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Isometric dilation and Sarason's commutant lifting theorem in several variables

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Abstract. The article deals with isometric dilation and commutant lifting for a class of n-tuples $(n \geq 3)$ of commuting contractions. We show that operator tuples in the class dilate to tuples of commuting isometries of BCL type. As a consequence of such an explicit dilation, we show that their von Neumann inequality holds on a one-dimensional variety of the closed unit polydisc. On the basis of such a dilation, we prove a commutant lifting theorem of Sarason's type by establishing that every commutant can be lifted to the dilation space in a commuting and norm-preserving manner. This further leads us to find yet another class of n-tuples $(n \geq 3)$ of commuting contractions each of which possesses isometric dilation.

1 Introduction

In the last century, one of the elegant results in operator theory, due to Sz.-Nagy and Foias ([31]), is that any contraction acting on a Hilbert space dilates to an isometry. That is, for a contraction T acting on a Hilbert space $\mathcal{H}$, there exists an isometry V acting on a Hilbert space $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$ and

$$T^m = P_{\mathcal{H}} V^m|_{\mathcal{H}} \quad (m \in \mathbb{Z}_+).$$

Generalizing this, Ando ([4]) proved that for any pair of commuting contractions (T_1, T_2) on $\mathcal{H}$, there exists a pair of commuting isometries (V_1, V_2) on $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$ and

$$T_1^{m_1} T_2^{m_2} = P_{\mathcal{H}} V_1^{m_1} V_2^{m_2}|_{\mathcal{H}} \quad ((m_1, m_2) \in \mathbb{Z}_+^2).$$

The pair (V_1, V_2) is called an isometric dilation of (T_1, T_2). Different proofs of the Ando dilation have surfaced recently; see [17, 23, 24, 35]. However, further generalization of Ando's result for an n-tuples of commuting contractions $(n \geq 3)$ fails, and a counterexample was discovered in [32]. It is still not very clear why the result fails in general, and it is essential to understand subtle differences between the case $n = 2$ and $n > 2$. As a matter of course, it is important to identify n-tuples of commuting contractions each of which has an isometric dilation. By an isometric dilation of an n-tuple of commuting contractions $(T_1, \ldots, T_n)$ on $\mathcal{H}$, we mean an n-tuple of commuting isometries $(V_1, \ldots, V_n)$ on $\mathcal{K}$ such that $\mathcal{H} \subseteq \mathcal{K}$ and

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$$T_1^{\alpha_1} \cdots T_n^{\alpha_n} = P_{\mathcal{H}} V_1^{\alpha_1} \cdots V_n^{\alpha_n}|_{\mathcal{H}},$$

for all $(\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$. A fair amount of research has been carried out to identify different classes of n-tuples of commuting contractions each of which possesses an isometric dilation. Often such a class is obtained by imposing certain positivity condition. For more details, see [12, 13, 14, 18, 20, 21, 22] for the polydisc setting, and for the setting of the unit ball in $\mathbb{C}^n$, see [7, 30], and references therein. Also see [3, 5] for some general domains in $\mathbb{C}^n$.

One of the main contributions of this article is the discovery of a new class of n-tuples of commuting contractions ($n \geq 3$), each of which possesses an isometric dilation. Our dilation result emerged from the study of a commutant lifting theorem (CLT), a very close neighbor of dilation theory. Indeed, this article is as much a study of dilation theory as of CLT. Commutant lifting theorem was first studied in the context of Nevanlinna-Pick interpolation by D. Sarason in [34]. Soon after, Sz-Nagy and Foias extended the result in a general setting and proved that if V on $\mathcal{K}$ is an isometric dilation of a contraction T acting on $\mathcal{H}$, and X is a contraction on $\mathcal{H}$ such that $XT = TX$, then there exists a contraction Y on $\mathcal{K}$ satisfying

$$YV = VY, \ Y^*|_{\mathcal{H}} = X^* \text{ and } \|Y\| = \|X\|.$$

Because of the second identity above, Y is called a *co-extension or lift* of X. This result is known as Sz-Nagy and Foias' commutant lifting theorem, and it is known to be equivalent to Ando dilation ([33]). Multivariable analogue of Sz-Nagy and Foias' commutant lifting theorem does not hold in general. See [29] and also see [6, 9, 10, 11, 25, 26, 27] for related results. The best result known in the polydisc setting is due to Ball, Li, Timotin, and Trent ([9]), where the authors considered a class of n-tuples of commuting contractions along with their specific isometric dilations and found a necessary and sufficient condition for the commutant lifting theorem to hold. This shows that even if we restrict ourselves to a suitable class of n-tuples of commuting contractions and their isometric dilations, it may not always possible to lift every commutant in a commuting and norm-preserving way. Therefore, it raises the following natural question:

Is there a class of n-tuples of commuting contractions so that each of the n-tuples has an isometric dilation and every commutant can be lifted to the isometric dilation space in a commuting and norm-preserving way?

Needless to say that, the class of n-tuples requires to be nice enough for the above question to have a positive answer. We do not know the existence of any such class in the literature. In this article, we answer this question affirmatively by exhibiting a class and prove an exact analogue of Sz-Nagy and Foias' commutant lifting theorem, which is another main contribution of this article. As a consequence, this leads us to identify a class of commuting contractions each of which possesses an isometric dilation.

We now proceed to describe some of our results. This requires to set few notations and terminologies. For a Hilbert space $\mathcal{H}$, we denote by $\mathcal{B}(\mathcal{H})$ the space of all bounded operators acting on $\mathcal{H}$. The set of all n-tuples of commuting contractions on $\mathcal{H}$ is denoted by $\mathcal{T}^n(\mathcal{H})$; that is,

$$\mathcal{T}^n(\mathcal{H}) = \{(T_1, \ldots, T_n) : T_i \in \mathcal{B}(\mathcal{H}), \|T_i\| \leq 1, T_i T_j = T_j T_i, 1 \leq i, j \leq n\}.$$

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For $T = (T_1, \dots, T_n) \in \mathcal{T}^n(\mathcal{H})$, we denote by P_T the product contraction of T, that is, $P_T := T_1 \cdots T_n$, and for $A \in \mathcal{B}(\mathcal{H})$, we denote by C_A the completely positive map on $\mathcal{B}(\mathcal{H})$ induced by A defined as

$$C_A : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}),\ X \mapsto AXA^*.$$

For any $\gamma = (\gamma_1 \dots, \gamma_n) \in \mathbb{Z}_+^n$ and $T = (T_1, \dots, T_n) \in \mathcal{T}^n(\mathcal{H})$, we define

$$\Delta_T^{\gamma} := (I - C_{T_1})^{\gamma_1} \cdots (I - C_{T_n})^{\gamma_n}(I_{\mathcal{H}}).$$

With this notation, it is easy to see that for $i \neq j$,

$$\Delta_T^{\mathbf{e}_i+\mathbf{e}_j} = (I - C_{T_i})(I - C_{T_j})(I_{\mathcal{H}}) = I - T_iT_i^* - T_jT_j^* + T_iT_jT_i^*T_j^*,$$

where for $i = 1, \dots, n$,

$$\mathbf{e}_i = (0, \dots, 0, \underbrace{1}_{i \text{ th place}}, 0, \dots, 0).$$

A contraction S acting on $\mathcal{H}$ is pure if $S^{*n} \to 0$ in the strong operator topology (SOT). The class of n-tuples of commuting contractions that we consider is defined as follows:

$$\mathfrak{B}_0^n(\mathcal{H}) := \left\{T \in \mathcal{T}^n(\mathcal{H}) : \Delta_T^{\mathbf{e}_i+\mathbf{e}_j} = 0 \text{ for all } i \neq j \text{ and } P_T \text{ is pure}\right\}.$$

