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Self-approaching curves*

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Abstract

We present a new class of curves which are self-approaching in the following sense. For any three consecutive points a, b, c on the curve the point b is closer to c than a to c. This is a generalisation of curves with increasing chords which are self-approaching in both directions. We show a tight upper bound of $5{\cdot}3331\ldots$ for the length of a self-approaching curve over the distance between its endpoints.

1. *Introduction*

Let C be an oriented curve in the plane. We call C self-approaching, if and only if for any three consecutive points a, b, c in oriented order on C, the inequality

$$d(a, c) \geqslant d(b, c)$$

holds. In other words, while walking along a self-approaching curve, one gets closer to each point of the curve that has not yet been passed.

In this paper we study the following question. Is this property strong enough to ensure that the detour a self-approaching curve makes while running from a to b, i.e. the ratio of the length of the connecting curve segment over the Euclidian distance from a to b, can be bounded, independently of a, b, and the curve C?

This problem has recently arisen in computational geometry, in analysing the performance of an on-line navigation strategy for a mobile robot; see [**3**].

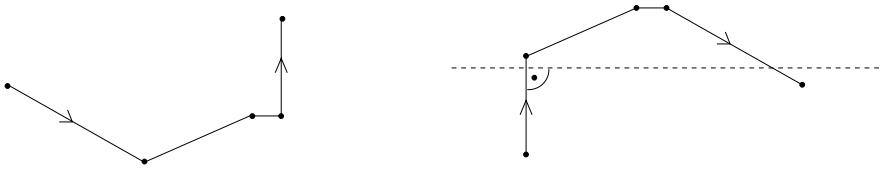
There is an interesting connection between self-approaching curves and curves with increasing chords that are defined by the property

$$d(a, d) \geqslant d(b, c)$$

for any four consecutive curve points a, b, c, d. Namely, a curve has increasing chords if and only if it is self-approaching in both directions. In [**2**] the problem analogous to ours has been posed for curves with increasing chords. A solution has been provided by Rote in [**4**]. He cuts a curve with increasing chords into small pieces, sorts them by slope and shows that the reassembled curve has again increasing chords. As it is also convex, by construction, its length can easily be bounded. This way, Rote shows

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Self-approaching Not self-approaching

Fig. 1. If the segments of a self-approaching polygonal line are sorted by slope, the result need not remain self-approaching.

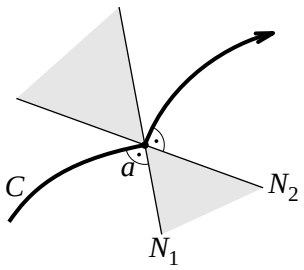


Fig. 2. The bundle of lines defined as normals to C at a.

that $2\pi/3 \approx 2{\cdot}094$ is the sharp upper bound for the detour of planar curves with increasing chords.

Unfortunately, Rote's method cannot be applied to self-approaching curves, because this weaker property is, in general, not preserved if segments are sorted by slope; see Fig. 1 for an example.

In fact, establishing a sharp upper bound for the detour of self-approaching curves seems to be more complicated.

In this paper, we proceed as follows. First, in Theorem 4 we prove that the length of a self-approaching curve cannot exceed the perimeter of its convex hull. Next, we circumscribe a self-approaching curve by a simple, closed convex curve whose length can be easily computed. This yields an upper bound for the perimeter of the convex hull; see Theorem 5. Finally in Theorem 6, we demonstrate that the resulting bound of $5{\cdot}3331\ldots$ is in fact the sharp upper bound for the detour of self-approaching curves.

2. *Definitions and properties*

The curves considered here are assumed to be piecewise smooth curves in the plane. For a curve C and a point a inside a smooth piece of C, the tangent to C at a and the normal to C at a, which is perpendicular to the tangent, are uniquely determined. Let a be a point of C such that two smooth pieces of C meet at a. The two normals N_1 and N_2 to the corresponding smooth pieces at a define a set of lines, each line of this set is regarded as a normal to C at a; see Fig. 2.

Let $d(a, b)$ denote the Euclidean distance between two points a and b. For two points $a \leqslant b$ ($a < b$) on a directed curve C, $C^{\geqslant a}$ ($C^{>a}$) denotes the part of C from a to the end (without a), $C[a, b]$ means the part of C between a and b, and length $(C[a, b])$ means its arc length.

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Definition 1. An oriented curve is called self-approaching if the inequality

$$d(a, c) \geqslant d(b, c)$$

is fulfilled for any three consecutive points a, b, c on the curve.

Let C be an oriented curve from a to b. Then the quantity

$$\frac{\text{length}\,(C[a, b])}{d(a, b)}$$

is called the detour of a curve from a to b.

The detour of a curve from a to b is the reciprocal of the minimum growth rate used in [**4**].

First, we give two equivalent definitions for self-approaching curves. The following lemma shows that the self-approaching property is equivalent to the fact that for any point a on the curve the rest of the curve lies fully on one side of any normal to C at a. We call this the normal property.

