_n]$ and $1\leq n\leq N$. For convenience, we also write

$$\hat{\eta}_\tau(t):=\bar{\eta}_\tau(t-\tau)=\eta_{n-1}\qquad\forall t\in(t_{n-1},t_n].$$

Lemma 4.3. *For any $1\leq n\leq N$ and for given $\boldsymbol{R}_n\in\boldsymbol{X}$, the weak formulation* (4.2b) *has a unique solution $\eta_n\in Y$.*

Proof. From (3.11), (4.2b) can be written equivalently into an operator equation

$$\mathcal{L}\eta_n=f_n, \tag{4.10}$$

where $f_n\in Y'$ is defined by

$$\langle f_n,\varphi\rangle=\frac{1}{\tau}\int_{\Omega_h}C_\rho\eta_{n-1}\varphi+\int_{\Omega_c}\left[r(\eta_{n-1})|\mathbf{curl}\,\boldsymbol{R}_n|^2\right]_\epsilon\varphi+\mathcal{F}(\varphi).$$

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From Lemma 3.3, Lemma 3.4, and Lemma 3.5, we conclude that $\mathcal{L}$ is a coercive, strictly monotone, and hemi-continuous operator on Y. From [21, Theorem 26.A], problem (4.10) has a solution $\eta_n \in Y$.

Now we prove the uniqueness of the solution. Let $\eta_n, \tilde{\eta}_n \in Y$ be two solutions of (4.2b). Using Lemma 3.3, we have

$$0 = \langle \mathcal{L}\eta_n - \mathcal{L}\tilde{\eta}_n, \eta_n - \tilde{\eta}_n \rangle \geq \min(\tau^{-1}C_\rho, \kappa_{\min}) \|\eta_n - \tilde{\eta}_n\|^2_{H^1(\Omega_h)} + \frac{h}{8}\|\eta_n - \tilde{\eta}_n\|^5_{L^5(\Gamma_r)}.$$

We conclude $\eta_n = \tilde{\eta}_n$ in Ω_h. Then the solution is unique. □

Lemma 4.4. *Then there exists a constants $C > 0$ depending only on ϵ, T, $\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{H}^1(\Omega_h))}$, $\|\theta_0\|_{H^1(\Omega_h)}$, and the material parameters such that*

$$\|\eta_\tau\|_{L^\infty(0,T;L^2(\Omega_h))} + \|\nabla\eta_\tau\|_{L^2(0,T;L^2(\Omega_h))} + \|\eta_\tau\|_{L^5(0,T;L^5(\Gamma_r))} \leq C, \tag{4.11}$$

$$\|\bar{\eta}_\tau\|_{L^\infty(0,T;L^2(\Omega_h))} + \|\nabla\bar{\eta}_\tau\|_{L^2(0,T;L^2(\Omega_h))} + \|\bar{\eta}_\tau\|_{L^5(0,T;L^5(\Gamma_r))} \leq C. \tag{4.12}$$

Proof. Recall from (3.13) and (3.14) that

$$\int_{\Omega_h} C_\rho(\boldsymbol{u}\cdot\nabla\eta_n)\eta_n \geq 0, \qquad \int_{\Gamma_r} [\Psi(\eta_n+\theta_0) - \Psi(\theta_0)]\eta_n \geq \frac{h}{8}|\eta_n|^5.$$

Taking $\varphi = \eta_n$ in (4.2b), we find that

$$\int_{\Omega_h} \frac{C_\rho}{\tau}(\eta_n - \eta_{n-1})\eta_n + \int_{\Omega_h} \kappa|\nabla\eta_n|^2 + \frac{h}{8}\int_{\Gamma_r}|\eta_n|^5 \leq \int_{\Omega_c}\Big[r(\eta_{n-1})|\mathbf{curl}\,\boldsymbol{R}_n|^2\Big]_\epsilon \eta_n + \mathcal{F}(\eta_n). \tag{4.13}$$

The first term on the left-hand side of (4.13) satisfies

$$2\int_{\Omega_h} \frac{C_\rho}{\tau}(\eta_n - \eta_{n-1})\eta_n \geq \int_{\Omega_h} \frac{C_\rho}{\tau}|\eta_n|^2 - \int_{\Omega_h} \frac{C_\rho}{\tau}|\eta_{n-1}|^2. \tag{4.14}$$

For any $\epsilon > 0$, the first term on the righthand side satisfies

$$\left|\int_{\Omega_c}\Big[r(\eta_{n-1})|\mathbf{curl}\,\boldsymbol{R}_n|^2\Big]_\epsilon \eta_n\right| \leq \frac{|\Omega_c|}{\epsilon}\|\eta_n\|_{L^2(\Omega_c)}. \tag{4.15}$$

The second term on the righthand side can be estimated as follows

$$\begin{aligned}|\mathcal{F}(\eta_n)| \leq & C\|\theta_0\|_{L^\infty(\Gamma_{\text{out}})}\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Gamma_{\text{out}})}\|\eta_n\|_{L^2(\Gamma_{\text{out}})} \\ & + C\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega_h)}\|\nabla\theta_0\|_{\boldsymbol{L}^2(\Omega_h)}\|\eta_n\|_{L^3(\Omega_h)} + C\|\nabla\theta_0\|_{\boldsymbol{L}^2(\Omega_h)}\|\nabla\eta_n\|_{\boldsymbol{L}^2(\Omega_h)} \\ \leq & C\|\eta_n\|_{H^1(\Omega_h)},\end{aligned} \tag{4.16}$$

where the constant C in the last inequality depends only on $\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{H}^1(\Omega_h))}$, $\|\theta_0\|_{H^1(\Omega_h)}$, and the material parameters.

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Now inserting (4.14), (4.15), and (4.16) into (4.13) and summing up the result for $1\leq n\leq m$, we get

$$\begin{aligned}&\int_{\Omega_h} C_\rho|\eta_m|^2+\kappa_{\min}\sum_{n=1}^m\tau\|\nabla\eta_n\|^2_{\boldsymbol{L}^2(\Omega_h)}+\frac{h}{8}\sum_{n=1}^m\tau\|\eta_n\|^5_{L^5(\Gamma_r)}\\&\leq\frac{|\Omega_c|}{\epsilon}\sum_{n=1}^m\tau\|\eta_n\|_{L^2(\Omega_c)}+C\sum_{n=1}^m\tau\|\eta_n\|_{H^1(\Omega_h)}.\end{aligned}$$

Then there exist two positive constants C_0,C_1 depending only on ϵ, T, $\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{H}^1(\Omega_h))}$, $\|\theta_0\|_{H^1(\Omega_h)}$, and the material parameters such that

$$\|\eta_m\|^2_{L^2(\Omega_h)}+\sum_{n=1}^m\tau\|\nabla\eta_n\|^2_{\boldsymbol{L}^2(\Omega_h)}+\sum_{n=1}^m\tau\|\eta_n\|^5_{L^5(\Gamma_r)}\leq C_0+C_1\sum_{n=1}^m\tau\|\eta_n\|^2_{L^2(\Omega_h)}. \tag{4.17}$$

By Gronwall's inequality, we have

$$\|\eta_m\|_{L^2(\Omega_h)}\leq C\qquad\forall 1\leq m\leq N.$$

Inserting the above estimate into (4.17) leads to

$$\sum_{n=1}^m\tau\|\nabla\eta_n\|^2_{\boldsymbol{L}^2(\Omega_h)}+\sum_{n=1}^m\tau\|\eta_n\|^5_{L^5(\Gamma_r)}\leq C.$$

