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RANDOM SUBSETS OF SELF-AFFINE FRACTALS

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Abstract. We find the almost-sure Hausdorff and box-counting dimensions of random subsets of self-affine fractals obtained by selecting subsets at each stage of the hierarchical construction in a statistically self-similar manner.

§1. *Introduction.* An iterated function system (IFS) consists of a finite family of contractions $S_1, \ldots, S_N : \mathbb{R}^n \to \mathbb{R}^n$ with $N \geq 2$. It is well-known that this defines a unique non-empty compact set $F \subseteq \mathbb{R}^n$,

$$F = \bigcup_{i=1}^{N} S_i(F), \tag{1.1}$$

called the *attractor* of the IFS; see, for example, [9, 13]. If the S_i are similarities, we say that F is *self-similar*; if the S_i are affine mappings, F is said to be *self-affine*.

The attractor F may be obtained by an iterative procedure. We index compositions of the S_i in the usual way by sequences $\mathbf{i} = (i_1 \ldots i_k)$ where $i_j \in \{1, \ldots, N\}$, so that $S_\mathbf{i} \equiv S_{i_1} \circ \cdots \circ S_{i_k}$ etc, and we write $|\mathbf{i}|$ for the length of the sequence $\mathbf{i}$. If B is some (large) ball or other compact set such that $S_i(B) \subseteq B$ for all i, then the sequence of compact sets $F_k \equiv \bigcup_{|\mathbf{i}|=k} S_\mathbf{i}(B)$ decreases to F as $k \to \infty$. The set F_{k+1} may be thought of as being obtained from F_k upon replacing each of its components $S_\mathbf{i}(B)$ by the N subsets $S_\mathbf{i}S_1(B), \ldots, S_\mathbf{i}S_N(B)$. In this paper, we instead consider the random compact subsets of F obtained by replacing each $S_\mathbf{i}(B)$ by a *random* subcollection of the $S_\mathbf{i}S_1(B), \ldots, S_\mathbf{i}S_N(B)$. If the probability distribution of the choice of the subcollection indices $\{1, \ldots, N\}$ is the same at each stage, this construction could be regarded as a particular example of a *statistically self-affine* set.

A great deal is known about the Hausdorff and box-counting dimensions of self-similar sets—see, for example, [9, 13]—and also of their natural random counterparts [6, 12, 17]. The random subset construction above has attracted particular attention in the context of the Mandelbrot percolation process and its variants [3, 16]. Dimensions of self-affine sets are more awkward to analyse [2, 7, 8, 13, 15]. Very little has been done to extend this theory to the random setting: working in a submultiplicative rather than a multiplicative setting presents many challenges. Gatzouras and Lalley [11] have considered a

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specific random construction based on generalized Sierpiński carpets, Jordan *et al* [14] looked at random perturbations of self-affine constructions while keeping the linear part of the affine maps fixed, and Jinghu and Lixin [23] worked in a setting that is close to self-similar and so avoids the main difficulties. Here we investigate the random subset variant indicated above, basing our treatment on random subtrees of the N-ary rooted tree that underlies the IFS construction.

§2. *Description of the model.* In this section we describe the random construction in detail.

We code the IFS construction in the usual way. For each $k = 0, 1, 2, \ldots$, let $\mathbf{J}_k = \{(i_1 \ldots i_k) : 1 \le i_j \le N\}$ be the set of sequences or words of length k, with $\mathbf{J}_0$ containing only the null sequence $\emptyset$. Let $\mathbf{J} = \bigcup_{k=0}^{\infty} \mathbf{J}_k$ be the set of all finite sequences, and let $\mathbf{J}_\infty = \{(i_1 i_2 \ldots) : 1 \le i_j \le N\}$ be the corresponding set of infinite sequences. We abbreviate members of $\mathbf{J}$ and $\mathbf{J}_\infty$ as $\mathbf{i} = (i_1 \ldots i_k)$ and $\mathbf{i} = (i_1 i_2 \ldots)$, and we denote the number of terms in $\mathbf{i} \in \mathbf{J}$ by $|\mathbf{i}|$. If $\mathbf{i}, \mathbf{j} \in \mathbf{J}$ or if $\mathbf{i} \in \mathbf{J}$ and $\mathbf{j} \in \mathbf{J}_\infty$, we denote by $\mathbf{ij}$ the sequence obtained by juxtaposition of the terms of $\mathbf{i}$ and $\mathbf{j}$. If $\mathbf{i}$ is a curtailment of $\mathbf{j}$, i.e. if $\mathbf{j} = \mathbf{ii}'$ for some $\mathbf{i}'$, we write $\mathbf{i} \preceq \mathbf{j}$. If $\mathbf{i}, \mathbf{j} \in \mathbf{J}$ or $\mathbf{J}_\infty$, then $\mathbf{i} \wedge \mathbf{j} \in \mathbf{J}$ denotes the maximal common initial subsequence of $\mathbf{i}$ and $\mathbf{j}$. We write $\mathbf{i}|_k = i_1 \ldots i_k$ for any $\mathbf{i} \in \mathbf{J}$ with $k \le |\mathbf{i}|$ or $\mathbf{i} \in \mathbf{J}_\infty$.

We may regard $\mathbf{J}$ as an N-ary tree rooted at $\emptyset$ in the obvious way, with each vertex $\mathbf{i}$ joined to N "children" $\{\mathbf{i}1, \ldots, \mathbf{i}N\}$. Then $\mathbf{J}_\infty$, which may be thought of as the boundary of $\mathbf{J}$, can be made into a compact metric space by using the metric $d(\mathbf{i}, \mathbf{j}) = 2^{-|\mathbf{i} \wedge \mathbf{j}|}$ for distinct $\mathbf{i}, \mathbf{j} \in \mathbf{J}_\infty$. The *cylinders* $\mathcal{C}_\mathbf{i} = \{\mathbf{j} \in \mathbf{J}_\infty : \mathbf{i} \preceq \mathbf{j}\}$ for $\mathbf{i} \in \mathbf{J}$ form a base of open and closed neighborhoods for $\mathbf{J}_\infty$.

We call $\mathbf{T} \subseteq \mathbf{J}$ a *tree* or a *subtree* of $\mathbf{J}$ if for all $\mathbf{i} \in \mathbf{T}$, we have $\mathbf{i}|_k \in \mathbf{T}$ for all $0 \le k \le |\mathbf{i}|$. Given a tree $\mathbf{T}$, we write $\mathbf{T}_k = \{\mathbf{i} \in \mathbf{T} : |\mathbf{i}| = k\}$ for its kth level vertices and $\mathbf{T}_\infty = \{\mathbf{i} \in \mathbf{J}_\infty : \mathbf{i}|_k \in \mathbf{T} \text{ for all } k\} \subseteq \mathbf{J}_\infty$ for its boundary, which we may think of as the limiting infinite tree.

Points of IFS attractors may be coded using this notation. We define projections from $\mathbf{J}_\infty$ to $\mathbb{R}^n$ by

$$x(\mathbf{i}) = \lim_{k \to \infty} S_{\mathbf{i}|_k}(y) \equiv \lim_{k \to \infty} S_{i_1} \circ \cdots \circ S_{i_k}(y); \tag{2.2}$$

since the S_i are contractions, this limit exists and is independent of $y \in \mathbb{R}^n$. (Note that $\mathbf{i} \mapsto x(\mathbf{i})$ is continuous but not necessarily an injection.) The attractor satisfying (1.1) is then given by

$$F = \bigcup_{\mathbf{i} \in \mathbf{J}_\infty} x(\mathbf{i}). \tag{2.3}$$

A tree $\mathbf{T}$ identifies a compact subset of F. Thus we write

$$F(\mathbf{T}) = \bigcup_{\mathbf{i} \in \mathbf{T}_\infty} x(\mathbf{i}). \tag{2.4}$$

From the constructive point of view, if $B \subset \mathbb{R}^n$ is a closed ball sufficiently large to ensure that $S_i(B) \subseteq B$ for all i, then $F(\mathbf{T})$ may be obtained by the usual

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iterative method of producing IFS attractors, with the proviso that iterated images $S_{\mathbf{i}}(B)$ corresponding to $\mathbf{i} \notin \mathbf{T}$ are omitted. Thus $F(\mathbf{T})$ is the intersection of a decreasing sequence of closed sets:

$$F(\mathbf{T}) = \bigcap_{k=0}^{\infty} \bigcup_{\mathbf{i} \in \mathbf{T}_k} S_{\mathbf{i}}(B).$$