Lemma 2. *An oriented curve C is self-approaching if and only if any normal to C at any point a does not cross $C^{>a}$.*

Proof. The normal property means that in point a we move closer or maintain the distance to every point in $C^{>a}$. This property holds continuously, so for any three consecutive points the self-approaching property holds.

If the normal property is not fulfilled then there exists a point a such that a normal to C at a crosses $C^{>a}$ in c'. Then in a we move away from some points in $C^{>c'}$. So there are points $b \in C^{>a}$ and $c \in C^{>c'}$ for which the self-approaching property is not true. □

Now we give another equivalent definition of the self-approaching property which we call the right angle property.

Lemma 3. *An oriented curve C is self-approaching if and only if for any point a on C there is a right angle at point a which contains $C^{\geqslant a}$.*

Proof. Let a be a point on a self-approaching curve C and consider the two tangents T_1 and T_2 from a to $C^{\geqslant a}$ such that $C^{\geqslant a}$ lies between T_1 and T_2 which span an angle φ. If one of T_1 and T_2 is a tangent to C at a then $\varphi \leqslant 90°$ follows from the normal property (Lemma 2), see Fig. 3(i). Otherwise there are two points $b, c \in C^{>a}$ with $b \in T_1$ and $c \in T_2$, see Fig. 3(ii). Let us assume that b appears before c on C. Then from Definition 1 we have $d(a, c) \geqslant d(b, c)$, in other words $\overline{bc}$ is not the largest edge of the triangle abc, which means that $\varphi \leqslant 90°$.

If a curve C is not self-approaching, then due to Lemma 2 the normal property is not fulfilled, i.e. there is a point a such that a normal to C at a crosses $C^{>a}$ in c'. Then in a we move away from some points in $C^{>c'}$. So there are points $b \in C^{>a}$ and $c \in C^{>c'}$ for which the right angle property is not true.

Example. The logarithmic spiral, directed to the center, is an interesting example for a self-approaching curve. In polar coordinates it is the set of all points $(\varphi, e^{\varphi \cot \alpha})$

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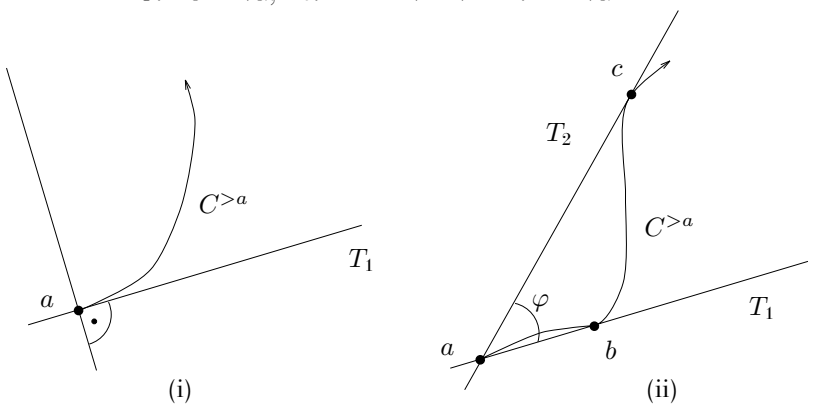


Fig. 3. For any point a on C, the rest of the curve is included in a right angle at a.

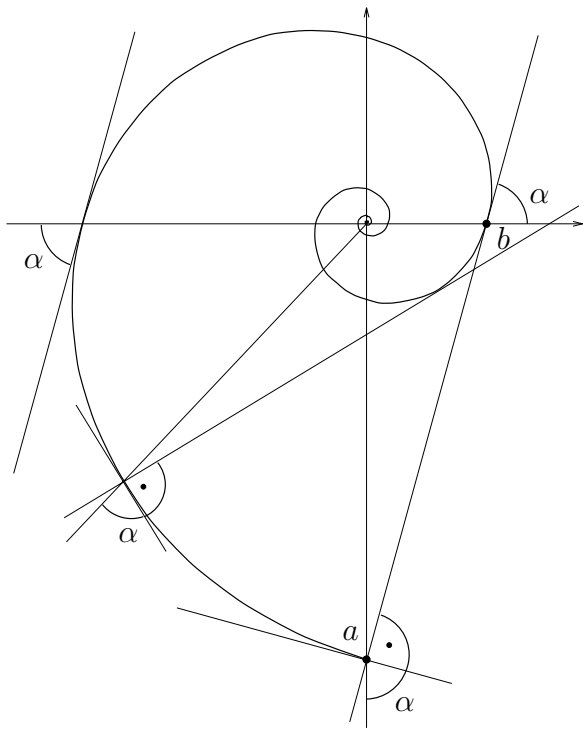


Fig. 4. The narrowest self-approaching logarithmic spiral.

with constant angle $\alpha < 90°$ between the tangent and the radius to each point on the curve, see Fig. 4. It is self-approaching if α fulfils

$$\alpha \leqslant \arctan\left(\frac{3\pi}{2W\left(\frac{3}{2}\pi\right)}\right) \approx 74{\cdot}66°$$

in which W denotes Lambert's W function [**1**] defined by the functional equation $W(x)\,e^{W(x)} = x$. Fig. 4 shows the limiting case where the normal at any point is tangent to the rest of the curve.