This shows (4.11). Similarly, (4.12) comes directly from (4.11). □

4.4 Existence of the solutions to the regularized problem

Since the solutions are discrete in time, Lemma 4.2 and Lemma 4.4 indicate that

$$\|\bar{\boldsymbol{R}}_\tau\|_{\boldsymbol{L}^2(Q)}+\|\mathbf{curl}\,\bar{\boldsymbol{R}}_\tau\|_{\boldsymbol{L}^2(Q)}+\|\boldsymbol{R}_\tau\|_{\boldsymbol{L}^2(Q)}+\|\mathbf{curl}\,\boldsymbol{R}_\tau\|_{\boldsymbol{L}^2(Q)}\leq C, \tag{4.18}$$

$$\|\eta_\tau\|_{\boldsymbol{L}^2(Q_h)}+\|\nabla\eta_\tau\|_{\boldsymbol{L}^2(Q_h)}+\|\eta_\tau\|_{L^5(S_r)}\leq C, \tag{4.19}$$

$$\|\bar{\eta}_\tau\|_{\boldsymbol{L}^2(Q_h)}+\|\nabla\bar{\eta}_\tau\|_{\boldsymbol{L}^2(Q_h)}+\|\bar{\eta}_\tau\|_{L^5(S_r)}\leq C, \tag{4.20}$$

where C depends only on ϵ, T, $\|\boldsymbol{H}_f\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}$, $\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{H}^1(\Omega_h))}$, $\|\theta_0\|_{H^1(\Omega_h)}$, and the material parameters. By (4.18), there exist a subsequence of $\{\bar{\boldsymbol{R}}_\tau\}$ and a subsequence of $\{\boldsymbol{R}_\tau\}$, which have common subscripts and are denoted by the same notations, such that

$$\bar{\boldsymbol{R}}_\tau\rightharpoonup\boldsymbol{R},\quad\boldsymbol{R}_\tau\rightharpoonup\boldsymbol{R}\quad\text{in } L^2(0,T;\boldsymbol{X}),$$

where $\boldsymbol{R}\in L^2(0,T;\boldsymbol{X})$ and "$\rightharpoonup$" denotes weak convergence of the sequences. Similarly, by (4.19) and (4.20), there exist a subsequence of $\{\bar{\eta}_\tau\}$ and a subsequence of $\{\eta_\tau\}$ with common subscripts such that

$$\bar{\eta}_\tau\rightharpoonup\eta,\quad\eta_\tau\rightharpoonup\eta\quad\text{in } L^2(0,T;Y).$$

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Furthermore, since $\boldsymbol{L}^2(Q)$, $L^2(Q_h)$ are embedded compactly into $\boldsymbol{L}^1(Q)$, $L^1(Q_h)$ respectively, we have

$$\bar{\boldsymbol{R}}_\tau \to \boldsymbol{R}, \quad \mathbf{curl}\,\bar{\boldsymbol{R}}_\tau \to \mathbf{curl}\,\boldsymbol{R} \quad \text{in } \boldsymbol{L}^1(Q), \qquad \bar{\eta}_\tau \to \eta \quad \text{in } L^1(Q_h), \tag{4.21}$$

where "$\to$" denotes strong convergence of the sequences. In this subsection, we shall prove that the limits $\boldsymbol{R}$, η solve the regularized problem (4.1). Without causing confusions, we shall always use $\{\bar{\boldsymbol{R}}_\tau\}$, $\{\boldsymbol{R}_\tau\}$, $\{\bar{\eta}_\tau\}$, $\{\eta_\tau\}$ to denote their convergent subsequences in the rest of this section.

Theorem 4.1. *The limit function $\boldsymbol{R}$ solves the weak form of eddy current model* (4.1a) *with the initial condition $\boldsymbol{R}(0)=0$.*

Proof. Remember that $\nabla C^\infty(\Omega)+\boldsymbol{C}_0^\infty(\Omega)\subset\boldsymbol{X}$. For any $\boldsymbol{v}(\boldsymbol{x},t)=\boldsymbol{w}(\boldsymbol{x})\xi(t)$ with $\boldsymbol{w}\in\nabla C^\infty(\Omega)+\boldsymbol{C}_0^\infty(\Omega)$ and $\xi\in\boldsymbol{C}_0^\infty(0,T)$, from (4.2a) we find that

$$\mu\int_Q \frac{\partial \boldsymbol{R}_\tau}{\partial t}\cdot\boldsymbol{v}+\int_Q r(\hat{\eta}_\tau)\,\mathbf{curl}\,\bar{\boldsymbol{R}}_\tau\cdot\mathbf{curl}\,\boldsymbol{v}=-\mu\int_Q \frac{\delta \boldsymbol{H}_f}{\delta t}\cdot\boldsymbol{v}. \tag{4.22}$$

It is clear that the first term on the left-hand side satisfies

$$\lim_{\tau\to0}\int_Q \frac{\partial \boldsymbol{R}_\tau}{\partial t}\cdot\boldsymbol{v}=-\lim_{\tau\to0}\int_Q \boldsymbol{R}_\tau\cdot\frac{\partial \boldsymbol{v}}{\partial t}=-\int_Q \boldsymbol{R}\cdot\frac{\partial \boldsymbol{v}}{\partial t}=\int_Q \frac{\partial \boldsymbol{R}}{\partial t}\cdot\boldsymbol{v}. \tag{4.23}$$

For the second term, since r is Lipschitz continuous, from (4.21) we know that

$$r(\hat{\eta}_\tau)\to r(\eta) \quad \text{a.e. in } Q.$$

Since r is bounded and $\mathbf{curl}\,\bar{\boldsymbol{R}}_\tau$ converges to $\mathbf{curl}\,\boldsymbol{R}$ strongly in $\boldsymbol{L}^1(Q)$, it follows that

$$\lim_{\tau\to0}\int_Q r(\hat{\eta}_\tau)\,\mathbf{curl}\,\bar{\boldsymbol{R}}_\tau\cdot\mathbf{curl}\,\boldsymbol{v}=\lim_{\tau\to0}\int_Q r(\hat{\eta}_\tau)\,\mathbf{curl}\,\boldsymbol{R}\cdot\mathbf{curl}\,\boldsymbol{v}=\int_Q r(\eta)\,\mathbf{curl}\,\boldsymbol{R}\cdot\mathbf{curl}\,\boldsymbol{v}. \tag{4.24}$$

Moreover, by the Lipschitz continuity of $\boldsymbol{f}$, the source magnetic field $\boldsymbol{H}_f$ is also Lipschitz-continuous in time. This yields

$$\lim_{\tau\to0}\int_Q \frac{\delta \boldsymbol{H}_f}{\delta t}\cdot\boldsymbol{v}=\int_Q \frac{\partial \boldsymbol{H}_f}{\partial t}\cdot\boldsymbol{v}. \tag{4.25}$$

From (4.22)-(4.25), we get

$$\mu\int_Q \frac{\partial \boldsymbol{R}}{\partial t}\cdot\boldsymbol{w}+\int_Q r(\eta)\,\mathbf{curl}\,\boldsymbol{R}\cdot\mathbf{curl}\,\boldsymbol{w}=-\mu\int_Q \frac{\partial \boldsymbol{H}_f}{\partial t}\cdot\boldsymbol{w}.$$

By the arbitrariness of ξ, the above equality yields

$$\mu\int_\Omega \frac{\partial \boldsymbol{R}}{\partial t}\cdot\boldsymbol{w}+\int_\Omega r(\eta)\,\mathbf{curl}\,\boldsymbol{R}\cdot\mathbf{curl}\,\boldsymbol{w}=-\mu\int_\Omega \frac{\partial \boldsymbol{H}_f}{\partial t}\cdot\boldsymbol{w} \qquad \forall\boldsymbol{w}\in\nabla C^\infty(\Omega)+\boldsymbol{C}_0^\infty(\Omega).$$

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By the density of $\nabla C^\infty(\Omega)+\boldsymbol{C}_0^\infty(\Omega)$ in $\boldsymbol{X}$, we conclude that

$$\mu\int_\Omega \frac{\partial \boldsymbol{R}}{\partial t}\cdot\boldsymbol{w}+\int_\Omega r(\eta)\,\mathbf{curl}\,\boldsymbol{R}\cdot\mathbf{curl}\,\boldsymbol{w}=-\mu\int_\Omega \frac{\partial \boldsymbol{H}_f}{\partial t}\cdot\boldsymbol{w}\qquad \forall \boldsymbol{w}\in\boldsymbol{X}.$$