Here we will be concerned with self-affine sets and their subsets. Fix non-singular linear contractions $T_1, \ldots, T_N : \mathbb{R}^n \to \mathbb{R}^n$. Let $\mathbf{a} \equiv (a_1, \ldots, a_N) \in \mathbb{R}^{nN}$ be a set of translation vectors and let $S_i : \mathbb{R}^n \to \mathbb{R}^n$ be the affine contractions

$$S_i(x) = T_i(x) + a_i, \quad i = 1, \ldots, N. \tag{2.5}$$

We write $F(\mathbf{a}) \equiv F$ for the self-affine attractor satisfying (1.1) to emphasize its dependence on $\mathbf{a}$. The projection maps depend on $\mathbf{a}$, so upon writing $x_{\mathbf{a}}(\mathbf{i}) = \lim_{k\to\infty} S_{\mathbf{i}|_k}(0)$ we have

$$F(\mathbf{a}) = \bigcup_{\mathbf{i} \in \mathbf{J}_\infty} x_{\mathbf{a}}(\mathbf{i}). \tag{2.6}$$

As before, trees specify subsets of $F(\mathbf{a})$, and for a tree $\mathbf{T}$ we write

$$F(\mathbf{a}, \mathbf{T}) = \bigcup_{\mathbf{i} \in \mathbf{T}_\infty} x_{\mathbf{a}}(\mathbf{i}). \tag{2.7}$$

Our ultimate aim is to find the dimensions of $F(\mathbf{a}, \mathbf{T})$ where $\mathbf{T}$ is a random tree.

We recall the basic results on the dimensions of self-affine sets. The *singular values* $\alpha_i \equiv \alpha_i(T)$ $(i = 1, \ldots, n)$ of a linear contraction $T : \mathbb{R}^n \to \mathbb{R}^n$ are the positive square roots of the eigenvalues of TT^*, where T^* is the adjoint of T. Equivalently, they are the lengths of the principal semi-axes of the image $T(B)$ of the unit ball B. With the convention that $1 > \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n > 0$, the *singular value function* $\phi^s(T)$ is defined for $0 \leq s \leq n$ as

$$\phi^s(T) = \alpha_1 \alpha_2 \cdots \alpha_{m-1} \alpha_m^{s-m+1},$$

where m is the integer such that $m - 1 < s \leq m$; for convenience, we set $\phi^s(T) = (\alpha_1 \alpha_2 \cdots \alpha_n)^{s/n}$ if $s \geq n$. Singular value functions are submultiplicative, that is, $\phi^s(TU) \leq \phi^s(T)\phi^s(U)$, a fact that has many important consequences. Under various conditions, the unique number s satisfying

$$\lim_{k\to\infty} \left[\sum_{\mathbf{i} \in \mathbf{J}_k} \phi^s(T_{\mathbf{i}}) \right]^{1/k} = 1 \tag{2.8}$$

gives the Hausdorff and box dimension of the self-affine set of the IFS with S_i given by (2.5). In particular, if the S_i have $\|T_i\| < 1/2$ for all i, then (2.8) gives the dimensions of $F(\mathbf{a})$ for almost all $\mathbf{a} = (a_1, \ldots, a_N) \in \mathbb{R}^{nN}$ in the sense of nN-dimensional Lebesgue measure; see [7, 8, 14, 21].

We will be concerned with the random subset $F(\mathbf{a}, \mathbf{T})$ of $F(\mathbf{a})$ defined in terms of the random subtree $\mathbf{T}$ of $\mathbf{J}$ constructed as follows. Fix probabilities $p_i \in (0, 1]$ for $i = 1, \ldots, N$. We construct $\mathbf{T}$ by defining $\mathbf{T}_k$ inductively. Let $\mathbf{T}_0$

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be the null sequence $\emptyset$. Suppose that the kth-level vertices $\mathbf{T}_k$ have been selected. The $(k+1)$-level vertices $\mathbf{T}_{k+1}$ are obtained as follows: if $\mathbf{i} \in \mathbf{J}_k \setminus \mathbf{T}_k$, then $\mathbf{i}i \notin \mathbf{T}_{k+1}$; if $\mathbf{i} \in \mathbf{T}_k$, then $\mathbf{i}i \in \mathbf{T}_{k+1}$ with probability p_i for $i = 1, \ldots, N$ (given $\mathbf{i}$, the events $\mathbf{i}i \in \mathbf{T}_{k+1}$ need not be independent for $i = 1, \ldots, N$). We assume that these selections are independent for $\mathbf{i} \neq \mathbf{j}$ (but not necessarily identically distributed). Continuing in this way, we obtain the random tree $\mathbf{T} = \bigcup_{k=0}^{\infty} \mathbf{T}_k$ with corresponding random infinite tree $\mathbf{T}_\infty$. (In some of our later results and examples, there will be further restrictions on the probability structure of the random trees.)

More formally, let Ω be the set of all subtrees of $\mathbf{J}$. For $k = 1, 2, \ldots$, let $\mathcal{F}_k$ be a sequence of σ-algebras of subsets of Ω, defined as follows. First, define an equivalence relation $\sim_k$ on Ω by $\mathbf{T} \sim_k \mathbf{T}'$ if $\mathbf{T}_m = \mathbf{T}'_m$ for $m = 0, 1, \ldots, k$, where $\mathbf{T}, \mathbf{T}' \in \Omega$ (so two trees are equivalent if they are identical down to the kth-level vertices). Then, let $\mathcal{F}_k$ be the (finite) set of finite unions of equivalence classes under $\sim_k$. Thus $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$, and we let $\mathcal{F}$ be the σ-algebra generated by $\bigcup_{k=0}^{\infty} \mathcal{F}_k$. We assume that the probability measure P on $(\Omega, \mathcal{F})$ which defines the random trees satisfies the condition that for all $\mathbf{i} \in \mathbf{J}$ and $i \in \{1, \ldots, N\}$,

$$\mathsf{P}(\mathbf{i}i \in \mathbf{T} | \mathcal{F}_k) = \begin{cases} p_i & \text{if } \mathbf{i} \in \mathbf{T}, \\ 0 & \text{if } \mathbf{i} \notin \mathbf{T}, \end{cases} \tag{2.9}$$

where $k = |\mathbf{i}|$. Thus, if $\mathbf{i} \notin \mathbf{T}$, then $\mathbf{i}i \notin \mathbf{T}$ for all $i \in \{1, \ldots, N\}$ almost surely. It is easy to see that there are many possibilities for P that satisfy this condition.

In §3, we show how to construct a random measure on the random tree $\mathbf{T}$ which will enable us to use the potential-theoretic method to find the Hausdorff and box-counting dimensions of the random set $F(\mathbf{a}, \mathbf{T})$ defined in terms of this random tree.

§3. *A random measure on the random tree.* In this section we construct a random measure on $\mathbf{J}_\infty$ that is supported by the random subset $\mathbf{T}_\infty$. The measure of the cylinders $\mathcal{C}_\mathbf{i}$ will be subordinate to the singular value functions of the $T_\mathbf{i}$ for $\mathbf{i} \in \mathbf{T}$. We write $T_\mathbf{i} = T_{i_1} \ldots T_{i_k}$ and $p_\mathbf{i} = p_{i_1} \ldots p_{i_k}$ for $\mathbf{i} = (i_1 \ldots i_k) \in \mathbf{J}$ in the usual way. Expectation will be denoted by E.

LEMMA 3.1. *With notation as above,*

$$\mathsf{E}\Bigg(\sum_{\mathbf{i} \in \mathbf{T}_k} \phi^s(T_\mathbf{i}) \Bigg) = \sum_{\mathbf{i} \in \mathbf{J}_k} \phi^s(T_\mathbf{i}) p_\mathbf{i}. \tag{3.10}$$

Proof. This is immediate, as each $\mathbf{i} \in \mathbf{J}_k$ has probability $p_\mathbf{i}$ of being in $\mathbf{T}_k$. □

The critical exponent d, which is defined by (3.11) in the following lemma, is central in the analysis that follows and turns out to be the almost-sure dimension of our random subsets.