For the base case $n = 2$, if (T_1, T_2) is a pair of commuting contractions on $\mathcal{H}$ such that T_1 is a co-isometry and T_2 is a pure contraction, then clearly, $(T_1, T_2) \in \mathfrak{B}_0^2(\mathcal{H})$. In fact, it turns out that if $\mathcal{H}$ is finite-dimensional, then an arbitrary element of $\mathfrak{B}_0^2(\mathcal{H})$ is a direct sum of pairs of the above form (see Proposition 5.2). However, in infinite dimension, it is no longer the case. We have demonstrated this by finding an example in Section 5. It can be shown without much difficulty that for any $T \in \mathfrak{B}_0^n(\mathcal{H})$ and for a subset $\{i_1, \dots, i_k\}$ of $\{1, \dots, n\}$,

$$\Delta_T^{\mathbf{e}_{i_1}+\cdots+\mathbf{e}_{i_k}} = 0. \tag{1.1}$$

In other words, $\mathfrak{B}_0^n(\mathcal{H})$ is a subclass of Brehmer n-tuples on $\mathcal{H}$ (see [20]). Therefore, any element of $\mathfrak{B}_0^n(\mathcal{H})$ has an isometric dilation. However, we show that any $T \in \mathfrak{B}_0^n(\mathcal{H})$ has an isometric dilation of the form $(M_{\Phi_1}, \dots, M_{\Phi_n})$ acting on the Hardy space $H_{\mathcal{E}}^2(\mathbb{D})$ for some co-efficient Hilbert space $\mathcal{E}$, where

$$\Phi_i(z) = U_i(P_i^{\perp} + zP_i)\ (z \in \mathbb{D})$$

and for all $i = 1, \dots, n$, $U_i \in \mathcal{B}(\mathcal{H})$ is a unitary, $P_i \in \mathcal{B}(\mathcal{E})$ is an orthogonal projection so that

$$\Phi_i(z)\Phi_j(z) = \Phi_j(z)\Phi_i(z) \text{ and } \Phi_1(z) \cdots \Phi_n(z) = zI_{\mathcal{E}} \quad (z \in \mathbb{D}). \tag{1.2}$$

Moreover, the tuple of isometries also belongs to the same class; that is,

$$(M_{\Phi_1}, \dots, M_{\Phi_n}) \in \mathfrak{B}_0^n(H_{\mathcal{E}}^2(\mathbb{D})).$$

In case the reader is wondering what properties are required for U_i and P_i so that (1.2) holds, see [16, 15] or Section 2 below. Here, the Hardy space $H_{\mathcal{E}}^2(\mathbb{D})$ is defined as

$$H_{\mathcal{E}}^2(\mathbb{D}) = \{f(z) = \sum_{k=0}^{\infty} a_k z^k : a_k \in \mathcal{E}, z \in \mathbb{D}, \|f\|^2 = \sum_{k=0}^{\infty} \|a_k\|^2 < \infty\},$$

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and the isometry $M_{\Phi_i} : H^2_{\mathcal{E}}(\mathbb{D}) \to H^2_{\mathcal{E}}(\mathbb{D})$ is the multiplication operator associated to Φ_i defined as

$$M_{\Phi_i} f(z) = \Phi_i(z) f(z) \quad (f \in H^2_{\mathcal{E}}(\mathbb{D})).$$

It is evident from (1.2) that $(M_{\Phi_1}, \dots, M_{\Phi_n})$ is an n-tuple of commuting isometries and $M_{\Phi_1} \cdots M_{\Phi_n} = M_z$, where M_z is the shift on $H^2_{\mathcal{E}}(\mathbb{D})$. Thus, M_z can be factorized as the product of the n-tuple $(M_{\Phi_1}, \dots, M_{\Phi_n})$, and it is a remarkable result of Berger, Coburn, and Lebow ([16]) that any commuting contractive factor of M_z arises in this way. For this reason, $(M_{\Phi_1}, \cdots, M_{\Phi_n})$ is commonly known as a BCL n-tuple. Now coming back to the present context, the explicit isometric dilation of $T \in \mathfrak{B}_0^n(\mathcal{H})$ to a BCL n-tuple plays a pivotal role in what follows and aids to establish a sharp von Neumann type inequality. For a contraction $T \in \mathcal{B}(\mathcal{H})$, the defect operator D_T and the defect space $\mathcal{D}_T$ of T are defined as

$$D_T := (I - TT^*)^{1/2} \text{ and } \mathcal{D}_T := \overline{\operatorname{ran}}\, D_T.$$

If an n-tuple of commuting contractions T has an isometric dilation, then its von Neumann inequality holds on $\overline{\mathbb{D}}^n$ (see [31]); that is, for all polynomials $p \in \mathbb{C}[z_1, \dots, z_n]$,

$$\|p(T)\|_{\mathcal{B}(\mathcal{H})} \leq \sup_{\mathbf{z} \in \overline{\mathbb{D}}^n} |p(\mathbf{z})|. \tag{1.3}$$

The above inequality is refined, and a significant reduction in terms of the dimension of the set where the inequality holds is obtained.

Theorem 1.1 *Let $T = (T_1, \dots, T_n) \in \mathfrak{B}_0^n(\mathcal{H})$ and $\dim \mathcal{D}_{T_i} < \infty$ $(i = 1, \dots, n)$. Then there exists a one-dimensional symmetric algebraic variety W in $\mathbb{C}^n$ such that for all polynomials $p \in \mathbb{C}[z_1, \dots, z_n]$,*

$$\|p(T)\|_{\mathcal{B}(\mathcal{H})} \leq \sup_{z \in W \cap \overline{\mathbb{D}}^n} |p(z)|.$$

This result is proved in Theorem 3.4 below. Such a refined von Neumann inequality has been studied earlier for various other class of n-tuples of commuting contractions (see [2, 13, 23]), and most often, such a result is obtained on the basis of finding explicit and nice enough isometric dilations.

Using the explicit isometric dilation, as indicated above, we answer the question raised earlier and prove an analogue of Sz-Nagy and Foias' commutant lifting theorem for $\mathfrak{B}_0^n(\mathcal{H})$.

Theorem 1.2 *Let $T = (T_1, \dots, T_n) \in \mathfrak{B}_0^n(\mathcal{H})$ and let $(M_{\Phi_1}, \dots, M_{\Phi_n}) \in \mathfrak{B}_0^n(H^2_{\mathcal{E}}(\mathbb{D}))$ be an isometric dilation of T for some Hilbert space $\mathcal{E}$. If $XT_i = T_i X$ $(i = 1, \dots, n)$ for some contraction $X \in \mathcal{B}(\mathcal{H})$, then there exists $Y \in \mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}))$ such that Y is a lift of X,*

$$YM_{\Phi_i} = M_{\Phi_i} Y \quad (i = 1, \dots, n), \text{ and } \|Y\| = \|X\|.$$

This result is proved in Theorem 4.7 below. The proof heavily relies on a detailed analysis on the structure of commutants of T as well as the structure of commutants of its isometric dilation $(M_{\Phi_1}, \dots, M_{\Phi_n})$. For instance, we show that if $T = (T_1, \dots, T_n) \in \mathfrak{B}_0^n(\mathcal{H})$ and $XT_i = T_i X$ for some contraction $X \in \mathcal{B}(\mathcal{H})$, then

$$I - XX^* = G_1 + \cdots + G_n \text{ and } (I - C_{T_i})(G_j) = G_j - T_i G_j T_i^* = 0 \quad (i \neq j),$$

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where G_i's are positive operators on $\mathcal{H}$ (see Theorem 4.4 below). The above identity has some resemblance to the necessary and sufficient condition for CLT obtained in [9]. In fact, Theorem 1.2 is motivated from the above identity and the result of [9]. However, the proof of the theorem is entirely different from that of [9], partly because the isometric dilation and the dilation map are different, and one cannot directly use the result of [9] in this context as well. Finally, as CLT and dilation are closely related, the above theorem provides a new class of n-tuples of commuting contractions such that each of the tuples has an isometric dilation. The following result is proved in Theorem 4.10 below.

