Cayley's problem of seven lines on a quartic surface. By H. D. URSELL, B.A., Trinity College.

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1. The following investigations were undertaken in connection with Cayley's problem of seven lines on a quartic surface*.

The condition that a straight line joining two points (x_i), (y_i) should lie wholly on a quadric $P(x)=0$ is of the form

$$\lambda^2 P_{xx} + 2\lambda\mu P_{xy} + \mu^2 P_{yy} = 0, \qquad \text{(all } \lambda, \mu\text{)}$$

or, as we may also write it (using the summation convention),

$$a_{ij}x_ix_j = a_{ij}x_iy_j = a_{ij}y_iy_j = 0,$$

where $a_{ij}=a_{ji}$ are the coefficients in P_{xx}. These conditions will hold if, for each k, we have

$$x_k \,.\, a_{ij}x_iy_j = y_k \,.\, a_{ij}x_ix_j, \quad x_k \,.\, a_{ij}y_iy_j = y_k \,.\, a_{ij}x_iy_j,$$

or if

$$a_{ij}x_i(jk)=0, \quad a_{ij}y_j(ik)=0,$$

where (jk) denotes the line coordinate $(x_jy_k - x_ky_j)$. Since $a_{ij}=a_{ji}$, we have

$$a_{ij}x_i(jk)=0, \quad a_{ij}y_i(jk)=0.$$

Treating this pair of equations in the same way we find

$$a_{ij}(il)(jk)=0 \quad \text{.....(1).}$$

The equations so found, one for each value of (k, l), give the necessary and sufficient conditions that a straight line should lie on a quadric. They are the *equations of the quadric in line coordinates* (in any number of dimensions).

In the same way we may show that the equations of the quadric in plane coordinates are

$$a_{ij}(imn)(jpq)=0 \qquad \text{(each } m, n, p, q\text{)}.$$

Similarly for flat spaces of any dimensionality. Moreover the analysis applies, not only to quadrics, but to primals of any order (including primes). The equations in plane coordinates of a cubic primal are

$$a_{ijk}(ii'i'')(jj'j'')(kk'k'')=0 \qquad \text{(all } i', \ldots, k'') \text{ ...(2).}$$

Example. The quadric

$$a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 = 0$$

has the line-equations

$$(k=l=1) \qquad a_2(12)^2 + a_3(13)^2 + a_4(14)^2 = 0, \text{ etc.},$$

$$(k=1, l=2) \qquad a_3(13)(23) + a_4(14)(24) = 0, \text{ etc.}$$

* See Wakeford, *Proc. Lond. Math. Soc.* (2) 21, 1922, 98–113; and Prof. Baker's commentary, *ibid.* 114–133.

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The equations of the second type give

$$\frac{(13)\sqrt{a_1a_3}}{(24)\sqrt{a_2a_4}} = \frac{(14)\sqrt{a_1a_4}}{(32)\sqrt{a_2a_3}} = \frac{(12)\sqrt{a_1a_2}}{(43)\sqrt{a_3a_4}} = \frac{(24)\sqrt{a_2a_4}}{(13)\sqrt{a_2a_3}},$$

and so $(13)\sqrt{a_1a_3} = (24)\sqrt{a_2a_4} = l$, say; .....; .....;

and now the equations of the first type all reduce to $l^2 + m^2 + n^2 = 0$, which is also the condition

$$(13)(24) + (12)(43) + (14)(32) = 0 \quad \text{.....(3).}$$

2. In the representation of lines of [3]-space as points of a quadric

$$ll' + mm' + nn' = 0 \quad \text{.....(3')}$$

lying in a [5], a point of the [3]-space is represented by a plane in [5] which lies wholly on this quadric. If the point is (x, y, z, t) the line coordinates of a straight line through it are the determinants of order 2 of some array

$$\left|\left| \begin{matrix} x & y & z & t \\ x_1 & y_1 & z_1 & t_1 \end{matrix} \right|\right|.$$

