J. Fluid Mech. (2014), *vol.* 741, *pp* 316–349. © Cambridge University Press 2014
doi:10.1017/jfm.2013.652

Wave turbulence in a rotating channel

Julian F. Scott†

Laboratoire de Mécanique des Fluides et d'Acoustique, Université de Lyon, Ecole Centrale de Lyon, 36 avenue Guy de Collongue, 69134 Ecully CEDEX, France

(Received 4 January 2013; revised 29 November 2013; accepted 5 December 2013; first published online 13 February 2014)

This paper describes wave-turbulence closure and its consequences for rapidly rotating (i.e. small Rossby number) turbulence confined by two infinite, parallel walls perpendicular to the rotation axis. Expressing the flow as a combination of inertial waveguide modes leads to a spectral matrix, whose diagonal elements express the distribution of energy over modes and whose off-diagonal elements represent correlations between modes of different orders. In preparation for wave-turbulence closure, the flow is decomposed into two-dimensional and wave components. The former is found to evolve as if it were a classical, two-dimensional, non-rotating flow, but with wall friction due to Ekman pumping by the boundary layers. Evolution equations for the wave-component elements of the spectral matrix are derived using a wave-turbulence approach. Detailed analysis of these equations shows that, surprisingly, the two-dimensional component has no effect on wave-component energetics. As expected for wave turbulence, energy transfer between wave modes is via resonant triads and takes place at times $O(\varepsilon^{-2})$ multiples of the rotational period, where ε is the Rossby number. Despite playing no role in wave-mode energetics, the two-dimensional component produces decay of the off-diagonal elements of the spectral matrix on a time scale that is small compared with $O(\varepsilon^{-2})$ rotation periods. There are thus three asymptotically distinct stages in the evolution of the turbulence in the limit of small Rossby number: the two-dimensional flow begins to evolve at the usual large-eddy turnover time scale ($O(\varepsilon^{-1})$ multiples of the rotation period) and continues to develop thereafter. This is followed by decorrelation of different wave orders and finally evolution of the wave energy spectra due to resonant interactions.

Key words: homogeneous turbulence, rotating turbulence, wave-turbulence interactions

1. Introduction

The spectral description of unbounded, homogeneous turbulence has a long history (see e.g. Batchelor 1953), more recently including extensions to externally imposed anisotropizing influences such as rotation and stratification. Rotating turbulence is a fundamental problem of fluid dynamics and has been the subject of many previous studies. Applications include geophysical flows, turbomachinery and astrophysical systems. Spectral closures, notably the eddy-damped quasi-normal Markovian (EDQNM) model, have been developed including rotation (see e.g. Cambon & Jacquin 1989; Waleffe 1993; Cambon, Mansour & Godeferd 1997). Direct numerical

†Email address for correspondence: julian.scott@ec-lyon.fr

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

simulation (DNS) has also played an increasingly important role in recent years (see e.g. Teitelbaum & Mininni 2012 and references therein).

Wave-turbulence theory (see Zakharov, L'vov & Falkovich 1992; Nazarenko 2011; Newell & Rumpf 2011) provides a natural closure in the limit of weak turbulence for a system, such as a rotating fluid, which involves dispersive waves. Weakness of the turbulence means that linear theory provides a first approximation and wave-turbulence theory leads to the conclusion that nonlinear effects (which take many wave periods to act) are dominated by resonances in which the sum of wave frequencies is zero (or nearly so, the precise condition depending on the strength of the turbulence). The wave-turbulence approach has been used by Galtier (2003) and Bellet *et al.* (2006) to study the evolution of unconfined, homogeneous rotating turbulence at small Rossby number (which is the measure of turbulence strength in a rotating fluid). Bellet *et al.* (2006) also showed that the EDQNM model becomes wave-turbulence theory in the limit of small Rossby number, the semi-empirical eddy damping being asymptotically unimportant, while quasi-normality and Markovianization follow naturally from wave-turbulence analysis.

However, turbulence is never really unbounded and homogeneous and it is of interest to study the effects of confinement. Many studies, such as Kartashova (1994), Pushkarev (1999), Zakharov *et al.* (2005), Lvov, Nazarenko & Pokorni (2006), Bourouiba (2008), Nazarenko (2011, chapter 10) and references therein, have been undertaken to investigate wave turbulence confined in three dimensions with comparable length scales of confinement in all directions. Such confinement leads to wave modes with *discrete* frequencies and the central question is to what extent nonlinear interactions are inhibited by the resulting difficulty in achieving resonance. For weak enough turbulence, near resonances are ineffective, thus nonlinear interactions between modes are strongly suppressed compared with the unconfined case. As the turbulence strength (the Rossby number in the rotating case) is increased for a given box, which is large compared with the turbulence scales, the system first enters the mesoscopic range, in which a limited set of near resonances are effective, before, finally, near resonances are fully expressed and confinement becomes unimportant.

There is, however, an interesting case, intermediate between unconfined and fully confined turbulence, in which the flow remains infinite in one or two dimensions. The modes are then continuous in frequency, so near resonances are always fully effective, though confinement means that the turbulence cannot remain homogeneous in the direction(s) of confinement. Of course, no real system is infinite in any direction and we have in mind one in which the length scales of confinement are much greater in certain directions, here treated as infinite.

With the above considerations in mind, we study the relatively simple problem in which decaying, rotating turbulence is confined between two infinite, parallel walls, $x_3 = 0, h$, which rotate about the x_3-axis. The turbulence is assumed to be statistically homogeneous and isotropic in x_1, x_2, but confinement means that it is inhomogeneous with respect to x_3.

Fourier transforms are used for spectral analysis in the unconfined, homogeneous case. A natural equivalent for the present problem is to write the velocity field as a linear combination of inviscid, linear inertial waveguide modes (similar ideas have been proposed for the non-rotating channel by Turner (1999) and by Yang, Su & Wu (2010) for pressure-gradient-driven channel flow with rotation axis parallel to the walls), whose amplitudes evolve in time due to viscosity and nonlinearity. These modes have $\exp[\mathrm{i}(k_1 x_1 + k_2 x_2)]$ dependence on x_1, x_2, where $\boldsymbol{k} = (k_1, k_2)$ is a two-dimensional wave vector and, depending on which velocity component is

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

considered, either $\cos(n\pi x_3/h)$ or $\sin(n\pi x_3/h)$ dependence on x_3, where the mode order, n, can take any integer value. Thus, the modes are parameterized by $\boldsymbol{k}$ and n: confinement can be thought of as leading to discretization of the x_3 component of a three-dimensional wave vector, $\boldsymbol{K} = (k_1, k_2, n\pi/h)$.

A spectral matrix, $A_{nm}(k, t)$, where $k = |\boldsymbol{k}|$, can be defined via the statistical average of products of the amplitudes of modes of orders n and m. The diagonal elements of this matrix are real and positive. They represent the distribution of energy over the modes. The off-diagonal elements can be complex and express correlations between modes of different orders. Our aim is to study the time evolution of $A_{nm}(k, t)$, in particular in the limit of small Rossby number, ε.

We might hope that wave-turbulence closure would apply in this limit. However, $n = 0$ modes are non-dispersive (their group velocity is zero) and one of the prerequisites of wave-turbulence theory is dispersive waves. For this reason, we separate the $n = 0$ and $n \neq 0$ components. A two-dimensional component of the flow is constructed by combining the contributions of all $n = 0$ modes. The remainder is referred to as the wave component.

In § 2 and appendix A we define the modes and derive equations for the time evolution of the modal amplitudes under viscous and nonlinear effects. In the limit of small Rossby number, nonlinear effects are small and, in order that viscous dissipation not kill the turbulence before nonlinearity has had time to intervene, the Ekman number, E, must also be small. When both numbers are small, it is found that viscosity leads to a linear damping term in the amplitude equations. The damping factor is the sum of two terms representing damping by Ekman pumping by the wall boundary layers and dissipation from outside the layers.

In § 3 we introduce the spectral matrix and obtain its evolution equation. As usual when the equation for the second-order moments is derived, it contains third-order moments. Evolution equations for the latter are also derived and, unsurprisingly, contain fourth-order moments. In preparation for wave-turbulence analysis, the fourth-order moments are expressed as the sum of quasi-normal and cumulant contributions.

In § 4 we define the two-dimensional and wave components of the flow. The governing equations of the two-dimensional component are derived. Apart from a term representing forcing by the wave component, these equations are those of two-dimensional, non-rotating flow with a linear term representing wall friction due to Ekman pumping. Because the wave component has oscillations of frequency comparable with the rotational angular velocity, the time average of the wave-forcing term over the long time scales needed for evolution is small. Thus, one can neglect forcing by the wave component and, apart from wall-friction damping, the two-dimensional component evolves as if it were classical, two-dimensional, non-rotating turbulence. It does so on a time scale $O(\varepsilon^{-1})$ longer than the rotational period, corresponding to the usual large-eddy turnover time. Two-dimensional turbulence has been much studied in previous work (see the reviews by Kraichnan & Montgomery (1980), Boffetta & Ecke (2012) and the much briefer description by Frisch (1995, § 9.7)) and we simply adopt the results of that work. The amplitudes of the $n \neq 0$ modes which make up the wave component evolve on much longer time scales than the initial development of the two-dimensional component. At such times, the two-dimensional component is in the final stage in which energy has cascaded to small wavenumbers and consists of well-separated vortices.

Section 5 is the centre of gravity of this paper and focuses on the evolution of the wave component, using wave-turbulence analysis to derive evolution equations for $A_{nm}(k, t)$ $(n, m \neq 0)$. The influence of the two-dimensional component is examined.

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

Surprisingly, it is found not to affect the $n \neq 0$ energy distributions, $A_{nn}(k, t)$, which evolve according to a closed system of equations on the usual wave-turbulence time scale, $O(\varepsilon^{-2})$ times the rotational period. On the other hand, the off-diagonal elements of $A_{nm}(k, t)$ $(n, m \neq 0)$ are strongly affected and decay on a time scale asymptotically small compared with ε^{-2} rotation periods. Thus, different $n \neq 0$ decorrelate before there is significant energy transfer between them.

2. Modal formulation

Consider incompressible, turbulent flow between two infinite, parallel walls, $x_3 = 0, h$, which rotate at angular velocity $\boldsymbol{\Omega} = (0, 0, \Omega)$ about the x_3-axis. From here on, we non-dimensionalize spatial coordinates using the length scale h, time using $(2\Omega)^{-1}$, velocity using $2\Omega h$ and pressure using $4\rho\Omega^2 h^2$. We use a frame of reference, x_1, x_2, x_3, rotating with the walls, so the Navier–Stokes equations take the form

$$\frac{\partial u_i}{\partial t} + \underbrace{\varepsilon_{i3j} u_j}_{\text{Coriolis term}} + \underbrace{\frac{\partial}{\partial x_j}(u_i u_j)}_{\text{Nonlinear term}} = -\frac{\partial p}{\partial x_i} + \underbrace{E \frac{\partial^2 u_i}{\partial x_j \partial x_j}}_{\text{Viscous term}}, \tag{2.1}$$

$$\frac{\partial u_i}{\partial x_i} = 0, \tag{2.2}$$

where, here and henceforth, the summation convention applies, $E = \nu/(2\Omega h^2)$ is the Ekman number and p is a pressure variable incorporating the centrifugal force. The boundary conditions are

$$u_i = 0, \quad x_3 = 0, 1. \tag{2.3}$$

The flow will be assumed statistically homogeneous and isotropic in x_1, x_2. It follows that the mean-flow velocity components $\bar{u}_1 = \bar{u}_2 = 0$, and hence $\bar{u}_3 = 0$ according to the statistical average of (2.2) and (2.3). Thus, there is no mean flow. Had we not assumed isotropy, inhomogeneity with respect to x_3 would generally lead to forcing of the mean flow by the Reynolds stresses. The initial flow is assumed to consist of turbulent scales of size comparable to the channel width, i.e. $O(1)$ in non-dimensional terms.

In what follows, we aim to develop a theory for small Rossby and Ekman numbers. This means that viscosity and nonlinearity can be neglected over time spans of $O(1)$. We first introduce the inertial modes which arise when viscosity and nonlinearity are dropped entirely.

2.1. *Inertial modes*

Dropping the nonlinear and viscous terms in (2.1), as well as the x_1 and x_2 components of (2.3) (inviscid boundary conditions), equations (2.1)–(2.3) have complex solutions (inertial waveguide modes) of the form

$$u_i = W_i^{(n)}(x_3; \boldsymbol{k}) \exp\{\mathrm{i}k_1 x_1 + \mathrm{i}k_2 x_2 - \mathrm{i}\omega_n(k)t\}, \tag{2.4}$$

$$p = \Phi^{(n)}(x_3; \boldsymbol{k}) \exp\{\mathrm{i}k_1 x_1 + \mathrm{i}k_2 x_2 - \mathrm{i}\omega_n(k)t\}, \tag{2.5}$$

where

$$W_1^{(n)}(x_3; \boldsymbol{k}) = \frac{k_1\omega_n(k) + \mathrm{i}k_2}{k}\cos(n\pi x_3), \quad W_2^{(n)}(x_3; \boldsymbol{k}) = \frac{k_2\omega_n(k) - \mathrm{i}k_1}{k}\cos(n\pi x_3), \tag{2.6}$$

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

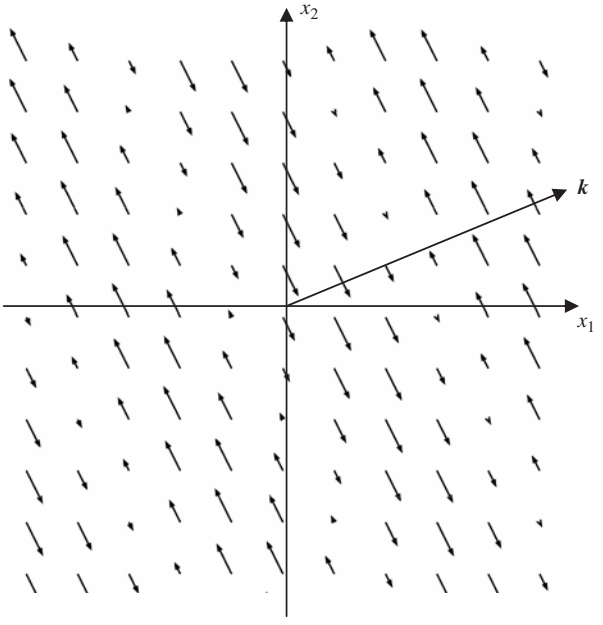


FIGURE 1. Illustration of the velocity field of an $n = 0$ mode.

$$W_3^{(n)}(x_3; \boldsymbol{k}) = -\frac{\mathrm{i}k}{(k^2 + n^2\pi^2)^{1/2}} \sin(n\pi x_3), \tag{2.7}$$

$$\Phi^{(n)}(x_3; \boldsymbol{k}) = -\frac{k}{k^2 + n^2\pi^2} \cos(n\pi x_3), \tag{2.8}$$

$$\omega_n(k) = \frac{n\pi}{(k^2 + n^2\pi^2)^{1/2}}. \tag{2.9}$$

The modes are parameterized by a real wave vector $\boldsymbol{k} = (k_1, k_2)$, of norm $k = |\boldsymbol{k}|$, and n, which can take any integer value. If one defines a three-dimensional wave vector, $\boldsymbol{K} = (k_1, k_2, n\pi)$, equation (2.9) can be thought of as the usual inertial-wave dispersion relation and the effect of confinement is to discretize the x_3 component of $\boldsymbol{K}$. Note that $-1 \leqslant \omega_n \leqslant 1$ and that the mode $-\boldsymbol{k}$, $-n$ is the complex conjugate of $\boldsymbol{k}$, n. Real (and hence physically meaningful) modal velocity and pressure fields require a linear combination of the mode and its conjugate (with complex conjugate coefficients). Conjugate modes are thus intimately associated and represent the same physical entity, but the mathematics is simpler if complex modes are employed, rather than attempting to combine conjugates.

Modes with $n = 0$ are special because $\omega_0(k) = 0$. (2.4) and its conjugate then represent a steady, two-dimensional flow ($u_3 = 0$, $u_1 = u_1(x_1, x_2)$, $u_2 = u_2(x_1, x_2)$). Figure 1 illustrates this flow: the velocity is perpendicular to $\boldsymbol{k}$ and varies sinusoidally in the direction of $\boldsymbol{k}$. The special nature of $n = 0$ modes will later have important repercussions.

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

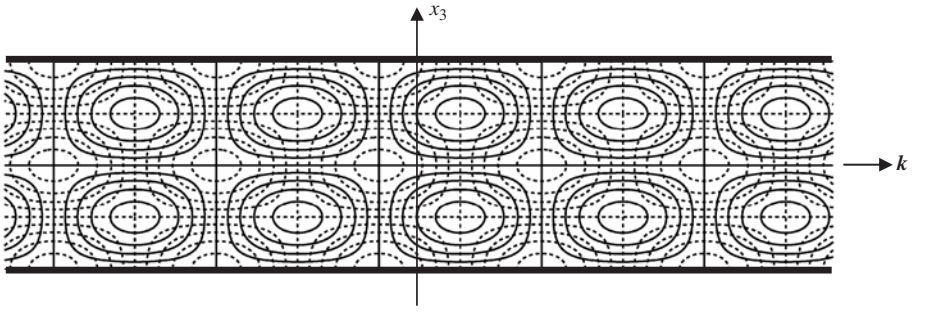


FIGURE 2. Snapshot of the velocity field of an $n=2$, $k=4$ mode in the plane containing $\boldsymbol{k}$ and the rotation axis. The solid curves are streamlines projected onto the given plane, while the dashed curves are contours of constant velocity component perpendicular to that plane.

The $n \neq 0$ modes are waves that propagate along the channel, having phase velocity in the direction of $\boldsymbol{k}$ if $n > 0$ and in the direction of $-\boldsymbol{k}$ if $n < 0$ (it is perhaps interesting to note that the group velocity, $\boldsymbol{\nabla}_{\boldsymbol{k}}\omega_n$, is in the opposite direction). Figure 2 illustrates the instantaneous velocity field for a particular mode (and its conjugate) in the plane defined by $\boldsymbol{k}$ and the rotation axis. There is a checkerboard pattern of vortices, neighbouring vortices being contrarotating. The velocity component perpendicular to the plane is represented by the dashed curves. As time increases, the whole pattern moves to the right or left, depending on the sign of n.