Theorem 1.3 *Let $(T_1, \ldots, T_n, T_{n+1})$ be an $(n+1)$-tuple of commuting contractions on $\mathcal{H}$ such that $(T_1, \ldots, T_n) \in \mathfrak{B}_0^n(\mathcal{H})$. Then $(T_1, \ldots, T_n, T_{n+1})$ has an isometric dilation and, consequently, satisfies the von Neumann inequality.*

This article is organized as follows. In the next section, we discuss some preparatory results. Section 3 is devoted to find an explicit isometric dilation for $\mathfrak{B}_0^n(\mathcal{H})$ and to obtain a sharper version of the von Neumann inequality for operator n-tuples in $\mathfrak{B}_0^n(\mathcal{H})$ with finite dimensional defect spaces. In Section 4, we prove CLT for $\mathfrak{B}_0^n(\mathcal{H})$, and, as a consequence, we find a new class of n-tuples of commuting contractions, each of which possesses an isometric dilation. Examples on n-tuples in $\mathfrak{B}_0^n(\mathcal{H})$ are considered in Section 5.

2 Preparatory results

We set few definitions and conventions and also establish some preliminary results, in this section, which we need subsequently. All Hilbert spaces are assumed to be separable and over the complex field $\mathbb{C}$.

2.1 Isometric dilation.

Let $T \in \mathcal{T}^n(\mathcal{H})$. If there exist an n-tuple $V = (V_1, \ldots, V_n)$ of commuting isometries on a Hilbert space $\mathcal{K}$ and an isometry $\Pi : \mathcal{H} \to \mathcal{K}$ such that

$$\Pi T_i^* = V_i^* \Pi, \quad (i = 1, \ldots, n),$$

then

$$(T_1, \ldots, T_n) \cong (P_{\mathcal{Q}} V_1|_{\mathcal{Q}}, \ldots, P_{\mathcal{Q}} V_n|_{\mathcal{Q}}),$$

where $\mathcal{Q} = \text{ran}\Pi$ is a joint $(V_1^*, \ldots, V_n^*)$-invariant subspace of $\mathcal{K}$ and the unitary equivalence is induced by Π. Here, two tuples $(A_1, \ldots, A_n) \in \mathcal{T}^n(\mathcal{H})$ and $(B_1, \ldots, B_n) \in \mathcal{T}^n(\mathcal{K})$ are said to be unitarily equivalent, denoted by $(A_1, \ldots, A_n) \cong (B_1, \ldots, B_n)$, if there exists a unitary $U : \mathcal{H} \to \mathcal{K}$ such that $UA_i = B_i U$ for all $i = 1, \ldots, n$. Since $(V_1, \ldots, V_n)$ is a co-extension of $(P_{\mathcal{Q}} V_1|_{\mathcal{Q}}, \ldots, P_{\mathcal{Q}} V_n|_{\mathcal{Q}})$ and co-extension is a special type of dilation, we say that V is an isometric dilation of T. In such a case, the isometry Π is called the dilation map. Moreover, V is a *minimal isometric dilation* of T if

$$\mathcal{K} = \bigvee_{\alpha \in \mathbb{Z}_+^n} V_1^{\alpha_1} \cdots V_n^{\alpha_n}(\mathcal{Q}).$$

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Throughout the article, most of the isometric dilations we obtained are of the above form.

2.2 Commuting tuples in $\mathfrak{B}_0^n(\mathcal{H})$

Let $T \in \mathcal{T}^n(\mathcal{H})$, and let $P_T := T_1 \cdots T_n$ be the product contraction of T. Recall that for all $k = 2, \ldots, n$, and $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$,

$$\Delta_T^{\mathbf{e}_{i_1}+\cdots+\mathbf{e}_{i_k}} = (I - C_{T_{i_1}}) \cdots (I - C_{T_{i_k}})(I_{\mathcal{H}}), \tag{2.1}$$

and

$$\mathfrak{B}_0^n(\mathcal{H}) := \left\{ T \in \mathcal{T}^n(\mathcal{H}) : \Delta_T^{\mathbf{e}_i+\mathbf{e}_j} = 0 \text{ for all } i \neq j \text{ and } P_T \text{ is pure} \right\}.$$

We consider a couple of elementary properties of $\mathfrak{B}_0^n(\mathcal{H})$ in the next two lemmas. Both the lemmas remain valid even if we do not assume P_T to be pure in the definition of $\mathfrak{B}_0^n(\mathcal{H})$. The first lemma shows that any element of $\mathfrak{B}_0^n(\mathcal{H})$ satisfies Brehmer positivity.

Lemma 2.1 *If $T = (T_1, \ldots, T_n) \in \mathfrak{B}_0^n(\mathcal{H})$, then for all $k = 3, \ldots, n$ and $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$,*

$$\Delta_T^{\mathbf{e}_{i_1}+\cdots+\mathbf{e}_{i_k}} = 0.$$

Proof Since $\Delta_T^{\mathbf{e}_i+\mathbf{e}_j} = 0$ for all $i \neq j$, the proof of the lemma follows from the definition of $\Delta_T^{\mathbf{e}_{i_1}+\cdots+\mathbf{e}_{i_k}}$; that is,

$$\begin{aligned} \Delta_T^{\mathbf{e}_{i_1}+\cdots+\mathbf{e}_{i_k}} &= (I - C_{T_{i_1}}) \cdots (I - C_{T_{i_k}})(I_{\mathcal{H}}) \\ &= (I - C_{T_{i_1}}) \cdots (I - C_{T_{i_{k-2}}})(\Delta_T^{\mathbf{e}_{i_{k-1}}+\mathbf{e}_{i_k}}). \end{aligned}$$

■

Lemma 2.2 *If $T = (T_1, \ldots, T_n) \in \mathfrak{B}_0^n(\mathcal{H})$, then for any $\varnothing \neq F \subseteq \{1, \ldots, i-1, i+1, \ldots, n\}$,*

$$\Delta_{(T_i, T_F)}^{(1,1)} = 0,$$

where for $F = \{n_1, \ldots, n_j\} \subseteq \{i_1, \ldots, i_k\}$, $T_F := T_{n_1} \cdots T_{n_j}$.

Proof We prove the lemma only for $F = \{2, \ldots, n\}$, as the proof for arbitrary F is similar. Using Agler's hereditary functional calculus ([1]) to the following identity, for $z_1, \ldots, z_n \in \mathbb{D}$ and $w_1, \ldots, w_n \in \mathbb{D}$,

$$\begin{aligned} (1 - z_1\bar{w}_1)(1 - z_2 \cdots z_n\bar{w}_1 \cdots \bar{w}_n) &= (1 - z_1\bar{w}_1)\Big(\sum_{i=2}^{n}(1 - z_i\bar{w}_i) - \sum_{\substack{i,j=2 \\ i<j}}^{n}(1 - z_i\bar{w}_i)(1 - z_j\bar{w}_j) \\ &+ \sum_{\substack{i,j,k=2 \\ i<j<k}}^{n}(1 - z_i\bar{w}_i)(1 - z_j\bar{w}_j)(1 - z_k\bar{w}_k) - \cdots + (-1)^{n-2}(1 - z_2\bar{w}_2)(1 - z_3\bar{w}_3) \cdots (1 - z_n\bar{w}_n)\Big), \end{aligned}$$

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we obtain

$$\Delta^{(1,1)}_{(T_i,T_F)} = \sum_{i=2}^{n} \Delta_T^{\mathbf{e}_1+\mathbf{e}_i} - \sum_{\substack{i,j=2\\ i<j}}^{n} \Delta_T^{\mathbf{e}_1+\mathbf{e}_i+\mathbf{e}_j} + \sum_{\substack{i,j,k=2\\ i<j<k}}^{n} \Delta_T^{\mathbf{e}_1+\mathbf{e}_i+\mathbf{e}_j+\mathbf{e}_k} - \cdots + (-1)^{n-2}\Delta_T^{\mathbf{e}_1+\cdots+\mathbf{e}_n}.$$

The proof now follows by Lemma 2.1. ∎

2.3 BCL tuple.

Let $\mathcal{E}$ be a Hilbert space. The space of all bounded $\mathcal{B}(\mathcal{E})$-valued holomorphic functions on $\mathbb{D}$ is denoted by $H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$. For any $\Psi \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$, the associated multiplication operator M_Ψ is a bounded operator on the Hardy space $H^2_{\mathcal{E}}(\mathbb{D})$ and defined as

$$M_\Psi f(z) = \Psi(z)f(z) \quad (f \in H^2_{\mathcal{E}}(\mathbb{D})).$$

A BCL n-tuple on $H^2_{\mathcal{E}}(\mathbb{D})$ is an n-tuple of commuting isometries $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{E}}(\mathbb{D})$ such that $M_{\Phi_1}\cdots M_{\Phi_n} = M_z$, where M_z is the shift on $H^2_{\mathcal{E}}(\mathbb{D})$. For a unitary U on $\mathcal{E}$ and an orthogonal projection $P \in \mathcal{B}(\mathcal{E})$, it is easy to verify that M_Φ is an isometry, where

$$\Phi(z) = U(P^\perp + zP) \ (z \in \mathbb{D}). \tag{2.2}$$

Berger, Coburn, and Lebow's theorem shows that any BCL n-tuple arises in the same way as (2.2). We recall the theorem for further use.