This special curve is in a sense the narrowest self-approaching logarithmic spiral.

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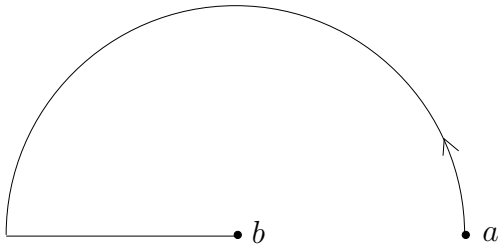


Fig. 5. A simple self-approaching curve with a detour of $\pi + 1$.

One can show that its detour equals $1/\cos\alpha_{\max} \approx 3{\cdot}78$, but despite its optimized form there are other self-approaching curves with a bigger detour. For example, the simple curve shown in Fig. 5 has a detour of $\pi + 1$.

However, there is something interesting about the logarithmic spiral. Let us suppose that we fix a string at point b of Fig. 4 and attach a pencil at point a. Now we move the pencil clockwise holding the string taut. Then the pencil draws the spiral while the string wraps around the inner part of the spiral. (Therefore this curve is its own involute.) This implies that the string is of the same length as the inner part of the spiral. Consequently, the total length of the spiral equals the perimeter of its convex hull. This fact will now be generalized to arbitrary self-approaching curves.

3. *Analysing the detour of self-approaching curves*

First we show that the length of self-approaching curves are bounded by the perimeter of their convex hull. Then we estimate the perimeter of their convex hull and prove, by giving an example, that the bound is tight.

Let $\mathrm{ch}(C)$ denote the convex hull of a curve C and $\mathrm{per}(C)$ the length of the perimeter of $\mathrm{ch}(C)$. For two points a and b let $R(a,b)$ denote the ray starting at a and passing through b.

THEOREM 4. *The length of a self-approaching curve C is less than or equal to the perimeter,* $\mathrm{per}(C)$*, of its convex hull.*

Proof. The length of a curve C is, by definition, the supremum of the lengths of all polygonal chains with vertices on C in the same order as they appear on C. Therefore, an upper bound for the length of all such chains is also an upper bound for the length of C.

We take an arbitrary polygonal chain Q whose vertices lie on C in the same order. By induction on the number of vertices of Q, we will prove that Q is shorter than the perimeter, $\mathrm{per}(Q)$, of its convex hull, $\mathrm{ch}(Q)$, which in turn is bounded by $\mathrm{per}(C)$. Note that the vertices of $\mathrm{ch}(Q)$ are also vertices of Q and are therefore points on C.

The assertion is true for Q being a line segment, so let us assume that Q has at least three vertices, the first two are called a and b. The induction hypothesis is that $\mathrm{length}(Q^{\geqslant b}) \leqslant \mathrm{per}(Q^{\geqslant b})$.

We distinguish two cases depending on whether b lies on the boundary of $\mathrm{ch}(Q)$ or not.

Case 1. The point b is on the boundary of $\mathrm{ch}(Q)$. We have a situation as depicted in Fig. 6. When passing from $\mathrm{ch}(Q^{\geqslant b})$ to $\mathrm{ch}(Q)$, the convex hull changes as follows.

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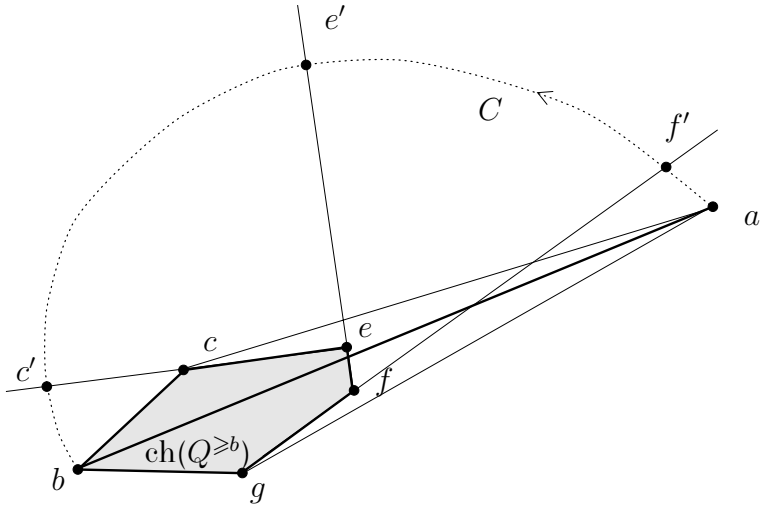


Fig. 6. $d(c,a) + d(g,a) \geqslant d(g,f) + d(f,e) + d(e,c) + d(a,b)$.