Here $W_i^{(n)}(x_3; \boldsymbol{k}) \exp\{\mathrm{i}k_1x_1 + \mathrm{i}k_2x_2\}$ form a basis set for solenoidal u_i which satisfy $u_3 = 0$ on $x_3 = 0, 1$. Thus, we can express the turbulent velocity as

$$u_i(\boldsymbol{x}, t) = \sum_{n=-\infty}^{\infty} \int \tilde{u}_n(\boldsymbol{k}, t) W_i^{(n)}(x_3; \boldsymbol{k}) \exp\{\mathrm{i}k_1x_1 + \mathrm{i}k_2x_2\} \, \mathrm{d}^2\boldsymbol{k}. \tag{2.10}$$

The $W_i^{(n)}(x_3; \boldsymbol{k})$ are orthonormal in the sense that

$$\int_0^1 W_i^{(n)*}(x_3; \boldsymbol{k}) W_i^{(n')}(x_3; \boldsymbol{k}) \, \mathrm{d}x_3 = \delta_{n-n'}, \tag{2.11}$$

where $*$ denotes complex conjugation, $\delta_0 = 1$ and $\delta_n = 0$ if $n \neq 0$. Equations (2.10), (2.11) and the Fourier identity

$$\int_{-\infty}^{\infty} \exp\{\mathrm{i}yz\} \, \mathrm{d}z = 2\pi\delta(y), \tag{2.12}$$

where $\delta(y)$ is the Dirac function, lead to

$$\tilde{u}_n(\boldsymbol{k}, t) = \frac{1}{4\pi^2} \int_0^1 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u_i(\boldsymbol{x}, t) \, W_i^{(n)*}(x_3; \boldsymbol{k}) \exp\{-\mathrm{i}k_1x_1 - \mathrm{i}k_2x_2\} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \, \mathrm{d}x_3. \tag{2.13}$$

Equation (2.13) provides a transform whose inverse is (2.10). Since u_i is real, $\tilde{u}_n^*(\boldsymbol{k}, t) = \tilde{u}_{-n}(-\boldsymbol{k}, t)$ follows from (2.13): modes and their conjugates are combined by (2.10) such that the resulting velocity field is real.

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

2.2. *Modal amplitude equation*

The equation governing the time evolution of $\tilde{u}_n$ follows from applying the transform (2.13) to (2.1). Details of the derivation can be found in appendix A: the result is

$$\frac{\partial \tilde{u}_n}{\partial t} + \left(\underbrace{\mathrm{i}\omega_n(k)}_{\text{Coriolis term}} + \underbrace{E(k^2 + n^2\pi^2)}_{\text{Volumetric viscous damping}} \right) \tilde{u}_n$$
$$= \underbrace{\sum_{n_p, n_q = -\infty}^{\infty} \int N_{nn_pn_q}(\boldsymbol{k}, \boldsymbol{p}) \tilde{u}^*_{n_p}(\boldsymbol{p}) \tilde{u}^*_{n_q}(-\boldsymbol{k} - \boldsymbol{p}) \, \mathrm{d}^2\boldsymbol{p}}_{\text{Nonlinearity}}$$
$$+ \underbrace{\frac{1}{4\pi^2} E \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{\partial u_\alpha}{\partial x_3} W^{(n)*}_\alpha \right]_0^1 \exp\{-\mathrm{i}k_1 x_1 - \mathrm{i}k_2 x_2\} \, \mathrm{d}x_1 \, \mathrm{d}x_2}_{\text{Wall viscous damping}} \tag{2.14}$$

where $[f(x_3)]_0^1 = f(1) - f(0)$ and, here and henceforth, α is an index which runs over the values 1, 2 and to which the summation convention applies. Note that the pressure-gradient term in (2.1) does not contribute to (2.14) and that the Coriolis term has a particularly simple form.

The left-hand side of (2.14) should be no surprise: the second term expresses modal oscillations and volumetric damping by viscosity. The right-hand side represents nonlinearity and viscous effects at the walls. The nonlinear term shows interactions between wave-vector triads $\boldsymbol{k}$, $\boldsymbol{p}$, $\boldsymbol{q}$ such that $\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = 0$. Furthermore, the nonlinear coefficient, $N_{nn_pn_q}(\boldsymbol{k}, \boldsymbol{p})$, is zero unless one of the four conditions $n \pm n_p \pm n_q = 0$ is met. This, along with $\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = 0$, are the conditions for mode–triad nonlinear coupling. Here $N_{nn_pn_q}(\boldsymbol{k}, \boldsymbol{p})$ has the symmetry property

$$N_{nn_pn_q}(\boldsymbol{k}, \boldsymbol{p}) = N_{nn_qn_p}(\boldsymbol{k}, -\boldsymbol{k} - \boldsymbol{p}). \tag{2.15}$$

In the absence of nonlinearity and viscosity, equation (2.14) yields the expected result that $\tilde{u}_n(\boldsymbol{k}, t) \propto \exp\{-\mathrm{i}\omega_n(k)t\}$.

With the small Rossby and Ekman numbers limit in mind, it is convenient to suppress the modal oscillations by writing

$$\tilde{u}_n = a_n(\boldsymbol{k}, t) \mathrm{e}^{-\mathrm{i}\omega_n(k)t}, \tag{2.16}$$

where $a_n(\boldsymbol{k}, t)$ are the mode amplitudes. Equation (2.14) becomes

$$\frac{\partial a_n}{\partial t} + E(k^2 + n^2\pi^2) a_n$$
$$= \sum_{n_p, n_q = -\infty}^{\infty} \int \mathrm{e}^{\mathrm{i}(\omega_n(k) + \omega_{n_p}(p) + \omega_{n_q}(|\boldsymbol{k}+\boldsymbol{p}|))t} N_{nn_pn_q}(\boldsymbol{k}, \boldsymbol{p}) a^*_{n_p}(\boldsymbol{p}) a^*_{n_q}(-\boldsymbol{k} - \boldsymbol{p}) \, \mathrm{d}^2\boldsymbol{p}$$
$$+ \frac{1}{4\pi^2} E \mathrm{e}^{\mathrm{i}\omega_n(k)t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{\partial u_\alpha}{\partial x_3} W^{(n)*}_\alpha \right]_0^1 \exp\{-\mathrm{i}k_1 x_1 - \mathrm{i}k_2 x_2\} \, \mathrm{d}x_1 \, \mathrm{d}x_2 \tag{2.17}$$

which is the mode amplitude equation, in a preliminary form because the wall integral term has not yet been expressed in terms of the a_n. In the absence of nonlinearity and viscosity, the a_n are time independent, which is the reason for their introduction. They are slowly varying when the Rossby and Ekman numbers are small. Note that $a^*_n(\boldsymbol{k}, t) = a_{-n}(-\boldsymbol{k}, t)$ follows from $\tilde{u}^*_n(\boldsymbol{k}, t) = \tilde{u}_{-n}(-\boldsymbol{k}, t)$.

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

2.3. *Small Rossby and Ekman numbers: final version of the amplitude equation*

Up to now, no use has been made of the supposed smallness of the Rossby and Ekman numbers and the results are exact. From here on, we suppose that these parameters are small. We then expect nonlinearity to be unimportant for time spans of $O(1)$ and thin boundary layers at the walls. Linearized, boundary-layer analysis is used in appendix B to evaluate the wall integral term in (2.17). The analysis is based on that of Racz & Scott (2008), who studied a quite different problem, but one which also involved modal damping in confined rotating flow at small Rossby and Ekman numbers. The result is

$$\frac{\partial a_n}{\partial t} + \Delta_n\, a_n = \sum_{n_p,n_q=-\infty}^{\infty} \int \mathrm{e}^{\mathrm{i}(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(|\boldsymbol{k}+\boldsymbol{p}|))t} N_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p}) a^*_{n_p}(\boldsymbol{p}) a^*_{n_q}(-\boldsymbol{k}-\boldsymbol{p})\,\mathrm{d}^2\boldsymbol{p}, \tag{2.18}$$

where

$$\Delta_n(k) = E^{1/2} D_n + E(k^2 + n^2\pi^2) \tag{2.19}$$

is the overall damping factor of the given mode,

$$\begin{aligned} D_n \;=\; & \tfrac{1}{2}(\gamma^{(n)}_+(k) + \gamma^{(n)}_-(k))(W^{(n)*}_\alpha(0;\boldsymbol{k}) W^{(n)}_\alpha(0;\boldsymbol{k}) + W^{(n)*}_\alpha(1;\boldsymbol{k}) W^{(n)}_\alpha(1;\boldsymbol{k})) \\ & + \tfrac{1}{2}\mathrm{i}(\gamma^{(n)}_+(k) - \gamma^{(n)}_-(k))\varepsilon_{\alpha 3\beta}(W^{(n)*}_\alpha(0;\boldsymbol{k}) W^{(n)}_\beta(0;\boldsymbol{k}) + W^{(n)*}_\alpha(1;\boldsymbol{k}) W^{(n)}_\beta(1;\boldsymbol{k})) \end{aligned} \tag{2.20}$$

is the (complex) damping factor due to the boundary layers (Ekman pumping) and

$$\gamma^{(n)}_\pm(k) = \frac{1 \mp \mathrm{i}}{\sqrt{2}}(1 \pm \omega_n(k))^{1/2}. \tag{2.21}$$

In (2.20) and what follows, β, like α, is an index which runs over the values 1, 2 and to which the summation convention applies. The real part of D_n represents actual damping, whereas its imaginary part yields a small ($O(E^{1/2})$) viscous correction to the inviscid modal frequency, ω_n, which is of lesser importance. Equation (2.18) is the final version of the amplitude equation. It governs evolution of the a_n due to nonlinearity and viscosity. Equation (2.19) shows that there are two viscous damping mechanisms: boundary layer and volumetric.

Using (2.6) and (2.21), equation (2.20) gives

$$\begin{aligned} D_n(k) \;&=\; (1-\omega_n(k))^2 \gamma^{(n)}_+(k) + (1+\omega_n(k))^2 \gamma^{(n)}_-(k) \\ &=\; \frac{(1-\omega_n^2(k))^{1/2}}{\sqrt{2}}\{(1+\omega_n(k))^{3/2} + (1-\omega_n(k))^{3/2} \\ &\quad + \mathrm{i}((1+\omega_n(k))^{3/2} - (1-\omega_n(k))^{3/2})\}. \end{aligned} \tag{2.22}$$

Employing (2.9), $D_0(k) = 2^{1/2}$ is real and constant, while $\mathrm{Re}(D_n(k))$, $n \neq 0$ (Re( ) denotes the real part of a complex number) is an increasing function of $k/|n|$, which takes the values $\mathrm{Re}(D_n) = 0$ at $k/|n| = 0$ and $\mathrm{Re}(D_n) = 2^{1/2}$ at $k/|n| = \infty$. In particular, boundary-layer damping is absent when $k = 0$ and $n \neq 0$. Another way of expressing these results uses the angle between the three-dimensional wave vector, $\boldsymbol{K} = (k_1, k_2, n\pi)$, and the rotation vector. According to the dispersion relation, (2.9), $\omega_n(k)$ depends only on this angle. Thus, (2.22) implies that $D_n(k)$ is also a function of the same angle. When the angle is zero, i.e. the two vectors are aligned, $D_n(k) = 0$, and there is no boundary-layer damping. Then $\mathrm{Re}(D_n(k))$ increases with angle until $D_n(k) = 2^{1/2}$ when the angle is $\pi/2$, yielding maximal damping equal to that for $n = 0$.

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

The relative importance of volumetric and boundary-layer damping depends on which mode is considered. The damping factor of the former is $E|\boldsymbol{K}|^2$ and is small, of $O(E)$, when $|\boldsymbol{K}|^2 = O(1)$, but increases with $|\boldsymbol{K}|$. This is the usual behaviour in unbounded turbulence: smaller scales are more subject to damping. On the other hand, as we have just seen, boundary-layer damping depends on the direction of $\boldsymbol{K}$, rather than its norm. The damping factor is $O(E^{1/2})$, apart from directions near the rotation vector, where it goes to zero. Since $E \ll 1$, $E^{1/2}$ is asymptotically large compared with E. This tends to favour boundary-layer over volumetric damping, but this is not the case if $|\boldsymbol{K}|^2$ is sufficiently large or the angle between $\boldsymbol{K}$ and the rotation vector is sufficiently small.

Dropping the nonlinear term in (2.18) leads to viscous decay of the form $a_n \propto \exp[-\Delta_n t]$. Here $\mathrm{Re}(\Delta_n(k))$ is an increasing function of k, so minimum damping occurs for $k = 0$. Thus, at sufficiently large times, linear theory predicts concentration of energy near $k = 0$ due to viscous dissipation. Recalling that $k = 0$ modes are independent of x_1, x_2, the turbulence should become 'horizontally' stratified in the absence of nonlinearity. For $n \neq 0$, $\mathrm{Re}(\Delta_n(0)) = En^2\pi^2$, so the least damped $n \neq 0$ mode is $n = 1$, $k = 0$. Its damping factor, $\mathrm{Re}(\Delta_n(0)) = E\pi^2$, can be compared with that, $\mathrm{Re}(\Delta_0(0)) = (2E)^{1/2}$, for $n = 0$. If $E < 2/\pi^4$, the least damped mode is $n = 1$, $k = 0$, while when $E > 2/\pi^4$ it is $n = 0$, $k = 0$.

3. Spectral analysis

As usual, the principal measures of turbulence are the second-order velocity moments. Statistical homogeneity of u_i with respect to x_1, x_2 implies

$$\overline{u_i(x_1, x_2, x_3, t)u_j(x_1 + r_1, x_2 + r_2, x_3', t)} = R_{ij}(r_1, r_2, x_3, x_3', t). \tag{3.1}$$

From (2.12), (2.13) and (3.1),

$$\overline{\tilde{u}_n^*(\boldsymbol{k}, t)\tilde{u}_m(\boldsymbol{k}', t)} = \Psi_{nm}(\boldsymbol{k}, t)\delta(\boldsymbol{k} - \boldsymbol{k}'), \tag{3.2}$$

where

$$\begin{aligned}\Psi_{nm}(\boldsymbol{k}, t) \;=\; & \frac{1}{4\pi^2}\int_0^1\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ij}(r_1, r_2, x_3, x_3', t)W_i^{(n)}(x_3; \boldsymbol{k})W_j^{(m)*}(x_3'; \boldsymbol{k}) \\ & \times \exp\{-\mathrm{i}k_1r_1 - \mathrm{i}k_2r_2\}\,\mathrm{d}r_1\,\mathrm{d}r_2\,\mathrm{d}x_3\,\mathrm{d}x_3'\end{aligned} \tag{3.3}$$

is a spectral matrix, which is Hermitian, positive semi-definite and such that $\Psi_{nm}(\boldsymbol{k}, t) = \Psi_{-m,-n}(-\boldsymbol{k}, t)$. Isotropy with respect to x_1, x_2 implies that $\Psi_{nm}(\boldsymbol{k}, t)$ is independent of the direction of $\boldsymbol{k}$, i.e. $\Psi_{nm}(\boldsymbol{k}, t) = \Psi_{nm}(k, t)$. It follows that $\Psi_{nm}(k, t) = \Psi_{-m,-n}(k, t)$ and $\Psi_{nn}(k, t) = \Psi_{-n,-n}(k, t)$ is real and non-negative.

The inverse relation to (3.3) follows from (2.10), (3.1) and (3.2):

$$\begin{aligned}R_{ij}(r_1, r_2, x_3, x_3', t) \;=\; & \sum_{n,m=-\infty}^{\infty}\int \Psi_{nm}(k, t)W_i^{(n)*}(x_3; \boldsymbol{k})W_j^{(m)}(x_3'; \boldsymbol{k}) \\ & \times \exp\{\mathrm{i}k_1r_1 + \mathrm{i}k_2r_2\}\,\mathrm{d}^2\boldsymbol{k},\end{aligned} \tag{3.4}$$

showing that $\Psi_{nm}(k, t)$ fully characterizes the second-order, two-point velocity moments. Setting $r_1 = r_2 = 0$ and $x_3' = x_3$ gives the one-point moments

$$\overline{u_iu_j} = \sum_{n,m=-\infty}^{\infty}\int \Psi_{nm}(k, t)W_i^{(n)*}(x_3; \boldsymbol{k})W_j^{(m)}(x_3; \boldsymbol{k})\,\mathrm{d}^2\boldsymbol{k}. \tag{3.5}$$

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

From (2.16) and (3.2),

$$\Psi_{nm}(k,t) = A_{nm}(k,t)\mathrm{e}^{\mathrm{i}(\omega_n(k)-\omega_m(k))t}, \tag{3.6}$$

where

$$\overline{a_n^*(\boldsymbol{k},t)a_m(\boldsymbol{k}',t)} = A_{nm}(k,t)\delta(\boldsymbol{k}-\boldsymbol{k}'), \tag{3.7}$$

and, like Ψ_{nm}, the matrix A_{nm} is Hermitian and positive semi-definite. Here $A_{nm}(k,t) = A_{-m,-n}(k,t)$ and $A_{nn}(k,t) = A_{-n,-n}(k,t) = \Psi_{nn}(k,t)$ is real and non-negative. The matrix A_{nm} evolves slowly with time since the Rossby and Ekman numbers are small.

Setting $j = i$ in (3.5) and integrating across the channel using (2.11) and $A_{nn} = \Psi_{nn}$,

$$\frac{1}{2}\int_0^1 \overline{u_i u_i}\,\mathrm{d}x_3 = \frac{1}{2}\sum_{n=-\infty}^{\infty}\int A_{nn}(k,t)\,\mathrm{d}^2\boldsymbol{k} \tag{3.8}$$

is the statistically and x_3-averaged kinetic energy. This result gives the average kinetic energy as a sum of real, non-negative modal contributions. We see that the diagonal elements of $A_{nm}(k)$ represent the energy distribution over $\boldsymbol{k}$ of modes with a given value of n, whereas the off-diagonal ones express correlations between modal contributions with different values of n. One can also define energy spectra

$$E_n(k,t) = \pi k A_{nn}(k,t), \tag{3.9}$$

which yield the average kinetic energy via

$$\frac{1}{2}\int_0^1 \overline{u_i u_i}\,\mathrm{d}x_3 = \sum_{n=-\infty}^{\infty}\int_0^{\infty} E_n(k,t)\,\mathrm{d}k. \tag{3.10}$$

Note that $E_{-n}(k,t) = E_n(k,t)$, i.e. n and $-n$ give equal contributions to the energy.