Theorem 2.3 (cf. [16]) *Let $\Phi_1, \ldots, \Phi_n$ be $\mathcal{B}(\mathcal{E})$-valued bounded analytic function on $\mathbb{D}$ for some Hilbert space $\mathcal{E}$. Then $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{E}}(\mathbb{D})$ is a BCL n-tuple if and only if for all $i = 1, \ldots, n$,*

$$\Phi_i(z) = U_i(P_i^\perp + zP_i) \ (z \in \mathbb{D})$$

for some unitaries U_i's on $\mathcal{E}$ and orthogonal projections P_i's in $\mathcal{B}(\mathcal{E})$ satisfying

(1) $U_iU_j = U_jU_i$ $(i, j = 1, \ldots, n)$;
(2) $U_1\cdots U_n = I_{\mathcal{E}}$;
(3) $P_i + U_i^*P_jU_i = P_j + U_j^*P_iU_j \le I_{\mathcal{E}}\,(i \ne j)$; *and*
(4) $P_1 + U_1^*P_2U_1 + \cdots + U_1^*\cdots U_{n-1}^*P_nU_1\cdots U_{n-1} = I_{\mathcal{E}}$.

It is evident from the above theorem that a BCL n-tuple is completely determined by a triple $(\mathcal{E}, \underset{\sim}{U}, \underset{\sim}{P})$, where $\mathcal{E}$ is a Hilbert space, $\underset{\sim}{U} = (U_1, \ldots, U_n)$ is an n-tuple of unitaries on $\mathcal{E}$, $\underset{\sim}{P} = (P_1, \ldots, P_n)$ is an n-tuple of projections in $\mathcal{B}(\mathcal{E})$, and U_i's and P_i's satisfy conditions (1)–(4) in the above theorem. *For the rest of the article,* we use a triple $(\mathcal{E}, \underset{\sim}{U}, \underset{\sim}{P})$ to refer to a BCL n-tuple, and it would always inherit the meaning that the commuting unitary tuple $\underset{\sim}{U} = (U_1, \ldots, U_n)$ and the projection tuple $\underset{\sim}{P} = (P_1, \ldots, P_n)$ satisfy conditions (1)– (4), and the corresponding BCL n-tuple is given by $\Phi_i(z) = U_i(P_i^\perp + zP_i)$ $(z \in \mathbb{D}, i = 1, \ldots, n)$. Few remarks are in order.

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Remarks 2.4

(i) Condition (3) in the above theorem implies mutual orthogonality of the projections P_i and $U_i^* P_j U_i$ $(i \neq j)$. That is,

$$P_i(U_i^* P_j U_i) = (U_i^* P_j U_i)P_i = 0,$$

which is also equivalent to

$$(U_i P_i U_i^*)P_j = P_j(U_i P_i U_i^*) = 0.$$

(ii) Conditions (2) and (4), in the above theorem, together imply that

$$M_{\Phi_1} \cdots M_{\Phi_n} = M_z.$$

(iii) For $n = 2$, if (M_{Φ_1}, M_{Φ_2}) is a BCL pair corresponding to $(\mathcal{E}, \underset{\sim}{U}, \underset{\sim}{P})$, then U_2 and P_2 can be recovered from U_1 and P_1. Indeed, by condition (2), $U_2 = U_1^*$, and by condition (4), $P_2 = I - U_1 P_1 U_1^*$. Thus, the BCL pair (M_{Φ_1}, M_{Φ_2}) is completely determined by $(\mathcal{E}, U, P)$, where $U = U_1$ and $P = I - U_1 P_1 U_1^*$, and consequently, Φ_1 and Φ_2 can be written as

$$\Phi_1(z) = (P + zP^\perp)U \text{ and } \Phi_2(z) = U^*(P^\perp + zP) \quad (z \in \mathbb{D}).$$

Lemma 2.5 *Let $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ be a BCL n-tuple corresponding to $(\mathcal{E}, \underset{\sim}{U}, \underset{\sim}{P})$. Then for all $i = 1, \ldots, n$, the orthogonal projection $(I_{H^2_\mathcal{E}(\mathbb{D})} - M_{\Phi_i} M_{\Phi_i}^*)$ has the block matrix representation*

$$\begin{bmatrix} U_i P_i U_i^* & 0 \\ 0 & 0 \end{bmatrix} : \mathcal{E} \oplus zH^2_\mathcal{E}(\mathbb{D}) \to \mathcal{E} \oplus zH^2_\mathcal{E}(\mathbb{D})$$

with respect to the decomposition $H^2_\mathcal{E}(\mathbb{D}) = \mathcal{E} \oplus zH^2_\mathcal{E}(\mathbb{D})$.

Proof Since M_{Φ_i} is an isometry, $(I_{H^2_\mathcal{E}(\mathbb{D})} - M_{\Phi_i} M_{\Phi_i}^*)$ is a projection onto $H^2_\mathcal{E}(\mathbb{D}) \ominus M_{\Phi_i} H^2_\mathcal{E}(\mathbb{D})$. Also since $M_z = M_{\Phi_1} \cdots M_{\Phi_n}$, $zH^2_\mathcal{E}(\mathbb{D}) \subseteq \text{ran} M_{\Phi_i}$, and therefore,

$$(I_{H^2_\mathcal{E}(\mathbb{D})} - M_{\Phi_i} M_{\Phi_i}^*)(zH^2_\mathcal{E}(\mathbb{D})) = 0.$$

However, for $\eta \in \mathcal{E}$,

$$(I_{H^2_\mathcal{E}(\mathbb{D})} - M_{\Phi_i} M_{\phi_i}^*)\eta = \eta - (U_i P_i^\perp + zU_i P_i)(P_i^\perp U_i^* \eta) = (I_\mathcal{E} - U_i P_i^\perp U_i^*)\eta = U_i P_i U_i^* \eta.$$

This completes the proof. ∎

Let $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ be a BCL n-tuple corresponding to $(\mathcal{E}, \underset{\sim}{U}, \underset{\sim}{P})$. Suppose that $U_i P_j = P_j U_i$ for all $i, j = 1, \ldots, n$. Then it follows from the condition (4) in Theorem 2.3 that

$$P_1 + P_2 + \cdots + P_n = I_\mathcal{E}.$$

In other words, P_i's are mutually orthogonal and $\mathcal{E} = \oplus_{i=1}^n \mathcal{E}_i$, where $\mathcal{E}_i = \text{ran}\, P_i (i = 1, \ldots, n)$. Since $U_i P_j = P_j U_i$, therefore, $\mathcal{E}_i$ is a U_j-reducing subspace of $\mathcal{E}$. Set $U_j^{(i)} := U_j|_{\mathcal{E}_i}$ for all $i, j = 1, \ldots, n$. Then it is clear that with respect to the

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decomposition $H^2_{\mathcal{E}}(\mathbb{D}) = H^2_{\mathcal{E}_1}(\mathbb{D}) \oplus \cdots \oplus H^2_{\mathcal{E}_n}(\mathbb{D})$, M_{Φ_i} has the following block diagonal representation:

(2.3)
$$M_{\Phi_i} = \begin{bmatrix} I_{H^2(\mathbb{D})} \otimes U_i^{(1)} & 0 & \dots & 0 & \dots & 0 \\ 0 & I_{H^2(\mathbb{D})} \otimes U_i^{(2)} & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & M_z \otimes U_i^{(i)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & I_{H^2(\mathbb{D})} \otimes U_i^{(n)} \end{bmatrix},$$

for all $i = 1, \dots, n$. The above block diagonal representation will be used many times in the sequel, as most of the the BCL n-tuples, in this paper, will have the property that $U_i P_j = P_j U_i$ for all $i, j = 1, \dots, n$. Finally, it follows from the commutativity of U_i's that $(M_{\Phi_1}, \dots, M_{\Phi_n})$ is doubly commuting; that is, $M^*_{\Phi_i} M_{\phi_j} = M_{\Phi_j} M^*_{\Phi_i}$ for all $i \neq j$. We summarize this in the next lemma for future reference.