The plane coordinates of the plane in [5] corresponding to (x, y, z, t) are proportional to determinants such as $|\, l_1 m_2 n_3 \,|$, the suffixes referring to three general lines through (x, y, z, t). If we substitute the values of $l_i, \ldots$ in terms of (x, y, z, t) and (x_i, y_i, z_i, t_i), the result in each case must contain a factor $|\, x y_1 z_2 t_3 \,|$, since when this vanishes the three points in [5] of coordinates $(l_i, m_i, n_i, l_i', m_i', n_i')$ $(i = 1, 2, 3)$ are in line. Removing this constant of proportionality, we see that the plane coordinates of the plane corresponding to (x, y, z, t) are proportional to *quadric functions of the coordinates* (x, y, z, t). Actually we find, if we return to a suffix notation,

$$[(23)(31)(12)] = 0,$$

$$[(14)(24)(34)] \propto x_4^2, \quad [(23)(14)(24)] \propto -x_2x_4.$$

If now we take coordinates in [5] so that the vertices of the simplex of reference are the representative points of six lines (1, 2, 3, 4, 5, 6) in the [3]-space, the coordinates (123) of a plane representing a point in [3] are proportional to the quadric functions of the coordinates of the point, whose vanishing represents the condition that the point lies on the quadric through the lines 4, 5, 6. For (123) is such a quadric function, and it vanishes if, and only if, the plane in [5] meets the plane (456), namely in a point of this plane on the quadric (3'); and the latter point corre-

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sponds to a generator of the same system as 4, 5 and 6 of the quadric in [3] which contains them. The identity

$$(123)(456)-(234)(156)+(341)(256)-(412)(356)=0 \ldots(4),$$

which is a typical identity holding between the coordinates of any plane in 5 dimensions (cf. (3) above), gives rise therefore to an identity between the quartic surfaces which consist of pairs of quadrics through complementary triads of the six lines in [3]. To prove (4), we pick out the coefficient of $(y_5z_6 - y_6z_5)$, which is

$$x_4(123)-x_1(234)+x_2(341)-x_3(412)=-\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}=0.$$

If we write for short

$$(ij)=(lmn)(ij6)=(ji),$$

where $(lmnij6)$ is an even permutation of (123456), then we obtain the following identities, analogous to (4), between the pairs of quadrics, by permuting 1, 2, 3, 4, 5.

$$(15)+(25)+(35)+(45)=0 \qquad \text{(v)},$$
$$(12)+(13)+(14)+(15)=0 \qquad \text{(i)},$$
$$(12)+(23)+(24)+(25)=0 \qquad \text{(ii)},$$
$$(13)+(23)+(34)+(35)=0 \qquad \text{(iii)},$$
$$(14)+(24)+(34)+(45)=0 \qquad \text{(iv)}.$$

Subtracting (iv) and (v) from the sum of (i), (ii) and (iii) we find

$$(12)+(13)+(23)-(45)=0.$$

Hence we may calculate (13), (35), (52), (24), (41) in terms of (12), (23), (34), (45), (51) by means of the identities (i) ... (v). Note also that the equation just found giving (13) is precisely that analogous to (4) in which (4, 5) take the place of (5, 6). It is easy to convince oneself that the plane equations of the quadric (3′) do not give any relations between the quantities (ij). Hence (12), (23), (34), (45), (51) (or any other set of five (ij) in which the i, j form a single cycle) form five linearly independent quartic surfaces through the six lines. The position of 6 is here special; if we had used 1 instead of 6, the five quartics would be described as (26), (45), (25), (23), (56).

3. I shall now find the explicit algebraical form of the condition that seven lines in [3] should lie on a quartic surface. This is the vanishing of a determinant Δ of 35 rows and columns. The first row of Δ consists of the quartic monomials of the coordinates

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(x_1, y_1, z_1, t_1) of a point on the first line: and the next four rows are obtained from the first by applying the operator

$$\vartheta_1 = x_1' \frac{\partial}{\partial x_1} + y_1' \frac{\partial}{\partial y_1} + z_1' \frac{\partial}{\partial z_1} + t_1' \frac{\partial}{\partial t_1}$$

one, two, three and four times, (x_1', y_1', z_1', t_1') being some other point of the line. In the same way five rows of Δ arise from each of the other six lines. The expansion of Δ depends only on the line coordinates

$$l_i = \begin{vmatrix} y_i & z_i \\ y_i' & z_i' \end{vmatrix}, \quad l_i' = \begin{vmatrix} x_i & t_i \\ x_i' & t_i' \end{vmatrix}, \quad \ldots\ldots$$

of the seven lines. It is of the tenth degree in x_i, y_i, z_i, t_i as also in x_i', y_i', z_i', t_i', hence also in $l_i, m_i, n_i, l_i', m_i', n_i'$.