It is left to show that $\boldsymbol{R}$ satisfies the initial condition. Take any $\boldsymbol{w}\in\boldsymbol{C}_0^\infty(\Omega)$ and let $\boldsymbol{v}(t)=(T-t)\boldsymbol{w}$. Using integration by part, we have

$$\begin{aligned}T\int_\Omega \boldsymbol{R}(0)\cdot\boldsymbol{w}&=-\int_0^T\int_\Omega \frac{\partial}{\partial t}(\boldsymbol{R}\cdot\boldsymbol{v})=\int_0^T\int_\Omega \boldsymbol{R}\cdot\boldsymbol{w}-\int_0^T\int_\Omega \frac{\partial \boldsymbol{R}}{\partial t}\cdot\boldsymbol{v}\\&=\lim_{\tau\to0}\int_0^T\int_\Omega \boldsymbol{R}_\tau\cdot\boldsymbol{w}+\int_0^T\int_\Omega \frac{1}{\mu}\left[r(\eta)\,\mathbf{curl}\,\boldsymbol{R}\cdot\mathbf{curl}\,\boldsymbol{v}+\mu\frac{\partial \boldsymbol{H}_f}{\partial t}\cdot\boldsymbol{v}\right].\end{aligned}$$

By arguments similar to (4.24)-(4.25), we can prove that

$$\begin{aligned}\int_0^T\int_\Omega\left[r(\eta)\,\mathbf{curl}\,\boldsymbol{R}\cdot\mathbf{curl}\,\boldsymbol{v}+\mu\frac{\partial \boldsymbol{H}_f}{\partial t}\cdot\boldsymbol{v}\right]&=\lim_{\tau\to0}\int_0^T\int_\Omega\left[r(\hat\eta_\tau)\,\mathbf{curl}\,\bar{\boldsymbol{R}}_\tau\cdot\mathbf{curl}\,\boldsymbol{v}+\mu\frac{\delta \boldsymbol{H}_f}{\delta t}\cdot\boldsymbol{v}\right]\\&=-\lim_{\tau\to0}\int_0^T\int_\Omega \mu\frac{\partial \boldsymbol{R}_\tau}{\partial t}\cdot\boldsymbol{v}.\end{aligned}$$

Combining the above two equations yields

$$T\int_\Omega \boldsymbol{R}(0)\cdot\boldsymbol{w}=\lim_{\tau\to0}\left[\int_0^T\int_\Omega \boldsymbol{R}_\tau\cdot\boldsymbol{w}-\int_0^T\int_\Omega \frac{\partial \boldsymbol{R}_\tau}{\partial t}\cdot\boldsymbol{v}\right]=\lim_{\tau\to0}T\int_\Omega \boldsymbol{R}_\tau(0)\cdot\boldsymbol{w}=0\quad\forall\boldsymbol{w}\in\boldsymbol{C}_0^\infty(\Omega).$$

This shows that $\boldsymbol{R}(0)=0$ in Ω. The proof is completed. □

Theorem 4.2. *The limit function η solves the weak heat equation* (4.1b) *with the initial condition* $\eta(0)=0$.

Proof. For any $\varphi(\boldsymbol{x},t)=v(\boldsymbol{x})\xi(t)$ with $\varphi\in Y\cap C^\infty(\Omega_h)$ and $\xi\in C_0^\infty(0,T)$, from (4.2b) we find that

$$\begin{aligned}&\int_{Q_h} C_\rho\left(\frac{\partial\eta_\tau}{\partial t}+\boldsymbol{u}\cdot\nabla\bar\eta_\tau\right)\varphi+\int_{Q_h}\kappa\nabla\bar\eta_\tau\cdot\nabla\varphi+\int_{S_r}\Psi(\bar\eta_\tau+\theta_0)\varphi+\int_{S_{\text{out}}}C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})\bar\eta_\tau\varphi\\&=\int_{Q_c}\left[r(\hat\eta_\tau)\left|\mathbf{curl}\,\bar{\boldsymbol{R}}_\tau\right|^2\right]_\epsilon\varphi+\int_{S_r}\Psi(\theta_0)\varphi+\int_0^T\mathcal{F}(\varphi).\end{aligned}\tag{4.26}$$

By arguments similar to (4.23), we have

$$\lim_{\tau\to0}\int_{Q_h}C_\rho\frac{\partial\eta_\tau}{\partial t}\varphi=\int_{Q_h}C_\rho\frac{\partial\eta}{\partial t}\varphi.\tag{4.27}$$

By the weak convergence of $\bar\eta_\tau$ in $L^2(0,T;Y)$, we have

$$\lim_{\tau\to0}\int_{Q_h}\left[C_\rho(\boldsymbol{u}\cdot\nabla\bar\eta_\tau)\varphi+\kappa\nabla\bar\eta_\tau\cdot\nabla\varphi\right]=\int_{Q_h}\left[C_\rho(\boldsymbol{u}\cdot\nabla\eta)\varphi+\kappa\nabla\eta\cdot\nabla\varphi\right].\tag{4.28}$$

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Since $L^5(S_r)$ is embedded compactly into $L^4(S_r)$, we know that $\bar{\eta}_\tau$ converges strongly to η in $L^4(S_r)$, and thus

$$\lim_{\tau\to 0}\int_{S_r}\Psi(\bar{\eta}_\tau+\theta_0)\varphi=\int_{S_r}\Psi(\eta+\theta_0)\varphi. \tag{4.29}$$

By Lemma 4.4, $\bar{\eta}_\tau$ is uniformly bounded in $L^2(0,T;L^2(\Gamma_{\text{out}}))$ and thus in $L^2(S_{\text{out}})$, so it has a subsequence which converges weakly to η in $L^2(S_{\text{out}})$. Then we have

$$\lim_{\tau\to 0}\int_{S_{\text{out}}}C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})\bar{\eta}_\tau\varphi=\int_{S_{\text{out}}}C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})\eta\varphi. \tag{4.30}$$

Now we consider the righthand side of (4.26). Since $\hat{\eta}_\tau$ converges strongly to η in $L^1(Q_c)$ and $\mathbf{curl}\,\bar{\boldsymbol{R}}_\tau$ converges strongly to $\boldsymbol{R}$ in $\boldsymbol{L}^1(Q_c)$, we know that

$$\lim_{\tau\to 0}\left[r(\hat{\eta}_\tau)|\mathbf{curl}\,\bar{\boldsymbol{R}}_\tau|^2\right]_\epsilon=\left[r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2\right]_\epsilon \quad \text{a.e. in } Q_c.$$

Since $\left[r(\hat{\eta}_\tau)|\mathbf{curl}\,\bar{\boldsymbol{R}}_\tau|^2\right]_\epsilon$ is bounded uniformly with respect to τ, we have

$$\lim_{\tau\to 0}\int_{Q_c}\left[r(\hat{\eta}_\tau)|\mathbf{curl}\,\bar{\boldsymbol{R}}_\tau|^2\right]_\epsilon\varphi=\int_{Q_c}\left[r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2\right]_\epsilon\varphi. \tag{4.31}$$

Inserting (4.27)-(4.31) into (4.26) shows that

$$\int_{Q_h}C_\rho\left(\frac{\partial\eta}{\partial t}+\boldsymbol{u}\cdot\nabla\eta\right)\varphi+\int_0^T a(\eta,\varphi)=\int_{Q_c}\left[r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2\right]_\epsilon\varphi+\int_0^T\mathcal{F}(\varphi).$$

Similarly, by the arbitrariness of ξ, it yields

$$\int_{\Omega_h}C_\rho\left(\frac{\partial\eta}{\partial t}+\boldsymbol{u}\cdot\nabla\eta\right)v+a(\eta,v)=\int_{\Omega_c}\left[r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2\right]_\epsilon v+\mathcal{F}(v)\qquad\forall v\in Y\cap C^\infty(\Omega_h).$$

By the density of $Y\cap C^\infty(\Omega_h)$ in Y, this shows that

$$\int_{\Omega_h}C_\rho\left(\frac{\partial\eta}{\partial t}+\boldsymbol{u}\cdot\nabla\eta\right)v+a(\eta,v)=\int_{\Omega_c}\left[r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2\right]_\epsilon v+\mathcal{F}(v)\qquad\forall v\in Y.$$

Moreover, by arguments similar to those in the proof of Theorem 4.1, we can prove that η also satisfies the initial condition $\eta(0)=0$. □

4.5 Stability of the solutions to the regularized problem

Now we prove some stability estimates for the solutions of the regularized problem. They will play the key role in the study of the well-posedness of (3.8).