Lemma 2.6 *Let $(M_{\Phi_1}, \dots, M_{\Phi_n})$ be a BCL n-tuple corresponding to $(\mathcal{E}, \underset{\sim}{U}, \underset{\sim}{P})$ such that $U_i P_j = P_j U_i$ for all $i, j = 1, \dots, n$. Then $(M_{\Phi_1}, \dots, M_{\Phi_n})$ is doubly commuting.*

We now characterize BCL n-tuples on $H^2_{\mathcal{E}}(\mathbb{D})$ which are also in $\mathfrak{B}^n_0(H^2_{\mathcal{E}}(\mathbb{D}))$. The result is proved for the case $n = 2$ in [28, Proposition 3.1], and we generalize it here for arbitrary n.

Proposition 2.7 *Let $M = (M_{\Phi_1}, \dots, M_{\Phi_n})$ on $H^2_{\mathcal{E}}(\mathbb{D})$ be a BCL n-tuple corresponding to $(\mathcal{E}, \underset{\sim}{U}, \underset{\sim}{P})$. Then the following are equivalent.*

(i) $M \in \mathfrak{B}^n_0(H^2_{\mathcal{E}}(\mathbb{D}))$;
(ii) $U_i P_j = P_j U_i$ *for* $i, j = 1, \dots, n$.

Proof of $(i) \implies (ii)$. For a fix i, set $\tilde{M}_{\Phi_i} := M_{\Phi_1} \dots M_{\Phi_{i-1}} M_{\Phi_{i+1}} \dots M_{\Phi_n}$. Then, $(M_{\Phi_i}, \tilde{M}_{\Phi_i})$ is a BCL pair and by Lemma 2.2, $\Delta^{(1,1)}_{(M_{\Phi_i}, \tilde{M}_{\Phi_i})} = 0$.

Therefore, by applying [28, Proposition 3.1] for the BCL pair $(M_{\Phi_i}, \tilde{M}_{\Phi_i})$, we get $U_i P_i = P_i U_i$ for all $i = 1, \dots, n$. Now, for $i \neq j$, it follows from Lemma 2.5 that for $\eta \in \mathcal{E}$,

$$\begin{aligned} &\Delta_M^{\mathbf{e}_i + \mathbf{e}_j} \eta \\ &= \Big((I_{H^2_{\mathcal{E}}(\mathbb{D})} - M_{\Phi_i} M^*_{\Phi_i}) - M_{\Phi_j} (I_{H^2_{\mathcal{E}}(\mathbb{D})} - M_{\Phi_i} M^*_{\Phi_i}) M^*_{\Phi_j} \Big) \eta \\ &= U_i P_i U_i^* \eta - U_j (P_j^\perp + M_z P_j)(I_{H^2_{\mathcal{E}}(\mathbb{D})} - M_{\Phi_i} M^*_{\Phi_i})(P_j^\perp + M_z^* P_j) U_j^* \eta \\ &= U_i P_i U_i^* \eta - U_j (P_j^\perp + M_z P_j) U_i P_i U_i^* (P_j^\perp U_j^* \eta) \quad \text{(by Lemma 2.5)} \\ &= U_i P_i U_i^* \eta - U_j U_i P_i U_i^* U_j^* \eta \quad \text{(as } U_i P_i U_i^* \perp P_j \text{ by part (i) of Remarks 2.4).} \end{aligned}$$

Since $\Delta_M^{\mathbf{e}_i + \mathbf{e}_j} = 0$, the above computation shows that $P_j - U_i P_j U_i^* = 0$. That is, $U_i P_j = P_j U_i$ for all $i \neq j$. This completes the proof. ∎

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Proof of $(ii) \implies (i)$. By Lemma 2.6, $M = (M_{\Phi_1}, \ldots, M_{\Phi_n})$ is doubly commuting. Then for any $i \neq j$,

$$\Delta_M^{\mathbf{e}_i+\mathbf{e}_j} = (I_{H^2_{\mathcal{E}}(\mathbb{D})} - M_{\Phi_i} M_{\Phi_i}^*)(I_{H^2_{\mathcal{E}}(\mathbb{D})} - M_{\Phi_j} M_{\Phi_j}^*).$$

We have already observed in the proof of Lemma 2.6 that in this case, P_i's are mutually orthogonal projections. The proof now follows from Lemma 2.5. ∎

It has been observed in [28, Proposition 3.6] that for a BCL pair (M_{Φ_1}, M_{Φ_2}) on $H^2_{\mathcal{E}}(\mathbb{D})$, if $\mathcal{E}$ is finite dimensional, then $\Delta^{(1,1)}_{(M_{\phi_1}, M_{\phi_2})} \geq 0$ is equivalent to $\Delta^{(1,1)}_{(M_{\phi_1}, M_{\phi_2})} = 0$. We also prove a similar result for BCL n-tuples and which shows that in the case of finite dimensional co-efficient space $\mathcal{E}$, the class $\mathfrak{B}_0^n$ is more relevant.

Proposition 2.8 *Let $M = (M_{\Phi_1}, \ldots, M_{\Phi_n})$ be a BCL n-tuple corresponding to $(\mathcal{E}, \underset{\sim}{U}, \underset{\sim}{P})$ such that for $i \neq j$, $\Delta_M^{\mathbf{e}_i+\mathbf{e}_j} \geq 0$. If $\dim \mathcal{E} < \infty$, then $M \in \mathfrak{B}_0^n(H^2_{\mathcal{E}}(\mathbb{D}))$.*

Proof As we have seen in the proof of the above proposition, for $\eta \in \mathcal{E}$ and $i \neq j$,

$$\Delta_M^{\mathbf{e}_i+\mathbf{e}_j} \eta = (U_i P_i U_i^* - U_j U_i P_i U_i^* U_j^*)\eta.$$

Then, $\Delta_M^{\mathbf{e}_i+\mathbf{e}_j} \geq 0$ $(i \neq j)$ implies $U_i P_i U_i^* - U_j U_i P_i U_i^* U_j^* \geq 0$; that is, $U_j P_i U_j^* \leq P_i$. Hence, ran P_i is a U_j-invariant subspace. Since $\dim \mathcal{E} < \infty$, ran P_i is also a U_j-reducing subspace. In other words, $U_j P_i = P_i U_j$ $(i \neq j)$. Therefore, by the above computation, $\Delta_M^{\mathbf{e}_i+\mathbf{e}_j}|_{\mathcal{E}} = 0$ $(i \neq j)$. However, it can be checked very easily that

$$\Delta_M^{\mathbf{e}_i+\mathbf{e}_j}|_{zH^2_{\mathcal{E}}(\mathbb{D})} = 0$$

even when $\mathcal{E}$ is infinite-dimensional. This completes the proof. ∎

We end the section with a result which ensures that BCL n-tuples are crucial while studying isometric dilations of n-tuples of commuting contractions. Observe that if $T = (T_1, \ldots, T_n) \in \mathcal{T}^n(\mathcal{H})$, then the product contraction $P_T = T_1 \cdots T_n$ is pure if and only if $T^{*\alpha} \to 0$ in SOT as $\alpha \to \infty$. Here, for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, $T^\alpha = T_1^{\alpha_1} \cdots T_n^{\alpha_n}$ and $\alpha \to \infty$ means that $\alpha_i \to \infty$ for each $i = 1, \ldots, n$. Indeed, it follows from the following inequalities

$$P_T^l P_T^{*l} \leq T^\alpha T^{*\alpha} \leq P_T^m P_T^{*m},$$

where $m = \min\{\alpha_i : i = 1, \ldots, n\}$ and $l = \max\{\alpha_i : i = 1, \ldots, n\}$.