The segments $\overline{gf}$, $\overline{fe}$ and $\overline{ec}$ which belong to $\mathrm{ch}\,(Q^{\geqslant b})$ are replaced by the segments $\overline{ca}$ and $\overline{ga}$. Since $\mathrm{length}\,(Q^{\geqslant b}) + d(a,b) = \mathrm{length}\,(Q)$ it suffices to prove that

$$d(c,a) + d(g,a) \geqslant d(g,f) + d(f,e) + d(e,c) + d(a,b).$$

We do not know which way C takes from a to b but there are either points $f' \in R(g,f)$, $e' \in R(f,e)$ and $c' \in R(e,c)$ in exactly this order on $C[a,b]$ or there are points $c' \in R(c,e)$, $e' \in R(e,f)$ and $f' \in R(f,g)$ in this order on $C[a,b]$. Without loss of generality we assume the first case.

While the curve C moves from a to f' it gets closer to f because f' arises before f on C. Therefore

$$d(g,a) \geqslant d(g,f') = d(g,f) + d(f,f').$$

By the same argument C gets closer to f while it runs from f' to e'. Therefore

$$d(f,f') \geqslant d(f,e') = d(f,e) + d(e,e').$$

Similarly we have $d(e,e') \geqslant d(e,c) + d(c,c')$ and also $d(c,c') \geqslant d(c,b)$. Altogether we conclude

$$d(c,a) + d(g,a) \geqslant d(g,f) + d(f,e) + d(e,c) + d(c,b) + d(c,a).$$

The triangle inequality $d(c,b) + d(c,a) \geqslant d(a,b)$ finishes the proof. This also works for $c = b$ or $g = b$.

Notice that this generalizes to any number of vertices of $\mathrm{ch}\,(Q^{\geqslant b})$, instead of e, f and c.

Case 2. The point b is not on the boundary of $\mathrm{ch}\,(Q)$. Then a must lie in the wedge included between the prolongations of the adjacent edges of $\mathrm{ch}\,(Q^{\geqslant b})$ at b, see Fig. 7. The neighbouring vertices of b in $\mathrm{ch}\,(Q^{\geqslant b})$ are called c and e. The angle φ between $\overline{bc}$ and $\overline{be}$ is less than or equal to 90° because of the right angle property (Lemma 3).

We assume a situation as in Fig. 7 where the points b and e of $\mathrm{ch}\,(Q^{\geqslant b})$ are not on the boundary of $\mathrm{ch}\,(Q)$. Let p be the intersection point of $\overline{ab}$ and $R(e,f)$. Using

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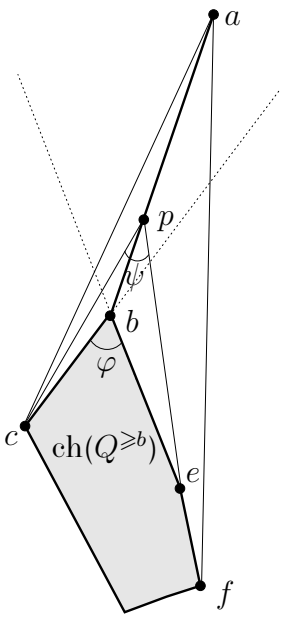


Fig. 7. $d(c,p) + d(e,p) \geqslant d(b,p) + d(b,c) + d(b,e)$.

the induction hypothesis and the fact $\varphi \leqslant 90°$ we first show that length$(Q^{\geqslant p})$ is not greater than per$(Q^{\geqslant p})$. Passing from ch$(Q^{\geqslant b})$ to ch$(Q^{\geqslant p})$, the convex hull changes as follows. The segments $\overline{bc}$ and $\overline{be}$ which belong to ch$(Q^{\geqslant b})$ are replaced by the segments $\overline{pc}$ and $\overline{pe}$ of ch$(Q^{\geqslant p})$. Since length$(Q^{\geqslant p})$ is equal to length$(Q^{\geqslant b}) + d(b,p)$ it is sufficient to show that

$$d(c,p) + d(e,p) \geqslant d(b,p) + d(b,c) + d(b,e)$$

holds, this is exactly the conclusion of Lemma 7 in the appendix below; the proof is elementary but lengthy.

Since we can use the assumption that length$(Q^{\geqslant p}) \leqslant$ per$(Q^{\geqslant p})$ and the fact $\psi \leqslant \varphi \leqslant 90°$ the same argument holds also for $Q^{\geqslant a} = Q$ and also for the case that more vertices of ch$(Q^{\geqslant b})$ than only b and e do not reappear as vertices in ch(Q). This concludes the proof. □

In the following, for two points p and q let $\mathrm{circ}_p(q)$ denote the circle with centre p passing through q.