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Lemma 4.5. *There is a constant C depending only on Ω_c, T, and material parameters such that*

$$\|\boldsymbol{R}\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{L}^2(\Omega))}+\|\mathbf{curl}\,\boldsymbol{R}\|_{\boldsymbol{L}^2(Q)}\leq C\left\|\boldsymbol{H}_f\right\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}.$$

Proof. Taking $\boldsymbol{v}=\boldsymbol{R}$ in (4.1a), we find that

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega \mu|\boldsymbol{R}|^2+\int_{\Omega_c} r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2=-\int_\Omega \mu\frac{\partial \boldsymbol{H}_f}{\partial t}\cdot\boldsymbol{R}.$$

Integrating the above equality with respect to $0\leq t\leq s$ and using the Cauchy-Schwarz inequality leads to

$$\int_\Omega \mu|\boldsymbol{R}(s)|^2+\int_0^s\int_{\Omega_c} r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2\leq C\left\|\boldsymbol{H}_f\right\|^2_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}+\int_0^s\int_\Omega \mu|\boldsymbol{R}(s)|^2\quad\forall s\in[0,T].$$

An application of Gronwall's inequality yields

$$\int_\Omega \mu|\boldsymbol{R}(s)|^2\leq C\left\|\boldsymbol{H}_f\right\|^2_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}\qquad\forall s\in[0,T].$$

Taking $s=T$, we also have

$$\int_{Q_c} r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2\leq C\left\|\boldsymbol{H}_f\right\|^2_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}.$$

The proof is completed. □

Lemma 4.6. *There exists a constant C only depending on $\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{H}^1(\Omega_h))}$ and the material parameters such that*

$$\|\eta\|_{L^\infty(0,T;L^1(\Omega_h))}+\|\eta\|^4_{L^4(S_r)}\leq C\left\|\boldsymbol{H}_f\right\|^2_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}+C\|\kappa\nabla\theta_0\|_{\boldsymbol{H}(\mathrm{div},\Omega_h)}.$$

Proof. First we define

$$g_\gamma(s)=\frac{1}{\gamma}\mathrm{sign}(s)\min(|s|,\gamma).$$

It is clear that g_γ is a bounded, absolutely continuous, and increasing function in $\mathbb{R}$. For any $w\in Y$, it is easy to see that $g_\gamma(w)\in Y$ and

$$\nabla g_\gamma(w)=\chi_{\{x\in\Omega_h:|w|<\gamma\}}\frac{1}{\gamma}\nabla w,\qquad \lim_{\gamma\to 0}g_\gamma(w)\to\mathrm{sign}(w)\quad\text{a.e. in }\Omega_h.\tag{4.32}$$

Here $\chi_{\{x\in\Omega_h:|w|<\gamma\}}$ is the characteristic function which satisfies

$$\chi_{\{x\in\Omega_h:|w|<\gamma\}}=\begin{cases}1 & \text{if } |w|<\gamma,\\ 0 & \text{if } |w|\geq\gamma.\end{cases}$$

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It is easy to see that $g_\gamma(w)\in Y$ for any $w\in Y$.

Taking the test function by $\varphi=g_\gamma(\eta)$ in (4.1b), we have

$$\int_{\Omega_h} C_\rho\left(\frac{\partial\eta}{\partial t}+\boldsymbol{u}\cdot\nabla\eta\right)g_\gamma(\eta)+a(\eta,g_\gamma(\eta))=\int_{\Omega_c}\left[r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2\right]_\epsilon g_\gamma(\eta)+\mathcal{F}(g_\gamma(\eta)). \quad (4.33)$$

We shall analyze (4.33) term by term. By the convergence in (4.32), we find that

$$\lim_{\gamma\to0}\int_{\Omega_h} C_\rho\frac{\partial\eta}{\partial t}g_\gamma(\eta)=\int_{\Omega_h} C_\rho\frac{\partial\eta}{\partial t}\mathrm{sign}(\eta)=\frac{\partial}{\partial t}\int_{\Omega_h} C_\rho|\eta|. \quad (4.34)$$

Let $G_\gamma(s)=\int_0^s g_\gamma(s')\mathrm{d}s'$ be one primitive function of $g_\gamma(s)$. It is easy to see $G_\gamma(\eta)=0$ on Γ_{in} and thus $G_\gamma(\eta)\in Y$. Since $\mathrm{div}\,\boldsymbol{u}=0$ and C_ρ is constant in Ω_f, we find that

$$\int_{\Omega_h} C_\rho\boldsymbol{u}\cdot\nabla\eta g_\gamma(\eta)=C_\rho\int_{\Omega_f}\boldsymbol{u}\cdot\nabla G_\gamma(\eta)=C_\rho\int_{\Gamma_{\mathrm{out}}}(\boldsymbol{u}\cdot\boldsymbol{n})G_\gamma(\eta).$$

Thus we deduce that

$$\lim_{\gamma\to0}\left|\int_{\Omega_h} C_\rho\boldsymbol{u}\cdot\nabla\eta g_\gamma(\eta)\right|\le C_\rho\lim_{\gamma\to0}\int_{\Gamma_{\mathrm{out}}}\boldsymbol{u}\cdot\boldsymbol{n}\int_0^{|\eta|}\min(1,s/\gamma)\mathrm{d}s\le\int_{\Gamma_{\mathrm{out}}} C_\rho\boldsymbol{u}\cdot\boldsymbol{n}|\eta|.$$

Now we consider the second term on the left hand side of (4.33). From (4.32) and (3.14), we have

$$\begin{aligned}
&\int_{\Omega_h}\kappa\nabla\eta\cdot\nabla g_\gamma(\eta)=\frac{1}{\gamma}\int_{\Omega_h}\chi_{\{\boldsymbol{x}\in\Omega_h:|w|<\gamma\}}\kappa|\nabla\eta|^2\ge0,\\
&\lim_{\gamma\to0}\int_{\Gamma_r}[\Psi(\eta+\theta_0)-\Psi(\theta_0)]g_\gamma(\eta)\ge\frac{h}{8}\|\eta\|^4_{L^4(\Gamma_r)}+\lambda\|\eta\|_{L^1(\Gamma_r)},\\
&\lim_{\gamma\to0}\int_{\Gamma_{\mathrm{out}}} C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})\eta g_\gamma(\eta)=\int_{\Gamma_{\mathrm{out}}} C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})|\eta|.
\end{aligned}$$

Combining all the terms above, we get

$$\lim_{\gamma\to0}a(\eta,g_\gamma(\eta))\ge\frac{h}{8}\|\eta\|^4_{L^4(\Gamma_r)}+\lambda\|\eta\|_{L^1(\Gamma_r)}+\int_{\Gamma_{\mathrm{out}}} C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})|\eta|. \quad (4.35)$$

From the definition of $[\cdot]_\epsilon$, it is easy to see that

$$\left|\int_{\Omega_c}\left[r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2\right]_\epsilon g_\gamma(\eta)\right|\le\int_{\Omega_c} r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2\le C\|\mathbf{curl}\,\boldsymbol{R}\|^2_{\boldsymbol{L}^2(\Omega_c)}. \quad (4.36)$$