LEMMA 3.2. *The limit* $\lim_{k\to\infty} [\mathsf{E}(\sum_{\mathbf{i}\in\mathbf{T}_k} \phi^s(T_\mathbf{i}))]^{1/k}$ *exists and is strictly decreasing in* s *for* $s \geq 0$. *Furthermore, if* $\sum_{i=1}^{N} p_i \geq 1$, *there is a unique*

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number $d \geq 0$ such that

$$\lim_{k\to\infty}\left[\mathsf{E}\Big(\sum_{\mathbf{i}\in\mathbf{T}_k}\phi^d(T_\mathbf{i})\Big)\right]^{1/k} = \lim_{k\to\infty}\left[\sum_{\mathbf{i}\in\mathbf{J}_k}\phi^d(T_\mathbf{i})p_\mathbf{i}\right]^{1/k} = 1. \tag{3.11}$$

Proof. The limit $\Phi(s) \equiv \lim_{k\to\infty}[\sum_{\mathbf{i}\in\mathbf{J}_k}\phi^s(T_\mathbf{i})p_\mathbf{i}]^{1/k}$ exists for all $s \geq 0$ by the standard property of submultiplicative sequences. We have that $\Phi(0) = \sum_{i=1}^N p_i$ and $\lim_{s\to\infty}\Phi(s) = 0$. Thus, if $\sum_{i=1}^N p_i \geq 1$, the continuity and strict monotonicity in s of $\Phi(s)$, see [7, Proposition 4.1], imply that there exists a unique $d \geq 0$ such that $\Phi(d) = 1$. The left-hand equation of (3.11) follows from Lemma 3.1. □

We assume henceforth that $\sum_{i=1}^N p_i \geq 1$; otherwise d is not defined and $\mathbf{T}$ becomes extinct (i.e. $\mathbf{T}_k = \emptyset$ for large k) with probability 1, a case that is obviously not of interest.

LEMMA 3.3. *If $s > d$, then, almost surely, $\sum_{k=0}^\infty\sum_{\mathbf{i}\in\mathbf{T}_k}\phi^s(T_\mathbf{i})$ is finite with $\sum_{\mathbf{i}\in\mathbf{T}_k}\phi^s(T_\mathbf{i}) \to 0$ as $k\to\infty$.*

Proof. Let $X_k^s = \sum_{\mathbf{i}\in\mathbf{T}_k}\phi^s(T_\mathbf{i})$. For $s > d$,

$$\lim_{k\to\infty}[\mathsf{E}(X_k^s)]^{1/k} = \lim_{k\to\infty}\left[\mathsf{E}\Big(\sum_{\mathbf{i}\in\mathbf{T}_k}\phi^s(T_\mathbf{i})\Big)\right]^{1/k} < 1,$$

so for some γ with $0 < \gamma < 1$ and $c > 0$, we have that $\mathsf{E}(X_k^s) \leq c\gamma^k$ for all $k \in \mathbb{N}$. Thus

$$\mathsf{E}\Big(\sum_{k=1}^\infty(X_k^s)\Big) = \sum_{k=1}^\infty \mathsf{E}(X_k^s) \leq c\sum_{k=1}^\infty \gamma^k < \infty,$$

and the conclusion follows. □

We first define, using Carathéodory's construction, deterministic measures $\mathcal{M}^s$ on $\mathbf{J}_\infty$ in terms of coverings of $\mathbf{T}$ by cylinders. For each s, define

$$\mathcal{M}_k^s(A) = \inf\left\{\sum_\mathbf{i} p_\mathbf{i}\phi^s(T_\mathbf{i}) : A \subseteq \bigcup \mathcal{C}_\mathbf{i},\ |\mathbf{i}| \geq k\right\}$$

for $k \geq 0$ and $A \subseteq \mathbf{J}_\infty$, and let

$$\mathcal{M}^s(A) = \lim_{k\to\infty}\mathcal{M}_k^s(A).$$

Then $\mathcal{M}^s$ is a regular Borel measure obtained by Carathéodory's construction; see [20]. The following lemma states that d is also the critical parameter for $\mathcal{M}^s(\mathbf{J}_\infty)$.

LEMMA 3.4. *If $0 < s < d$, then $\mathcal{M}^s(\mathbf{J}_\infty) > 0$.*

Proof. The proof, which depends on the submultiplicativity of $p_\mathbf{i}\phi^s(T_\mathbf{i})$, is virtually identical to that of [7, Proposition 4.1], except that here we work with $p_\mathbf{i}\phi^s(T_\mathbf{i})$ rather than $\phi^s(T_\mathbf{i})$. □

We fix $0 < t < d$ for the rest of the section. We now replace $\mathcal{M}^t$ by a subordinate measure μ that is positive and finite.

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PROPOSITION 3.5. *Let $0 < t < d$. Then $\mathcal{M}^t(\mathbf{J}_\infty) > 0$ and there exists a Borel measure μ on $\mathbf{J}_\infty$ such that $0 < \mu(\mathbf{J}_\infty) < \infty$ and*

$$\mu(\mathcal{C}_\mathbf{i}) \le p_\mathbf{i}\phi^t(T_\mathbf{i}) \quad \textit{for } \mathbf{i} \in \mathbf{J}. \tag{3.12}$$

Proof. The proof follows the standard method of reducing positive measures to positive finite measures and may be proved in a similar way to [5, Theorem 5.4] or deduced from [20, Theorem 54]. Full details of the proof in this context can be found in [18, Proposition 1.11]. □

We use a martingale argument to obtain random measures μ_∞ that are supported by the random trees $\mathbf{T}_\infty$.

For $k \in \mathbb{N}$, we define random measures μ_k on the cylinders $\{\mathcal{C}_\mathbf{q} : |\mathbf{q}| \le k\}$ by setting

$$\mu_k(\mathcal{C}_\mathbf{q}) = \sum_{\mathbf{q} \le \mathbf{i} \in \mathbf{T}_k} \frac{\mu(\mathcal{C}_\mathbf{i})}{p_\mathbf{i}}; \tag{3.13}$$

so if $\mathbf{q} \notin \mathbf{T}$, then $\mu_k(\mathcal{C}_\mathbf{q}) = 0$. Clearly, μ_k is $\mathcal{F}_k$-measurable and additive on $\{\mathcal{C}_\mathbf{q} : |\mathbf{q}| \le k\}$. In the following, we write $q = |\mathbf{q}|$ for $\mathbf{q} \in \mathbf{J}$.

LEMMA 3.6. *For all $\mathbf{q} \in \mathbf{J}$, $\{\mu_k(\mathcal{C}_\mathbf{q})\}_{k=q}^\infty$ is an L^2 martingale with respect to $\mathcal{F}_k$. Thus we may define $\mu_\infty(\mathcal{C}_\mathbf{q}) = \lim_{k\to\infty} \mu_k(\mathcal{C}_\mathbf{q})$, with the limit existing almost surely for all $\mathbf{q} \in \mathbf{J}$. Then μ_∞ is a random measure on the Borel subsets of $\mathbf{J}_\infty$ that is supported by $\mathbf{T}_\infty$. In particular, for all $\mathbf{q} \in \mathbf{J}$,*

$$\mathsf{E}(\mu_\infty(\mathcal{C}_\mathbf{q})) = \mu(\mathcal{C}_\mathbf{q})$$

and

$$\mathsf{E}(\mu_\infty(\mathcal{C}_\mathbf{q})^2) \le c\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q}), \tag{3.14}$$

where c is a constant independent of $\mathbf{q}$ and $0 < \mu_\infty(\mathbf{J}_\infty) < \infty$ with positive probability.

Proof. For each $\mathbf{q} \in \mathbf{T}$,

$$\begin{aligned}
\mathsf{E}(\mu_{k+1}(\mathcal{C}_\mathbf{q})|\mathcal{F}_k) &= \mathsf{E}\Bigg(\sum_{\mathbf{q}\le\mathbf{i}i\in\mathbf{T}_{k+1}} \frac{\mu(\mathcal{C}_{\mathbf{i}i})}{p_{\mathbf{i}i}}\Bigg|\mathcal{F}_k\Bigg) \quad \text{by (3.13)}\\
&= \sum_{\mathbf{q}\le\mathbf{i}\in\mathbf{T}_k} p_\mathbf{i}^{-1}\mathsf{E}\Bigg(\sum_{i:\mathbf{i}i\in\mathbf{T}_{k+1}} \frac{\mu(\mathcal{C}_{\mathbf{i}i})}{p_i}\Bigg)\\
&= \sum_{\mathbf{q}\le\mathbf{i}\in\mathbf{T}_k} p_\mathbf{i}^{-1}\sum_{i=1}^N \frac{\mu(\mathcal{C}_{\mathbf{i}i})}{p_i}\cdot p_i \quad \text{by (2.9)}\\
&= \sum_{\mathbf{q}\le\mathbf{i}\in\mathbf{T}_k} \mu(\mathcal{C}_\mathbf{i})/p_\mathbf{i}\\
&= \mu_k(\mathcal{C}_\mathbf{q}),
\end{aligned}$$

so $(\mu_k(\mathcal{C}_\mathbf{q}), \mathcal{F}_k)_{k\ge q}$ is a non-negative martingale.