Δ is *antisymmetric* in the seven lines: it changes sign when two of the lines are interchanged. From the representation of lines, points, and planes in [3] by points, planes α, and planes β of the quadric in [5], we see also that Δ is invariant under projective transformations of l, m, n, l', m', n' which leave the quadratic form (3′) unaltered. Thus Δ is expressible algebraically in terms of the mutual invariants of the seven lines

$$(ij) = l_i l_j' + l_i' l_j + m_i m_j' + m_i' m_j + n_i n_j' + n_i' n_j.$$

We show now that the quantities (ij) are subject to precisely one identity. Let us take coordinates in [5] so that the first six lines are the vertices of the simplex of reference (the non-vanishing coordinate being in each case unity). The seventh line we now suppose to have coordinates $(x_1, x_2, x_3, x_4, x_5, x_6)$. The quadratic form (3′) is now

$$\Omega \equiv a_{ij} x_i x_j,$$

where $a_{ij} = (ij)$ and $a_{ii} = (ii) = 0$.

Also $a_{ij} x_i = (j7) \qquad (j = 1, 2, 3, 4, 5, 6).$

If we regard the (ij) $(i, j = 1, \ldots, 7)$ as known, these equations give the values of x_i; in order that these may correspond to a point on the quadric it is necessary and sufficient that

$$(j7) x_j = a_{ij} x_i x_j = 0.$$

Thus we have $\Delta' \equiv |(ij)| = 0 \qquad (i, j = 1, 2, \ldots 7)$

as the necessary and sufficient condition that the quantities (ij) shall be the invariants of some set of seven lines.

To sum up, then, Δ is an antisymmetric function, of the tenth degree in the coordinates of each line, of the bilinear invariants (ij). By elementary considerations it contains the (symmetric) factor $\prod_{i \neq j} (ij)$. The residual factor, in which we are interested, we may also denote by Δ; it is antisymmetric and of the fourth degree in

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the coordinates of each line; and it is unique (in algebraic form), *mod* Δ'.

Now Δ' is symmetrical: and there is no antisymmetric function of the (ij) of degree 2 in each line. For the general term of any such function can involve no square $(12)^2$, since this would give symmetry in 1 and 2. It would then be of one of the forms

$$(12)(23)(31).(45)(56)(67)(74),$$

$$(12)(23)(34)(45)(56)(67)(71).$$

If in the first we interchange 1 and 2, we change the sign of the antisymmetric function: but the particular term written is unaltered. Similarly the second term written is invariant under the (odd) permutation of (123) with (765). Thus Δ has in fact a unique antisymmetric expression in terms of the (ij).

4. I shall now discuss the possible antisymmetric forms of the appropriate degree in the invariants (ij). We have to catalogue the (types of) products of (ij) of the right degree in each line, and reject all which are invariant under some odd permutation. Of course no term can have a factor $(12)^4$. Suppose that we have a factor $(12)^3$; then we must also suppose, for example, that there are factors (13) (24), as the remaining factors involving 1, 2 cannot have their other constituents the same. If we now add $(34)^3$ we get symmetry in 5, 6 and 7. Again, if we have

$(12)^3(13)(24)(34)^2(35)(45)$, we get symmetry in 6 and 7.

$(12)^3(13)(24)(34)^2(35)(46)$ gives $(56)(57)^2(67)^2$ and symmetry in the triads (135)(246),

$(12)^3(13)(24)(35)^3(45)$ gives $(46)(47)(67)^3$, and symmetry in 6 and 7,

$(12)^3(13)(24)(35)^3(56)$ gives $(47)^2(67)^2(46)$ and symmetry in the triads (124)(356),

$(12)^3(13)(24)(35)^2(34)$ gives $(45)(46)(57)(67)^3$ and symmetry in the triads (765)(123),

or else $(46)^2(57)^2(67)^2$ and symmetry in the triads (135)(246),

$(12)^3(13)(24)(35)^2(36)(45)^2$ gives $(47)(67)^3$ and symmetry in the triads (123)(764).