Remember that $\kappa\nabla\theta_0\in\boldsymbol{H}(\mathrm{div},\Omega_h)$ and $|g_\gamma(\eta)|\le1$. There exists a constant $C>0$ depending only on Ω_h such that

$$\left|\int_{\Omega_h}\kappa\nabla\theta_0\cdot\nabla g_\gamma(\eta)\right|\le\left|\int_{\Omega_h}\nabla(\kappa\nabla\theta_0)g_\gamma(\eta)\right|+\left|\int_{\Gamma_h}\kappa\frac{\partial\theta_0}{\partial\boldsymbol{n}}g_\gamma(\eta)\right|\le C\|\kappa\nabla\theta_0\|_{\boldsymbol{H}(\mathrm{div},\Omega_h)}.$$

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With a constant $C>0$ depending only on Ω_h and the material parameters, we have

$$\begin{aligned}|\mathcal{F}(g_\gamma(\eta))| &\le \int_{\Gamma_{\text{out}}} C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})\theta_0 + \int_{\Omega_h} C_\rho|\boldsymbol{u}\cdot\nabla\theta_0| + \left|\int_{\Omega_h}\kappa\nabla\theta_0\cdot\nabla g_\gamma(\eta)\right| \\ &\le C\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\|\theta_0\|_{H^1(\Omega_h)} + C\|\kappa\nabla\theta_0\|_{\boldsymbol{H}(\text{div},\Omega_h)}. \end{aligned} \tag{4.37}$$

Finally, inserting (4.34), (4.35), (4.36), and (4.37) into (4.33) leads to

$$\frac{\partial}{\partial t}\int_{\Omega_h} C_\rho|\eta| + \frac{h}{8}\|\eta\|^4_{L^4(\Gamma_r)} \le C\|\mathbf{curl}\,\boldsymbol{R}\|^2_{\boldsymbol{L}^2(\Omega_c)} + C\|\kappa\nabla\theta_0\|_{\boldsymbol{H}(\text{div},\Omega_h)}.$$

The proof is completed by integrating the above inequality with respect to $0\le t\le s$ for any $s\in[0,T]$ and using Lemma 4.5. □

Lemma 4.7. *There exists an constant $C>0$ depending only on $\boldsymbol{u}$ and the material parameters such that*

$$\int_{Q_h}\frac{\gamma\kappa|\nabla\eta|^2}{(1+|\eta|)^{1+\gamma}} \le C\|\boldsymbol{H}_f\|^2_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))} + C\|\kappa\nabla\theta_0\|^2_{\boldsymbol{L}^2(\Omega)}, \quad 0<\gamma<1.$$

Proof. First we define $g(s)=\text{sign}(s)[1-(1+|s|)^{-\gamma}]$. Clearly $|g|\le 1$ and we have $g(w)\in Y$ for all $w\in Y$. Then it is easy to see

$$\nabla g(w) = \frac{\gamma\nabla w}{(1+|w|)^{\gamma+1}}.$$

Let G be the primitive function of g defined by

$$G(s) = \int_0^s g(s')\mathrm{d}s' = \frac{1}{1-\gamma} + |s| + \frac{1}{\gamma-1}(1+|s|)^{-\gamma+1} \ge 0.$$

This shows that

$$\int_0^T\int_{\Omega_h} C_\rho\frac{\partial\eta}{\partial t}g(\eta) = \int_0^T\frac{\partial}{\partial t}\int_{\Omega_h} C_\rho G(\eta) = \int_{\Omega_h} C_\rho G(\eta(T)) \ge 0.$$

Since $\text{div}\,\boldsymbol{u}=0$ and $\text{supp}(\boldsymbol{u})=\bar{\Omega}_f$, the second term can be estimated as follows

$$\int_0^T\int_{\Omega_h} C_\rho(\boldsymbol{u}\cdot\nabla\eta)g(\eta) = C_\rho\int_0^T\int_{\Omega_f}\boldsymbol{u}\cdot\nabla G(\eta) = C_\rho\int_0^T\int_{\Gamma_{\text{out}}}(\boldsymbol{u}\cdot\boldsymbol{n})G(\eta) \ge 0.$$

Taking the test function by $\varphi=g(\eta)$ in (4.1b) and using the two inequalities, we get

$$\begin{aligned}&\int_{Q_h}\kappa\nabla\eta\cdot\nabla g(\eta) + \int_{S_r}[\Psi(\eta+\theta_0)-\Psi(\theta_0)]g(\eta) + \int_{S_{\text{out}}} C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})\eta g(\eta) \\ &\le \int_{Q_c}\left[r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2\right]_\epsilon g(\eta) + \int_0^T\mathcal{F}(g(\eta)). \end{aligned} \tag{4.38}$$

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The first term on the left hand side satisfies

$$\int_{Q_h} \kappa \nabla \eta \cdot \nabla g(\eta) = \int_{Q_h} \frac{\gamma \kappa |\nabla \eta|^2}{(1+|\eta|)^{\gamma+1}}. \tag{4.39}$$

By (3.14), the second term is also non-negative, in fact, the integrands satisfy

$$[\Psi(\eta+\theta_0)-\Psi(\theta_0)]g(\eta) \geq \left[1-(1+|\eta|)^{-\gamma}\right]\left(\frac{h}{8}|\eta|^4+\lambda|\eta|\right) \geq 0. \tag{4.40}$$

It is easy to see that

$$\int_{\Gamma_{\text{out}}} C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})\eta g(\eta) = \int_{\Gamma_{\text{out}}} C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})|\eta|\left[1-(1+|\eta|)^{-\gamma}\right] \geq 0. \tag{4.41}$$

It is left to estimate the righthand side of (4.38). From (4.5), the first term satisfies

$$\int_{Q_c} \left[r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2\right]_\epsilon g(\eta) \leq \int_{Q_c} r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2 \leq C\left\|\boldsymbol{H}_f\right\|^2_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}. \tag{4.42}$$

The second term on the righthand side is estimated as follows

$$\int_0^T \mathcal{F}(g(\eta)) \leq \int_{Q_h} C_\rho|\boldsymbol{u}\cdot\nabla\theta_0| + \|\kappa\nabla\theta_0\|_{\boldsymbol{L}^2(\Omega_h)}\left[\int_{Q_h}\frac{\gamma|\nabla\eta|^2}{(1+|\eta|)^{\gamma+1}}\right]^{\frac{1}{2}}. \tag{4.43}$$

Finally, inserting (4.39)-(4.43) into (4.38) completes the proof. □

Lemma 4.8. *Suppose $1\leq q<5/4$. There exists a constant $C>0$ depending only on $\boldsymbol{u}$, q, and the material parameters such*

$$\|\eta\|_{L^{4q/3}(Q_h)} + \|\nabla\eta\|_{\boldsymbol{L}^q(Q_h)} \leq C.$$

Proof. Take $p=4q/3$ and $q_1=3q/(3-q)$. By the Cauchy-Schwarz inequality, we have

$$\int_{\Omega_h}|\xi(t)|^p \leq C\|\eta(t)\|^{q/3}_{L^1(\Omega_h)}\left[\int_{\Omega_h}|\eta(t)|^{q_1}\right]^{1-q/3}. \tag{4.44}$$

By the embedding of $W^{1,q}(\Omega)$ into $L^{q_1}(\Omega)$ and using Poincáre's inequality, we have

$$\left[\int_{\Omega_h}|\eta(t)|^{q_1}\right]^{1-q/3} \leq C\|\eta\|^q_{W^{1,q}(\Omega_h)} \leq C\|\nabla\eta\|^q_{\boldsymbol{L}^q(\Omega_h)}.$$