As an immediate consequence of Theorem 4, we have an upper bound of 2π for the detour of self-approaching curves, because any such curve from point a to point b must be contained in $\mathrm{circ}_b(a)$. The following theorem refines this argument to a smaller bound, which will be shown to be tight afterwards in Theorem 6.

Theorem 5. *The perimeter of the convex hull of a self-approaching curve is not greater than*

$$c_{\max} := \max_{\beta \in [0..\frac{1}{2}\pi]} \frac{2\beta + \pi + 2}{\sqrt{(5 - 4\cos\beta)}} \approx 5{\cdot}3331\ldots$$

times the distance of its endpoints.

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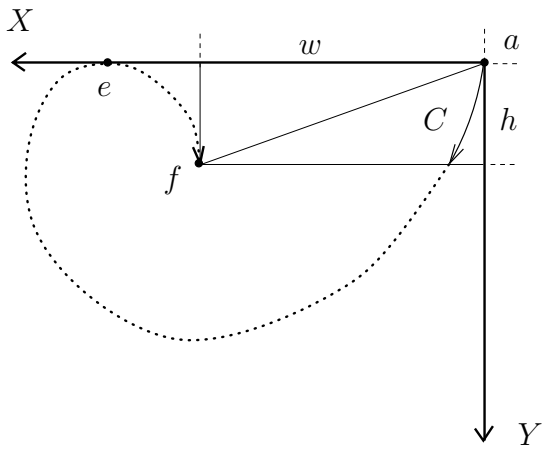


Fig. 8. Any self-approaching curve is contained in a wedge of 90°.

Proof. Let a and f denote the first and final points of a self-approaching curve C. The proof works as follows: We show that $\mathrm{per}(C) \leqslant c_{\max} d(a, f)$ holds if the curve does *not* cross the line segment $\overline{af}$. If it does, we apply this bound for each subcurve between two successive curve points on $\overline{af}$ and add up the length. Due to the self-approaching property, the curve points on $\overline{af}$ appear in the same order as on C. Then we know that the sum of the perimeters of all such subcurves is less than $c_{\max}$ times the distance between a and f. Now consider two subsequent subcurves C_1 and C_2. The two convex sets $\mathrm{per}(C_1)$ and $\mathrm{per}(C_2)$ intersect at least in one point and therefore we conclude $\mathrm{per}(C_1 \cup C_2) \leqslant \mathrm{per}(C_1) + \mathrm{per}(C_2)$. This argument can be applied successively, so the perimeter of the whole curve is smaller than the sum of the perimeters of all the subcurves which in turn is smaller than $c_{\max}$ times the distance between a and f.

So for the rest of this section we may assume that C does not cross the line segment $\overline{af}$. Because of the right angle property(Lemma 3), the whole of C lies between two orthogonal halflines X and Y starting at a. Without loss of generality we assume that the initial part of C lies on the left side of the edge directed from a to f, and, if necessary, we rotate X and Y such that the halfline on the other (right) side of $\overline{af}$ touches C at a point e, see Fig. 8. Let h and w be the height and width of the bounding box of a and f according to the rectangle with sides parallel to X and Y and with diagonal $\overline{af}$.

We will construct a convex area, A, that contains C and we will show that the perimeter of this area divided by $d(a, f) = \sqrt{(h^2 + w^2)}$ is bounded by $c_{\max}$.

This construction goes as follows, refer to Fig. 9. First we know that C is contained in the right angle at a, and $C[a, e]$ is contained in $\mathrm{circ}_e(a)$ and $C[e, f]$ is contained in $\mathrm{circ}_f(e)$, due to the self-approaching property.

We also know that $C[a, e]$ must avoid $\mathrm{circ}_f(e)$, but pass around it to reach e because it must not cross $\overline{af}$. We conclude that $\mathrm{circ}_f(e)$ is contained in $\mathrm{circ}_e(a)$.

Now, we will enlarge these circles to a certain extent. Instead of e, we use a point e' on X with $d(a, e') \geqslant d(a, e)$, such that $\mathrm{circ}_{e'}(a)$ still contains $\mathrm{circ}_f(e')$ and touches it in one point c'. This is possible because for every position of e' on X with $d(a, e') \geqslant d(a, e)$, the whole circle $\mathrm{circ}_{e'}(a)$ is always on one side of Y, while $\mathrm{circ}_f(e')$ must