3.1. *Spectral evolution equation*

Using (2.12), (2.13), (2.16) and the homogeneity of u_i, it can be shown that

$$\overline{a_n(\boldsymbol{k},t)a_{n_p}(\boldsymbol{p},t)a_{n_q}(\boldsymbol{q},t)} = \Theta_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p},t)\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}). \tag{3.11}$$

Equation (2.18) implies

$$\begin{aligned}
&\frac{\partial a_n^*(\boldsymbol{k})}{\partial t}a_m(\boldsymbol{k}') + \Delta_n^*(k)\,a_n^*(\boldsymbol{k})a_m(\boldsymbol{k}') \\
&\quad = \sum_{n_p,n_q=-\infty}^{\infty}\int \mathrm{e}^{-\mathrm{i}(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(|\boldsymbol{k}+\boldsymbol{p}|))t}N^*_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p}) \\
&\qquad \times a_m(\boldsymbol{k}')a_{n_p}(\boldsymbol{p})a_{n_q}(-\boldsymbol{k}-\boldsymbol{p})\,\mathrm{d}^2\boldsymbol{p}
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&a_n^*(\boldsymbol{k})\frac{\partial a_m(\boldsymbol{k}')}{\partial t} + \Delta_m(k')\,a_n^*(\boldsymbol{k})a_m(\boldsymbol{k}') \\
&\quad = \sum_{n_p,n_q=-\infty}^{\infty}\int \mathrm{e}^{\mathrm{i}(\omega_m(k')+\omega_{n_p}(p)+\omega_{n_q}(|\boldsymbol{k}'+\boldsymbol{p}|))t}N_{mn_pn_q}(\boldsymbol{k}',\boldsymbol{p}) \\
&\qquad \times a_n^*(\boldsymbol{k})a_{n_p}^*(\boldsymbol{p})a_{n_q}^*(-\boldsymbol{k}'-\boldsymbol{p})\,\mathrm{d}^2\boldsymbol{p}.
\end{aligned} \tag{3.13}$$

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

Taking the sum of (3.12) and (3.13) and averaging,

$$\delta(\boldsymbol{k}-\boldsymbol{k}')\left(\frac{\partial A_{nm}(k)}{\partial t}+(\Delta_n^*(k)+\Delta_m(k'))A_{nm}(k)\right)=\delta(\boldsymbol{k}-\boldsymbol{k}')$$
$$\left\{\sum_{n_p,n_q=-\infty}^{\infty}\int \mathrm{e}^{-\mathrm{i}(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(|\boldsymbol{k}+\boldsymbol{p}|))t}N^*_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})\Theta_{mn_pn_q}(\boldsymbol{k}',\boldsymbol{p})\,\mathrm{d}^2\boldsymbol{p}\right.$$
$$\left.+\sum_{n_p,n_q=-\infty}^{\infty}\int \mathrm{e}^{\mathrm{i}(\omega_m(k')+\omega_{n_p}(p)+\omega_{n_q}(|\boldsymbol{k}'+\boldsymbol{p}|))t}N_{mn_pn_q}(\boldsymbol{k}',\boldsymbol{p})\Theta^*_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})\,\mathrm{d}^2\boldsymbol{p}\right\}, \tag{3.14}$$

where we have used (3.7) and (3.11) to express the spectral moments. Integrating over $\boldsymbol{k}'$ gives

$$\frac{\partial A_{nm}(k)}{\partial t}+(\Delta_n^*(k)+\Delta_m(k))A_{nm}(k)=\tau_{nm}(k)+\tau^*_{mn}(k), \tag{3.15}$$

where

$$\tau_{nm}(k)=\sum_{n_p,n_q=-\infty}^{\infty}\int \mathrm{e}^{-\mathrm{i}(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(|\boldsymbol{k}+\ \boldsymbol{p}|))t}N^*_{nn_pn_q}(\ \boldsymbol{k},\boldsymbol{p})\Theta_{mn_pn_q}(\boldsymbol{k},\boldsymbol{p})\,\mathrm{d}^2\boldsymbol{p}. \tag{3.16}$$

Equation (3.15) describes the time evolution of the spectral matrix A_{nm}. It suffers from the usual closure problem that third-order moments enter via (3.16). Isotropy has been employed to deduce that $\tau_{nm}=\tau_{nm}(k)$. Note that terms in (3.16), other than those for which $n\pm n_p\pm n_q=0$, are zero.

3.2. *Evolution equation of the third-order spectral moments*

An equation for $\Theta_{mn_pn_q}$ can also be derived from (2.18). The reasoning is similar to that used to obtain (3.15) and gives

$$\delta(\boldsymbol{k}+\boldsymbol{k}'+\boldsymbol{k}'')\left(\frac{\partial\Theta_{mn_pn_q}(\ \boldsymbol{k},\boldsymbol{k}')}{\partial t}+(\Delta_m(k)+\Delta_{n_p}(k')+\Delta_{n_q}(k''))\Theta_{mn_pn_q}(\boldsymbol{k},\boldsymbol{k}')\right)$$
$$=\sum_{m_p,m_q=-\infty}^{\infty}\int \mathrm{e}^{\mathrm{i}\omega_{m_p}(p)t}(\mathrm{e}^{\mathrm{i}(\omega_m(k)+\omega_{m_q}(|\boldsymbol{k}+\boldsymbol{p}|))t}N_{mm_pm_q}(\boldsymbol{k},\boldsymbol{p})\overline{a_{n_p}(\boldsymbol{k}')a_{n_q}(\boldsymbol{k}'')a^*_{m_p}(\boldsymbol{p})a^*_{m_q}(-\boldsymbol{k}-\boldsymbol{p})}$$
$$+\mathrm{e}^{\mathrm{i}(\omega_{n_p}(k')+\omega_{m_q}(|\boldsymbol{k}'+\boldsymbol{p}|))t}N_{n_pm_pm_q}(\boldsymbol{k}',\boldsymbol{p})\overline{a_m(\boldsymbol{k})a_{n_q}(\boldsymbol{k}'')a^*_{m_p}(\boldsymbol{p})a^*_{m_q}(-\boldsymbol{k}'-\boldsymbol{p})}$$
$$+\mathrm{e}^{\mathrm{i}(\omega_{n_q}(k'')+\omega_{m_q}(|\boldsymbol{k}''+\boldsymbol{p}|))t}N_{n_qm_pm_q}(\boldsymbol{k}'',\boldsymbol{p})\overline{a_m(\boldsymbol{k})a_{n_p}(\boldsymbol{k}')a^*_{m_p}(\boldsymbol{p})a^*_{m_q}(-\boldsymbol{k}''-\boldsymbol{p})})\,\mathrm{d}^2\ \boldsymbol{p}. \tag{3.17}$$

The appearance of fourth-order moments in (3.17) reflects the closure problem.

In physical space, unlike their two- and three-point counterparts, fourth-order, four-point velocity moments do not necessarily tend to zero as the distance between points increases. Thus, if two pairs of points become infinitely separated, but each pair maintains a finite separation, the four-point moments approach a product of two-point moments, rather than zero. Of course, here we make the assumption (usual in theoretical studies of turbulence) of asymptotic statistical independence in the limit of infinite separation. The four-point moments then approach quasi-normality at large separations and can thus be written as the sum of their quasi-normal value and a cumulant correction which tends to zero with increasing separation.

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

The equivalent statement in spectral space is (see Newell & Rumpf 2011 and references therein)

$$\begin{aligned}\overline{a_{n_p}(\boldsymbol{k}')a_{n_q}(\boldsymbol{k}'')a^*_{m_p}(\boldsymbol{p})a^*_{m_q}(-\boldsymbol{k}-\boldsymbol{p})} &= \overline{a_{n_p}(\boldsymbol{k}')a_{n_q}(\boldsymbol{k}'')}\ \overline{a^*_{m_p}(\boldsymbol{p})a^*_{m_q}(-\boldsymbol{k}-\boldsymbol{p})}\\ &+ \overline{a_{n_p}(\boldsymbol{k}')a^*_{m_p}(\boldsymbol{p})}\ \overline{a_{n_q}(\boldsymbol{k}'')a^*_{m_q}(-\boldsymbol{k}-\boldsymbol{p})} + \overline{a_{n_p}(\boldsymbol{k}')a^*_{m_q}(-\boldsymbol{k}-\boldsymbol{p})}\ \overline{a_{n_q}(\boldsymbol{k}'')a^*_{m_p}(\boldsymbol{p})}\\ &+ K_{n_pn_qm_pm_q}(\boldsymbol{k}',\boldsymbol{k}'',\ \boldsymbol{p})\delta(\boldsymbol{k}+\boldsymbol{k}'+\boldsymbol{k}''),\end{aligned} \tag{3.18}$$

in which the first three terms on the right-hand side correspond to the quasi-normal approximation and the final term is the cumulant correction. Using (3.7) for the second-order moments in (3.18) and similar expressions for the other fourth-order moments in (3.17), it is shown in appendix C that

$$\begin{aligned}&\frac{\partial\Theta_{mn_pn_q}(\boldsymbol{k},\boldsymbol{p})}{\partial t} + (\Delta_m(k)+\Delta_{n_p}(p)+\Delta_{n_q}(|\boldsymbol{k}+\ \boldsymbol{p}|))\Theta_{mn_pn_q}(\boldsymbol{k},\boldsymbol{p})\\ &= 2\sum_{m_p,m_q=-\infty}^{\infty}\left.\begin{aligned}&\{\mathrm{e}^{\mathrm{i}(\omega_m(k)+\omega_{m_p}(p)+\omega_{m_q}(|\boldsymbol{k}+\boldsymbol{p}|))t}N_{mm_pm_q}(\boldsymbol{k},\boldsymbol{p})A_{m_pn_p}(p)A_{m_qn_q}(|\boldsymbol{k}+\ \boldsymbol{p}|)\\ &+\mathrm{e}^{\mathrm{i}(\omega_{m_p}(k)+\omega_{n_p}(p)+\omega_{m_q}(|\boldsymbol{k}+\boldsymbol{p}|))t}N_{n_pm_pm_q}(\boldsymbol{p},\boldsymbol{k})A_{m_pm}(k)A_{m_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|)\\ &+\mathrm{e}^{\mathrm{i}(\omega_{m_q}(k)+\omega_{m_p}(p)+\omega_{n_q}(|\boldsymbol{k}+\boldsymbol{p}|))t}N_{n_qm_pm_q}(-\boldsymbol{k}-\boldsymbol{p},\boldsymbol{p})A_{m_pn_p}(p)A_{m_qm}(k)\}+\end{aligned}\right\}\ \mathrm{QN}\\ &\sum_{m_p,m_q=-\infty}^{\infty}\left.\begin{aligned}&\int\mathrm{e}^{\mathrm{i}\omega_{m_p}(q)t}\{\mathrm{e}^{\mathrm{i}(\omega_m(k)+\omega_{m_q}(|\boldsymbol{k}+\boldsymbol{q}|))t}N_{mm_pm_q}(\boldsymbol{k},\boldsymbol{q})K_{n_pn_qm_pm_q}(\boldsymbol{p},-\boldsymbol{k}-\boldsymbol{p},\boldsymbol{q})\\ &+\mathrm{e}^{\mathrm{i}(\omega_{n_p}(p)+\omega_{m_q}(|\boldsymbol{p}+\boldsymbol{q}|))t}N_{n_pm_pm_q}(\boldsymbol{p},\boldsymbol{q})K_{mn_qm_pm_q}(\boldsymbol{k},-\boldsymbol{k}-\boldsymbol{p},\boldsymbol{q})\\ &+\mathrm{e}^{\mathrm{i}(\omega_{n_q}(|\boldsymbol{k}+\boldsymbol{p}|)+\omega_{m_q}(|\boldsymbol{k}+\boldsymbol{p}-\boldsymbol{q}|))t}N_{n_qm_pm_q}(-\boldsymbol{k}-\boldsymbol{p},\boldsymbol{q})K_{mn_pm_pm_q}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\}\,\mathrm{d}^2\boldsymbol{q}\end{aligned}\right\}\ \mathrm{K}\end{aligned} \tag{3.19}$$

as the evolution equation for the third-order moments occurring in (3.16). Here, the quasi-normal and cumulant contributions have been indicated by QN and K, respectively. As usual, a closure problem arises from the cumulants.

4. Two-dimensional/wave decomposition

At this stage, we would like to adopt the wave-turbulence approach. There is however a problem: the group velocity

$$\boldsymbol{c}_n(\boldsymbol{k}) = \nabla_{\boldsymbol{k}}\omega_n = -\frac{n\pi\boldsymbol{k}}{(k^2+n^2\pi^2)^{3/2}} \tag{4.1}$$

is zero for $n=0$. Since wave-turbulence theory requires dispersive waves, it does not apply to the $n=0$ part of the flow.

Let us define the two-dimensional component of the flow by combining all $n=0$ modes:

$$U_i = \int\tilde{u}_0(\boldsymbol{k},t)W_i^{(0)}(\boldsymbol{k})\exp\{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2\}\,\mathrm{d}^2\boldsymbol{k} \tag{4.2}$$

and write the total velocity as

$$u_i = U_i + v_i, \tag{4.3}$$

where

$$v_i = \sum_{\substack{n=-\infty\\(n\neq0)}}^{\infty}\int\tilde{u}_n(\boldsymbol{k},t)W_i^{(n)}(x_3;\boldsymbol{k})\exp\{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2\}\,\mathrm{d}^2\boldsymbol{k} \tag{4.4}$$

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

represents all $n \neq 0$ modes and will be called the wave component. Both $\boldsymbol{U}$ and $\boldsymbol{v}$ are real. It follows from (2.6) that

$$\int_0^1 \boldsymbol{v}_\perp \, \mathrm{d}x_3 = 0, \tag{4.5}$$

where the symbol $\perp$ indicates the projection of a vector perpendicular to the rotation axis, e.g. $\boldsymbol{v}_\perp = (v_1, v_2, 0)$.

Note that $U_3 = 0$ and U_1, U_2 are independent of x_3, hence the name: two-dimensional component. According to (4.3) and (4.5),

$$\boldsymbol{U} = \int_0^1 \boldsymbol{u}_\perp \, \mathrm{d}x_3, \tag{4.6}$$

which shows that $\boldsymbol{U}$ is the average of $\boldsymbol{u}_\perp$ over x_3. Since $\omega_0(k) = 0$, the two-dimensional component is steady in the absence of viscosity and nonlinearity. At small Rossby and Ekman numbers, $\boldsymbol{U}$ will evolve slowly. On the other hand, the wave component contains modal oscillations of frequency $O(1)$. Since modes are solenoidal, so are $\boldsymbol{U}$ and $\boldsymbol{v}$. Being solenoidal and two-dimensional, $\boldsymbol{U}$ can be expressed using a stream function, $\psi(x_1, x_2, t)$:

$$U_1 = \frac{\partial \psi}{\partial x_2}, \quad U_2 = -\frac{\partial \psi}{\partial x_1}. \tag{4.7}$$

Integrating (2.1) across the channel, it is shown in appendix D that

$$\frac{\partial U_\alpha}{\partial t} + \frac{\partial}{\partial x_\beta}(U_\alpha U_\beta) = -\frac{\partial \Pi}{\partial x_\alpha} + E\frac{\partial^2 U_\alpha}{\partial x_\beta \partial x_\beta} - (2E)^{1/2} U_\alpha - \frac{\partial}{\partial x_\beta}\left(\int_0^1 v_\alpha v_\beta \, \mathrm{d}x_3\right), \tag{4.8}$$

where

$$\Pi = \int_0^1 p \, \mathrm{d}x_3 + (1 + (2E)^{1/2})\psi. \tag{4.9}$$

Recalling that $\boldsymbol{U}$ is solenoidal,

$$\frac{\partial U_\alpha}{\partial x_\alpha} = 0. \tag{4.10}$$

Equations (4.8) and (4.10) govern the two-dimensional component, $\boldsymbol{U}$, Π. Equation (4.8) shows that it is subject to wall viscous friction (due to Ekman pumping) and nonlinear forcing by the wave component. Note that the factor $2^{1/2}$ in the wall-friction term corresponds to the damping factor $D_0(k) = 2^{1/2}$ of $n = 0$ modes. Note also that the Coriolis term does not appear in (4.8) (it has been incorporated into the pressure via (4.9)). Thus, there is no direct effect of rotation on the two-dimensional component (it does however contribute through the wall-friction term).

Consider the initial evolution of $\boldsymbol{U}$ according to (4.8) and (4.10). Given the velocity non-dimensionalization, $\boldsymbol{U}$ and $\boldsymbol{v}$ are of $O(\varepsilon)$, where $\varepsilon \ll 1$ is the initial Rossby number. Both nonlinear terms in (4.8) are $O(\varepsilon^2)$, but, thanks to the temporal oscillations of $\boldsymbol{v}$, the final term in (4.8) contributes little over the long times needed for evolution of $\boldsymbol{U}$ and can be neglected. The result is

$$\frac{\partial U_\alpha}{\partial t} + \frac{\partial}{\partial x_\beta}(U_\alpha U_\beta) = -\frac{\partial \Pi}{\partial x_\alpha} + E\frac{\partial^2 U_\alpha}{\partial x_\beta \partial x_\beta} - (2E)^{1/2} U_\alpha. \tag{4.11}$$

This, with (4.10), governs the two-dimensional component during the initial phase and is decoupled from the wave component.

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

Assuming sufficiently small E that the turbulence is not killed by viscosity before nonlinearity intervenes, the two-dimensional component evolves on the time scale $t = O(\varepsilon^{-1})$, which is the usual large-scale turnover time and defines the initial phase of evolution. The most interesting case is when E is small enough that the turbulence persists to large εt and we assume this in what follows. The initial phase of evolution of the two-dimensional energy-containing scales is then described by (4.11) without the viscous terms, i.e. we have classical high-Reynolds-number, two-dimensional turbulence.

The long-time evolution of freely decaying, high-Reynolds-number, two-dimensional turbulence has been the object of much study (see the references cited in the introduction). Based on numerical and experimental results, it has been found that, after many large-eddy turnover times (large εt), the flow becomes a collection of well-separated vortices of growing separation and size. These vortices move around and occasionally merge, hence their increase in size and sparseness. This process reflects an inverse cascade of energy towards larger scales. Energy is very nearly conserved, while being transferred to larger and larger scales. As a result, the dominant part of the energy spectrum, $E_0(k)$, is to be found at small k when εt is large.

It is difficult to know whether the wave-component term in (4.8) has a significant, cumulative effect on the two-dimensional flow, in particular on its spectrum, $E_0(k)$, at the much larger times ($t = O(\varepsilon^{-2})$) at which we expect the wave-component spectra, A_{nn} ($n \neq 0$), to evolve. Later (from § 5.4 onwards), we will adopt the simplest hypothesis and suppose that the above description of the two-dimensional component remains at least qualitatively valid up to $t = O(\varepsilon^{-2})$. In particular, we will assume that the dominant part of the two-dimensional energy lies at small k.

5. Wave-turbulence closure

The $n \neq 0$ modes are dispersive and we expect wave-turbulence theory to apply. The objective of this section is to derive the resulting evolution equations for A_{nm}, assuming $n, m \neq 0$ throughout. In so doing, we implicitly suppose that E is sufficiently small that viscous damping has not killed the turbulence at the times (up to $t = O(\varepsilon^{-2})$) considered. Given dominance of boundary-layer damping (whose damping rate is $O(E^{1/2})$) for the large scales of turbulence, this requires $E \leqslant O(\varepsilon^4)$. If this condition is not met, the turbulence is damped by viscosity before significant energy transfer between wave modes can occur.