By the martingale convergence theorem (see [22]), for each $\mathbf{q}$ there exists a random variable $\mu_\infty(\mathcal{C}_\mathbf{q}) \ge 0$ such that $\mu_k(\mathcal{C}_\mathbf{q}) \to \mu_\infty(\mathcal{C}_\mathbf{q})$ almost surely.

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Then μ_k is additive on any finite set of cylinders for k sufficiently large; so μ_∞ is almost surely finitely additive on the cylinders and extends to a measure on the Borel subsets of $\mathbf{T}_\infty$.

To show that $\mu_\infty(\mathbf{T}_\infty) > 0$ with positive probability, it is enough to show that $\mu_k(\mathcal{C}_\mathbf{q})$ is L^2 bounded, i.e. that $\sup_k \mathsf{E}(\mu_k(\mathcal{C}_\mathbf{q})^2) < \infty$. We show more generally that

$$\mathsf{E}(\mu_k(\mathcal{C}_\mathbf{q})^2) \le c\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q}) \quad \text{for } k \ge q,$$

where c does not depend on k or $\mathbf{q}$.

For $k \ge q$, upon taking conditional expectations of $\mu_{k+1}(\mathcal{C}_\mathbf{q})^2$ we get

$$\begin{aligned}
\mathsf{E}(\mu_{k+1}(\mathcal{C}_\mathbf{q})^2|\mathcal{F}_k) &= \mathsf{E}\Bigg(\Bigg(\sum_{\substack{\mathbf{i}i\in\mathbf{T}_{k+1}\\ \mathbf{q}\preceq\mathbf{i}}} \frac{\mu(\mathcal{C}_{\mathbf{i}i})}{p_{\mathbf{i}i}}\Bigg)^2\Bigg|\mathcal{F}_k\Bigg) \quad \text{by (3.13)}\\
&= \mathsf{E}\Bigg(\sum_{\substack{\mathbf{i}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i}}} p_\mathbf{i}^{-2} \sum_{i:\mathbf{i}i\in\mathbf{T}_{k+1}} \sum_{j:\mathbf{i}j\in\mathbf{T}_{k+1}} \frac{\mu(\mathcal{C}_{\mathbf{i}i})\mu(\mathcal{C}_{\mathbf{i}j})}{p_i\cdot p_j}\Bigg|\mathcal{F}_k\Bigg)\\
&\quad + \mathsf{E}\Bigg(\sum_{\substack{\mathbf{i}\ne\mathbf{j}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i},\mathbf{j}}} p_\mathbf{i}^{-1}p_\mathbf{j}^{-1} \sum_{i:\mathbf{i}i\in\mathbf{T}_{k+1}} \sum_{j:\mathbf{j}j\in\mathbf{T}_{k+1}} \frac{\mu(\mathcal{C}_{\mathbf{i}i})\mu(\mathcal{C}_{\mathbf{j}j})}{p_i\cdot p_j}\Bigg|\mathcal{F}_k\Bigg)\\
&\le \sum_{\substack{\mathbf{i}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i}}} p_\mathbf{i}^{-2}\Bigg(\sum_{1\le i,j\le N} \frac{\mu(\mathcal{C}_\mathbf{i})\mu(\mathcal{C}_\mathbf{i})}{p_i\cdot p_j}\Bigg)\\
&\quad + \sum_{\substack{\mathbf{i}\ne\mathbf{j}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i},\mathbf{j}}} p_\mathbf{i}^{-1}p_\mathbf{j}^{-1}\mathsf{E}\Bigg(\sum_{i:\mathbf{i}i\in\mathbf{T}_{k+1}} \sum_{j:\mathbf{j}j\in\mathbf{T}_{k+1}} \frac{\mu(\mathcal{C}_{\mathbf{i}i})\mu(\mathcal{C}_{\mathbf{j}j})}{p_i\cdot p_j}\Bigg)\\
&\le c_1 \sum_{\substack{\mathbf{i}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i}}} (\mu(\mathcal{C}_\mathbf{i})/p_\mathbf{i})^2 + \sum_{\substack{\mathbf{i}\ne\mathbf{j}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i},\mathbf{j}}} \mu(\mathcal{C}_\mathbf{i})\mu(\mathcal{C}_\mathbf{j})p_\mathbf{i}^{-1}p_\mathbf{j}^{-1}\\
&\qquad \text{by (2.9) and independence}\\
&\le c_1 \sum_{\substack{\mathbf{i}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i}}} (\mu(\mathcal{C}_\mathbf{i})/p_\mathbf{i})^2 + \Bigg(\sum_{\substack{\mathbf{i}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i}}} \mu(\mathcal{C}_\mathbf{i})/p_\mathbf{i}\Bigg)^2\\
&\le c_1 \sum_{\substack{\mathbf{i}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i}}} (\mu(\mathcal{C}_\mathbf{i})/p_\mathbf{i})^2 + \mu_k(\mathcal{C}_\mathbf{q})^2 \quad \text{by (3.13)},
\end{aligned}$$

where $c_1 = \sum_{i,j=1}^N p_i^{-1}p_j^{-1}$. Taking conditional expectations again, it follows that, for $k \ge q$,

$$\begin{aligned}
\mathsf{E}(\mu_{k+1}(\mathcal{C}_\mathbf{q})^2|\mathcal{F}_q) &= \mathsf{E}(\mathsf{E}(\mu_{k+1}(\mathcal{C}_\mathbf{q})^2|\mathcal{F}_k)|\mathcal{F}_q)\\
&\le c_1\mathsf{E}\Bigg(\sum_{\substack{\mathbf{i}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i}}} (\mu(\mathcal{C}_\mathbf{i})/p_\mathbf{i})^2\Bigg|\mathcal{F}_q\Bigg) + \mathsf{E}(\mu_k(\mathcal{C}_\mathbf{q})^2|\mathcal{F}_q). \qquad (3.15)
\end{aligned}$$

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Consider the first term of this sum for $k \geq q$. Set

$$\alpha_+ = \max_{i=1,\dots,N}\{\alpha_1(T_i)\} < 1.$$

If $\mathbf{q} \notin \mathbf{T}_q$, then

$$\mathsf{E}\Big(\sum_{\mathbf{i}\in\mathbf{T}_k,\mathbf{q}\preceq\mathbf{i}} \mu(\mathcal{C}_\mathbf{i})^2 \Big| \mathcal{F}_q\Big) = 0.$$

For $\mathbf{q} \in \mathbf{T}_q$, we have

$$\begin{aligned}
&\mathsf{E}\Big(\sum_{\substack{\mathbf{i}\in\mathbf{T}_k\\ \mathbf{q}\preceq\mathbf{i}}} (\mu(\mathcal{C}_\mathbf{i})/p_\mathbf{i})^2 \Big| \mathcal{F}_q\Big)\\
&= \sum_{\substack{\mathbf{i}\in\mathbf{J}_k\\ \mathbf{q}\preceq\mathbf{i}}} (\mu(\mathcal{C}_\mathbf{i})/p_\mathbf{i})^2 p_{i_{q+1}}\dots p_{i_k}\\
&\leq \sum_{\substack{\mathbf{i}\in\mathbf{J}_k\\ \mathbf{q}\preceq\mathbf{i}}} \mu(\mathcal{C}_\mathbf{i}) p_\mathbf{i}^{-1}\cdot \phi^t(T_\mathbf{i})\ p_\mathbf{i}\ p_\mathbf{q}^{-1} \quad \text{by (3.12)}\\
&\leq p_\mathbf{q}^{-1} \sum_{\mathbf{i}'\in\mathbf{J}_{k-q}} \mu(\mathcal{C}_{\mathbf{q}\mathbf{i}'})\phi^t(T_{\mathbf{q}\mathbf{i}'}) \qquad (3.16)\\
&\leq p_\mathbf{q}^{-1} \sum_{\mathbf{i}'\in\mathbf{J}_{k-q}} \mu(\mathcal{C}_{\mathbf{q}\mathbf{i}'})\phi^t(T_\mathbf{q})\phi^t(T_{\mathbf{i}'})\\
&\leq p_\mathbf{q}^{-1}\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q}) \sum_{\mathbf{i}'\in\mathbf{J}_{k-q}} \frac{\mu(\mathcal{C}_{\mathbf{q}\mathbf{i}'})}{\mu(\mathcal{C}_\mathbf{q})}\phi^t(T_{\mathbf{i}'})\\
&\leq p_\mathbf{q}^{-1}\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q})\alpha_+^{t(k-q)} \qquad (3.17)
\end{aligned}$$