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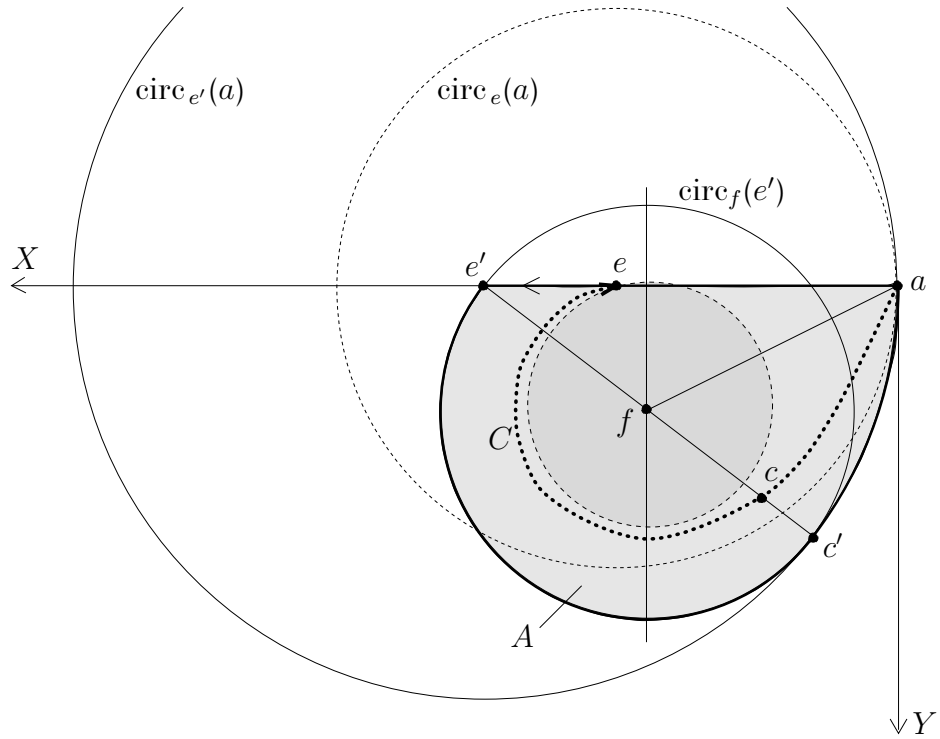


Fig. 9. Curve C is contained in the shaded area A.

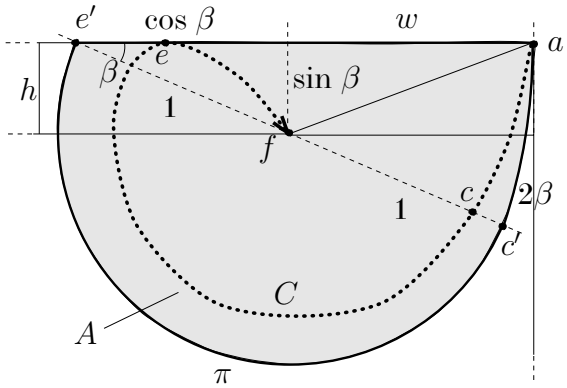


Fig. 10. Measuring the perimeter of the convex area A.

eventually cross Y. Note that $d(e', a) = d(e', c') = d(c', f) + d(f, e') = 2d(f, e')$ holds, in other words the radius of circ $_{e'}(a)$ equals the diameter of circ $_f(e')$.

Now let c be the point at which curve C crosses $\overline{fc'}$ first. We know that $C^{\leqslant c}$ is contained in circ $_e(a) \subseteq$ circ $_{e'}(a)$, $C^{\geqslant c}$ is contained in circ $_f(c) \subseteq$ circ $_f(c') =$ circ $_f(e')$. So the curve C is included in the convex area A limited by $\overline{e'a}$, the circular arc from a to c' about e', and the halfcircle from c' to e' about f, see Fig. 9.

We choose a scale such that $d(f, c') = 1$. Let β be the angle between $\overline{e'a}$ and $\overline{e'c'}$, see Fig. 10. Then $d(e', c') = 2 = d(e', a)$, $w = 2 - \cos\beta$ and $h = \sin\beta$. The length of the circular arc from a to c' equals 2β while the halfcircle from c' to e' has length π.

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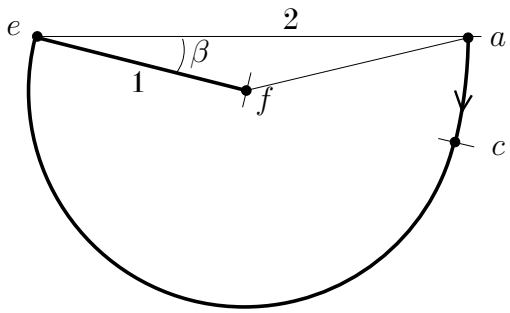


Fig. 11. This self-approaching curve has a detour of 4·38.