From Lemma 4.6, it is clear that

$$\int_{Q_h}|\eta|^p \leq C\|\eta\|^{q/3}_{L^\infty(0,T;L^1(\Omega_h))}\|\nabla\eta\|^q_{\boldsymbol{L}^q(Q_h)} \leq C\|\nabla\eta\|^q_{\boldsymbol{L}^q(Q_h)}. \tag{4.45}$$

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Take $\gamma=\frac{5-4q}{3}$. Then clearly $p=\frac{(1+\gamma)q}{2-q}$. From Lemma 4.7 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_{Q_h}|\nabla\eta|^q &\le \left[\int_{Q_h}\frac{|\nabla\eta|^2}{(1+|\eta|)^{1+\gamma}}\right]^{q/2}\left[\int_{Q_h}(1+|\eta|)^{\frac{(1+\gamma)q}{2-q}}\right]^{1-q/2}\\ &\le C\left[1+\|\eta\|^p_{L^p(Q_h)}\right]^{1-q/2}\le C_1+C_2\|\nabla\eta\|^{p(1-q/2)}_{\boldsymbol{L}^q(Q_h)},\end{aligned}$$

where C_1,C_2 are positive constants depending only on q, $\boldsymbol{u}$, and the material parameters. If $C_2\|\nabla\eta\|^{p(1-q/2)}_{\boldsymbol{L}^q(Q_h)}\le C_1$, we have

$$\|\nabla\eta\|^q_{\boldsymbol{L}^q(Q_h)}\le 2C_1.$$

Otherwise, we have

$$\|\nabla\eta\|^{(2q-1)/3}_{\boldsymbol{L}^q(Q_h)}\le 2C_2.$$

This shows that $\|\nabla\eta\|_{\boldsymbol{L}^q(Q_h)}\le C$. Finally, (4.45) implies $\|\eta\|_{\boldsymbol{L}^p(Q_h)}\le C$. □

5 Well-posedness of problem (3.8)

The purpose of this section is to prove the well-posedness of problem (3.8). The main idea is to study the limit of the solution of (4.1) as the regularization parameter $\epsilon\to 0$. To emphasize the dependency of the solutions on ϵ, we append them with the subscript ϵ, that is, denote the solutions by $(\boldsymbol{R}_\epsilon,\eta_\epsilon)$. Then problem (4.1) can be rewritten as follows: Find $\boldsymbol{R}_\epsilon\in\boldsymbol{L}^2(0,T;\boldsymbol{X})$ and $\eta_\epsilon\in L^2(0,T;Y)$ such that $\boldsymbol{R}_\epsilon(0)=0$, $\eta_\epsilon(0)=0$, and

$$\mu\int_\Omega\frac{\partial\boldsymbol{R}_\epsilon}{\partial t}\cdot\boldsymbol{v}+\int_{\Omega_c}r(\eta_\epsilon)\,\mathbf{curl}\,\boldsymbol{R}_\epsilon\cdot\mathbf{curl}\,\boldsymbol{v}=-\mu\int_\Omega\frac{\partial\boldsymbol{H}_f}{\partial t}\cdot\boldsymbol{v}\qquad\forall\boldsymbol{v}\in\boldsymbol{X},\tag{5.1a}$$

$$\int_{\Omega_h}C_\rho\left(\frac{\partial\eta_\epsilon}{\partial t}+\boldsymbol{u}\cdot\nabla\eta_\epsilon\right)\varphi+a(\eta_\epsilon,\varphi)=\int_{\Omega_h}\left[r(\eta_\epsilon)|\mathbf{curl}\,\boldsymbol{R}_\epsilon|^2\right]_\epsilon\varphi+\mathcal{F}(\varphi)\quad\forall\varphi\in Y.\tag{5.1b}$$

From Lemma 4.5, there exist an $\boldsymbol{R}\in L^2(0,T;\boldsymbol{X})$ and a subsequence of $\boldsymbol{R}_\epsilon$ denoted by the same notation such that

$$\boldsymbol{R}_\epsilon\rightharpoonup\boldsymbol{R}\quad\text{in }\boldsymbol{L}^2(0,T;\boldsymbol{X}),\qquad\boldsymbol{R}_\epsilon\to\boldsymbol{R},\quad\mathbf{curl}\,\boldsymbol{R}_\epsilon\to\mathbf{curl}\,\boldsymbol{R}\quad\text{in }\boldsymbol{L}^1(Q).\tag{5.2}$$

Furthermore, from Lemma 4.8, there exists an $\eta\in W^{1,q}(Q_h)$ and a subsequence of η_ϵ denoted by the same notation such that

$$\eta_\epsilon\to\eta\quad\text{in }W^{1,q}(Q_h)\qquad\forall q\in[1,5/4).\tag{5.3}$$

Since r is bounded and Lipschitz continuous, we know that

$$r(\eta_\epsilon)\to r(\eta)\quad\text{a.e. in }Q_h.$$

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Theorem 5.1. *Let $\boldsymbol{R}$ be the limit of the approximate solutions $\boldsymbol{R}_\epsilon$ as $\epsilon\to 0$. Then $\boldsymbol{R}$ satisfies the weak formulation* (3.8a) *together with the initial condition $\boldsymbol{R}(0)=0$.*

Proof. The proof is parallel to that of Lemma 4.1. We omit the details here. □

Lemma 5.1. *There exists a subsequence of $\boldsymbol{R}_\epsilon$ denoted still by the same notation such that*

$$\lim_{\epsilon\to 0}\left\| r(\eta_\epsilon)|\mathbf{curl}\,\boldsymbol{R}_\epsilon|^2 - r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2\right\|_{\boldsymbol{L}^1(Q_c)}=0.$$

Proof. Let $\psi_\epsilon\in H^1(\Omega)/\mathbb{R}$ solve the elliptic problem

$$\int_\Omega \nabla\psi_\epsilon\cdot\nabla\varphi=\int_\Omega(\boldsymbol{R}_\epsilon-\boldsymbol{R})\cdot\nabla\varphi \qquad \forall\,\varphi\in H^1(\Omega)/\mathbb{R}.$$

It is clear that $\boldsymbol{w}_\epsilon:=\boldsymbol{R}_\epsilon-\boldsymbol{R}-\nabla\psi_\epsilon\in\boldsymbol{H}_0(\mathrm{div},\Omega)\cap\boldsymbol{H}(\mathbf{curl},\Omega)$ and $\mathrm{div}\,\boldsymbol{w}_\epsilon=0$ in Ω. There exists a constant $C>0$ independent of ϵ such that

$$\|\boldsymbol{w}_\epsilon(T)\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}+\|\psi_\epsilon(T)\|_{H^1(\Omega)}\le C\|\boldsymbol{R}_\epsilon(T)-\boldsymbol{R}(T)\|_{\boldsymbol{H}(\mathbf{curl},\Omega)}\le C.$$

By the compact embedding of $\boldsymbol{H}_0(\mathrm{div},\Omega)\cap\boldsymbol{H}(\mathbf{curl},\Omega)$ into $\boldsymbol{L}^2(\Omega)$ [2], there exist a $\boldsymbol{w}\in\boldsymbol{X}$ and a subsequence of $\boldsymbol{w}_\epsilon(T)$ denoted by the same notation such that

$$\lim_{\epsilon\to 0}\|\boldsymbol{w}_\epsilon(T)-\boldsymbol{w}\|_{\boldsymbol{L}^2(\Omega)}=0. \tag{5.4}$$

Since $\boldsymbol{R}_\epsilon\rightharpoonup\boldsymbol{R}$ in $\boldsymbol{L}^2(0,T;\boldsymbol{X})$, we know that

$$\psi_\epsilon\rightharpoonup 0 \quad \text{in } L^2(0,T;H^1(\Omega)), \qquad \boldsymbol{w}_\epsilon\rightharpoonup 0 \quad \text{in } \boldsymbol{L}^2(0,T;\boldsymbol{X}).$$

This shows that $\boldsymbol{w}=0$ in Ω.