We write

$$S_1 = \Sigma \pm (12)^3(13)(24)(35)^2(36)(45)(46)(47)(57)(67)^2,$$

$$S_2 = \Sigma \pm (12)^3(13)(24)(35)^2(36)(45)(47)^2(56)(67)^2,$$

$$S_3 = \Sigma \pm (12)^3(13)(24)(35)^2(36)(46)^2(47)(67)(57)^2.$$

On the other hand $(12)^3$ (13) (24) $(35)^2$ (36) (46) $(47)^2$ gives (56)(57)(67) and symmetry in (135)(247).

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Thus S_1, S_2, S_3 exhaust the possible forms which contain a factor $(12)^3$, and, with $(13)(24)$, double factors in 3 or 4. But if 3. 4 occur again in single factors only, we have

$(12)^3(13)(24)(35)(36)(37)(45)(46)(47)$ and symmetry in 5, 6, and 7,

or else

$(12)^3(13)(24)(34)(35)(36)(45)(46)$ and symmetry in 5 and 6,

or else

$(12)^3(13)(24)(34)(35)(36)(45)(47)(67)^2(56)(57)$ and symmetry in $(136)(247)$.

We may henceforth reject all cubed factors. Suppose next that we have $(12)^2(13)^2(23)$. Then we may also suppose that we have $(24)(35)$. If we add a factor $(45)^2$ we get $(46)(57)(67)^3$; if we add (45) we get $(46)^2(57)^2(67)^2$ or $(46)(47)(56)(57)(67)^2$ which give symmetry in $(246)(357)$, as does also $(46)^2(47)(56)(57)^2(67)$, to which we are reduced in the absence of (45). Thus we can have no factor (23) with $(12)^2(13)^2$.

If we take $(12)^2(13)^2(24)^2(35)^2$ we get again either $(46)^2(57)^2(67)^2$ or $(46)(47)(56)(57)(67)^2$ and symmetry in the triads $(246)(357)$ [the latter case giving in fact symmetry in 6 and 7]. We take now

$(12)^2(13)^2(24)^2(35)(36)$:

$(45)^2$ gives $(57)(67)^3$,

$(45)(46)$ gives $(57)^2(67)^2$ and symmetry in 5 and 6,

$(45)(47)$ gives $(56)(57)(67)^2$ and symmetry in the triads $(136)(247)$,

$(47)^2$ gives $(56)^2(57)(67)$ and symmetry in 5 and 6.

Hence with $(12)^2(13)^2(24)^2$ we must have $(34)(35)$. If however we add (45) we get $(56)(57)(67)^3$, and if we add (46) we get $(56)(57)^2(67)^2$, which is of the form $(23)(21)^2(31)^2$, already excluded.

We are reduced to $(12)^2(13)^2(24)(25)$, and 3 also must appear again in single factors. If we add $(36)(37)$ then we get either $(45)^2(46)(57)(67)^2$ or $(45)(46)^2(57)^2(67)$ or $(46)^2(47)(56)(57)^2$, all of which give symmetry in $(245)(367)$. Now

$$(12)^2(13)^2(24)(25)(34)(36)\ (45)(46)(57)^2(67)^2$$

gives symmetry in $(123)(765)$, while $(45)^2$ gives $(57)(67)^3$. We then write

$$S_4 = \Sigma \pm (12)^2(13)^2(24)(25)(34)(36)(47)^2(56)^2(57)(67),$$

$$S_5 = \Sigma \pm (12)^2(13)^2(24)(25)(34)(36)(45)(47)(56)(57)(67)^2,$$

and there are no other terms in $(12)^2(13)^2$.

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Suppose that there is no squared factor at all. Then if we write $(12)(13)(14)(15).(23)(24)(26)$ we get $(37)(47)(57)(67)$ and so $(35)(46)(56)$, which gives symmetry in (135), (246). If however we put $(23)(26)(27)$ we cannot complete 3 without bringing it to a product of the previous type with (12) replaced by (13) or (23).