Then the perimeter of the constructed convex area A divided by $d(a, f) = \sqrt{(h^2 + w^2)}$ is given by the following function g in β.

$$g(\beta) := \frac{2\beta + \pi + 2}{\sqrt{[\sin^2 \beta + (2 - \cos \beta)^2]}} = \frac{2\beta + \pi + 2}{\sqrt{(5 - 4\cos \beta)}}$$

The maximum value, $c_{\max}$, of $g(\beta)$ equals aproximately $5{\cdot}3331\ldots$ for $0 \leqslant \beta \leqslant \pi/2$ and is attained at $\beta_{\text{opt}} \approx 11{\cdot}22^\circ$ with $(2\beta_{\text{opt}} + \pi + 2)\sin \beta_{\text{opt}} = 5 - 4\cos \beta_{\text{opt}}$. □

Surprisingly, it turns out that there cannot be a smaller upper bound for self-approaching curves.

Theorem 6. *The constant $c_{\max}$ is a tight upper bound for the detour of self-approaching curves.*

Proof. That $c_{\max}$ is an upper bound for the detour of self-approaching curves follows directly from Theorems 4 and 5. To prove tightness we construct a curve with a convex hull similar to the bounding area A in Theorem 5.

As a first step we consider the curve in Fig. 11. From the start point, a, to the end, f, it consists of a circular arc of radius 2 and angle β, a half circle of radius 1, and a line segment of length 1. This curve is self-approaching, its length equals $2\beta + \pi + 1$ while $d(a, f) = \sqrt{(5 - 4\cos \beta)}$. The ratio takes on a maximum value of approximately 4·38.

But there is some room for improvement in the last step, i. e. the line segment from e to f. Instead of walking straight from e to f we use a sequence of pieces of small cycloids. (A cycloid is known to be the orbit of a point on the boundary of a rolling circle and it has another cycloid as its involute.) For an odd number $n \in \mathbb{N}$ we can fill a rectangle of height h and width $w = 2nh/\pi$ with n successive congruent pieces of cycloids such that they form a curve from the lower left to the upper right corner, see Fig. 12. Each piece is a cycloid generated by a circle of radius h/π rolling on a vertical line, and each one is the involute of its successor. The resulting curve is self-approaching, and its length is exactly $2w$ since the length of a piece is twice its width.

Now let us replace the line segment in our first step by such a construction in a rectangle of height $h = 1/[(2n/\pi) + 1]$ and width $w = 1 - h = 2nh/\pi$, see Fig. 13.

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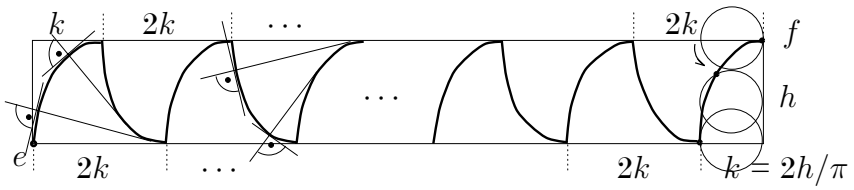


Fig. 12. How to fill a rectangle of width w and height h with a self-approaching curve of length $2w$ using pieces of cycloids.

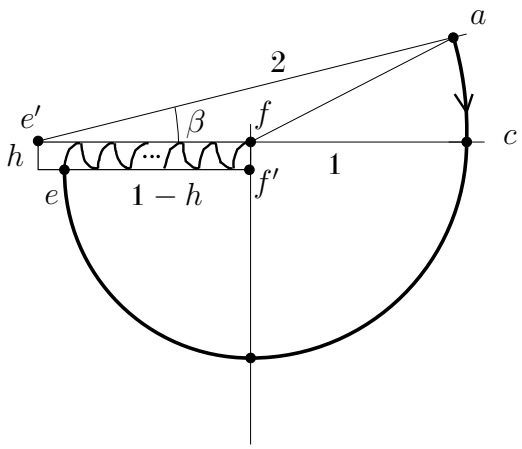


Fig. 13. Replacing the line segment by a rectangle that gives room for improvement.

The curve consists of a circular arc of radius 2 about e' of length 2β, a quartercircle of radius 1 about f of length $\pi/2$, a quartercircle of radius $(1-h)$ about f' of length $(1-h)\pi/2$ and a sequence of cycloids from e to f of overall length $2(1-h)$.

Now we choose the constant $\beta = \beta_{\text{opt}}$ from Theorem 5 and for $n = 1, 3, 5\ldots$ we have a sequence of self-approaching curves with a detour of at least

$$\frac{2\beta_{\text{opt}} + \pi/2 + (1-h)(\pi/2+2)}{\sqrt{(5 - 4\cos\beta_{\text{opt}})}}$$

which converges to $c_{\max}$ for $n \to \infty$ (i. e. $h \to 0$). □

Actually, as n tends to infinity, one could think of the curve of Fig. 12 as a thick line segment of length 2, while its endpoints are only distance 1 apart. One might wonder if a factor bigger than 2 can be achieved by a different technique. Note that this is not possible as a direct consequence of Theorem 5. Also note that there are many other ways of constructing the self-approaching thick line segment of length 2.