Proposition 2.9 *Let $T = (T_1, \ldots, T_n) \in \mathcal{T}^n(\mathcal{H})$ be such that the product contraction P_T is pure and T admits an isometric dilation. Then any minimal isometric dilation of T is unitarily equivalent to a BCL n-tuple $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ on $H^2_{\mathcal{E}}(\mathbb{D})$ for some Hilbert space $\mathcal{E}$.*

Proof Let $V = (V_1, \ldots, V_n)$ on $\mathcal{K}$ be a minimal isometric dilation of T. That is,

$$T^\alpha = P_{\mathcal{H}} V^\alpha|_{\mathcal{H}} \quad (\alpha \in \mathbb{Z}_+^n) \text{ and } \mathcal{K} = \bigvee_{\beta \in \mathbb{Z}_+^n} V^\beta(\mathcal{H}).$$

Since an n-tuple of commuting isometries with the product isometry being pure is unitarily equivalent to a BCL n-tuple [16, Theorem 3.1], it suffices to prove that the

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product isometry of V is pure or equivalently, by the observation made prior to this proposition, $V^{*\alpha} \to 0$ in SOT as $\alpha \to \infty$. Since the product contraction $P_T := T_1 \cdots T_n$ is pure and the minimal isometric dilation is always a co-extension ([31]), we have for $h \in \mathcal{H}$,

$$\|V^{*\alpha}h\| = \|T^{*\alpha}h\| \to 0 \text{ in SOT} \quad \text{as} \quad \alpha \to \infty.$$

Let $h \in \mathcal{K}$ be such that $h = \sum_{0 \le \beta \le \gamma} V^\beta h_\beta$ ($\gamma \in \mathbb{Z}_+^n, h_\beta \in \mathcal{H}$). Then for $\alpha \ge \gamma$, we have

$$V^{*\alpha}h = \sum_{0 \le \beta \le \gamma} V^{*(\alpha-\beta)}h_\beta,$$

and

$$\|V^{*\alpha}h\| = \|\sum_{0 \le \beta \le \gamma} V^{*(\alpha-\beta)}h_\beta\| \le \sum_{0 \le \beta \le \gamma} \|V^{*(\alpha-\beta)}h_\beta\| \to 0 \text{ in SOT as } \alpha \to \infty.$$

Since $\{\sum_{0 \le \beta \le \gamma} V^\beta h_\beta : \gamma \in \mathbb{Z}_+^n, h_\beta \in \mathcal{H}\}$ is a dense subspace of $\mathcal{K}$ and $\{V^{*\alpha}\}_{\alpha \in \mathbb{Z}_+^n}$ is a uniformly bounded family, it follows that $V^{*\alpha} \to 0$ in SOT as $\alpha \to \infty$. This completes the proof. ∎

3 Isometric dilation and von Neumann inequality for $\mathfrak{B}_0^n(\mathcal{H})$

This section is devoted to find explicit BCL type isometric dilations for tuples in $\mathfrak{B}_0^n(\mathcal{H})$. Isometric dilations of such tuples are known due to a result of Brehmer ([20]). However, we find explicit BCL type dilations of such tuples and prove a sharp von Neumann inequality in certain cases. Explicit BCL type dilations are the key to prove the commutant lifting theorem in the subsequent section. For a contraction $T \in \mathcal{B}(\mathcal{H})$, recall that the defect operator and the defect space of T are defined as

$$D_T := (I - TT^*)^{1/2} \quad \text{and} \quad \mathcal{D}_T := \overline{\text{ran}}\, D_T.$$

For an n-tuple of commuting contractions $T = (T_1, \ldots, T_n) \in \mathcal{T}^n(\mathcal{H})$, the product contraction is denoted by $P_T := T_1 \cdots T_n$. We are now ready to prove the dilation result for the case when the defect spaces of T_i's are finite dimensional.

Theorem 3.1 *Let $T = (T_1, \ldots, T_n) \in \mathfrak{B}_0^n(\mathcal{H})$ be such that $\dim \mathcal{D}_{T_i} < \infty$ ($i = 1, \ldots, n$). Let $\mathcal{E} = \bigoplus_{i=1}^n \mathcal{D}_{T_i}$. Then there exist an isometry $\Pi : \mathcal{H} \to H^2_{\mathcal{E}}(\mathbb{D})$ and a BCL n-tuple $(M_{\Phi_1}, \ldots, M_{\Phi_n}) \in \mathfrak{B}_0^n(H^2_{\mathcal{E}}(\mathbb{D}))$ such that*

$$\Pi T_i^* = M_{\Phi_i}^* \Pi$$

for all $i = 1, \ldots, n$.

Proof Since $\Delta_T^{\mathbf{e}_i + \mathbf{e}_j} = 0$ for all $i \ne j$,

$$D_{T_j}^2 - T_i D_{T_j}^2 T_i^* = D_{T_i}^2 - T_j D_{T_i}^2 T_j^* = 0 \quad (i \ne j).$$

As the defect spaces are assumed to be finite dimensional, the above identities induce unitaries $W_j^{(i)} : \mathcal{D}_{T_j} \to \mathcal{D}_{T_j}$ for all $i \ne j$, defined by

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$$W_j^{(i)}(D_{T_j}h) = D_{T_j}T_i^*h \quad (h \in \mathcal{H}). \tag{3.1}$$

Then observe that for $i, k = 1, \ldots, n$ and $i \neq k \neq j$, $W_j^{(i)}W_j^{(k)} = W_j^{(k)}W_j^{(i)}$. Indeed,

$$W_j^{(i)}W_j^{(k)}(D_{T_j}h) = W_j^{(i)}(D_{T_j}T_k^*h) = D_{T_j}T_i^*T_k^*h = W_j^{(k)}W_j^{(i)}(D_{T_j}h).$$

Now for $i = j$, we set $W_j^{(j)} := W_j^{*(1)} \cdots W_j^{*(j-1)}W_j^{*(j+1)} \cdots W_j^{*(n)}$. Then it is clear from the definition that

$$W_j^{(j)}(D_{T_j}P_T^*h) = D_{T_j}T_j^*h \quad (h \in \mathcal{H}),$$

and for each $j = 1, \ldots, n$, the tuple $(W_j^{(1)}, \ldots, W_j^{(n)})$ is a tuple of commuting unitaries and

$$W_j^{(1)} \cdots W_j^{(n)} = I_{\mathcal{D}_{T_j}}. \tag{3.2}$$

We now have all the ingredients to define the triple $(\mathcal{E}, \underset{\sim}{U}, \underset{\sim}{P})$ corresponding to the dilating BCL n-tuple. We take $\mathcal{E} = \oplus_{i=1}^n \mathcal{D}_{T_i}$, $\underset{\sim}{U} = (U_1, \ldots, U_n)$ and $\underset{\sim}{P} = (P_1, \ldots, P_n)$, where for $i = 1, \ldots, n$,

$$U_i^* = \bigoplus_{j=1}^n W_j^{(i)} \quad \text{and} \quad P_i(x_1, \ldots, x_n) = (0, \ldots 0, x_i, 0, \ldots 0) \quad (x_l \in \mathcal{D}_{T_l}, l = 1, \ldots, n).$$

By the construction of U_i and P_i and the identity 3.2, it easy to observe that for $i, j = 1, \ldots, n$,

$$U_iU_j = U_jU_i, \quad U_iP_j = P_jU_i, \quad P_1 + \cdots + P_n = I_{\mathcal{E}}, \quad \text{and} \quad U_1 \cdots U_n = I_{\mathcal{E}}.$$