5.1. *Cumulant contribution*

At the large t necessary for significant evolution of A_{nm}, the integrand in (3.19) has rapid, nearly self-cancelling oscillations as a function of $\boldsymbol{q}$ unless $m_p = m_q = 0$. Neglecting other terms and recalling that $N_{lm_pm_q} = 0$ is zero unless $l \pm m_p \pm m_q = 0$, the cumulant contribution to (3.19) becomes

$$\int (\delta_{n_p} N_{000}(\boldsymbol{p}, \boldsymbol{q}) K_{mn_q00}(\boldsymbol{k}, -\boldsymbol{k} - \boldsymbol{p}, \boldsymbol{q}) + \delta_{n_q} N_{000}(-\boldsymbol{k} - \boldsymbol{p}, \boldsymbol{q}) K_{mn_p00}(\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q})) \, \mathrm{d}^2\boldsymbol{q}. \tag{5.1}$$

Since (5.1) does not display rapid oscillations with respect to $\boldsymbol{p}$, following long-time evolution according to (3.19), the same is true of the corresponding contribution to $\Theta_{mn_pn_q}(\boldsymbol{k}, \boldsymbol{p})$. It follows that the resulting contribution to (3.16) can be neglected due to rapid, self-cancelling oscillations of the integrand unless $n_p = n_q = 0$, but then it is zero since $N_{nn_pn_q} = 0$ (recall that $n \neq 0$). We conclude that the cumulant contribution is negligible.

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

Note that it has been assumed (as in all wave-turbulence analyses) that the quantity $K_{n_pn_qm_pm_q}$, defined via (3.18) and representing the cumulants, does not have rapid oscillations which match those of the exponentials in the cumulant term of (3.19) in such a way as to destroy self-cancellation. One way of achieving this is to suppose the initial turbulence statistically independent at separations large compared with the channel width (this is stronger than the earlier assumption of statistical independence at infinite separation and is in keeping with initial turbulence scales comparable with the width). Of course, inertial-wave propagation will disperse the initial statistical dependencies over large distances at the long times considered here. Such propagation is represented by the rapidly oscillating exponentials and the need to take it into account is the underlying reason for the change of modal amplitude variables, (2.16). Note that the above assumption concerning the initial turbulence is sufficient, but not necessary. It would require very specific (hence improbable) initialization of the long-range statistics of the turbulence for rapid oscillations in $K_{n_pn_qm_pm_q}$ to match and avoid self-cancellation of the exponentials in the cumulant term of (3.19).

5.2. *Quasi-normal contributions*

The first term in the quasi-normal sum of (3.19) is

$$\mathrm{e}^{\mathrm{i}(\omega_m(k)+\omega_{m_p}(p)+\omega_{m_q}(|\boldsymbol{k}+\boldsymbol{p}|))t}N_{mm_pm_q}(\boldsymbol{k},\boldsymbol{p})A_{m_pn_p}(p)A_{m_qn_q}(|\ \boldsymbol{k}+\boldsymbol{p}|). \tag{5.2}$$

Given that $A_{m_pn_p}(p)A_{m_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|)$ is slowly varying, $\Theta_{mn_pn_q}(\boldsymbol{k},\boldsymbol{p})$ inherits oscillations of the form $\mathrm{e}^{\mathrm{i}(\omega_m(k)+\omega_{m_p}(p)+\omega_{m_q}(|\boldsymbol{k}+\boldsymbol{p}|))t}$. Rapid oscillations with respect to $\boldsymbol{p}$ in (3.16) make the result negligible unless $m_p=n_p$ and $m_q=n_q$. Restricting attention to this term in the sum, $\tau_{nm}(k)$ has oscillations of the form $\mathrm{e}^{\mathrm{i}(\omega_m(k)-\omega_n(k))t}$. Unless $n=m$, the temporal oscillations of $\tau_{nm}(k)$ and $\tau^*_{mn}(k)$ will have little effect on the long-time evolution of A_{nm} according to (3.15). Thus, we focus on the case $n=m$, $m_p=n_p$ and $m_q=n_q$, yielding the contribution

$$2\delta_{n-m}\mathrm{e}^{\mathrm{i}(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(|\boldsymbol{k}+\ \boldsymbol{p}|))t}N_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{n_pn_p}(p)A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|) \tag{5.3}$$

to (3.19).

The second term in the quasi-normal sum of (3.19) is

$$\mathrm{e}^{\mathrm{i}(\omega_{m_p}(k)+\omega_{n_p}(p)+\omega_{m_q}(|\boldsymbol{k}+\boldsymbol{p}|))t}N_{n_pm_pm_q}(\boldsymbol{p},\boldsymbol{k})A_{m_pm}(k)A_{m_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|). \tag{5.4}$$

Arguing as for (5.2), we focus on the term $m_p=n$, $m_q=n_q$:

$$2\mathrm{e}^{\mathrm{i}(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(|\boldsymbol{k}+\boldsymbol{p}|))t}N_{n_pnn_q}(\boldsymbol{p},\boldsymbol{k})A_{nm}(k)A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|). \tag{5.5}$$

Similar reasoning applies to the third term in the quasi-normal sum of (3.19), leading to

$$2\mathrm{e}^{\mathrm{i}(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(|\boldsymbol{k}+\boldsymbol{p}|))t}N_{n_qn_pn}(-\boldsymbol{k}-\ \boldsymbol{p},\boldsymbol{p})A_{n_pn_p}(p)A_{nm}(k). \tag{5.6}$$

Combining (5.3), (5.5), (5.6) and (3.19) gives

$$\begin{aligned}&\frac{\partial\Theta_{mn_pn_q}(\boldsymbol{k},\boldsymbol{p})}{\partial t}+(\Delta_m(k)+\Delta_{n_p}(p)+\Delta_{n_q}(|\boldsymbol{k}+\ \boldsymbol{p}|))\Theta_{mn_pn_q}(\ \boldsymbol{k},\boldsymbol{p})\\&\quad=2\mathrm{e}^{\mathrm{i}(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(|\boldsymbol{k}+\boldsymbol{p}|))t}[\delta_{n-m}N_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{n_pn_p}(p)A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|)\\&\qquad+N_{n_pnn_q}(\boldsymbol{p},\boldsymbol{k})A_{nm}(k)A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|)+N_{n_qn_pn}(-\boldsymbol{k}-\ \boldsymbol{p},\boldsymbol{p})A_{n_pn_p}(p)A_{nm}(k)]+\cdots,\end{aligned} \tag{5.7}$$

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

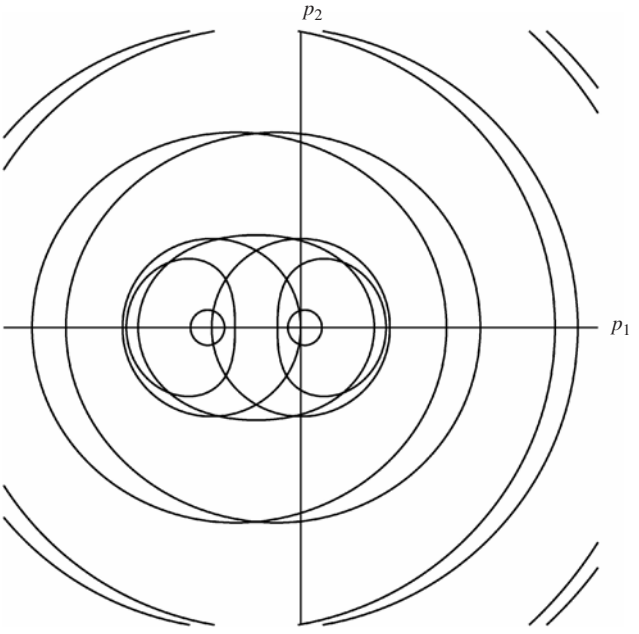


FIGURE 3. Resonance curves in the $\boldsymbol{p}$ plane ($|p_1| < 10$, $|p_2| < 10$) for $n = 2$, $k_1 = 3$, $k_2 = 0$ and all n_p, n_q such that $n \pm n_p \pm n_q = 0$.

in which $\cdots$ denotes asymptotically insignificant contributions which have been neglected as described above.

5.3. *The resonance curve and evaluation of* $\tau_{nm}(k)$

Triad resonance occurs when

$$\omega_n(k) + \omega_{n_p}(p) + \omega_{n_q}(|\boldsymbol{k} + \boldsymbol{p}|) = 0. \tag{5.8}$$

For given $\boldsymbol{k}$, n, n_p and n_q ($n \pm n_p \pm n_q = 0$), the resonance curve, $C_{nn_pn_q}(\boldsymbol{k})$, is the locus of points in the $\boldsymbol{p}$ plane for which (5.8) is satisfied. Note that there may not exist such points, in which case $C_{nn_pn_q}(\boldsymbol{k})$ is the empty set.

Figure 3 shows the resonance curves for particular $\boldsymbol{k}$, n and all n_p, n_q. There are an infinite number of closed loops extending to $|\boldsymbol{p}| = \infty$, each of which corresponds to a particular choice of n_p and n_q (though not all n_p, n_q satisfying $n \pm n_p \pm n_q = 0$ yield solutions of (5.8)). Each loop can thought of as the projection onto the (p_1, p_2) plane of a cut through the resonance surface in the three-dimensional space, (p_1, p_2, p_3), of the unconfined problem (see Smith & Waleffe 1999; Bellet *et al.* 2006), the cut being taken parallel to the (p_1, p_2) plane at $p_3 = \pm n_p \pi$.

The significance of the resonance curve is that the frequency of the oscillatory, exponential factor in (5.7) is zero. Thus, at or near the curve, the solution, $\Theta_{mn_pn_q}(\boldsymbol{k}, \boldsymbol{p})$, grows over long time spans (a three-wave resonance), whereas it oscillates with frequency $\omega_n(k) + \omega_{n_p}(p) + \omega_{n_q}(|\boldsymbol{k} + \boldsymbol{p}|)$ away from the curve. We do not go into the technical details, which are standard in wave-turbulence studies and described by Newell & Rumpf (2011, and references therein), but the asymptotically important

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

part of the solution is

$$\Theta_{mn_pn_q}(\boldsymbol{k},\boldsymbol{p}) = -2\mathrm{i}\frac{\mathrm{e}^{\mathrm{i}(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(|\boldsymbol{k}+\boldsymbol{p}|))t}-1}{\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(|\boldsymbol{k}+\boldsymbol{p}|)} \times [\delta_{n-m}N_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{n_pn_p}(p)A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|) + N_{n_pnn_q}(\boldsymbol{p},\boldsymbol{k})A_{nm}(k)A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|) + N_{n_qn_pn}(-\boldsymbol{k}-\boldsymbol{p},\boldsymbol{p})A_{n_pn_p}(p)A_{nm}(k)] + \cdots . \tag{5.9}$$

The oscillatory term in (5.9) has amplitude which tends to infinity as the resonance curve is approached, while the overall factor in front of the square brackets approaches a limit which grows like t. These properties reflect expected results of resonance.

Using (5.9) in (3.16) and taking the limit $t\to\infty$, it is shown in appendix E that

$$\tau_{nm}(k)+\tau^*_{mn}(k) \sim (\mu_n(k)+\mu^*_m(k))A_{nm}(k) + \delta_{n-m}\sum_{n_p,n_q=-\infty}^{\infty}\oint_{C_{nn_pn_q}(\boldsymbol{k})}\frac{\lambda_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{n_pn_p}(p)A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|)}{\Gamma_{n_pn_q}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\boldsymbol{p}|, \tag{5.10}$$

where

$$\mu_n(k) = \frac{1}{2}\sum_{n_p,n_q=-\infty}^{\infty}\oint_{C_{nn_pn_q}(\boldsymbol{k})}\frac{\eta_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{n_pn_p}(p)}{\Gamma_{n_pn_q}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\boldsymbol{p}| - \frac{1}{2}\mathrm{i}\sum_{n_p,n_q=-\infty}^{\infty}\int\frac{\eta_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{n_pn_p}(p)}{\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(|\boldsymbol{k}+\boldsymbol{p}|)}\,\mathrm{d}^2\boldsymbol{p}, \tag{5.11}$$

$$\lambda_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p}) = 4|N_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})|^2, \tag{5.12}$$

$$\eta_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p}) = 8\mathrm{Re}(N^*_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})N_{n_qn_pn}(-\boldsymbol{k}-\boldsymbol{p},\boldsymbol{p})), \tag{5.13}$$

$$\Gamma_{n_pn_q}(\boldsymbol{k},\boldsymbol{p}) = \left|\frac{n_p\boldsymbol{p}}{(p^2+n_p^2\pi^2)^{3/2}}+\frac{n_q(\boldsymbol{k}+\boldsymbol{p})}{(|\boldsymbol{k}+\boldsymbol{p}|^2+n_q^2\pi^2)^{3/2}}\right|. \tag{5.14}$$

Equations (5.10) and (5.11) contain line integrals taken around the resonance curve. Note that in the case when (5.8) has no solutions ($C_{nn_pn_q}(\boldsymbol{k})$ is the empty set), integrals over the resonance curve should be interpreted as zero. Equation (5.11) also contains a surface integral, which should be interpreted here and henceforth as a Cauchy principal value, defined as follows. In the case when (5.8) has no solutions, the surface integral is non-singular and the Cauchy principal value is simply the usual integral. Otherwise, the principal value is obtained by removing a thin strip of the $\boldsymbol{p}$ plane from the region of integration. The strip edges are located either side of and equidistant from $C_{nn_pn_q}(\boldsymbol{k})$. This removes the singularity and the Cauchy principal value follows from taking the limit in which the strip width tends to zero.

Using (5.10) and (5.11), equation (3.15) yields

$$\frac{\partial A_{nn}(k)}{\partial t}+2\mathrm{Re}(\Delta_n(k))A_{nn}(k) = \sum_{n_p,n_q=-\infty}^{\infty}\oint_{C_{nn_pn_q}(\boldsymbol{k})}\frac{A_{n_pn_p}(p)(\eta_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{nn}(k)+\lambda_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|))}{\Gamma_{n_pn_q}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\boldsymbol{p}|, \tag{5.15}$$

$$\frac{\partial A_{nm}(k)}{\partial t}+(\Xi^*_n(k)+\Xi_m(k))A_{nm}(k)=0 \quad n\neq m, \tag{5.16}$$

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

where

$$\Xi_n(k) = \Delta_n(k) - \frac{1}{2}\sum_{n_p,n_q=-\infty}^{\infty}\oint_{C_{nn_pn_q}(\boldsymbol{k})}\frac{\eta_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{n_pn_p}(p)}{\Gamma_{n_pn_q}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\boldsymbol{p}| - \frac{1}{2}\mathrm{i}\sum_{n_p,n_q=-\infty}^{\infty}\int\frac{\eta_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{n_pn_p}(p)}{\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(|\boldsymbol{k}+\boldsymbol{p}|)}\,\mathrm{d}^2\boldsymbol{p}. \tag{5.17}$$

Equations (5.15)–(5.17) express the final result of wave-turbulence analysis. Equation (5.15) describes evolution of the wave energy distributions A_{nn} $(n \neq 0)$, while (5.16) governs the correlations between different orders of wave modes. Observe that these equations are not closed because A_{00} appears in the sums on the right-hand sides of (5.15) and (5.17). As we shall see in the next subsection, the two-dimensional contributions to (5.15) have the remarkable property of being self-cancelling. Thus, the two-dimensional component does not affect the energetics of the waves, leading to closure of (5.15). Note that only those terms in the sums of (5.15) and (5.17) for which $n \pm n_p \pm n_q = 0$ are non-zero.

Double resonance occurs if $n \pm n_p \pm n_q = 0$ and $\Gamma_{n_pn_q}(\boldsymbol{k},\boldsymbol{p}) = 0$ somewhere on the resonance curve. This happens if the components $\boldsymbol{p}$ and $-\boldsymbol{k}-\boldsymbol{p}$ of a resonant triad have equal group velocities. If it occurred, double resonance would create problems of divergence of integrals in the above equations. Systematic numerical determination of simultaneous solutions of (5.8) and $\Gamma_{n_pn_q}(\boldsymbol{k},\ \boldsymbol{p}) = 0$ for $|n_p| \leqslant 20$, $|n_q| \leqslant 20$ turned up many pairs, n_p, n_q, for which such solutions occur at certain discrete values of k. These solutions can potentially be of two types, depending on the sign of the determinant of the matrix of second derivatives with respect to p_1, p_2 of $\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(|\boldsymbol{k}+\boldsymbol{p}|)$. The first type occurs if the determinant is negative. This type represents crossing of the resonance curve by itself and gives a double resonance. On the other hand, if the determinant is positive, the solution does not lead to double resonance, but corresponds instead to the appearance or disappearance of part of the resonance curve as k is varied. It turns out that all solutions are of the second type, so double resonance and its associated problems do not arise.

5.4. *Effects of the two-dimensional component*

As noted above, although n, $m \neq 0$, the two-dimensional component enters (5.15)–(5.17) via the corresponding terms of the sums in (5.15) and (5.17) and we now investigate its effects on the evolution of A_{nm}.