Since $\mathbf{curl}\,\boldsymbol{w}_\epsilon=\mathbf{curl}(\boldsymbol{R}_\epsilon-\boldsymbol{R})$ is supported only in Ω_c, taking $\boldsymbol{v}=\boldsymbol{w}_\epsilon$ in both (3.8a) and (5.1a), we find that

$$\int_\Omega\mu\frac{\partial(\boldsymbol{R}_\epsilon-\boldsymbol{R})}{\partial t}\cdot\boldsymbol{w}_\epsilon+\int_\Omega r(\eta_\epsilon)|\mathbf{curl}\,\boldsymbol{w}_\epsilon|^2=\int_\Omega[r(\eta)-r(\eta_\epsilon)]\mathbf{curl}\,\boldsymbol{R}\cdot\mathbf{curl}\,\boldsymbol{w}_\epsilon.$$

Since $\mathrm{div}(\boldsymbol{w}_\epsilon-\boldsymbol{w})=0$, the first term on the left hand side satisfies

$$\int_\Omega\mu\frac{\partial(\boldsymbol{R}_\epsilon-\boldsymbol{R})}{\partial t}\cdot\boldsymbol{w}_\epsilon=\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega\mu|\boldsymbol{w}_\epsilon|^2+\mu\int_\Omega\nabla\frac{\partial\psi_\epsilon}{\partial t}\cdot\boldsymbol{w}_\epsilon=\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega\mu|\boldsymbol{w}_\epsilon|^2.$$

This shows that

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega\mu|\boldsymbol{w}_\epsilon|^2+\int_\Omega r(\eta_\epsilon)|\mathbf{curl}\,\boldsymbol{w}_\epsilon|^2=\int_\Omega[r(\eta)-r(\eta_\epsilon)]\mathbf{curl}\,\boldsymbol{R}\cdot\mathbf{curl}\,\boldsymbol{w}_\epsilon.$$

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Integrating the above equality in time on $[0,T]$, we get

$$\frac{1}{2}\int_\Omega \mu|\boldsymbol{w}_\epsilon(T)|^2+\int_Q r(\eta_\epsilon)|\mathbf{curl}\,\boldsymbol{w}_\epsilon|^2=\int_Q [r(\eta)-r(\eta_\epsilon)]\,\mathbf{curl}\,\boldsymbol{R}\cdot\mathbf{curl}\,\boldsymbol{w}_\epsilon.$$

Since $r(\eta_\epsilon)$ converges to $r(\eta)$ a.e. in Q and $\mathbf{curl}\,\boldsymbol{R}\in\boldsymbol{L}^2(Q)$, we know that

$$\lim_{\epsilon\to 0}\int_Q |r(\eta)-r(\eta_\epsilon)||\mathbf{curl}\,\boldsymbol{R}|^2=0.$$

Therefore, the righthand side tends to zero as $\epsilon\to 0$. Together with (5.4), this yields

$$\lim_{\epsilon\to 0}\int_{Q_c} r(\eta_\epsilon)|\mathbf{curl}\,\boldsymbol{w}_\epsilon|^2=0.$$

Finally, we conclude that

$$\begin{aligned}&\lim_{\epsilon\to 0}\int_Q \Big|r(\eta_\epsilon)|\mathbf{curl}\,\boldsymbol{R}_\epsilon|^2-r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2\Big|\\ \le&\lim_{\epsilon\to 0}\int_Q r(\eta_\epsilon)|\mathbf{curl}\,\boldsymbol{w}_\epsilon|^2+\lim_{\epsilon\to 0}\int_{Q_c}|r(\eta_\epsilon)-r(\eta)||\mathbf{curl}\,\boldsymbol{R}|^2=0.\end{aligned}$$

This completes the proof. □

Theorem 5.2. *Let η be the limit of the approximate solutions η_ϵ as $\epsilon\to 0$. Then η satisfies the weak formulation* (3.8b) *together with the initial condition $\eta(0)=0$.*

Proof. First we define, for any $\gamma>0$,

$$g_\gamma(s):=\frac{1}{1+\gamma s^4},\quad G_\gamma(s):=\int_0^s g_\gamma(s')\mathrm{d}s',\qquad s\in(0,+\infty).$$

Clearly G_γ is a primitive function of g_γ and they satisfy $|g_\gamma|\le 1$, $|G_\gamma|\le T$. Since $\eta_\epsilon\to\eta$ in $W^{1,6/5}(Q_h)$ as $\epsilon\to 0$, it is easy to see $g_\gamma(\eta_\epsilon)\to g_\gamma(\eta)$ in $W^{1,6/5}(Q_h)$. Then $G_\gamma(\eta_\epsilon)\to G_\gamma(\eta)$ in $W^{1,6/5}(Q_h)$. Moreover, since $g_\gamma(\eta_\epsilon)$ and $G_\gamma(\eta_\epsilon)$ are uniformly bounded with respect to ϵ, we infer that

$$g_\gamma(\eta_\epsilon)\to g_\gamma(\eta),\quad G_\gamma(\eta_\epsilon)\to G_\gamma(\eta)\qquad\text{a. e. in } Q_h.$$

For any $v\in C_0^\infty(0,T;Y\cap C^\infty(\Omega_h))$, we write

$$\varphi_\epsilon=v g_\gamma(\eta_\epsilon),\qquad \varphi=v g_\gamma(\eta).$$

Clearly we have $\varphi_\epsilon\to\varphi$ a. e. in Q_h. The proof consists of two steps.

Step 1. Passage to limit as $\epsilon\to 0$. From (5.1b), we have

$$\begin{aligned}&\int_{Q_h} C_\rho\left(\frac{\partial\eta_\epsilon}{\partial t}+\boldsymbol{u}\cdot\nabla\eta_\epsilon\right)\varphi_\epsilon+\int_{Q_h}\kappa\nabla\eta_\epsilon\cdot\nabla\varphi_\epsilon+\int_{S_r}\Psi(\eta_\epsilon+\theta_0)\varphi_\epsilon+\int_{S_{\text{out}}}C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})\eta_\epsilon\varphi_\epsilon\\ =&\int_{Q_c}\Big[r(\eta_\epsilon)|\mathbf{curl}\,\boldsymbol{R}_\epsilon|^2\Big]_\epsilon\varphi_\epsilon+\int_{S_r}\Psi(\theta_0)\varphi_\epsilon+\int_0^T\mathcal{F}(\varphi_\epsilon).\end{aligned}\tag{5.5}$$

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It is easy to see that

$$\lim_{\epsilon\to 0}\int_{Q_h} C_\rho \frac{\partial \eta_\epsilon}{\partial t}\varphi_\epsilon = \lim_{\epsilon\to 0}\int_{Q_h} C_\rho \frac{\partial G_\gamma(\eta_\epsilon)}{\partial t} v = -\lim_{\epsilon\to 0}\int_{Q_h} C_\rho G_\gamma(\eta_\epsilon)\frac{\partial v}{\partial t} = -\int_{Q_h} C_\rho G_\gamma(\eta)\frac{\partial v}{\partial t}$$
$$= \int_{Q_h} C_\rho \frac{\partial \eta}{\partial t}\varphi. \tag{5.6}$$

Since $G_\gamma(\eta_\epsilon)\to G_\gamma(\eta)$ in $W^{1,6/5}(Q_h)$ and $\boldsymbol{u}\in \boldsymbol{L}^6(Q_h)$, we have

$$\lim_{\epsilon\to 0}\int_{Q_h} C_\rho \boldsymbol{u}\cdot\nabla\eta_\epsilon\varphi_\epsilon = \lim_{\epsilon\to 0}\int_{Q_h} C_\rho \boldsymbol{u}\cdot\nabla G_\gamma(\eta_\epsilon) v = \int_{Q_h} C_\rho \boldsymbol{u}\cdot\nabla G_\gamma(\eta) v = \int_{Q_h} C_\rho \boldsymbol{u}\cdot\nabla\eta\varphi. \tag{5.7}$$