provided that $\mu(\mathcal{C}_\mathbf{q}) \neq 0$; otherwise (3.17) follows from (3.16). Hence, by iterating (3.15) and using (3.17) with $\mathbf{q} \in \mathbf{T}_k$, we obtain

$$\begin{aligned}
\mathsf{E}(\mu_{k+1}(\mathcal{C}_\mathbf{q})^2|\mathcal{F}_q) &\leq c_1 p_\mathbf{q}^{-1}\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q})\alpha_+^{t(k-q)} + \mathsf{E}(\mu_k(\mathcal{C}_\mathbf{q})^2|\mathcal{F}_q)\\
&\leq c_1 p_\mathbf{q}^{-1}\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q}) \sum_{l=q}^{k} \alpha_+^{t(l-q)} + \mathsf{E}(\mu_q(\mathcal{C}_\mathbf{q})^2|\mathcal{F}_q)\\
&\leq c_2 p_\mathbf{q}^{-1}\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q}) + \mathsf{E}(\mu_q(\mathcal{C}_\mathbf{q})^2|\mathcal{F}_q), \qquad (3.18)
\end{aligned}$$

where $c_2 = c_1 \sum_{l=q}^{\infty} \alpha_+^{t(l-q)} < \infty$. By the definition (3.13) and the inequality (3.12), we have

$$\begin{aligned}
\mathsf{E}(\mu_q(\mathcal{C}_\mathbf{q})^2|\mathcal{F}_q) &= \mu_q(\mathcal{C}_\mathbf{q})^2 = p_\mathbf{q}^{-2}\mu(\mathcal{C}_\mathbf{q})^2\\
&\leq p_\mathbf{q}^{-1}\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q}). \qquad (3.19)
\end{aligned}$$

Thus, by (3.18) we have that, for $k \geq q$,

$$\mathsf{E}(\mu_k(\mathcal{C}_\mathbf{q})^2|\mathcal{F}_q) \leq c p_\mathbf{q}^{-1}\phi^t(T_\mathbf{q})\mu(\mathcal{C}_\mathbf{q}),$$

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where $c = (c_2 + 1)$. Finally, we get the unconditional expectation

$$\begin{aligned}\mathsf{E}(\mu_k(\mathcal{C}_{\mathbf{q}})^2) &= p_{\mathbf{q}}\,\mathsf{E}(\mu_k(\mathcal{C}_{\mathbf{q}})^2|\mathcal{F}_q)\\ &\le c\phi^t(T_{\mathbf{q}})\mu(\mathcal{C}_{\mathbf{q}}).\end{aligned}$$

In particular, $\mu_k(\mathcal{C}_\emptyset)$ is L^2 bounded, so the standard property of L^2 martingales (see [22]) gives that $\mu_\infty(\mathbf{J}_\infty) > 0$ with positive probability, and $\mathsf{E}(\mu_\infty(\mathcal{C}_\emptyset)) = \mathsf{E}(\mu_k(\mathcal{C}_\emptyset)) = \mu(\mathcal{C}_\emptyset) > 0$.

Since $\mu_k(\mathcal{C}_{\mathbf{q}}) \to \mu_\infty(\mathcal{C}_{\mathbf{q}})$ almost surely for all $\mathbf{q}$, by Fatou's lemma we have

$$\mathsf{E}(\mu_\infty(\mathcal{C}_{\mathbf{q}})^2) \le \varliminf_{k\to\infty} \mathsf{E}(\mu_k(\mathcal{C}_{\mathbf{q}})^2) \le c\phi^t(T_{\mathbf{q}})\mu(\mathcal{C}_{\mathbf{q}}). \tag{3.20}$$

This completes the proof of Lemma 3.6. □

§4. *The dimensions of random subsets of self-affine fractals.* We use the results of §3 to find almost-sure dimensions of random subsets of self-affine attractors. Keeping the notation of §2, $S_i(x) = T_i(x) + a_i$ are affine maps with linear parts T_i and translations a_i, and $x_{\mathbf{a}}(\mathbf{i}) = \lim_{k\to\infty} S_{\mathbf{i}|_k}(0)$ are the projections from $\mathbf{J}_\infty$ to $\mathbb{R}^n$. For a tree $\mathbf{T}$ with boundary $\mathbf{T}_\infty$, recall (2.7), i.e.

$$F(\mathbf{a}, T) = \bigcup_{\mathbf{i}\in\mathbf{T}_\infty} x_{\mathbf{a}}(\mathbf{i}). \tag{4.21}$$

We take $\mathbf{T}$ to be a random tree constructed as in (2.9) so that for each $\mathbf{a}$, $F(\mathbf{a}, T)$ is a random subset of the deterministic self-affine attractor $F(\mathbf{a})$ of the IFS $\{S_1, \dots, S_N\}$.

We investigate the Hausdorff and box-counting dimensions of $F(\mathbf{a}, T)$. The proof of the upper bound is a random version of that for the deterministic self-affine set $F(\mathbf{a})$ in [7, Theorem 5.4]. As usual, $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{B}}$ denote the Hausdorff and box-counting dimensions, respectively, with $\underline{\dim}_{\mathrm{B}}$ and $\overline{\dim}_{\mathrm{B}}$ denoting lower and upper box-counting dimensions.

THEOREM 4.1. *Let $\mathbf{T}$ be the random tree as in (2.9), and let $F(\mathbf{a}, T)$ be the random subset (4.21) of the self-affine set $F(\mathbf{a})$. Then, almost surely, $\dim_{\mathrm{H}} F(\mathbf{a}, T) \le \underline{\dim}_{\mathrm{B}} F(\mathbf{a}, T) \le \overline{\dim}_{\mathrm{B}} F(\mathbf{a}, T) \le \min\{d, n\}$ for all $\mathbf{a} \in \mathbb{R}^{nN}$, where d is given by (3.11).*

Proof. The result is clear if $d \ge n$. Let s be such that $d < s < n$. By Lemma 3.3, there exists almost surely a finite random constant C_1 such that

$$\sum_{k=1}^{\infty} \sum_{\mathbf{i}\in\mathbf{T}_k} \phi^s(T_{\mathbf{i}}) \le C_1. \tag{4.22}$$

Let B be a ball large enough to ensure that $S_i(B) \subseteq B$ for $i = 1, \dots, N$. Let m be the smallest integer greater than or equal to s. Given $0 < r < \sqrt{n}|B|$, for each $\mathbf{i} \in \mathbf{T}_\infty$ let $\mathbf{i}' = i_1 \dots i_q$ be the finite sequence obtained by curtailing $\mathbf{i}$ after q terms, where q is the smallest positive integer such that $r \ge \sqrt{n}|B|\alpha_m(T_{i_1\dots i_q}) > r\alpha_-^n$, with $\alpha_m(T_{i_1\dots i_q})$ being the mth singular value of $T_{\mathbf{i}'}$ and $\alpha_- = \min_{1\le i\le N}\{\alpha_n(T_i)\}$. Letting $A = \{\mathbf{i}' \in \mathbf{T} : \mathbf{i} \in \mathbf{T}_\infty\}$, we have, for

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all $\mathbf{i} \in \mathbf{T}_\infty$, that $\mathbf{i}|_k \in A$ for some k, and so $F(\mathbf{a}, \mathbf{T}) \subseteq \bigcup_{\mathbf{i}' \in A} S_{\mathbf{i}'}(B)$. Moreover, by (4.22) we have that, almost surely,

$$\sum_{\mathbf{i}' \in A} \phi^s(T_{\mathbf{i}'}) \le C_1 < \infty.$$

Just as in [7, Theorem 5.4], we note that each ellipsoid $S_{\mathbf{i}'}(B)$ for $\mathbf{i}' \in A$ may be covered by at most $c_1 \phi^s(T_{\mathbf{i}'}) \big(\alpha_m(T_{\mathbf{i}'})\big)^{-s} \le c_2 \phi^s(T_{\mathbf{i}'}) r^{-s}$ sets of diameter r, where c_1 and c_2 are independent of $\mathbf{i}'$ and r. The aggregate of these covers provides a cover of $F(\mathbf{a}, \mathbf{T})$ by $c_2 \sum_{\mathbf{i}' \in A} \phi^s(T_{\mathbf{i}'}) \le c_2 C_1 r^{-s}$ sets of diameter r. This implies that $\overline{\dim}_{\mathrm{B}} F(\mathbf{a}, T) \le s$, and the conclusion follows since this is true for all $d < s < n$. □