Thus, $\underset{\sim}{U}$ and $\underset{\sim}{P}$ satisfy all the conditions (1)–(4) as in Theorem 2.3, and by Proposition 2.7, the corresponding BCL n-tuple $(M_{\Phi_1}, \ldots, M_{\Phi_n}) \in \mathfrak{B}_0^n(H_{\mathcal{E}}^2(\mathbb{D}))$. In the rest of the proof, we construct the dilation map and show that $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ is an isometric dilation of $(T_1, \ldots, T_n)$. The following identity,

$$D_{P_T}^2 = D_{T_1}^2 + T_1D_{T_2}^2T_1^* + T_1T_2D_{T_3}^2T_1^*T_2^* + \cdots + T_1 \cdots T_{n-1}D_{T_n}^2T_1^* \cdots T_{n-1}^*h \quad (h \in \mathcal{H}),$$

implies that the map $\iota : \mathcal{D}_{P_T} \to \mathcal{E}$, defined by

$$D_{P_T}h \mapsto (D_{T_1}h, D_{T_2}T_1^*h, D_{T_3}T_1^*T_2^*h, \ldots, D_{T_n}T_1^* \cdots T_{n-1}^*h), \tag{3.3}$$

is an isometry. Since P_T is a pure contraction, the map $\pi : \mathcal{H} \to H_{\mathcal{D}_{P_T}}^2(\mathbb{D})$, defined by $\pi h = \sum_{k \in \mathbb{Z}_+} z^k(D_{P_T}P_T^{*k}h)$ $(z \in \mathbb{D}, h \in \mathcal{H})$, is an isometry. In fact, π is the well-known dilation map for P_T. Finally, the dilation map for us is the isometry $\Pi := (I \otimes \iota)\pi : \mathcal{H} \to H_{\mathcal{E}}^2(\mathbb{D})$. To complete the proof, we show that for all $i = 1, \ldots, n$,

$$\Pi T_i^* = M_{\Phi_i}^*\Pi.$$

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Indeed, it is established by the rigorous computation below. For $k \in \mathbb{Z}_+$, $\eta \in \mathcal{E}$, $h \in \mathcal{H}$,

$$\begin{aligned}
&\langle M_{\Phi_i}^* \Pi h, z^k \eta\rangle \\
&\quad = \langle (I \otimes \iota) \sum_{m\in\mathbb{Z}_+} z^m (D_{P_T} P_T^{*m} h), (U_i P_i^\perp + z U_i P_i) z^k \eta\rangle \\
&\quad = \langle (P_i^\perp U_i^* \iota D_{P_T} P_T^{*k} h + P_i U_i^* \iota D_{P_T} P_T^{*k+1} h), \eta\rangle \\
&\quad = \langle P_i^\perp U_i^* (D_{T_1} P_T^{*k} h, D_{T_2} T_1^* P_T^{*k} h, \ldots, D_{T_n} T_1^* \cdots T_{n-1}^* P_T^{*k} h), \eta\rangle \\
&\qquad\qquad + \langle P_i U_i^* (D_{T_1} P_T^{*k+1} h, D_{T_2} T_1^* P_T^{*k+1} h, \ldots, D_{T_n} T_1^* \cdots T_{n-1}^* P_T^{*k+1} h), \eta\rangle \\
&\quad = \langle U_i^* (D_{T_1} P_T^{*k} h, \ldots, D_{T_{i-1}} T_1^* \cdots T_{i-2}^* P_T^{*k} h, 0, D_{T_{i+1}} T_1^* \cdots T_i^* P_T^{*k} h, \ldots, D_{T_n} T_1^* \cdots T_{n-1}^* P_T^{*k} h), \eta\rangle \\
&\qquad\qquad + \langle U_i^* (0, \ldots, 0, D_{T_i} T_1^* \cdots T_{i-1}^* P_T^{*k+1} h, 0, \ldots, 0), \eta\rangle \text{ (as } U_i P_i = P_i U_i) \\
&\quad = \langle (D_{T_1} T_i^* P_T^{*k} h, \ldots, D_{T_n} T_1^* \cdots T_{n-1}^* T_i^* P_T^{*k} h), \eta\rangle.
\end{aligned}$$

However, for $\eta \in \mathcal{E}$, $h \in \mathcal{H}$, $k \in \mathbb{Z}_+$,

$$\begin{aligned}
\langle \Pi T_i^* h, z^k \eta\rangle &= \langle (I \otimes \iota) \sum_{m\in\mathbb{Z}_+} z^m (D_{P_T} P_T^{*m} T_i^* h), z^k \eta\rangle \\
&= \langle \iota D_{P_T} P_T^{*k} T_i^* h, \eta\rangle \\
&= \langle (D_{T_1} T_i^* P_T^{*k} h, \ldots, D_{T_n} T_1^* \cdots T_{n-1}^* T_i^* P_T^{*k} h), \eta\rangle.
\end{aligned}$$

Hence, $\Pi T_i^* = M_{\Phi_i}^* \Pi$ $(i = 1, \ldots, n)$, as asserted. ∎

An isometric dilation for arbitrary $T = (T_1, \ldots, T_n) \in \mathfrak{B}_0^n(\mathcal{H})$ can also be obtained in a similar way. Since the defect space $\mathcal{D}_{T_j}$ may not be necessarily finite dimensional, the operator $W_j^{(i)}$ $(j \neq i)$, as defined in 3.1, is an isometry and may not be necessarily a unitary. This forces us to take minimal unitary extension of the commuting tuple of isometries $(W_j^{(1)}, \ldots, W_j^{(j-1)}, W_j^{(j+1)}, \ldots, W_j^{(n)})$ defined on $\mathcal{D}_{T_j}$. We denote the extended tuple of unitaries on $\mathcal{K}_j$ again by $(W_j^{(1)}, \ldots, W_j^{(j-1)}, W_j^{(j+1)}, \ldots, W_j^{(n)})$. For $i = j$, we let the unitary $W_j^{(j)} : \mathcal{K}_j \to \mathcal{K}_j$ be

$$W_j^{(j)} := W_j^{(1)*} \cdots W_j^{(j-1)*} W_j^{(j+1)*} \cdots W_j^{(n)*}.$$

Then, by the same arguments as before, the tuple $(W_j^{(1)}, \ldots, W_j^{(n)})$ is a tuple of commuting unitaries,

$$W_j^{(j)} (D_{T_j} P_T^* h) = D_{T_j} T_j^* h \ (j = 1, \ldots, n, h \in \mathcal{H}), \text{ and } W_j^{(1)} \cdots W_j^{(n)} = I_{\mathcal{D}_{T_j}}.$$

The triple $(\mathcal{E}, \underset{\sim}{U}, \underset{\sim}{P})$ corresponding to the dilating BCL n-tuple is defined similarly by taking $\mathcal{E} = \bigoplus_{i=1}^n \mathcal{K}_i$, $\underset{\sim}{U} = (U_1, \ldots, U_n)$, and $\underset{\sim}{P} = (P_1, \ldots, P_n)$, where for $i = 1, \ldots, n$,

$$U_i^* = \bigoplus_{j=1}^n W_j^{(i)} \quad \text{and} \quad P_i(x_1, \ldots, x_n) = (0, \ldots 0, x_i, 0, \ldots 0) \quad (x_l \in \mathcal{K}_l, l = 1, \ldots, n).$$

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Since $\bigoplus_{i=1}^{n} \mathcal{D}_{T_i} \subseteq \mathcal{E}$, the isometry ι, as in 3.3, can be viewed as an isometry from $\mathcal{H}$ to $\mathcal{E}$. Then by the same computation as done in the proof of the previous theorem, one also has

$$\Pi T_i^* = M_{\Phi_i}^* \Pi \ (i = 1, \ldots, n),$$

where $\Pi := (I \otimes \iota)\pi : \mathcal{H} \to H^2_{\mathcal{E}}(\mathbb{D})$ and the isometry $\pi : \mathcal{H} \to H^2_{\mathcal{D}_{P_T}}(\mathbb{D})$ is the dilation map of P_T defined by $\pi h = \sum_{k \in \mathbb{Z}_+} z^k (D_{P_T} P_T^{*k} h)$ $(z \in \mathbb{D}, h \in \mathcal{H})$. Thus, we have proved the following dilation result.