First consider the right-hand side of (5.15). The contribution from $n_p = 0$ and $n_q = 0$ is

$$\sum_{n_p=\pm n}\oint_{C_{nn_p0}(\boldsymbol{k})}\frac{A_{n_pn_p}(p)(\eta_{nn_p0}(\boldsymbol{k},\boldsymbol{p})A_{nn}(k)+\lambda_{nn_p0}(\boldsymbol{k},\boldsymbol{p})A_{00}(|\boldsymbol{k}+\boldsymbol{p}|))}{\Gamma_{n_p0}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\boldsymbol{p}| + \sum_{n_q=\pm n}\oint_{C_{n0n_q}(\boldsymbol{k})}\frac{A_{00}(p)(\eta_{n0n_q}(\boldsymbol{k},\boldsymbol{p})A_{nn}(k)+\lambda_{n0n_q}(\boldsymbol{k},\boldsymbol{p})A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|))}{\Gamma_{0n_q}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\boldsymbol{p}|, \tag{5.18}$$

in which the sums have been reduced to just two terms using the condition $n \pm n_p \pm n_q = 0$. When $n_p = n$ and $n_q = 0$, there are no solutions of (5.8) because $\omega_{n_q}(|\boldsymbol{k}+\ \boldsymbol{p}|) = 0$, while $\omega_n(k)$ and $\omega_{n_p}(p)$ are non-zero and have the same sign. Thus, only the $n_p = -n$ term in the first sum of (5.18) is non-zero. When $n_p = -n$ and $n_q = 0$, equation (5.8) implies that the resonance curve, $C_{nn_p0}(\boldsymbol{k})$, is the circle $|\boldsymbol{p}| = k$. Similar reasoning

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

applies to the second sum in (5.18), leading to

$$\sum_{n_p=-n}\oint_{|\boldsymbol{p}|=k}\frac{A_{n_pn_p}(p)(\eta_{nn_p0}(\boldsymbol{k},\boldsymbol{p})A_{nn}(k)+\lambda_{nn_p0}(\boldsymbol{k},\boldsymbol{p})A_{00}(|\ \boldsymbol{k}+\boldsymbol{p}|))}{\Gamma_{n_p0}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\boldsymbol{p}|$$
$$+\sum_{n_q=-n}\oint_{|\boldsymbol{k}+\ \boldsymbol{p}|=k}\frac{A_{00}(p)(\eta_{n0n_q}(\boldsymbol{k},\boldsymbol{p})A_{nn}(k)+\lambda_{n0n_q}(\boldsymbol{k},\boldsymbol{p})A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|))}{\Gamma_{0n_q}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\boldsymbol{p}|. \quad (5.19)$$

Numerical calculation of $\eta_{nn_p0}(\boldsymbol{k},\ \boldsymbol{p})$ (using (5.13), (A 8) and (A 9)) for $|\boldsymbol{p}|=k$ and $n_p=-n$ shows that it is zero to machine precision (16 decimal places) and, although we have been unable to demonstrate it analytically, we assume that this is exactly true, hence

$$\sum_{n_p=-n}\oint_{|\boldsymbol{p}|=k}\frac{A_{n_pn_p}(p)\lambda_{nn_p0}(\ \boldsymbol{k},\boldsymbol{p})A_{00}(|\boldsymbol{k}+\ \boldsymbol{p}|)}{\Gamma_{n_p0}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\boldsymbol{p}|$$
$$+\sum_{n_q=-n}\oint_{|\boldsymbol{k}+\ \boldsymbol{p}|=k}\frac{A_{00}(p)(\eta_{n0n_q}(\boldsymbol{k},\boldsymbol{p})A_{nn}(k)+\lambda_{n0n_q}(\boldsymbol{k},\boldsymbol{p})A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|))}{\Gamma_{0n_q}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\boldsymbol{p}|. \quad (5.20)$$

Transforming to integration variable $\boldsymbol{q}=-\boldsymbol{k}-\boldsymbol{p}$ in the first sum gives

$$\sum_{n_p=-n}\oint_{|\boldsymbol{k}+\boldsymbol{q}|=k}\frac{A_{n_pn_p}(|\boldsymbol{k}+\ \boldsymbol{q}|)\lambda_{nn_p0}(\boldsymbol{k},-\ \boldsymbol{k}-\boldsymbol{q})A_{00}(q)}{\Gamma_{n_p0}(\boldsymbol{k},-\boldsymbol{k}-\boldsymbol{q})}|\mathrm{d}\ \boldsymbol{q}|. \quad (5.21)$$

Then $\lambda_{nn_p0}(\boldsymbol{k},-\boldsymbol{k}-\boldsymbol{q})=\lambda_{n0n_p}(\boldsymbol{k},\boldsymbol{q})$ and $\Gamma_{n_p0}(\boldsymbol{k},-\boldsymbol{k}-\boldsymbol{q})=\Gamma_{0n_p}(\boldsymbol{k},\boldsymbol{q})$ follow from (2.15), (5.12) and (5.14), hence (5.21) can be rewritten as

$$\sum_{n_p=-n}\oint_{|\boldsymbol{k}+\boldsymbol{q}|=k}\frac{A_{n_pn_p}(|\boldsymbol{k}+\ \boldsymbol{q}|)\lambda_{n0n_p}(\boldsymbol{k},\boldsymbol{q})A_{00}(q)}{\Gamma_{0n_p}(\boldsymbol{k},\boldsymbol{q})}|\mathrm{d}\boldsymbol{q}|. \quad (5.22)$$

Renaming the summation index and integration variable as n_q and $\boldsymbol{p}$ and combining the result with the second sum of (5.20) gives

$$\sum_{n_q=-n}\oint_{|\boldsymbol{k}+\boldsymbol{p}|=k}\frac{A_{00}(p)(\eta_{n0n_q}(\boldsymbol{k},\boldsymbol{p})A_{nn}(k)+2\lambda_{n0n_q}(\boldsymbol{k},\boldsymbol{p})A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|))}{\Gamma_{0n_q}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\ \boldsymbol{p}| \quad (5.23)$$

for the $n_p=0$ and $n_q=0$ contribution to the right-hand side of (5.15). Finally, numerical calculation shows that $\eta_{n0n_q}(\ \boldsymbol{k},\boldsymbol{p})=-2\lambda_{n0n_q}(\boldsymbol{k},\boldsymbol{p})$ (to machine precision; we have again been unable to prove this analytically, but assume that it is exactly true) when $|\boldsymbol{k}+\boldsymbol{p}|=k$ and $n_q=-n$. Employing this result and $A_{-n,-n}=A_{nn}$, equation (5.23) becomes

$$\sum_{n_q=-n}\oint_{|\boldsymbol{k}+\boldsymbol{p}|=k}\frac{\eta_{n0n_q}(\boldsymbol{k},\boldsymbol{p})A_{00}(p)(A_{nn}(k)-A_{nn}(|\boldsymbol{k}+\boldsymbol{p}|))}{\Gamma_{0n_q}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\boldsymbol{p}|=0. \quad (5.24)$$

It follows that $n_p=0$ and $n_q=0$ make no net contribution and so

$$\frac{\partial A_{nn}(k)}{\partial t}+2\mathrm{Re}(\Delta_n(k))A_{nn}(k)$$
$$=\sum_{n_p,n_q\neq0}\oint_{C_{nn_pn_q}(\boldsymbol{k})}\frac{A_{n_pn_p}(p)(\eta_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{nn}(k)+\lambda_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|))}{\Gamma_{n_pn_q}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\boldsymbol{p}| \quad (5.25)$$

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

is exactly equivalent to (5.15). We thus arrive at the surprising conclusion that, owing to cancellation of different terms, the two-dimensional component does not influence the evolution of A_{nn} $(n \neq 0)$ according to wave turbulence theory. Equation (5.25) is the final, closed evolution equation of the wave energy spectra. Given the initial $A_{nn} = O(\varepsilon^2)$, order-of-magnitude estimation and (5.25) imply evolution on the time scale $t = O(\varepsilon^{-2})$ (the usual scale of wave-turbulence theory).

Turning attention to (5.17), the $n_p = 0$ contribution to the real part is

$$-\frac{1}{2}\sum_{n_q=\pm n}\oint_{|\boldsymbol{k}+\boldsymbol{p}|=k}\frac{\eta_{n0n_q}(\boldsymbol{k},\boldsymbol{p})A_{00}(p)}{\Gamma_{0n_q}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\boldsymbol{p}|. \tag{5.26}$$

Because there is no resonance curve when $n_q = n$, only the $n_q = -n$ term is non-zero and we focus on that term. As discussed in § 4, we expect the energy of the two-dimensional component to be dominated by small p at the large εt considered here. For small p, the resonance circle, $|\boldsymbol{k}+\boldsymbol{p}| = k$, lies close to the straight line through $\boldsymbol{p} = 0$, perpendicular to $\boldsymbol{k}$. As $\boldsymbol{p} \to 0$ along the resonance circle, it can be shown that $\eta_{n0n_q}(\boldsymbol{k},\boldsymbol{p})/\Gamma_{0n_q}(\boldsymbol{k},\boldsymbol{p}) \to -2|n|^{-1}k(k^2+n^2\pi^2)^{3/2}$. We approximate $\eta_{n0n_q}(\boldsymbol{k},\boldsymbol{p})/\Gamma_{0n_q}(\boldsymbol{k},\boldsymbol{p})$ by this limiting value and take it outside the integral. Since the resonance circle is asymptotically a straight line, this leads to

$$2|n|^{-1}k(k^2+n^2\pi^2)^{3/2}\int_0^\infty A_{00}(p)\,\mathrm{d}p \tag{5.27}$$

for the $n_p = 0$ contribution to the real part of (5.17). From (3.9), $A_{00}(p) = \pi^{-1}p^{-1}E_0(p)$ and so

$$\int_0^\infty A_{00}(p)\,\mathrm{d}p = \pi^{-1}\int_0^\infty p^{-1}E_0(p)\,\mathrm{d}p. \tag{5.28}$$

In the absence of p^{-1}, the integral on the right of (5.28) would be the energy, $O(\varepsilon^2)$, of the two-dimensional component. The weighting by p^{-1} and the fact that p is small mean that (5.28) is large compared with $O(\varepsilon^2)$. On the other hand, order-of-magnitude estimation of the $n_p \neq 0$ contribution to the real part of (5.17) shows it to be $O(\varepsilon^2)$, hence

$$\mathrm{Re}(\Xi_n(k,t)) \sim \mathrm{Re}(\Delta_n(k)) + 2|n\pi|^{-1}k(k^2+n^2\pi^2)^{3/2}\int_0^\infty p^{-1}E_0(p)\,\mathrm{d}p \tag{5.29}$$

is positive and asymptotically large compared with $O(\varepsilon^2)$. As a result, the off-diagonal elements of A_{nm} decay according to (5.16) on a shorter time scale than $O(\varepsilon^{-2})$.

To summarize the results of this subsection, the diagonal elements of A_{nm} $(n, m \neq 0)$ evolve on the time scale ε^{-2} according to the closed system (5.25), whereas its off-diagonal elements decay more rapidly due to the presence of the two-dimensional component. This suggests that scattering of $n \neq 0$ inertial waves by $n = 0$ vortices leads to relatively rapid decorrelation of different $n \neq 0$, but is energy conserving. In any case, the spectral distribution of energy of the wave component is predicted to evolve independently of the two-dimensional one and is governed by (5.25).

5.5. *Realizability and energy conservation*

It is straightforward to show that (5.25) preserves positive $A_{nn}(k)$. Equation (5.25) can be rewritten as

$$\frac{\partial A_{nn}}{\partial t} + g_n A_{nn} = f_n, \tag{5.30}$$

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

where

$$g_n = 2\mathrm{Re}(\Delta_n(k)) - \sum_{n_p,n_q\neq 0} \oint_{C_{nn_pn_q}(\boldsymbol{k})} \frac{\eta_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{n_pn_p}(p)}{\Gamma_{n_pn_q}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\boldsymbol{p}|, \tag{5.31}$$

$$f_n = \sum_{n_p,n_q\neq 0} \oint_{C_{nn_pn_q}(\ \boldsymbol{k})} \frac{\lambda_{nn_pn_q}(\boldsymbol{k},\ \boldsymbol{p})A_{n_pn_p}(p)A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|)}{\Gamma_{n_pn_q}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\boldsymbol{p}|. \tag{5.32}$$

Defining $X_n = A_{nn}\exp[\int_0^t g_n(t')\,\mathrm{d}t']$, equation (5.30) implies

$$\frac{\partial X_n}{\partial t} = \exp\left[\int_0^t g_n(t')\,\mathrm{d}t'\right] f_n. \tag{5.33}$$

Suppose that $A_{nn}(k)$ are initially positive for all n and k, hence a positive integrand in (5.32), implying $f_n \geqslant 0$. Thus, X_n initially increases with time, leading to

$$A_{nn}(t) \geqslant A_{nn}(0)\exp\left[-\int_0^t g_n(t')\,\mathrm{d}t'\right] > 0. \tag{5.34}$$

This constraint stops A_{nn} from becoming negative and it remains positive for all n and k and for all time.

According to (3.8), the average energy is

$$\frac{1}{2}\int_0^1 \overline{u_iu_i}\,\mathrm{d}x_3 = \frac{1}{2}\sum_{n=-\infty}^{\infty}\int A_{nn}(k)\,\mathrm{d}^2\boldsymbol{k} \tag{5.35}$$

of which the $n = 0$ term in the sum gives the two-dimensional contribution. The relation between the integral of $A_{00}(k)$ and the two-dimensional part of the energy has already been noted in the previous subsection. The wave component contributes the remainder, i.e.

$$\frac{1}{2}\sum_{n\neq 0}\int A_{nn}(\boldsymbol{k})\,\mathrm{d}^2\boldsymbol{k}. \tag{5.36}$$

The energy equation for the wave component follows from integrating (5.25) over $\boldsymbol{k}$ and summing over $n \neq 0$. The contribution from the right-hand side of (5.25) is

$$\sum_{n,n_p,n_q\neq 0}\int\oint_{C_{nn_pn_q}(\ \boldsymbol{k})} \frac{A_{n_pn_p}(p)(\eta_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{nn}(k) + \lambda_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|))}{\Gamma_{n_pn_q}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\boldsymbol{p}|\,\mathrm{d}^2\boldsymbol{k}. \tag{5.37}$$

It is shown in appendix F that (5.37) is zero. Thus, other than viscous damping, equation (5.25) conserves the $n \neq 0$ part of the energy.

6. Conclusion

As far as we are aware, this is the first attempt to apply the wave-turbulence approach to rotating turbulence, confined in a single direction. As we have seen in §§ 2 and 3, well-adapted equivalents of the mathematical armoury of three-dimensional spectral theory exist for turbulence which is homogeneous and isotropic in the directions parallel to the walls, the energy spectrum being replaced by a spectral matrix, A_{nm}, and the wave vector discretized in the third direction.

One of the discrete components, $n = 0$, is distinguished because it is non-dispersive if considered as a wave. This has led us to decompose the flow into two-dimensional

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

and wave components, as described in § 4. The former first evolves on a time scale $O(\varepsilon^{-1})$, independently of the wave component, and becomes the set of well-separated vortices described in the literature on two-dimensional turbulence before the wave component has had the chance for nonlinear evolution. The statistical characteristics of the wave component are expressed by $A_{nm}(n, m \neq 0)$, for which a set of evolution equations, (5.15)–(5.17), was derived in § 5 using wave-turbulence analysis.

Equations (5.15)–(5.17) involve the two-dimensional component. Analysis of (5.15) has shown that, remarkably, the two-dimensional contributions cancel out, so there is no net effect on the evolution of A_{nn} $(n \neq 0)$. Thus, the diagonal elements of A_{nm} $(n, m \neq 0)$, representing the energy distribution of the wave component, evolve independently of the two-dimensional component, according to the closed system (5.25) and on a time scale $O(\varepsilon^{-2})$. On the other hand, the off-diagonal elements of A_{nm} $(n, m \neq 0)$, tend to zero on a time scale asymptotically smaller than $O(\varepsilon^{-2})$, due to the presence of the two-dimensional component.

There are thus three phases in the small-ε evolution. In the first, of duration $O(\varepsilon^{-1})$, only the two-dimensional component undergoes nonlinear evolution, and does so as for a classical, two-dimensional, non-rotating flow (apart from wall friction due to Ekman pumping). In the second, the different modal orders of the wave component become decorrelated by energy-conserving scattering by the vortices of the two-dimensional component. Finally, there are nonlinear energy transfers between $n \neq 0$ modes described by (5.25) and which take place on the time scale $O(\varepsilon^{-2})$.

Equation (5.25) has been shown to maintain positivity of A_{nn} $(n \neq 0)$. It has also been found that terms other than the viscous one conserve the energy of the wave component, a result consistent with energy-conserving scattering by the two-dimensional vortices. It remains to implement (5.25) numerically in future work.

The assumptions behind this work should be borne in mind. We have supposed a small Rossby number and initial turbulence scales comparable with the channel width. The Ekman number is also taken as small, otherwise turbulence is damped by viscosity before nonlinearity can intervene. More particularly, $E \leqslant O(\varepsilon^4)$ is required to reach the time scale, $O(\varepsilon^{-2})$, for energy transfer between wave modes. Finally, real systems are finite and infinite x_1, x_2 in the present problem implicitly supposes that inertial waves do not propagate over the lateral confinement scales in the time scale $O(\varepsilon^{-2})$. This means that the lateral scales should be large compared with ε^{-2} multiples of the channel width. Since ε is small, this condition is stronger than requiring a box which is large compared with the correlation length, the classical criterion for neglect of confinement in non-rotating turbulence. Note that a similar condition is required of Fourier DNS of unconfined rotating turbulence if effects of periodicity are to be avoided at small Rossby number.

Finally, appendix G is an attempt to clarify the above requirements of the theory via a particular choice of dimensional parameters.

Acknowledgements

This work was carried out as part of the French National Research Agency (ANR) contract 08-BLAN-0076 ('ANISO').

Appendix A. Derivation of (2.14)

Each term in (2.1) is multiplied by $W_i^{(n)*}(x_3; \boldsymbol{k}) \exp\{-\mathrm{i}k_1x_1 - \mathrm{i}k_2x_2\}/4\pi^2$ and integrated over the channel volume. The time derivative is taken outside the integral of the first term, which thus contributes $\partial\tilde{u}_n/\partial t$ according to (2.13). Using the fact

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

that the conjugates of (2.4) and (2.5) satisfy (2.1) without the nonlinear and viscous terms and $\varepsilon_{i3j} = -\varepsilon_{j3i}$, the Coriolis term yields

$$\frac{1}{4\pi^2}\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varepsilon_{i3j}u_j W_i^{(n)^*}\exp\{-\mathrm{i}k_1x_1-\mathrm{i}k_2x_2\}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$$
$$=\frac{1}{4\pi^2}\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\mathrm{i}\omega_n u_i W_i^{(n)^*}-\mathrm{i}k_\alpha u_\alpha \Phi^{(n)^*}+u_3\frac{\mathrm{d}\Phi^{(n)^*}}{\mathrm{d}x_3}\right)$$
$$\times\exp\{-\mathrm{i}k_1x_1-\mathrm{i}k_2x_2\}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3, \tag{A 1}$$

where, here and henceforth, α is an index which runs over the values 1, 2 and to which the summation convention applies. Integration by parts gives

$$\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\mathrm{i}k_\alpha u_\alpha\Phi^{(n)^*}-u_3\frac{\mathrm{d}\Phi^{(n)^*}}{\mathrm{d}x_3}\right)\exp\{-\mathrm{i}k_1x_1-\mathrm{i}k_2x_2\}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$$
$$=\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial u_\alpha}{\partial x_\alpha}+\frac{\partial u_3}{\partial x_3}\right)\Phi^{(n)^*}\exp\{-\mathrm{i}k_1x_1-\mathrm{i}k_2x_2\}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3=0, \tag{A 2}$$

where we have used (2.2) and (2.3). Note that, in deriving (A 2), u_i has been treated as if it were zero at infinity (so that boundary terms arising from integration by parts with respect to x_1 and x_2 are dropped). This is standard procedure in spectral analysis and is employed repeatedly in what follows. Combining (A 1) and (A 2), the contribution of the Coriolis term to the amplitude equation is $\mathrm{i}\omega_n\tilde{u}_n$.