PROPOSITION 4.2. *Assume that* $\max_{1 \le i \le N} \|T_i\| < 1/2$. *Let* $F(\mathbf{a}, \mathbf{T})$ *be as above. For all* $t < \min\{d, n\}$, *conditional on* $\mu_\infty(\mathbf{T}_\infty) > 0$, *almost surely* $\dim_{\mathrm{H}} F(\mathbf{a}) \ge t$ *for almost all* $\mathbf{a} \in \mathbb{R}^{nN}$. *(Note that* μ_∞, *defined by Lemma 3.6, depends on* t, *and for each such* t *there is a positive probability that* $\mu_\infty(\mathbf{T}_\infty) > 0$.*)*

Proof. This is an adaptation of the proof of [7, Theorem 5.3] to the random setting. Let $0 < t < \min\{n, d\}$, and let μ be as in (3.12) with $0 < \mu(\mathbf{J}_\infty) < \infty$. By Lemma 3.6, there exists a random measure μ_∞ supported by $\mathbf{T}_\infty$ such that $0 < \mu_\infty(\mathbf{J}_\infty) < \infty$ with positive probability and $\mathsf{E}(\mu_\infty(\mathcal{C}_\mathbf{q})^2) \le c \phi^t(T_\mathbf{q}) \mu(\mathcal{C}_\mathbf{q})$ for all $\mathbf{q}$.

Let $0 < s < t$ with s not an integer. It was shown in [7, 21] that, for such s and $\|T_i\| < 1/2$ for all $1 \le i \le N$, there is a number c_1 such that

$$\int_{\mathbf{a} \in B(0,\rho)} \frac{d\mathbf{a}}{|x_\mathbf{a}(\mathbf{i}) - x_\mathbf{a}(\mathbf{j})|^s} \le \frac{c_1}{\phi^s(T_{\mathbf{i} \wedge \mathbf{j}})}$$

for all distinct $\mathbf{i}, \mathbf{j} \in \mathbf{J}_\infty$. Writing $\mathbf{q} = \mathbf{i} \wedge \mathbf{j}$, we find that

$$\begin{aligned}
&\mathsf{E}\left(\int_{\mathbf{J}_\infty} \int_{\mathbf{J}_\infty} \int_{\mathbf{a} \in B(0,\rho)} \frac{d\mu_\infty(\mathbf{i})\, d\mu_\infty(\mathbf{j})\, d\mathbf{a}}{|x_\mathbf{a}(\mathbf{i}) - x_\mathbf{a}(\mathbf{j})|^s} \right) \\
&\le c_1 \mathsf{E}\left(\int_{\mathbf{J}_\infty} \int_{\mathbf{J}_\infty} \frac{d\mu_\infty(\mathbf{i})\, d\mu_\infty(\mathbf{j})}{\phi^s(T_{\mathbf{i} \wedge \mathbf{j}})} \right) \\
&\le c_1 \mathsf{E}\left(\sum_{\mathbf{q} \in \mathbf{J}} \sum_{i \ne j} \frac{\mu_\infty(\mathcal{C}_{\mathbf{q}i}) \mu_\infty(\mathcal{C}_{\mathbf{q}j})}{\phi^s(T_\mathbf{q})} \right) \\
&\le c_1 \mathsf{E}\left(\sum_{\mathbf{q} \in \mathbf{J}} \frac{\mu_\infty(\mathcal{C}_\mathbf{q})^2}{\phi^s(T_\mathbf{q})} \right) \\
&\le c_1 c \sum_{k=0}^{\infty} \sum_{\mathbf{q} \in \mathbf{J}_k} \frac{\phi^t(T_\mathbf{q}) \mu(\mathcal{C}_\mathbf{q})}{\phi^s(T_\mathbf{q})} \\
&\le c_1 c \sum_{k=0}^{\infty} \sum_{\mathbf{q} \in \mathbf{J}_k} \alpha_+^{k(t-s)} \mu(\mathcal{C}_\mathbf{q})
\end{aligned}$$

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$$\leq c_1 c\mu(\mathbf{J}_\infty) \sum_{k=0}^{\infty} \alpha_+^{k(t-s)}$$
$$< \infty,$$

where $\alpha_+ = \max_{1\leq i\leq N}\{\alpha_1(T_i)\}$ and we have used Lemma 3.6. It follows that, with probability 1,

$$\int_{\mathbf{J}_\infty}\int_{\mathbf{J}_\infty}\int_{\mathbf{a}\in B(0,\rho)} \frac{d\mu_\infty(\mathbf{i})\, d\mu_\infty(\mathbf{j})\, d\mathbf{a}}{|x_{\mathbf{a}}(\mathbf{i}) - x_{\mathbf{a}}(\mathbf{j})|^s} < \infty. \tag{4.23}$$

Applying Fubini's theorem gives

$$\int_{\mathbf{J}_\infty}\int_{\mathbf{J}_\infty} \frac{d\mu_\infty(\mathbf{i})\, d\mu_\infty(\mathbf{j})}{|x_{\mathbf{a}}(\mathbf{i}) - x_{\mathbf{a}}(\mathbf{j})|^s} < \infty$$

for almost all $\mathbf{a} \in B(0, \rho)$. Given $\mathbf{a} \in B(0, \rho)$, for each realization of μ_∞ we define a measure on $\mathbb{R}^n$ by

$$\nu(A) = \mu_\infty\{\mathbf{i} \in \mathbf{J}_\infty : x_{\mathbf{a}}(\mathbf{i}) \in A\} \quad \text{for } A \subseteq \mathbb{R}^n.$$

By the continuity of $\mathbf{i} \mapsto x_{\mathbf{a}}(\mathbf{i})$, ν is a random Borel measure on $\mathbb{R}^n$ supported by the random set $F(\mathbf{a}, \mathbf{T})$ such that $\nu(\mathbb{R}^n) = \mu_\infty(\mathbf{J}_\infty)$. Conditional on $\mu_\infty(\mathbf{T}_\infty) > 0$, the potential-theoretic characterization of Hausdorff dimension (see [9]) gives that $\dim_{\mathrm{H}} F(\mathbf{a}, \mathbf{T}) \geq s$ for almost all $\mathbf{a}$. □

We now assume more regularity for the random tree construction than just (2.9), namely, that the probability distribution of $\{i : \mathbf{i}i \in \mathbf{T}\}$ is the same for all $\mathbf{i} \in \mathbf{T}$. The random sets $F(\mathbf{a}, \mathbf{T})$ may then be thought of as *statistically self-affine*. In particular,

$$\mathsf{P}(\sharp\{i : \mathbf{i}i \in \mathbf{T}_{k+1} \mid \mathbf{i} \in \mathbf{T}_k\} = r) \equiv \mathsf{P}(\sharp\mathbf{T}_1 = r) \tag{4.24}$$

for all $\mathbf{i} \in \mathbf{T}_k$ and $r = 1, \ldots, N$, so that $\sharp\mathbf{T}_k$ is a Galton–Watson branching process; see [1]. By standard theory, the probability η of extinction, i.e. of $\mathbf{T}_k = 0$ for all sufficiently large k, is given by the smallest non-negative root of the generating function equation

$$\eta = \sum_{r=0}^{\infty} \mathsf{P}(\sharp(\mathbf{T}_1) = r)\eta^r. \tag{4.25}$$

THEOREM 4.3. *Assume that* $\max_{1\leq i\leq N} \|T_i\| < 1/2$. *Let* $F(\mathbf{a}, \mathbf{T})$ *be as above. With probability* η *we have* $F(\mathbf{a}, \mathbf{T}) = \emptyset$ *for all* $\mathbf{a}$, *and with probability* $1 - \eta > 0$ *we have* $\dim_{\mathrm{H}} F(\mathbf{a}, \mathbf{T}) = \dim_{\mathrm{B}} F(\mathbf{a}, \mathbf{T}) = \min\{d, n\}$ *for almost all* $\mathbf{a} \in \mathbb{R}^{nN}$, *where* d *is given by (3.11).*

Proof. Clearly, if the branching process becomes extinct, then $F(\mathbf{a}, \mathbf{T}) = \emptyset$.