Appendix

LEMMA 7. *Let v be a point inside a triangle abc. We connect each vertex to v using segments $l_1 = \overline{bv}$, $r_1 = \overline{cv}$ and $z = \overline{av}$. Let $l_2 = \overline{ab}$ and $r_2 = \overline{ac}$ be two edges of the triangle; see Fig.* 14. *If the angle $\varphi \leqslant \pi$ between l_1 and r_1 is less than or equal to $\pi/2$ then for the lengths of the segments $l_1 + r_1 + z \leqslant l_2 + r_2$ holds.*

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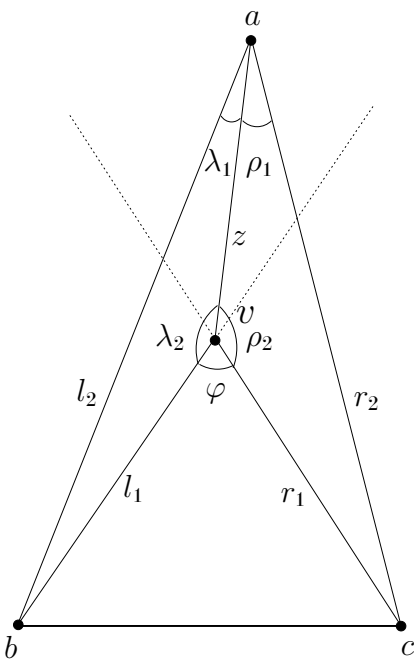


Fig. 14. $l_1 + r_1 + z \leqslant l_2 + r_2$ holds for $\varphi \leqslant \pi/2$.

Proof. The assumption is obviously true for $z = 0$. Let $z \neq 0$. We have to prove the inequality $l_2 - l_1 + r_2 - r_1 \geqslant z$. Let $\lambda_1 \leqslant \pi$ be the angle between l_2 and z, $\lambda_2 \leqslant \pi$ be the angle between l_1 and z, $\rho_1 \leqslant \pi$ be the angle between r_2 and z and $\rho_2 \leqslant \pi$ be the angle between r_1 and z. Using the law of sines we substitute l_1 with $z \sin \lambda_1/[\sin(\lambda_1 + \lambda_2)]$ and l_2 with $z \sin \lambda_2/[\sin(\lambda_1 + \lambda_2)]$. We transform r_1 and r_2 analogously and divide the whole expression by z. So we have to show that

$$\frac{\sin \lambda_2 - \sin \lambda_1}{\sin(\lambda_1 + \lambda_2)} + \frac{\sin \rho_2 - \sin \rho_1}{\sin(\rho_1 + \rho_2)} \geqslant 1$$

is true. We consider some simple transformations:

$$\begin{aligned}
\frac{\sin \lambda_2 - \sin \lambda_1}{\sin(\lambda_1 + \lambda_2)} &= \frac{\sin \lambda_2 - \sin[(\lambda_1 + \lambda_2) - \lambda_2]}{\sin(\lambda_1 + \lambda_2)} \\
&= \frac{\sin \lambda_2 - \sin(\lambda_1 + \lambda_2)\cos \lambda_2 + \cos(\lambda_1 + \lambda_2)\sin \lambda_2}{\sin(\lambda_1 + \lambda_2)} \\
&= -\cos \lambda_2 + \underbrace{\frac{[1 + \cos(\lambda_1 + \lambda_2)]\sin \lambda_2}{\sin(\lambda_1 + \lambda_2)}}_{(*)} \\
&\geqslant -\cos \lambda_2
\end{aligned}$$

Note that $(*) \geqslant 0$ holds because of $0 \leqslant \lambda_1 + \lambda_2 \leqslant \pi$. Similarly we have $(\sin \rho_2 - \sin \rho_1)/[\sin(\rho_1 + \rho_2)] \geqslant -\cos \rho_2$ so it is sufficient to show that $-\cos \lambda_2 - \cos \rho_2 \geqslant 1$ is true. We conclude $(\lambda_2 + \rho_2)/2 = \pi - \varphi/2$ from $\lambda_2 + \rho_2 + \varphi = 2\pi$. Since λ_2 and ρ_2 are inner angles of a triangle we know $\lambda_2, \rho_2 \leqslant \pi$ and from $\varphi \leqslant \pi/2$ we

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conclude $\lambda_2, \rho_2 \geqslant \pi/2$. Therefore $|\lambda_2 - \rho_2| \leqslant \pi/2$ is true. Now

$$-\cos\lambda_2 - \cos\rho_2 = -2\cos\left(\frac{\lambda_2+\rho_2}{2}\right)\cos\left(\frac{|\lambda_2-\rho_2|}{2}\right) \geqslant 1$$

holds since $-\cos[(\lambda_2+\rho_2)/2] = \cos(\varphi/2) \geqslant \cos(\pi/4) = \frac{1}{\sqrt{2}}$ and $\cos[(|\lambda_2-\rho_2|)/2] \geqslant \frac{1}{\sqrt{2}}$ are fulfilled.

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