The remaining terms in (2.1) are also treated using integration by parts. Given that $W_3^{(n)}=0$ on $x_3=0,1$ and the fact that the conjugate of (2.4) satisfies (2.2), the pressure term gives

$$-\frac{1}{4\pi^2}\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial p}{\partial x_\alpha}W_\alpha^{(n)^*}+\frac{\partial p}{\partial x_3}W_3^{(n)^*}\right)\exp\{-\mathrm{i}k_1x_1-\mathrm{i}k_2x_2\}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$$
$$=\frac{1}{4\pi^2}\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\mathrm{d}W_3^{(n)^*}}{\mathrm{d}x_3}-\mathrm{i}k_\alpha W_\alpha^{(n)^*}\right)p\exp\{-\mathrm{i}k_1x_1-\mathrm{i}k_2x_2\}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3=0. \tag{A 3}$$

Using $\mathrm{d}^2W_i^{(n)}/\mathrm{d}x_3^2=-n^2\pi^2W_i^{(n)}$, equation (2.3) and $W_3^{(n)}=0$ at $x_3=0,1$, the viscous term contributes

$$\frac{1}{4\pi^2}\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial^2u_i}{\partial x_\alpha\partial x_\alpha}+\frac{\partial^2u_i}{\partial x_3^2}\right)W_i^{(n)^*}\exp\{-\mathrm{i}k_1x_1-\mathrm{i}k_2x_2\}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$$
$$=-(k^2+n^2\pi^2)\tilde{u}_n+\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{\partial u_\alpha}{\partial x_3}W_\alpha^{(n)^*}\right]_0^1\exp\{-\mathrm{i}k_1x_1-\mathrm{i}k_2x_2\}\,\mathrm{d}x_1\,\mathrm{d}x_2. \tag{A 4}$$

Employing (2.3) and the conjugates of (2.10) and (2.12), the nonlinear contribution is

$$\frac{1}{4\pi^2}\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{\partial}{\partial x_\alpha}(u_iu_\alpha)+\frac{\partial}{\partial x_3}(u_iu_3)\right)W_i^{(n)^*}\exp\{-\mathrm{i}k_1x_1-\mathrm{i}k_2x_2\}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$$
$$=\frac{1}{4\pi^2}\int_0^1\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\mathrm{i}k_\alpha u_\alpha W_i^{(n)^*}-u_3\frac{\mathrm{d}W_i^{(n)^*}}{\mathrm{d}x_3}\right)u_i\exp\{-\mathrm{i}k_1x_1-\mathrm{i}k_2x_2\}\,\mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$$

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

$$= - \sum_{n_p,n_q=-\infty}^{\infty} \iint I_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\tilde{u}^*_{n_p}(\boldsymbol{p})\tilde{u}^*_{n_q}(\boldsymbol{q})\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})\,\mathrm{d}^2\boldsymbol{p}\,\mathrm{d}^2\boldsymbol{q}, \tag{A 5}$$

where

$$I_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) = \int_0^1 \left(W_3^{(n_p)^*}(x_3;\ \boldsymbol{p})\frac{\mathrm{d}W_i^{(n)^*}}{\mathrm{d}x_3}(x_3;\boldsymbol{k}) - \mathrm{i}k_\alpha W_\alpha^{(n_p)^*}(x_3;\boldsymbol{p}) \times\ W_i^{(n)^*}(x_3;\boldsymbol{k}) \right) W_i^{(n_q)^*}(x_3;\boldsymbol{q})\,\mathrm{d}x_3. \tag{A 6}$$

Since $\tilde{u}^*_{n_p}(\boldsymbol{p})\tilde{u}^*_{n_q}(\boldsymbol{q})\delta(\boldsymbol{k}+\ \boldsymbol{p}+\boldsymbol{q})$ is symmetric with respect to the permutation $n_p \leftrightarrow n_q$, $\boldsymbol{p}\leftrightarrow\boldsymbol{q}$, the right-hand side of (A 5) is unchanged when $I_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$ is replaced by the symmetrized version $(I_{nn_pn_q}(\boldsymbol{k},\ \boldsymbol{p},\boldsymbol{q}) + I_{nn_qn_p}(\boldsymbol{k},\ \boldsymbol{q},\boldsymbol{p}))/2$. Performing this replacement and integrating with respect to $\boldsymbol{q}$, the nonlinear contribution to the amplitude equation is

$$- \sum_{n_p,n_q=-\infty}^{\infty} \int N_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})\tilde{u}^*_{n_p}(\boldsymbol{p})\tilde{u}^*_{n_q}(-\boldsymbol{k}-\boldsymbol{p})\,\mathrm{d}^2\boldsymbol{p}, \tag{A 7}$$

where

$$N_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p}) = \tfrac{1}{2}(I_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p},-\boldsymbol{k}-\boldsymbol{p}) + I_{nn_qn_p}(\boldsymbol{k},-\boldsymbol{k}-\boldsymbol{p},\boldsymbol{p})) \tag{A 8}$$

has the property (2.15). Finally, combining the contributions of the different terms in (2.1) gives (2.14).

When needed, $N_{nn_pn_q}$ can be numerically determined from (A 8), where $I_{nn_pn_q}$ is obtained by analytical evaluation of the integral in (A 6) using (2.6) and (2.7). Carrying out the calculation in detail,

$$\begin{aligned}
I_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) = {} & \frac{k_1p_2-k_2p_1+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{p}\,\omega_{n_p}(p)}{4kpq}(\delta_{n-n_p-n_q}+\delta_{n+n_p-n_q}+\delta_{n-n_p+n_q}+\delta_{n+n_p+n_q}) \\
& \times(\boldsymbol{k}\cdot\boldsymbol{q}\,(1-\omega_n(k)\omega_{n_q}(q))+\mathrm{i}(k_1q_2-k_2q_1)(\omega_n(k)-\omega_{n_q}(q))) \\
& + \frac{kq(\,k_1p_2-k_2p_1+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{p}\,\omega_{n_p}(p))}{4p(k^2+n^2\pi^2)^{1/2}(q^2+n_q^2\pi^2)^{1/2}}(\delta_{n-n_p-n_q}+\delta_{n+n_p-n_q}-\delta_{n-n_p+n_q}-\delta_{n+n_p+n_q}) \\
& + \frac{\mathrm{i}n\pi p}{4kq(p^2+n_p^2\pi^2)^{1/2}}(\delta_{n-n_p-n_q}-\delta_{n+n_p-n_q}+\delta_{n-n_p+n_q}-\delta_{n+n_p+n_q}) \\
& \times(\boldsymbol{k}\cdot\boldsymbol{q}\,(1-\omega_n(k)\omega_{n_q}(q))+\mathrm{i}(k_1q_2-k_2q_1)(\omega_n(k)-\omega_{n_q}(q))) \\
& + \frac{\mathrm{i}n\pi kpq}{4(k^2+n^2\pi^2)^{1/2}(p^2+n_p^2\pi^2)^{1/2}(q^2+n_q^2\pi^2)^{1/2}} \\
& \times(\delta_{n-n_p-n_q}-\delta_{n+n_p-n_q}-\delta_{n-n_p+n_q}+\delta_{n+n_p+n_q}).
\end{aligned} \tag{A 9}$$

Observe that $I_{nn_pn_q}$, and hence $N_{nn_pn_q}$, is zero unless $n\pm n_p\pm n_q=0$, as noted in the main text.

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

Appendix B. Evaluation of the wall-viscous term

The linearized, boundary-layer approximation of (2.1) implies

$$\frac{\partial u_\alpha}{\partial t} + \varepsilon_{\alpha 3\beta} u_\beta = -\frac{\partial p}{\partial x_\alpha} + \frac{\partial^2 u_\alpha}{\partial \xi^2}, \tag{B 1}$$

where, like α, β is an index which runs over the values 1, 2 and to which the summation convention applies and $\xi = x_3/E^{1/2}$ or $\xi = (1-x_3)/E^{1/2}$ depending on which wall ($x_3 = 0$ or $x_3 = 1$) is considered. Thus, ξ is a wall-normal coordinate, scaled appropriately for the boundary layer, whose thickness is $O(E^{1/2})$. The boundary conditions at the wall imply

$$u_\alpha = 0, \quad \xi = 0, \tag{B 2}$$

while matching to the external flow is expressed by

$$u_\alpha \to u_\alpha^\infty, \quad \xi \to +\infty, \tag{B 3}$$

where $u_\alpha^\infty(x_1, x_2, t)$ is the velocity just outside the layer.

The boundary-layer approximation also implies that the tangential pressure gradient, $\partial p/\partial x_\alpha$, is independent of ξ and imposed from outside the layer. Equations (B 1) and (B 3) give

$$\frac{\partial p}{\partial x_\alpha} = -\frac{\partial u_\alpha^\infty}{\partial t} - \varepsilon_{\alpha 3\beta} u_\beta^\infty. \tag{B 4}$$

Using this result in (B 1),

$$\frac{\partial \hat{u}_\alpha}{\partial t} + \varepsilon_{\alpha 3\beta} \hat{u}_\beta = \frac{\partial^2 \hat{u}_\alpha}{\partial \xi^2}, \tag{B 5}$$

where $\hat{u}_\alpha = u_\alpha - u_\alpha^\infty$. Equation (B 5) is to be solved for $\hat{u}_\alpha$, subject to the wall and matching conditions:

$$\hat{u}_\alpha = -u_\alpha^\infty, \quad \xi = 0, \tag{B 6}$$

$$\hat{u}_\alpha \to 0, \quad \xi \to +\infty. \tag{B 7}$$

To implement the solution, we need to know u_α^∞.

According to (2.6), (2.10) and (2.16),

$$u_\alpha = \sum_{n=-\infty}^{\infty} u_\alpha^{(n)}(x_1, x_2, t) \cos n\pi x_3, \tag{B 8}$$

where

$$u_\alpha^{(n)} = \int \mathrm{e}^{-\mathrm{i}\omega_n(k)t} a_n(\boldsymbol{k}, t)\, Z_\alpha^{(n)}(\boldsymbol{k}) \exp\{\mathrm{i}k_1 x_1 + \mathrm{i}k_2 x_2\}\, \mathrm{d}^2\, \boldsymbol{k}, \tag{B 9}$$

$$Z_1^{(n)}(\boldsymbol{k}) = \frac{k_1 \omega_n(k) + \mathrm{i}k_2}{k}, \quad Z_2^{(n)}(\boldsymbol{k}) = \frac{k_2 \omega_n(k) - \mathrm{i}k_1}{k}. \tag{B 10}$$

Equation (B 8) is a Fourier cosine series representation of u_α, of which a truncated version is

$$\sum_{n=-N}^{N} u_\alpha^{(n)}(x_1, x_2, t) \cos n\pi x_3. \tag{B 11}$$

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

As N increases, equation (B 11) approaches u_α: first outside the boundary layer, then the boundary layer begins to appear when $N = O(E^{-1/2})$ and is fully expressed when $N \gg E^{-1/2}$. The velocity just outside the boundary layer can be represented by limiting the sum in (B 8) to $|n| \ll E^{-1/2}$ and setting $x_3 = 0$ or $x_3 = 1$, depending on which wall is considered. Thus, we obtain

$$u_\alpha^\infty = \sum_{|n|\ll E^{-1/2}} \int \mathrm{e}^{-\mathrm{i}\omega_n(k)t} a_n(\boldsymbol{k}, t)\, W_\alpha^{(n)}(x_3 = 0, 1; \boldsymbol{k}) \exp\{\mathrm{i}k_1 x_1 + \mathrm{i}k_2 x_2\}\, \mathrm{d}^2\boldsymbol{k}. \quad \text{(B 12)}$$

Given the supposed small Rossby and Ekman numbers, $a_n(\boldsymbol{k}, t)$ is slowly varying. Treating it as if it were time independent as regards the solution of (B 5)–(B 7) with (B 12) gives

$$\begin{aligned}\hat{u}_\alpha \;=\; & -\frac{1}{2} \sum_{|n|\ll E^{-1/2}} \int \mathrm{e}^{-\mathrm{i}\omega_n(k)t} a_n(\boldsymbol{k}, t)[(\mathrm{e}^{-\gamma_+^{(n)}\xi} + \mathrm{e}^{-\gamma_-^{(n)}\xi})\, W_\alpha^{(n)}(x_3 = 0, 1; \boldsymbol{k}) \\ & + \mathrm{i}(\mathrm{e}^{-\gamma_+^{(n)}\xi} - \mathrm{e}^{-\gamma_-^{(n)}\xi})\, \varepsilon_{\alpha 3\beta} W_\beta^{(n)}(x_3 = 0, 1; \boldsymbol{k})] \exp\{\mathrm{i}k_1 x_1 + \mathrm{i}k_2 x_2\}\, \mathrm{d}^2\boldsymbol{k}. \quad \text{(B 13)}\end{aligned}$$

Taking the ξ derivative and setting $\xi = 0$,

$$\begin{aligned}\left.\frac{\partial u_\alpha}{\partial \xi}\right|_{\xi=0} = \left.\frac{\partial \hat{u}_\alpha}{\partial \xi}\right|_{\xi=0} = & \frac{1}{2} \sum_{|n|\ll E^{-1/2}} \int \mathrm{e}^{-\mathrm{i}\omega_n(k)t} a_n(\boldsymbol{k}, t)[(\gamma_+^{(n)} + \gamma_-^{(n)})\, W_\alpha^{(n)}(x_3 = 0, 1; \boldsymbol{k}) \\ & + \mathrm{i}\,(\gamma_+^{(n)} - \gamma_-^{(n)})\, \varepsilon_{\alpha 3\beta} W_\beta^{(n)}(x_3 = 0, 1; \boldsymbol{k})] \exp\{\mathrm{i}k_1 x_1 + \mathrm{i}k_2 x_2\}\, \mathrm{d}^2\boldsymbol{k}. \quad \text{(B 14)}\end{aligned}$$

Using (2.12) and the definition of ξ gives

$$\begin{aligned}& \frac{1}{4\pi^2} E \mathrm{e}^{\mathrm{i}\omega_n(k)t} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{\partial u_\alpha}{\partial x_3} W_\alpha^{(n)*}\right]_0^1 \exp\{-\mathrm{i}k_1 x_1 - \mathrm{i}k_2 x_2\}\, \mathrm{d}x_1\, \mathrm{d}x_2 \\ & = -\frac{1}{2} E^{1/2} \sum_{|m|\ll E^{-1/2}} \mathrm{e}^{\mathrm{i}(\omega_n(k) - \omega_m(k))t} a_m(\boldsymbol{k}, t) \\ & \quad \times \{(\gamma_+^{(m)}(k) + \gamma_-^{(m)}(k))(W_\alpha^{(n)*}(0; \boldsymbol{k}) W_\alpha^{(m)}(0; \boldsymbol{k}) + W_\alpha^{(n)*}(1; \boldsymbol{k}) W_\alpha^{(m)}(1; \boldsymbol{k})) \\ & \quad + \mathrm{i}(\gamma_+^{(m)}(k) - \gamma_-^{(m)}(k)) \varepsilon_{\alpha 3\beta} (W_\alpha^{(n)*}(0; \boldsymbol{k}) W_\beta^{(m)}(0; \boldsymbol{k}) + W_\alpha^{(n)*}(1; \boldsymbol{k}) W_\beta^{(m)}(1; \boldsymbol{k}))\}. \quad \text{(B 15)}\end{aligned}$$

Since the coefficients a_m are slowly varying functions of time, $m \neq n$ terms on the right of (B 15) are oscillatory with an $O(1)$ period and hence, unlike the $m = n$ term, do not have a cumulative effect on the evolution of a_n over the long time spans of interest to us here. Neglecting $m \neq n$ terms in (B 15), (2.17) gives (2.18).

Appendix C. Derivation of (3.19)

Using (3.7), $a_{n_p}(\boldsymbol{k}') = a_{-n_p}^*(-\boldsymbol{k}')$ and $a_{m_q}^*(-\boldsymbol{k} - \boldsymbol{p}) = a_{-m_q}(\boldsymbol{k} + \boldsymbol{p})$, equation (3.18) becomes

$$\begin{aligned}\overline{a_{n_p}(\boldsymbol{k}') a_{n_q}(\boldsymbol{k}'') a_{m_p}^*(\boldsymbol{p}) a_{m_q}^*(-\boldsymbol{k} - \boldsymbol{p})} = & A_{-n_p, n_q}(k')\delta(\boldsymbol{k}' + \boldsymbol{k}'') A_{m_p, -m_q}(p)\delta(\boldsymbol{k}) \\ & + A_{m_p n_p}(k')\delta(\boldsymbol{k}' - \boldsymbol{p}) A_{m_q n_q}(k'')\delta(\boldsymbol{k}'' + \boldsymbol{k} + \boldsymbol{p}) \\ & + A_{m_q n_p}(k')\delta(\boldsymbol{k}' + \boldsymbol{k} + \boldsymbol{p}) A_{m_p n_q}(k'')\delta(\boldsymbol{k}'' - \boldsymbol{p}) \\ & + K_{n_p n_q m_p m_q}(\boldsymbol{k}', \boldsymbol{k}'', \boldsymbol{p})\delta(\boldsymbol{k} + \boldsymbol{k}' + \boldsymbol{k}'') \quad \text{(C 1)}\end{aligned}$$

with similar expressions for the other fourth-order moments in (3.17).