Otherwise, conditional on non-extinction, $\sharp\mathbf{T}_k \to \infty$ almost surely. Let $t < \min\{d, n\}$. For $\mathbf{q} \in \mathbf{T}_k$, write

$$F(\mathbf{a}, \mathbf{T}_{\mathbf{q}}) = \bigcup\{x_{\mathbf{a}}(\mathbf{i}) : \mathbf{i} \in \mathbf{T} \text{ and } \mathbf{q} \preceq \mathbf{i}\}.$$

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By Proposition 4.2, with positive probability p, say, $\dim_{\mathrm{H}} F(\mathbf{a}, \mathbf{T}) \geq t$ for almost all $\mathbf{a} \in \mathbb{R}^{nN}$; so, by the statistical self-affinity of the construction,

$$\mathsf{P}(\dim_{\mathrm{H}} F(\mathbf{a}, \mathbf{T_q}) \geq t \text{ for almost all } \mathbf{a}) = p.$$

Given $\epsilon > 0$, there is an integer m such that $(1-p)^m < \epsilon$. Conditional on non-extinction, there exists q such that $\sharp \mathbf{T}_q \geq m$. Then $F(\mathbf{a}, \mathbf{T}) = \bigcup_{|\mathbf{q}|=q} F(\mathbf{a}, \mathbf{T_q})$, with $\dim_{\mathrm{H}} F(\mathbf{a}, \mathbf{T}) = \max_{|\mathbf{q}|=q}\{\dim_{\mathrm{H}} F(\mathbf{a}, \mathbf{T_q})\}$ and $\dim_{\mathrm{H}} F(\mathbf{a}, \mathbf{T_q})$ independent for each such $\mathbf{q}$. Thus

$$\mathsf{P}(\dim_{\mathrm{H}} F(\mathbf{a}, \mathbf{T}) \geq t \mid \text{non-extinction}) \geq 1 - (1-p)^m > 1 - \epsilon.$$

This holds for all $\epsilon > 0$ and also for all $t < \min\{d, n\}$, so

$$\mathsf{P}(\dim_{\mathrm{H}} F(\mathbf{a}, \mathbf{T}) \geq t \mid \text{non-extinction}) = 1.$$

Combining this with Theorem 4.1 gives $\dim_{\mathrm{H}} F(a) = \dim_{\mathrm{B}} F(a) = \min\{d, n\}$, conditional on non-extinction. □

One could consider analogues of Theorem 4.3 for random sets based on random subtrees of a rooted tree with countably many offspring at each node, a situation that might be regarded as more natural in the context of branching processes. In this case we have $\mathbf{J}_k = \{(i_1, \ldots, i_k) : i_j \in \mathbb{N}\}$, with $\mathbf{J} = \bigcup_{k=0}^{\infty} \mathbf{J}_k$ and $\mathbf{J}_\infty = \{(i_1 i_2 \ldots) : i_j \in \mathbb{N}\}$. Whilst $\mathbf{J}_\infty$ is no longer compact under the metric $d(\mathbf{i}, \mathbf{j}) = 2^{-|\mathbf{i} \wedge \mathbf{j}|}$, a subtree $\mathbf{T} \subseteq \mathbf{J}$ will have compact boundary $\mathbf{T}_\infty$ if it has a finite, though not necessarily bounded, number of offspring at each node.

We take a sequence of non-singular linear contractions $T_1, T_2, \ldots : \mathbb{R}^n \to \mathbb{R}^n$ and translation vectors $\mathbf{a} \equiv (a_1, a_2, \ldots)$ with $a_i \in \mathbb{R}^n$, and write $S_i : \mathbb{R}^n \to \mathbb{R}^n$ for the corresponding affine contractions $S_i(x) = T_i(x) + a_i$. The resulting sets $F(\mathbf{a}, \mathbf{T})$ are defined in just the same way as in (2.2)–(2.7); thus $F(\mathbf{a}, \mathbf{T}) = \bigcup_{\mathbf{i} \in \mathbf{T}_\infty} x_{\mathbf{a}}(\mathbf{i})$, where $x_{\mathbf{a}}(\mathbf{i}) = \lim_{k \to \infty} S_{i_1} \circ \cdots \circ S_{i_k}(y)$. If the boundary $\mathbf{T}_\infty$ of the subtree $\mathbf{T}$ is compact, then $F(\mathbf{a}, \mathbf{T})$ is compact for all $\mathbf{a}$; however, there is no longer a fixed compact IFS attractor $F(\mathbf{a})$ that contains $F(\mathbf{a}, \mathbf{T})$ for all $\mathbf{T}$.

Analogously to the N-ary branching case, we may consider random subtrees $\mathbf{T}$ of this infinite branching tree $\mathbf{J}$. Let $p_i \in (0, 1]$ be probabilities with $1 < \sum_{i=1}^{\infty} p_i < \infty$. Let $\mathbf{T}$ be the random subtree of $\mathbf{J}$ defined by the probability condition (2.9) as before, and assume that the probability distribution of $\{i : \mathbf{i}i \in \mathbf{T}\}$ is the same for all $\mathbf{i} \in \mathbf{T}$ so that, in particular, (4.24) holds and $\sharp \mathbf{T}_k$ is a Galton–Watson branching process with extinction probability η given by (4.25). The condition $\sum_{i=1}^{\infty} p_i < \infty$ ensures that almost surely each node of $\mathbf{T}$ has finitely many children and hence that the random fractal $F(\mathbf{a}, \mathbf{T})$ is compact for all $\mathbf{a}$.

The recursive nature of the construction guarantees that, conditional on non-extinction, there is a specific almost-sure value for the Hausdorff dimension of the random fractal $F(\mathbf{a}, \mathbf{T})$, which one might hope would equal $\min\{d, n\}$ with d being the unique solution of the equation $\Phi(d) = 1$, where $\Phi(s) \equiv \lim_{k \to \infty} [\sum_{\mathbf{i} \in \mathbf{J}_k} \phi^s(T_\mathbf{i}) p_\mathbf{i}]^{1/k}$ and the sums are now over the kth-level vertex sets $\mathbf{J}_k$ of the infinite branching tree. One problem occurs at this point: whilst

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the condition $\sup_{1\le i<\infty}\|T_i\|<1$ implies that $\Phi(s)$ is strictly decreasing, it is not clear that Φ must be continuous. However, continuity is guaranteed if, in addition, there is a lower bound on the contraction rates, $0<\inf_{1\le i<\infty}\|T_i^{-1}\|$. In any case, the proof of Theorem 4.1 extends without difficulty to show that

$$\dim_{\mathrm{H}} F(\mathbf{a},\mathbf{T})\le \overline{\dim}_{\mathrm{B}} F(\mathbf{a},\mathbf{T})\le \min\{d_+,n\} \tag{4.26}$$

where $d_+=\inf\{s:\Phi(s)<1\}$.

The derivation of the lower bound for the N-ary case, which depends on Lemma 3.6, does not extend directly to the case of infinitely many contractions. One difficulty is that the constant c in (3.14), which is a multiple of $c_1=\sum_{i,j=1}^N p_i^{-1}p_j^{-1}$, will no longer be finite.

An alternative approach is to approximate $F(\mathbf{a},\mathbf{T})$ from below by sets $F(\mathbf{a},\mathbf{T}^N)$ based on random subtrees $\mathbf{T}^N$ with bounded offspring, to which we may apply Theorem 4.3. Thus, for each $N\ge 2$ and each tree $\mathbf{T}\subseteq\mathbf{J}$, setting

$$\mathbf{T}^N=\{(i_1,\dots,i_k)\in\mathbf{T}:1\le i_j\le N \text{ for all } j \text{ and all } k\in\mathbb{N}\}$$

gives a random subtree of $\mathbf{J}$ with probability distribution such that for all $\mathbf{i}\in\mathbf{J}$ and $i\in\mathbb{N}$,

$$\mathsf{P}(\mathbf{i}i\in\mathbf{T}^N\mid\mathcal{F}_k)=\begin{cases}p_i & \text{if } \mathbf{i}\in\mathbf{T}^N \text{ and } 1\le i\le N,\\ 0 & \text{if } i>N \text{ or } \mathbf{i}\notin\mathbf{T}^N,\end{cases}$$