It remains to discuss terms involving a single square $(12)^2$ at least, but no pair of squares such as $(12)^2(13)^2$. Now with $(12)^2(13)(14)(25)(26)$ we get either $(37)(47)(57)(67)$ and hence also

$$(34)^2(56)^2 \text{ or } (35)^2(46)^2 \text{ or } (35)(36)(45)(46) \text{ or } (34)(35)(46)(56),$$

all of which give forbidden symmetries; or else $(37)^2(67)(57)$ and hence also $(34)(45)^2(46)^2(56)^2$ and symmetry in 5 and 6; or else $(37)^2(47)(57)$, say: if we complete with $(36)(45)(56)(46)$ we get symmetry in $(124)(376)$.

Now $(12)^2(13)(14)(25)(26)(37)^2(47)(57)(35)(46)^2(56)$

is symmetrical in $(123)(467)$: and we are reduced to

$$S = \Sigma \pm (12)^2(13)(14)(25)(26)(37)^2(47)(57)(34)(46)(56)^2.$$

There are no other types. For if we reject all factors such as $(12)^2(13)(14)(25)(26)$, let us suppose first that there are three double factors $(12)^2(34)^2(56)^2$. Then either we have $(17)(27)(37)(47)$: in this case we get $(15)(26)(35)(46)$ and symmetry in $(135)(246)$: or else $(17)(27)(37)(57)$, say. In this case we may not put (13), since then 3, 4 would not appear with the same complement in two factors, and we should be reduced to S. Hence we have $(14)(26)$, say: and now again we shall be reduced to S or else a forbidden symmetry.

Next suppose that there are just two double factors $(12)^2(34)^2$. Then either we have no factor such as (13) and two factors $(56)(57)$: or else one factor (13) and three factors $(56)(57)(67)$. In the latter case we may suppose that there are factors $(25)(26)(15)$ and hence $(37)(47)$ and (46): but this gives symmetry in $(125)(347)$. In the former case we may suppose $(16)(26)(36)$ and $(17)(37)(47)$ (avoiding symmetry in 6 and 7): but then we have $(25)(45)$ and symmetry in $(126)(347)$.

Finally, suppose that there is just one square $(12)^2$, and also $(13)(14)(23)(25)$. Then we must have $(37)(47)(57)(67)$ and $(36)(46)(56)$ and so also (45): and this gives symmetry in 5 and 6.

5. It follows that the only antisymmetric functions of the (ij) of the required order are those whose typical terms have been found above. I shall now show that Cayley's problem is in fact solved by

$$S = 0.$$

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For $\Delta = 0$ is satisfied identically if 7 lies on the quadric through 4, 5 and 6. Thus Δ vanishes identically if we put $7 = 4 + 5 + 6$ (symbolically) with the condition $(56) + (64) + (45) = 0$, which corresponds to $\Delta' = 0$ and gives $(47) = -(56)$, etc.

If we make this substitution we see that the coefficient of $(12)^3(13).(24)(34)(35)(36)$ is zero except in S_1, S_2, S_3. In these it is

$$(56)^3(45)^2(46) - (56)^3(46)^2(45), \quad (56)^3(46)^3 - (56)^3(45)^3$$

and

$$(45)^4(46)(56) - (56)^4(45)(56),$$

respectively. In each case the relevant terms are obtained from those written out by leaving 1, 2, 3, 4 unaltered, replacing 5 by 7 and (6, 7) by either (6, 5) or (5, 6). Now no linear combination of the above quantities contains a factor $(56) + (64) + (45)$, as is easily verified. Thus Δ depends only on S_4, S_5 and S.

Again, S can give no terms in $(12)^2(13)^2$. When we make the substitution already used we find that the coefficient of $(12)^2(13)^2(25)^2(36)^2$ in S_4 is $\pm(56)^4(45)(46)$; in S_5 it is $\pm(56)^2[(64)^3(45) + (45)^3(46)]$.

In S_4 we simply have to interchange 4 and 7; in S_5 we interchange 4 and 7 and either leave the others unaltered or simultaneously interchange 2 and 3, 5 and 6. This latter (an *even* permutation) leaves the general term of S_4 invariant. Once more, we easily verify that no linear combination of the coefficients contains $(56) + (64) + (45)$ as a factor. Thus Δ is a constant multiple of S: and Cayley's complex is given explicitly by

$$0 = S \equiv \Sigma \pm (12)^2(13)(14)(25)(26)(37)^2(47)(57)(34)(46)(56)^2.$$

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