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

The first term on the right of (C 1) contributes

$$A_{-n_p,n_q}(k')\mathrm{e}^{\mathrm{i}\omega_m(k)t}\delta(\boldsymbol{k}'+\boldsymbol{k}'')\delta(\ \boldsymbol{k}) \times \sum_{m_p,m_q=-\infty}^{\infty}\int \mathrm{e}^{\mathrm{i}(\omega_{m_p}(p)+\omega_{m_q}(|\boldsymbol{k}+\boldsymbol{p}|))t}N_{mm_pm_q}(\boldsymbol{k},\boldsymbol{p})A_{m_p,-m_q}(p)\,\mathrm{d}^2\boldsymbol{p} \tag{C 2}$$

to the right-hand side of (3.17). As $\boldsymbol{k}\to 0$, $\mathrm{e}^{\mathrm{i}\omega_{m_q}(|\boldsymbol{k}+\boldsymbol{p}|)t}\to\mathrm{e}^{\mathrm{i}\omega_{m_q}(p)t}$, while (A 8) and (A 9) imply

$$\begin{aligned}N_{mm_pm_q}(\boldsymbol{k},\boldsymbol{p}) \sim &-\frac{\mathrm{i}m\pi}{8k(p^2+m_p^2\pi^2)^{1/2}}(\delta_{m-m_p-m_q}-\delta_{m+m_p-m_q}+\delta_{m-m_p+m_q}-\delta_{m+m_p+m_q})\\ &\times(\boldsymbol{k}\cdot\boldsymbol{p}\,(1-\omega_m(k)\omega_{m_q}(p))+\mathrm{i}(k_1p_2-k_2p_1)(\omega_m(k)-\omega_{m_q}(p)))\\ &+\frac{\mathrm{i}m\pi}{8k(p^2+m_q^2\pi^2)^{1/2}}(\delta_{m-m_q-m_p}-\delta_{m+m_q-m_p}+\delta_{m-m_q+m_p}-\delta_{m+m_q+m_p})\\ &\times(\boldsymbol{k}\cdot\boldsymbol{p}\,(1-\omega_m(k)\omega_{m_p}(p))+\mathrm{i}(k_1p_2-k_2p_1)(\omega_m(k)-\omega_{m_p}(p))).\end{aligned} \tag{C 3}$$

Thus, the integrand in (C 2) approaches an odd function of $\boldsymbol{p}$, so the integral tends to zero. Thanks to the factor $\delta(\ \boldsymbol{k})$, equation (C 2) is effectively zero, i.e. the first term on the right of (C 1) does not contribute to (3.17). Similar results hold for the other fourth-order moments in (3.17).

The second and third terms on the right of (C 1) contribute

$$\begin{aligned}&\delta(\boldsymbol{k}+\boldsymbol{k}'+\boldsymbol{k}'')\mathrm{e}^{\mathrm{i}\omega_m(k)t}\\ &\times\left(\sum_{m_p,m_q=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(\omega_{m_p}(k')+\omega_{m_q}(|\boldsymbol{k}+\boldsymbol{k}'|))t}N_{mm_pm_q}(\boldsymbol{k},\boldsymbol{k}')A_{m_pn_p}(k')A_{m_qn_q}(k'')\right.\\ &\left.+\sum_{m_p,m_q=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(\omega_{m_p}(k'')+\omega_{m_q}(|\boldsymbol{k}+\boldsymbol{k}''|))t}N_{mm_pm_q}(\boldsymbol{k},\boldsymbol{k}'')A_{m_qn_p}(k')A_{m_pn_q}(k'')\right)\end{aligned} \tag{C 4}$$

to the right-hand side of (3.17). Given the factor $\delta(\ \boldsymbol{k}+\boldsymbol{k}'+\boldsymbol{k}'')$, equation (C 4) is equivalent to

$$\begin{aligned}&\delta(\boldsymbol{k}+\boldsymbol{k}'+\boldsymbol{k}'')\mathrm{e}^{\mathrm{i}\omega_m(k)t}\\ &\times\left(\sum_{m_p,m_q=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(\omega_{m_p}(k')+\omega_{m_q}(k''))t}N_{mm_pm_q}(\boldsymbol{k},\boldsymbol{k}')A_{m_pn_p}(k')A_{m_qn_q}(k'')\right.\\ &\left.+\sum_{m_p,m_q=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(\omega_{m_p}(k'')+\omega_{m_q}(k'))t}N_{mm_pm_q}(\boldsymbol{k},-\boldsymbol{k}-\boldsymbol{k}')A_{m_qn_p}(k')A_{m_pn_q}(k'')\right).\end{aligned} \tag{C 5}$$

Permuting the indices m_p and m_q inside the second sum does not change its value. Thus, using (2.15), equation (C 5) gives

$$2\delta(\boldsymbol{k}+\boldsymbol{k}'+\boldsymbol{k}'')\sum_{m_p,m_q=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(\omega_m(k)+\omega_{m_p}(k')+\omega_{m_q}(k''))t}N_{mm_pm_q}(\ \boldsymbol{k},\boldsymbol{k}')A_{m_pn_p}(k')A_{m_qn_q}(k'') \tag{C 6}$$

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

for the contribution of the second and third terms on the right of (C 1) to (3.17). The other fourth-order moments in (3.17) give

$$2\delta(\boldsymbol{k}+\boldsymbol{k}'+\boldsymbol{k}'')\sum_{m_p,m_q=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(\omega_{m_p}(k)+\omega_{n_p}(k')+\omega_{m_q}(k''))t}N_{n_pm_pm_q}(\boldsymbol{k}',\boldsymbol{k})A_{m_pm}(k)A_{m_qn_q}(k'') \tag{C 7}$$

and

$$2\delta(\boldsymbol{k}+\boldsymbol{k}'+\boldsymbol{k}'')\sum_{m_p,m_q=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}(\omega_{m_q}(k)+\omega_{m_p}(k')+\omega_{n_q}(k''))t}N_{n_qm_pm_q}(\boldsymbol{k}'',\boldsymbol{k}')A_{m_pn_p}(k')A_{m_qm}(k). \tag{C 8}$$

The sum of (C 6)–(C 8) gives the quasi-normal contribution to the right-hand side of (3.17). Including the cumulant correction and integrating over $\boldsymbol{k}''$,

$$\begin{aligned}
&\frac{\partial\Theta_{mn_pn_q}(\boldsymbol{k},\boldsymbol{k}')}{\partial t}+(\Delta_m(k)+\Delta_{n_p}(k')+\Delta_{n_q}(|\boldsymbol{k}+\boldsymbol{k}'|))\Theta_{mn_pn_q}(\boldsymbol{k},\boldsymbol{k}')\\
&\quad=2\sum_{m_p,m_q=-\infty}^{\infty}[\mathrm{e}^{\mathrm{i}(\omega_m(k)+\omega_{m_p}(k')+\omega_{m_q}(|\boldsymbol{k}+\boldsymbol{k}'|))t}N_{mm_pm_q}(\boldsymbol{k},\boldsymbol{k}')A_{m_pn_p}(k')A_{m_qn_q}(|\boldsymbol{k}+\boldsymbol{k}'|)\\
&\qquad+\mathrm{e}^{\mathrm{i}(\omega_{m_p}(k)+\omega_{n_p}(k')+\omega_{m_q}(|\boldsymbol{k}+\boldsymbol{k}'|))t}N_{n_pm_pm_q}(\boldsymbol{k}',\boldsymbol{k})A_{m_pm}(k)A_{m_qn_q}(|\boldsymbol{k}+\boldsymbol{k}'|)\\
&\qquad+\mathrm{e}^{\mathrm{i}(\omega_{m_q}(k)+\omega_{m_p}(k')+\omega_{n_q}(|\boldsymbol{k}+\boldsymbol{k}'|))t}N_{n_qm_pm_q}(-\boldsymbol{k}-\boldsymbol{k}',\boldsymbol{k}')A_{m_pn_p}(k')A_{m_qm}(k)]\\
&\qquad+\sum_{m_p,m_q=-\infty}^{\infty}\int\mathrm{e}^{\mathrm{i}\omega_{m_p}(p)t}[\mathrm{e}^{\mathrm{i}(\omega_m(k)+\omega_{m_q}(|\boldsymbol{k}+\boldsymbol{p}|))t}N_{mm_pm_q}(\boldsymbol{k},\boldsymbol{p})K_{n_pn_qm_pm_q}(\boldsymbol{k}',-\boldsymbol{k}-\boldsymbol{k}',\boldsymbol{p})\\
&\qquad+\mathrm{e}^{\mathrm{i}(\omega_{n_p}(k')+\omega_{m_q}(|\boldsymbol{k}'+\boldsymbol{p}|))t}N_{n_pm_pm_q}(\boldsymbol{k}',\boldsymbol{p})K_{mn_qm_pm_q}(\boldsymbol{k},-\boldsymbol{k}-\boldsymbol{k}',\boldsymbol{p})\\
&\qquad+\mathrm{e}^{\mathrm{i}(\omega_{n_q}(|\boldsymbol{k}+\boldsymbol{k}'|)+\omega_{m_q}(|\boldsymbol{k}+\boldsymbol{k}'-\boldsymbol{p}|))t}N_{n_qm_pm_q}(-\boldsymbol{k}-\boldsymbol{k}',\boldsymbol{p})K_{mn_pm_pm_q}(\boldsymbol{k},\boldsymbol{k}',\ \boldsymbol{p})]\,\mathrm{d}^2\boldsymbol{p}.
\end{aligned} \tag{C 9}$$

Finally, replacing the integration variable $\boldsymbol{p}$ by $\boldsymbol{q}$, then $\boldsymbol{k}'$ by $\boldsymbol{p}$ gives (3.19).

Appendix D. Derivation of (4.8)

Equation (2.18) may be used to derive an equation for $\boldsymbol{U}$, but it is simpler to obtain it directly from (2.1). Thus, we take the integral over x_3 of the x_1 and x_2 components of (2.1). Using (2.3), (4.3), (4.5) and (4.7),

$$\frac{\partial U_\alpha}{\partial t}+\frac{\partial}{\partial x_\beta}(U_\alpha U_\beta)=-\frac{\partial P}{\partial x_\alpha}+E\frac{\partial^2U_\alpha}{\partial x_\beta\partial x_\beta}+E\left[\frac{\partial u_\alpha}{\partial x_3}\right]_0^1-\frac{\partial}{\partial x_\beta}\left(\int_0^1 v_\alpha v_\beta\,\mathrm{d}x_3\right), \tag{D 1}$$

where

$$P=\int_0^1 p\,\mathrm{d}x_3+\psi \tag{D 2}$$

incorporates the Coriolis term into the pressure, thus removing direct effects of rotation on the two-dimensional component.

Equation (B 14) can be used to evaluate the wall term in (D 1):

$$\begin{aligned}
E\left[\frac{\partial u_\alpha}{\partial x_3}\right]_0^1=&-\frac{1}{2}E^{1/2}\sum_{|n|\ll E^{-1/2}}\int\mathrm{e}^{-\mathrm{i}\omega_n(k)t}a_n(\boldsymbol{k},t)[(\gamma_+^{(n)}+\gamma_-^{(n)})(W_\alpha^{(n)}(0;\boldsymbol{k})+W_\alpha^{(n)}(1;\ \boldsymbol{k}))\\
&+\mathrm{i}\,(\gamma_+^{(n)}-\gamma_-^{(n)})\varepsilon_{\alpha3\beta}(W_\beta^{(n)}(0;\boldsymbol{k})+W_\beta^{(n)}(1;\boldsymbol{k}))]\exp\{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2\}\,\mathrm{d}^2\boldsymbol{k}.
\end{aligned} \tag{D 3}$$

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

Employing $\omega_0 = 0$, equations (2.21) and (4.2) give

$$-(2E)^{1/2}(U_\alpha + \varepsilon_{\alpha 3\beta} U_\beta) \tag{D 4}$$

for the $n = 0$ contribution to (D 3). Terms other than $n = 0$ have sinusoidal oscillations over time spans of $O(1)$. Recalling that $\boldsymbol{U}$ evolves slowly according to (D 1), the cumulative effect of the $n \neq 0$ terms in (D 3) over the long time spans necessary for evolution of $\boldsymbol{U}$ will be small compared with that of the (non-oscillatory) $n = 0$ term. Neglecting the $n \neq 0$ contributions to (D 3) and using (4.7) and (D 4), equation (D 1) becomes (4.8), where

$$\Pi = P + (2E)^{1/2}\psi. \tag{D 5}$$

Equations (D 2) and (D 5) yield (4.9).

Appendix E. Derivation of (5.10)

Using (5.9) in (3.16),

$$\begin{aligned}\tau_{nm}(k) \sim\ & -2\mathrm{i} \sum_{n_p, n_q = -\infty}^{\infty} \int N^*_{nn_pn_q}(\boldsymbol{k}, \boldsymbol{p}) \frac{1 - \mathrm{e}^{-\mathrm{i}(\omega_n(k) + \omega_{n_p}(p) + \omega_{n_q}(|\boldsymbol{k}+\ \boldsymbol{p}|))t}}{\omega_n(k) + \omega_{n_p}(p) + \omega_{n_q}(|\boldsymbol{k}+\boldsymbol{p}|)} \\ & \times [\delta_{n-m} N_{nn_pn_q}(\boldsymbol{k}, \boldsymbol{p}) A_{n_pn_p}(p) A_{n_qn_q}(|\ \boldsymbol{k}+\boldsymbol{p}|) + N_{n_pnn_q}(\ \boldsymbol{p}, \boldsymbol{k}) A_{nm}(k) A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|) \\ & + N_{n_qn_pn}(-\boldsymbol{k}-\boldsymbol{p},\ \ \boldsymbol{p}) A_{n_pn_p}(p) A_{nm}(k)]\, \mathrm{d}^2\boldsymbol{p}. \end{aligned} \tag{E 1}$$

Changing integration variable from $\boldsymbol{p}$ to $\boldsymbol{q} = -\boldsymbol{k} - \boldsymbol{p}$ and permuting the summation indices n_p and n_q yields the identity

$$\begin{aligned} & \sum_{n_p, n_q = -\infty}^{\infty} \int N^*_{nn_pn_q}(\boldsymbol{k}, \boldsymbol{p}) \frac{1 - \mathrm{e}^{-\mathrm{i}(\omega_n(k) + \omega_{n_p}(p) + \omega_{n_q}(|\boldsymbol{k}+\boldsymbol{p}|))t}}{\omega_n(k) + \omega_{n_p}(p) + \omega_{n_q}(|\boldsymbol{k}+\boldsymbol{p}|)} \\ & \quad \times N_{n_pnn_q}(\boldsymbol{p}, \boldsymbol{k}) A_{nm}(k) A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|)\, \mathrm{d}^2\boldsymbol{p} \\ & = \sum_{n_p, n_q = -\infty}^{\infty} \int N^*_{nn_qn_p}(\boldsymbol{k}, -\boldsymbol{k}-\boldsymbol{q}) \frac{1 - \mathrm{e}^{-\mathrm{i}(\omega_n(k) + \omega_{n_p}(q) + \omega_{n_q}(|\boldsymbol{k}+\boldsymbol{q}|))t}}{\omega_n(k) + \omega_{n_p}(q) + \omega_{n_q}(|\boldsymbol{k}+\boldsymbol{q}|)} \\ & \quad \times N_{n_qnn_p}(-\boldsymbol{k}-\boldsymbol{q}, \boldsymbol{k}) A_{nm}(k) A_{n_pn_p}(q)\, \mathrm{d}^2\ \boldsymbol{q}. \end{aligned} \tag{E 2}$$

Using (2.15), the right-hand side of (E 2) becomes

$$\begin{aligned} & \sum_{n_p, n_q = -\infty}^{\infty} \int N^*_{nn_pn_q}(\boldsymbol{k}, \boldsymbol{q}) \frac{1 - \mathrm{e}^{-\mathrm{i}(\omega_n(k) + \omega_{n_p}(q) + \omega_{n_q}(|\boldsymbol{k}+\boldsymbol{q}|))t}}{\omega_n(k) + \omega_{n_p}(q) + \omega_{n_q}(|\boldsymbol{k}+\boldsymbol{q}|)} \\ & \quad \times N_{n_qn_pn}(-\boldsymbol{k}-\boldsymbol{q}, \boldsymbol{q}) A_{nm}(k) A_{n_pn_p}(q)\, \mathrm{d}^2\ \boldsymbol{q}. \end{aligned} \tag{E 3}$$

It follows that the second and third terms inside the square brackets of (E 1) yield the same result. Thus,

$$\tau_{nm}(k) \sim \sum_{n_p, n_q = -\infty}^{\infty} \int \frac{1 - \mathrm{e}^{-\mathrm{i}\sigma_{nn_pn_q}(\boldsymbol{k}, \boldsymbol{p})t}}{\mathrm{i}\sigma_{nn_pn_q}(\boldsymbol{k},\ \ \boldsymbol{p})} f_{nn_pn_q}(\boldsymbol{k}, \boldsymbol{p})\, \mathrm{d}^2\boldsymbol{p}, \tag{E 4}$$

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

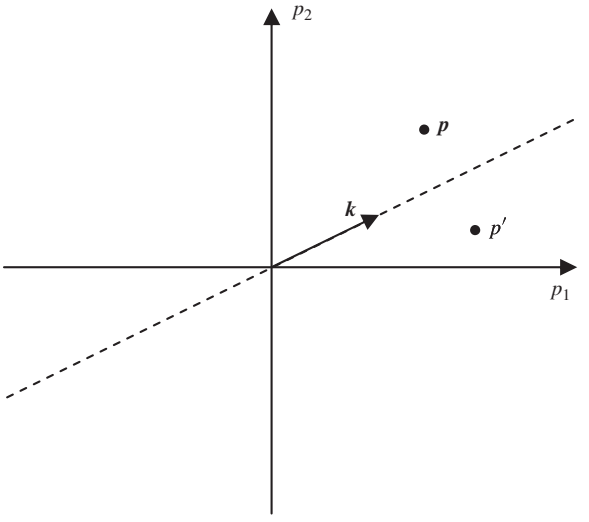


FIGURE 4. Reflection in the $\boldsymbol{p}$ plane. The point $\boldsymbol{p}$ becomes $\boldsymbol{p}'$ and vice versa.

where

$$\sigma_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p}) = \omega_n(k) + \omega_{n_p}(p) + \omega_{n_q}(|\boldsymbol{k}+\boldsymbol{p}|), \tag{E 5}$$

$$f_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p}) = \tfrac{1}{2}A_{n_pn_p}(p)(\delta_{n-m}\lambda_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|) + \eta_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{nm}(k)), \tag{E 6}$$

$$\eta_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p}) = 8N^*_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})N_{n_qn_pn}(-\boldsymbol{k}-\boldsymbol{p},\boldsymbol{p}). \tag{E 7}$$

It can be shown, using (A 8) and (A 9), that reflection of the point $\boldsymbol{p}$ in the line through the origin running parallel to $\boldsymbol{k}$ (see figure 4) leads to complex conjugation of $\eta_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})$. Since $\sigma_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})$ and $A_{n_pn_p}(p)$ are symmetric under such reflection, the net contribution of the imaginary part of $\eta_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})$ to (E 4) is zero. Thus, the same end result is achieved if we replace (E 7) by its real part, i.e. we use (5.13) instead of (E 7). This modified definition is used from here on.