where $k=|\mathbf{i}|$. Moreover, $F(\mathbf{a},\mathbf{T}^N)\subseteq F(\mathbf{a},\mathbf{T})$ for all $\mathbf{a}$. Let $\mathbf{J}_k^N=\{(i_1,\dots,i_k)\in\mathbf{J}:1\le i_j\le N\}$, and let d_N be the solution of $\Phi^N(d_N)=1$ where $\Phi^N(s)\equiv\lim_{k\to\infty}[\sum_{\mathbf{i}\in\mathbf{J}_k^N}\phi^s(T_\mathbf{i})p_\mathbf{i}]^{1/k}$. Assuming that $\sup_{1\le i<\infty}\|T_i\|<1/2$, it follows from Theorem 4.3 that, with positive probability,

$$\min\{d_N,n\}=\dim_{\mathrm{H}} F(\mathbf{a},\mathbf{T}^N)\le\dim_{\mathrm{H}} F(\mathbf{a},\mathbf{T})$$

for almost all $\mathbf{a}$ and for all N large enough so that $\sum_{i=1}^N p_i>1$. Non-extinction of the branching process underlying the unbounded offspring tree $\mathbf{T}$ occurs with probability $1-\eta$, where η is given by (4.25). Given non-extinction, the statistical self-affine nature of the construction implies that for all N, almost surely $\min\{d_N,n\}\le\dim_{\mathrm{H}} F(\mathbf{a},\mathbf{T})$ for almost all $\mathbf{a}$. We conclude that, conditional on non-extinction,

$$\min\left\{n,\ \sup_{1\le N<\infty} d_N\right\}\le\dim_{\mathrm{H}} F(\mathbf{a},\mathbf{T}) \tag{4.27}$$

for almost all $\mathbf{a}$.

Of course, we would like the estimates in (4.26) and (4.27) to be the same. Whilst this is the case for many families of contractions $(T_1,T_2,\dots)$, in general it does not follow easily from submultiplicative properties. In particular, it is not clear that $\lim_{N\to\infty}\Phi^N(s)=\Phi(s)$ for each s, although (local) uniform convergence would be enough for equality of the upper and lower bounds.

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§5. *Examples.* We illustrate Theorem 4.3 with some examples. Firstly, if each vertex of the random tree has a constant number of children, the possibility of extinction is avoided.

Example 5.1. Fix an integer m with $2 \le m \le N$. Let $\mathbf{T}$ be a "constant valence" random tree, so that each of the $\binom{N}{m}$ possible m-vertex subsets of $\{\mathbf{i}i \in \mathbf{T} : 1 \le i \le N\}$ occurs with equal probability $1/\binom{N}{m}$. In particular,

$$\mathsf{P}(\mathbf{i}i \in \mathbf{T} | \mathcal{F}_k) = \begin{cases} \dfrac{m}{N} & \text{if } \mathbf{i} \in \mathbf{T}, \\ 0 & \text{if } \mathbf{i} \notin \mathbf{T}. \end{cases} \tag{5.28}$$

By Theorem 4.3, with probability 1, $\dim_{\mathrm{H}} F(\mathbf{a}, \mathbf{T}) = \dim_{\mathrm{B}} F(\mathbf{a}, \mathbf{T}) = \min\{d, n\}$ for almost all $\mathbf{a}$, where d satisfies

$$\lim_{k\to\infty} \Bigg[\sum_{\mathbf{i}\in\mathbf{J}_k} \phi^d(T_{\mathbf{i}}) \Bigg]^{1/k} = \frac{N}{m}.$$

When the linear parts T_i are given by triangular matrices, the almost-sure dimensions are given by a closed formula.

Example 5.2. Suppose that the linear parts of the affine mappings (2.5) are simultaneously represented by upper triangular matrices T_i with respect to some basis, where

$$T_i = \begin{pmatrix} t_1^i & t_{12}^i & \cdots & t_{1n}^i \\ 0 & t_2^i & \cdots & t_{2n}^i \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & t_n^i \end{pmatrix}$$

with $0 < |t_r^i| < 1/2$ for $r = 1, \ldots, n$. Conditional on non-extinction of the construction, almost surely $\dim_{\mathrm{H}} F(\mathbf{a}, \mathbf{T}) = \dim_{\mathrm{B}} F(\mathbf{a}, \mathbf{T}) = \min\{s, n\}$ for almost all $\mathbf{a}$, where s satisfies

$$\max_{\substack{\{j_1,\ldots,j_{m-1}\} \\ \{j_1',\ldots,j_m'\} \\ m-1<s\le m}} \Big\{ |t_{j_1}^1 \cdots t_{j_{m-1}}^1|^{m-s} \, |t_{j_1'}^1 \cdots t_{j_m'}^1|^{s-m+1} \, p_1 + \cdots \\ + |t_{j_1}^N \cdots t_{j_{m-1}}^N|^{m-s} \, |t_{j_1'}^N \cdots t_{j_m'}^N|^{s-m+1} \, p_N \Big\} = 1, \tag{5.29}$$

where m is the integer such that $m - 1 < s \le m$ and the maximum is over all sets $\{j_1, \ldots, j_{m-1}\}$ and $\{j_1', \ldots, j_m'\}$ of $m-1$, respectively m, distinct integers from $\{1, \ldots, n\}$.

Attractors of IFSs represented by upper triangular matrices are studied in [**10**], where an explicit formula is obtained for (2.8). A trivial modification to include the p_i gives (5.29) as a closed form of (3.11), which by Theorem 4.3 is the almost-sure dimension.

Finally, we give an example to show that if the condition $\max_{1\le i\le N} \|T_i\| < 1/2$ in Theorem 4.3 is omitted, the dimension of the random set can be less than the value given by (3.11) almost surely.

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Example 5.3. Let

$$T_1 = T_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \lambda \end{pmatrix},$$

where $1/2 < \lambda < 1$ is the reciprocal of a Pisot–Vijayaraghavan number. Let a_1 and a_2 be vectors not lying on the same horizontal or vertical line. Let $p_1 = p_2 = p > 1/2$, and let F be the random set constructed as above based on the binary random tree with each offspring selected independently with probability p. Then there is a number $\epsilon > 0$ (depending on λ) such that, conditional on non-extinction,

$$\dim_{\mathrm{H}} F(\mathbf{a}, \mathbf{T}) \leq 2 - \frac{\log(1/\lambda)}{\log 2} + \frac{\log p}{\log 2} - \epsilon = d_p - \epsilon, \tag{5.30}$$

where d_p is the value given by (3.11) (which reduces to $2p\phi^{d_p}(T_1) = 1$ upon using the fact that the T_i are equal and diagonal).

To see this note in the non-random case (that is, with $p = 1$), provided that a_1 and a_2 do not lie on the same horizontal or vertical line, all the attractors are equivalent up to affine transformation and have Hausdorff dimension $s_1 \equiv 2 - \log(1/\lambda)/\log 2 - \epsilon = d_1 - \epsilon$ for some $\epsilon > 0$, where d_1 is given by (2.8); see [**4, 19, 21**].

Thus it is enough to consider the canonical case of $a_1 = (0, 0)^T$ and $a_2 = (1/2, 1 - \lambda)^T$, where the $S_i = T_i + a_i$ $(i = 1, 2)$ map the unit square onto $(1/2) \times \lambda$ rectangles in the bottom left and upper right of the unit square; let F denote the non-random attractor in this case.

If $s > s_1$, then for all $K \in \mathbb{N}$ and $0 < \delta < s - s_1$ we can find a covering of F by binary squares of sidelengths at most 2^{-K} such that $\sum_{k=K}^{\infty} 2^{-ks} N(k) \leq 2^{-K\delta}$, where there are $N(k)$ squares of sidelength 2^{-k} in the covering. This follows from the definitions of Hausdorff dimension and measure, noting that coverings by binary squares give values for Hausdorff measures equivalent to those obtained from coverings by arbitrary sets. By considering the random iterated construction of $F(\mathbf{a}, \mathbf{T})$ from the rectangles, there is a probability of p^k that $F \cap S$ will remain after the kth step of the iteration if S is a binary square of sidelength 2^{-k}. Thus the expected sum of $(s + \log p/\log 2)$th powers of sidelengths of the remaining squares, which cover $F(\mathbf{a}, \mathbf{T})$, is at most

$$\sum_{k=K}^{\infty} 2^{-k(s+\log p/\log 2)} p^k N(k) = \sum_{k=K}^{\infty} 2^{-ks} N(k) \leq 2^{-K\delta}.$$

Using the Borel–Cantelli lemma, it follows that $\dim_{\mathrm{H}} F(\mathbf{a}, \mathbf{T}) \leq s + \log p/\log 2$ for all $s > s_1$, giving (5.30).

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