From (5.3), the second term on the left hand side of (5.5) satisfies

$$\lim_{\epsilon\to 0}\int_{Q_h} \kappa\nabla\eta_\epsilon\cdot\nabla\varphi_\epsilon = \lim_{\epsilon\to 0}\int_{Q_h} \kappa g_\gamma(\eta_\epsilon)\nabla\eta_\epsilon\cdot\nabla v - \lim_{\epsilon\to 0}\int_{Q_h}\frac{4\gamma\kappa\eta_\epsilon^3}{(1+\gamma\eta_\epsilon^4)^2}|\nabla\eta_\epsilon|^2 v$$
$$= \int_{Q_h}\kappa g_\gamma(\eta)\nabla\eta\cdot\nabla v - \int_{Q_h}\frac{4\gamma\kappa\eta^3}{(1+\gamma\eta^4)^2}|\nabla\eta|^2 v = \int_{Q_h}\kappa\nabla\eta\cdot\nabla\varphi. \tag{5.8}$$

From Lemma 4.6, there exists a subsequence denoted by the same notation such that $\eta_\epsilon\to\eta$ in $L^3(S_r)$. This implies that

$$\big(h|\eta_\epsilon+\theta_0|^3+\lambda\big)g_\gamma(\eta_\epsilon)\to\big(h|\eta+\theta_0|^3+\lambda\big)g_\gamma(\eta) \qquad \text{a. e. in } S_r.$$

The third term on the left hand side of (5.5) satisfies

$$\lim_{\epsilon\to 0}\int_{S_r}\Psi(\eta_\epsilon+\theta_0)\varphi_\epsilon = \int_{S_r}\Psi(\eta+\theta_0)\varphi. \tag{5.9}$$

Moreover, Lemma 4.8 indicates that η_ϵ is uniformly bounded in $L^{8/5}(S_{\text{out}})$, thus attains a subsequence converging to η in $L^{4/3}(S_{\text{out}})$. Since $\boldsymbol{u}\in\boldsymbol{L}^\infty(0,T;\boldsymbol{H}^1(\Omega_h))$, we have $\boldsymbol{u}\in\boldsymbol{L}^\infty(0,T;\boldsymbol{H}^{1/2}(\Gamma_{\text{out}}))\hookrightarrow\boldsymbol{L}^4(S_{\text{out}})$. Then

$$\lim_{\epsilon\to 0}\int_{S_{\text{out}}} C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})\eta_\epsilon\varphi_\epsilon = \int_{S_{\text{out}}} C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})\eta\varphi. \tag{5.10}$$

For the righthand side of (5.5), from Lemma 5.1, we easily get

$$\lim_{\epsilon\to 0}\left[\int_{Q_c}\Big[r(\eta_\epsilon)|\mathbf{curl}\,\boldsymbol{R}_\epsilon|^2\Big]_\epsilon\varphi_\epsilon + \int_{S_r}\Psi(\theta_0)\varphi_\epsilon\right] = \int_{Q_c} r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2\varphi + \int_{S_r}\Psi(\theta_0)\varphi. \tag{5.11}$$

From (3.10) and the convergence of $G_\gamma(\eta_\epsilon)$, we have

$$\lim_{\epsilon\to 0}\int_0^T \mathcal{F}(\varphi_\epsilon) = \int_0^T \mathcal{F}(\varphi). \tag{5.12}$$

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Inserting (5.6)-(5.12) into (5.5) leads to

$$\begin{aligned}&\int_{Q_h} C_\rho\left(\frac{\partial\eta}{\partial t}+\boldsymbol{u}\cdot\nabla\eta\right)\varphi+\int_{Q_h}\kappa\nabla\eta\cdot\nabla\varphi+\int_{S_r}\Psi(\eta+\theta_0)\varphi+\int_{S_{\text{out}}}C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})\eta\varphi\\&=\int_{Q_c} r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2\varphi+\int_{S_r}\Psi(\theta_0)\varphi+\int_0^T\mathcal{F}(\varphi).\end{aligned}\tag{5.13}$$

The initial condition $\eta(0)=0$ can be proved similarly as in the proof of Theorem 4.1. We do not elaborate on the details here.

Step 2. Passage to limit as $\gamma\to 0$. Notice that $g_\gamma(\eta)\to 1$ a. e. in Q_h as $\gamma\to 0$. This implies

$$\lim_{\gamma\to 0}\int_{Q_h}\kappa\nabla\eta\cdot\nabla g_\gamma(\eta)=\lim_{\gamma\to 0}\int_{Q_h}\frac{-4\gamma\kappa\eta^3}{(1+\gamma\eta^4)^2}|\nabla\eta|^2=0.$$

We find that

$$\lim_{\gamma\to 0}\left[\int_{Q_h}C_\rho\left(\frac{\partial\eta}{\partial t}+\boldsymbol{u}\cdot\nabla\eta\right)\varphi+\int_{Q_h}\kappa\nabla\eta\cdot\nabla\varphi\right]=\int_{Q_h}C_\rho\left(\frac{\partial\eta}{\partial t}+\boldsymbol{u}\cdot\nabla\eta\right)v+\int_{Q_h}\kappa\nabla\eta\cdot\nabla v.$$

Since $\|\eta_\epsilon\|_{L^4(S_r)}\le C$, there exists a subsequence such that $\|\eta\|_{L^4(S_r)}\le\underline{\lim}_{\epsilon\to 0}\|\eta_\epsilon\|_{L^4(S_r)}\le C$. We conclude that

$$\lim_{\gamma\to 0}\left[\int_{S_r}\Psi(\eta+\theta_0)\varphi+\int_{S_{\text{out}}}C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})\eta\varphi\right]=\int_{S_r}\Psi(\eta+\theta_0)v+\int_{S_{\text{out}}}C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})\eta v.$$

Similarly, the righthand side converges to

$$\int_{Q_c} r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2 v+\int_{S_r}\Psi(\theta_0)v+\int_0^T\mathcal{F}(v).$$

Collecting all the above equalities and using (5.13), we finally get

$$\begin{aligned}&\int_{Q_h} C_\rho\left(\frac{\partial\eta}{\partial t}+\boldsymbol{u}\cdot\nabla\eta\right)v+\int_{Q_h}\kappa\nabla\eta\cdot\nabla v+\int_{S_r}\Psi(\eta+\theta_0)\varphi+\int_{S_{\text{out}}}C_\rho(\boldsymbol{u}\cdot\boldsymbol{n})\eta v\\&=\int_{Q_c} r(\eta)|\mathbf{curl}\,\boldsymbol{R}|^2 v+\int_{S_r}\Psi(\theta_0)v+\int_0^T\mathcal{F}(v)\qquad\forall v\in C_0^\infty(0,T;Y\cap C^\infty(\Omega_h)).\end{aligned}$$

By the arbitrariness of v, we conclude (3.8b). □

Now we present the main result of this paper.

Theorem 5.3. *Let $\boldsymbol{R},\eta$ be the limits of $\boldsymbol{R}_\epsilon,\eta_\epsilon$ given* (5.2) *and* (5.3) *respectively. Then $(\boldsymbol{R},\eta)$ solves the weak problem* (3.8)*. Furthermore,*

$$\|\eta\|_{L^{4q/3}(Q_h)}+\|\nabla\eta\|_{\boldsymbol{L}^q(Q_h)}+\|\boldsymbol{R}\|_{\boldsymbol{L}^2(0,T;\boldsymbol{H}(\mathbf{curl},\Omega))}\le C\qquad\forall q\in[1,5/4),$$

where the constant C depends only on $\|\kappa\nabla\theta_0\|_{\boldsymbol{H}(\text{div},\Omega_h)}$, $\|\boldsymbol{u}\|_{\boldsymbol{L}^\infty(0,T;\boldsymbol{H}^1(\Omega_h))}$, $\|\boldsymbol{H}_f\|_{\boldsymbol{H}^1(0,T;\boldsymbol{L}^2(\Omega))}$, T, q, and the material parameters.

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