Integrals of the form in (E 4) always arise in wave-turbulence analyses and are evaluated asymptotically for large t:

$$\int \frac{1-\mathrm{e}^{-\mathrm{i}\sigma_{nn_pn_q}(\boldsymbol{k},\ \boldsymbol{p})t}}{\mathrm{i}\sigma_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})} f_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})\,\mathrm{d}^2\boldsymbol{p} \;\to\; \oint_{C_{nn_pn_q}(\boldsymbol{k})} \frac{f_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})}{\Gamma_{n_pn_q}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\ \boldsymbol{p}| - \mathrm{i}\int \frac{f_{nn_pn_q}(\boldsymbol{k},\ \boldsymbol{p})}{\sigma_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})}\,\mathrm{d}^2\boldsymbol{p} \tag{E 8}$$

as $t\to\infty$, where

$$\Gamma_{n_pn_q}(\boldsymbol{k},\boldsymbol{p}) = \frac{1}{\pi}|\nabla_{\boldsymbol{p}}\sigma_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})| = \left|\frac{n_p\boldsymbol{p}}{(p^2+n_p^2\pi^2)^{3/2}} + \frac{n_q(\boldsymbol{k}+\boldsymbol{p})}{(|\boldsymbol{k}+\boldsymbol{p}|^2+n_q^2\pi^2)^{3/2}}\right| \tag{E 9}$$

and the surface integral on the right-hand side of (E 8) should be interpreted as a Cauchy principal value.

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

Using (E 6) and (E 8), equation (E 4) gives

$$\tau_{nm}(k) \sim \mu_n(k)A_{nm}(k) + \frac{1}{2}\delta_{n-m}\sum_{n_p,n_q=-\infty}^{\infty}\oint_{C_{nn_pn_q}(\boldsymbol{k})}\frac{\lambda_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{n_pn_p}(p)A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|)}{\Gamma_{n_pn_q}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\boldsymbol{p}|$$
$$-\frac{1}{2}\mathrm{i}\delta_{n-m}\sum_{n_p,n_q=-\infty}^{\infty}\int\frac{\lambda_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{n_pn_p}(p)A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|)}{\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(|\boldsymbol{k}+\boldsymbol{p}|)}\,\mathrm{d}^2\boldsymbol{p}. \tag{E 10}$$

Equation (E 10) and the fact that A_{nm} is Hermitian imply (5.10).

Appendix F. Evaluation of (5.37)

Recalling (E 9) and using standard properties of the Dirac delta function, equation (5.37) can be expressed as

$$\sum_{n,n_p,n_q\neq0}\int\oint_{C_{nn_pn_q}(\boldsymbol{k})}\frac{A_{n_pn_p}(p)(\eta_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{nn}(k)+\lambda_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})A_{n_qn_q}(|\boldsymbol{k}+\boldsymbol{p}|))}{\Gamma_{n_pn_q}(\boldsymbol{k},\boldsymbol{p})}|\mathrm{d}\boldsymbol{p}|\,\mathrm{d}^2\boldsymbol{k}$$
$$=\pi\sum_{n,n_p,n_q\neq0}\iiint B_{n_p}(p)(\eta_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})B_n(k)+\lambda_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})B_{n_q}(q))$$
$$\times\delta(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(q))\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})\,\mathrm{d}^2\boldsymbol{k}\,\mathrm{d}^2\,\boldsymbol{p}\,\mathrm{d}^2\boldsymbol{q}, \tag{F 1}$$

where $B_n(k)=A_{nn}(k)$. Using (5.12) and (5.13), this is the real part of

$$4\pi\sum_{n,n_p,n_q\neq0}\iiint N^*_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})(2N_{n_qn_pn}(\boldsymbol{q},\boldsymbol{p})B_n(k)B_{n_p}(p)+N_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})B_{n_p}(p)B_{n_q}(q))$$
$$\times\delta(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(q))\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})\,\mathrm{d}^2\boldsymbol{k}\,\mathrm{d}^2\,\boldsymbol{p}\,\mathrm{d}^2\boldsymbol{q}. \tag{F 2}$$

The relation

$$\sum_{n,n_p,n_q\neq0}\iiint N^*_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})N_{n_qn_pn}(\boldsymbol{q},\boldsymbol{p})B_n(k)B_{n_p}(p)$$
$$\times\delta(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(q))\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})\,\mathrm{d}^2\boldsymbol{k}\,\mathrm{d}^2\,\boldsymbol{p}\,\mathrm{d}^2\boldsymbol{q}$$
$$=\sum_{n,n_p,n_q\neq0}\iiint N^*_{nn_qn_p}(\boldsymbol{k},\boldsymbol{q})N_{n_pn_qn}(\boldsymbol{p},\boldsymbol{q})B_n(k)B_{n_q}(q)$$
$$\times\delta(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(q))\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})\,\mathrm{d}^2\boldsymbol{k}\,\mathrm{d}^2\,\boldsymbol{p}\,\mathrm{d}^2\boldsymbol{q} \tag{F 3}$$

follows from the permutation $n_p\leftrightarrow n_q$, $\boldsymbol{p}\leftrightarrow\boldsymbol{q}$ of summation indices and integration variables. Using $N^*_{nn_qn_p}(\boldsymbol{k},\boldsymbol{q})=N^*_{nn_pn_q}(\boldsymbol{k},-\boldsymbol{k}-\boldsymbol{q})$ in (F 3), equation (F 2) can be rewritten as

$$4\pi\sum_{n,n_p,n_q\neq0}\iiint N^*_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})\delta(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(q))\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})$$
$$\times[N_{n_qn_pn}(\boldsymbol{q},\boldsymbol{p})B_n(k)B_{n_p}(p)+N_{n_pn_qn}(\boldsymbol{p},\boldsymbol{q})B_n(k)B_{n_q}(q)$$
$$+N_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})B_{n_p}(p)B_{n_q}(q)]\,\mathrm{d}^2\boldsymbol{k}\,\mathrm{d}^2\,\boldsymbol{p}\,\mathrm{d}^2\boldsymbol{q}. \tag{F 4}$$

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

Employing (A 8), this becomes

$$\begin{aligned}
&2\pi \sum_{n,n_p,n_q\neq 0} \int\int\int (I^*_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) + I^*_{nn_qn_p}(\boldsymbol{k},\boldsymbol{q},\ \boldsymbol{p}))\\
&\quad\times \delta(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(q))\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})\\
&\quad\times [N_{n_qn_pn}(\boldsymbol{q},\boldsymbol{p})B_n(k)B_{n_p}(p)\\
&\quad+N_{n_pn_qn}(\ \boldsymbol{p},\boldsymbol{q})B_n(k)B_{n_q}(q)\\
&\quad+N_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})B_{n_p}(p)B_{n_q}(q)]\,\mathrm{d}^2\boldsymbol{k}\,\mathrm{d}^2\ \boldsymbol{p}\,\mathrm{d}^2\boldsymbol{q}. \qquad \text{(F 5)}
\end{aligned}$$

The relation

$$\begin{aligned}
&\sum_{n,n_p,n_q\neq 0} \iiint I^*_{nn_qn_p}(\boldsymbol{k},\boldsymbol{q},\boldsymbol{p})\delta(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(q))\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})\\
&\quad\times [N_{n_qn_pn}(\boldsymbol{q},\boldsymbol{p})B_n(k)B_{n_p}(p)+N_{n_pn_qn}(\ \boldsymbol{p},\boldsymbol{q})B_n(k)B_{n_q}(q)\\
&\quad+N_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})B_{n_p}(p)B_{n_q}(q)]\,\mathrm{d}^2\boldsymbol{k}\,\mathrm{d}^2\ \boldsymbol{p}\,\mathrm{d}^2\boldsymbol{q}\\
&=\sum_{n,n_p,n_q\neq 0} \iiint I^*_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\delta(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(q))\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})\\
&\quad\times [N_{n_qn_pn}(\boldsymbol{q},\boldsymbol{p})B_n(k)B_{n_p}(p)+N_{n_pn_qn}(\ \boldsymbol{p},\boldsymbol{q})B_n(k)B_{n_q}(q)\\
&\quad+N_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})B_{n_p}(p)B_{n_q}(q)]\,\mathrm{d}^2\boldsymbol{k}\,\mathrm{d}^2\ \boldsymbol{p}\,\mathrm{d}^2\boldsymbol{q} \qquad \text{(F 6)}
\end{aligned}$$

is obtained using $N_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p}) = N_{nn_qn_p}(\boldsymbol{k},-\boldsymbol{k}-\ \boldsymbol{p})$, followed by the permutation $n_p \leftrightarrow n_q$, $\boldsymbol{p}\leftrightarrow\boldsymbol{q}$ of summation indices and integration variables. Thus, equation (F 5) yields

$$\begin{aligned}
&4\pi \sum_{n,n_p,n_q\neq 0} \iiint I^*_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})\delta(\omega_n(k)+\omega_{n_p}(p)+\omega_{n_q}(q))\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})\\
&\quad\times [N_{n_qn_pn}(\boldsymbol{q},\boldsymbol{p})B_n(k)B_{n_p}(p)+N_{n_pn_qn}(\ \boldsymbol{p},\boldsymbol{q})B_n(k)B_{n_q}(q)\\
&\quad+N_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p})B_{n_p}(p)B_{n_q}(q)]\,\mathrm{d}^2\boldsymbol{k}\,\mathrm{d}^2\ \boldsymbol{p}\,\mathrm{d}^2\boldsymbol{q}. \qquad \text{(F 7)}
\end{aligned}$$

Using integration by parts in (A 6),

$$\begin{aligned}
I_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) \ =\ &-\int_0^1 \left(\frac{\mathrm{d}W_3^{(n_p)^*}}{\mathrm{d}x_3}(x_3;\boldsymbol{p}) + \mathrm{i}k_\alpha W_\alpha^{(n_p)^*}(x_3;\boldsymbol{p})\right) W_i^{(n_q)^*}(x_3;\boldsymbol{q})W_i^{(n)^*}(x_3;\boldsymbol{k})\,\mathrm{d}x_3\\
&-\int_0^1 W_3^{(n_p)^*}(x_3;\boldsymbol{p})\frac{\mathrm{d}W_i^{(n_q)^*}}{\mathrm{d}x_3}(x_3;\boldsymbol{q})W_i^{(n)^*}(x_3;\boldsymbol{k})\,\mathrm{d}x_3. \qquad \text{(F 8)}
\end{aligned}$$

Multiplying by $\delta(\boldsymbol{k}+\boldsymbol{p}+\ \boldsymbol{q})$,

$$\begin{aligned}
&\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})I_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) = -\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})\\
&\quad\times\int_0^1 \left(\frac{\mathrm{d}W_3^{(n_p)^*}}{\mathrm{d}x_3}(x_3;\boldsymbol{p}) - \mathrm{i}(p_\alpha+q_\alpha)W_\alpha^{(n_p)^*}(x_3;\boldsymbol{p})\right) W_i^{(n_q)^*}(x_3;\ \boldsymbol{q})W_i^{(n)^*}(x_3;\boldsymbol{k})\,\mathrm{d}x_3\\
&\quad-\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})\int_0^1 W_3^{(n_p)^*}(x_3;\ \boldsymbol{p})\frac{\mathrm{d}W_i^{(n_q)^*}}{\mathrm{d}x_3}(x_3;\boldsymbol{q})W_i^{(n)^*}(x_3;\ \boldsymbol{k})\,\mathrm{d}x_3. \qquad \text{(F 9)}
\end{aligned}$$

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

The incompressibility relation of the modes implies

$$\frac{\mathrm{d}W_3^{(n_p)^*}}{\mathrm{d}x_3}(x_3;\boldsymbol{p}) - \mathrm{i}p_\alpha W_\alpha^{(n_p)^*}(x_3;\boldsymbol{p}) = 0, \tag{F 10}$$

hence (F 9) gives

$$\begin{aligned}\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})I_{nn_pn_q}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) &= -\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})\\ &\times \int_0^1 \left(W_3^{(n_p)^*}(x_3;\ \boldsymbol{p})\frac{\mathrm{d}W_i^{(n_q)^*}}{\mathrm{d}x_3}(x_3;\boldsymbol{q}) - \mathrm{i}q_\alpha W_\alpha^{(n_p)^*}(x_3;\boldsymbol{p})W_i^{(n_q)^*}(x_3;\boldsymbol{q}) \right) W_i^{(n)^*}(x_3;\boldsymbol{k})\,\mathrm{d}x_3\\ &= -\delta(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q})I_{n_qn_pn}(\boldsymbol{q},\boldsymbol{p},\boldsymbol{k}).\end{aligned} \tag{F 11}$$

Then $N_{n_pn_qn}(\boldsymbol{p},\boldsymbol{q}) = N_{n_pnn_q}(\boldsymbol{p},-\boldsymbol{p}-\boldsymbol{q})$ and (F 11) imply that the integrand in (F 7) is antisymmetric under the permutation $n \leftrightarrow n_q$, $\boldsymbol{k} \leftrightarrow \boldsymbol{q}$. We conclude that (F 7), and hence (5.37), is zero.

Appendix G. Prerequisites of the theory from an experimental/DNS point of view

The aim of this appendix is to illustrate the order-of-magnitude prerequisites of the theory via a particular choice of dimensional parameters. These requirements, discussed in the conclusion, are as follows:

(i) small Rossby number, $\varepsilon = u'/2\Omega h$, where u' is a measure of the initial turbulent velocity fluctuations;

(ii) Ekman number, $E = \nu/2\Omega h^2$, of $O(\varepsilon^4)$ or smaller, otherwise viscous damping kills the turbulence before the interesting regime, $t = O(\varepsilon^{-2})$, of wave-turbulence interactions is reached;

(iii) ratio of lateral dimensions, L, to channel width, h, large compared with ε^{-2}, otherwise lateral confinement is significant.

Taking $u' = 10$ cm s^{-1} and water ($\nu = 10^{-6}$ m^2 s^{-1}) in a channel of width $h = 10$ cm, rotating at the angular velocity $\Omega = 5$ rad s^{-1}, leads to $\varepsilon = 0.1$ and $E = 10^{-5}$. Thus, the requirements of small ε and $E \leqslant O(\varepsilon^4)$ are both met.

Taken at face value, the requirement that $L/h \gg \varepsilon^{-2}$ implies a lateral scale $L \gg 10$ m, rather large from an experimental/DNS point of view. However, $L/h \gg \varepsilon^{-2}$ is an order-of-magnitude condition and a closer look shows that there is a purely numerical factor which makes this condition less restrictive. Consider a laterally circular channel of radius L. The time taken for wave modes in the channel to propagate from the lateral walls to the centre is L/c_g, where c_g is the radial group velocity. It can be shown, using (4.1), that the maximum non-dimensional group velocity, occurring for $n = 1$, $k = \pi/\sqrt{2}$, is only 0.12. Assuming that the time scale for nonlinear evolution of the wave component is well represented by ε^{-2} (as suggested by experience with the unconfined problem; see e.g. Bellet *et al.* (2006)), the condition for neglect of lateral confinement becomes $L/h \gg 0.12\varepsilon^{-2}$, i.e. $L \gg 1.2$ m for the present example. A radius of, say, 5 m should suffice, but represents a challenge for future experimental and DNS work. This issue should become clearer once numerical integration of (5.25) has been achieved, yielding a more precise value for the numerical factor in the order-of-magnitude estimate, ε^{-2}, of the wave-turbulence time scale for nonlinear evolution.

This example shows that the main problem facing experimental or DNS verification of the present theory would be to maintain a small Rossby number, while also

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press

avoiding effects of lateral confinement. Note that exactly the same difficulty arises in the classical problem of unbounded rotating turbulence, indeed it is there made worse by the need to accommodate a third dimension of non-confinement and the fact that higher non-dimensional group velocities arise.

REFERENCES

BATCHELOR, G. K. 1953 *The Theory of Homogeneous Turbulence*. Cambridge University Press.

BELLET, F., GODEFERD, F. S., SCOTT, J. F. & CAMBON, C. 2006 Wave turbulence in rapidly rotating flows. *J. Fluid Mech.* **562**, 83–121.

BOFFETTA, G. & ECKE, R. E. 2012 Two-dimensional turbulence. *Annu. Rev. Fluid Mech.* **44**, 427–451.

BOUROUIBA, L. 2008 Discreteness and resolution effects in rapidly rotating turbulence. *Phys. Rev.* E **78**, 056309.

CAMBON, C. & JACQUIN, L. 1989 Spectral approach to non-isotropic turbulence subjected to rotation. *J. Fluid Mech.* **202**, 295–317.

CAMBON, C., MANSOUR, N. N. & GODEFERD, F. S. 1997 Energy transfer in rotating turbulence. *J. Fluid Mech.* **337**, 302–332.

FRISCH, U. 1995 *Turbulence*. Cambridge University Press.

GALTIER, S. 2003 Weak inertial-wave turbulence theory. *Phys. Rev.* E **68**, 015301.

KARTASHOVA, E. A. 1994 Weakly nonlinear theory of finite-size effects in resonators. *Phys. Rev. Lett.* **72**, 2013–2016.

KRAICHNAN, R. H. & MONTGOMERY, D. 1980 Two-dimensional turbulence. *Rep. Prog. Phys.* **43**, 547–619.

LVOV, Y. V., NAZARENKO, S. & POKORNI, B. 2006 Discreteness and its effects on water-wave turbulence. *Physica* D **218**, 24–35.

NAZARENKO, S. 2011 *Wave Turbulence*. Springer.

NEWELL, A. C. & RUMPF, B. 2011 Wave turbulence. *Annu. Rev. Fluid Mech.* **43**, 59–78.

PUSHKAREV, A. N. 1999 On the Kolmogorov and frozen turbulence in numerical simulation of capillary waves. *Eur. J. Mech. (*B/*Fluids)* **18**, 345–351.

RACZ, J.-P. & SCOTT, J. F. 2008 Parametric instability in a rotating cylinder of gas subject to sinusoidal axial compression. Part 1. Linear theory. *J. Fluid Mech.* **595**, 265–290.

SMITH, L. M. & WALEFFE, F. 1999 Transfer of energy to two-dimensional large-scales in forced, rotating three-dimensional turbulence. *Phys. Fluids* **11**, 1608–1622.

TEITELBAUM, T. & MININNI, P. D. 2012 Decay of Batchelor and Saffman rotating turbulence. *Phys. Rev.* E **86**, 066320.

TURNER, L. 1999 Macroscopic structures of inhomogeneous, Navier–Stokes turbulence. *Phys. Fluids* **11**, 2367–2380.

WALEFFE, F. 1993 Inertial transfers in the helical decomposition. *Phys. Fluids* **A5**, 677–685.

YANG, Y.-T., SU, W.-D. & WU, J.-Z. 2010 Helical-wave decomposition and applications to channel turbulence with streamwise rotation. *J. Fluid Mech.* **662**, 91–122.

ZAKHAROV, V. E., KOROTKEVICH, A. O., PUSHKAREV, A. N. & DYACHENKO, A. I. 2005 Mesoscopic Wave Turbulence. *J. Expl Theor. Phys. Lett.* **82** (8), 487–491.

ZAKHAROV, V. E., L'VOV, V. S. & FALKOVICH, G. 1992 *Kolmogorov Spectra of Turbulence I*. Springer.

https://doi.org/10.1017/jfm.2013.652 Published online by Cambridge University Press