Theorem 3.2 *Let $T = (T_1, \ldots, T_n) \in \mathfrak{B}_0^n(\mathcal{H})$. Then there exist a Hilbert space $\mathcal{E}$, an isometry $\Pi : \mathcal{H} \to H^2_{\mathcal{E}}(\mathbb{D})$, and a BCL n-tuple $(M_{\Phi_1}, \ldots, M_{\Phi_n}) \in \mathfrak{B}_0^n(H^2_{\mathcal{E}}(\mathbb{D}))$ such that*

$$\Pi T_i^* = M_{\Phi_i}^* \Pi$$

for all $i = 1, \ldots, n$.

Remark 3.3 For $T \in \mathfrak{B}_0^n(\mathcal{H})$, by the definition, the product contraction P_T is pure. The above dilation result remains valid even if we remove the requirement that the product contraction P_T is pure. In such a case, the isometric dilation we get would not be a BCL n-tuple. Instead, it would be direct sums of a BCL n-tuple and an n-tuple of commuting unitaries. Indeed, if

$$Q := \text{SOT} - \lim_{n \to \infty} P_T^n P_T^{*n} \ \text{ and } \ \mathcal{Q} = \overline{\text{ran}}\, Q,$$

then for each $i = 1, \ldots, n$, the operator $\tilde{T}_i : \mathcal{Q} \to \mathcal{Q}$ defined by $\tilde{T}_i^{\,*}(Qh) = QT_i^* h$ is a co-isometry and $\tilde{T}_i \tilde{T}_j = \tilde{T}_j \tilde{T}_i$. Let $(W_1, \ldots W_n)$ on $\mathcal{K}$ be the minimal unitary co-extension of $(\tilde{T}_1, \ldots, \tilde{T}_n)$. Since the minimal isometric dilation map of P_T can be taken to be

$$\tilde{\pi} : \mathcal{H} \to H^2_{\mathcal{D}_{P_T}}(\mathbb{D}) \oplus \mathcal{K}, \quad h \mapsto (\pi(h), Qh) = \Big(\sum_{k \in \mathbb{Z}_+} z^k D_{P_T} P_T^{*k} h, Qh \Big),$$

one can check that

$$\Pi T_i^* = (M_{\Phi_i}^* \oplus W_i^*)\Pi \quad (i = 1, \ldots, n),$$

where $(M_{\Phi_1}, \ldots M_{\Phi_n}) \in \mathfrak{B}_0^n(H^2_{\mathcal{E}}(\mathbb{D}))$ is the BCL n-tuple as constructed in Theorem 3.2 and $\Pi : \mathcal{H} \to H^2_{\mathcal{E}}(\mathbb{D}) \oplus \mathcal{K}$ is given by $\Pi(h) = ((I \otimes \iota)\pi(h), Qh)$ with ι as in (3.3).

In the rest of this section, we prove a sharper version of the von Neumann inequality. Since any element T of $\mathfrak{B}_0^n(\mathcal{H})$ has an isometric dilation, T satisfies the von Neumann inequality for $\mathbb{C}[z_1, \ldots, z_n]$, the ring of polynomials in n variables with complex coefficients. That is, for all polynomials $p \in \mathbb{C}[z_1, \ldots, z_n]$,

$$\|p(T)\|_{\mathcal{B}(\mathcal{H})} \leq \sup_{\mathbf{z} \in \overline{\mathbb{D}}^n} |p(\mathbf{z})|.$$

However, we show that, in this particular case, one only needs to consider a one-dimensional variety instead of whole $\overline{\mathbb{D}}^n$. Before going into further details, we need to recall some results. Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting bounded operators on $\mathcal{H}$. The Taylor joint spectrum ([36]) of T is denoted by $\sigma(T)$. In the

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case when T consists of commuting matrices, $\sigma(T)$ is the set of all joint eigenvalues. Let $(M_{\Phi_1}, \ldots, M_{\Phi_n})$ be a BCL n-tuple on $H^2_{\mathcal{E}}(\mathbb{D})$ such that $\mathcal{E}$ is finite dimensional. Then, it is proved in [19, Theorem 7.3] that the set

$$W := \{\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n : (z_1, \ldots, z_n) \in \sigma(\Phi_1(\underline{\mathbf{z}}), \ldots, \Phi_n(\underline{\mathbf{z}}))\} \tag{3.4}$$

is a one-dimensional symmetric algebraic variety, where for $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$, $\underline{\mathbf{z}} := z_1 \cdots z_n$. Here, the symmetric property of W simply means that for non-zero $z_i (i = 1, \ldots, n)$,

$$(z_1, \ldots, z_n) \in W \quad \text{if and only if} \quad (\frac{1}{\bar{z}_1}, \ldots, \frac{1}{\bar{z}_n}) \in W.$$

We are now ready to prove the sharp von Neumann inequality.

Theorem 3.4 *Let $T = (T_1, \ldots, T_n) \in \mathfrak{B}_0^n(\mathcal{H})$ and $\dim \mathcal{D}_{T_i} < \infty$ for all $i = 1, \ldots, n$. Then there exists a one-dimensional symmetric algebraic variety W in $\mathbb{C}^n$ such that for all polynomials $p \in \mathbb{C}[z_1, \ldots, z_n]$,*

$$\|p(T)\|_{\mathcal{B}(\mathcal{H})} \leq \sup_{z \in W \cap \overline{\mathbb{D}}^n} |p(z)|.$$

Proof Let the BCL n-tuple $(M_{\Phi_1}, \ldots, M_{\Phi_n}) \in \mathfrak{B}_0^n(H^2_{\mathcal{E}}(\mathbb{D}))$ be an isometric dilation of T such that $\mathcal{E}$ is finite dimensional. The existence of such a dilation is shown in Theorem 3.1. Moreover, it also follows from Theorem 3.1 that

$$T \cong (P_{\mathcal{Q}} M_{\Phi_1}|_{\mathcal{Q}}, \ldots, P_{\mathcal{Q}} M_{\Phi_n}|_{\mathcal{Q}}),$$

where $\mathcal{Q} = \Pi(\mathcal{H}) \subseteq H^2_{\mathcal{E}}(\mathbb{D})$ is a co-invariant subspace. Then, for $p \in \mathbb{C}[z_1, \ldots, z_n]$,

$$\begin{aligned} \|p(T)\|_{\mathcal{B}(\mathcal{H})} &= \|P_{\mathcal{Q}} p(M_{\Phi_1}, \ldots, M_{\Phi_n})|_{\mathcal{Q}}\|_{\mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}))} \\ &\leq \|p(M_{\Phi_1}, \ldots, M_{\Phi_n})\|_{\mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}))} \\ &= \|M_{p(\Phi_1(z), \ldots, \Phi_n(z))}\|_{\mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}))}. \end{aligned}$$

Now, it is easy to observe that

$$\|M_{p(\Phi_1(z), \ldots, \Phi_n(z))}\|_{\mathcal{B}(H^2_{\mathcal{E}}(\mathbb{D}))} = \|p(\Phi_1(z), \ldots, \Phi_n(z))\|_{H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})}.$$

Clearly, the right-hand side is equal to

$$\sup_{\theta \in [0, 2\pi]} \|p(\Phi_1(e^{i\theta}), \ldots, \Phi_n(e^{i\theta}))\|_{\mathcal{B}(\mathcal{E})}.$$

Thus, we have

$$\|p(T)\|_{\mathcal{B}(\mathcal{H})} \leq \sup_{\theta \in [0, 2\pi]} \|p(\Phi_1(e^{i\theta}), \ldots, \Phi_n(e^{i\theta}))\|_{\mathcal{B}(\mathcal{E})}.$$

Since $\dim(\mathcal{E}) < \infty$, for $z \in \mathbb{T}$, $(\Phi_1(z), \ldots, \Phi_n(z))$ is an n-tuple of commuting unitary matrices. Also since $\Phi_1(z) \cdots \Phi_n(z) = zI_{\mathcal{E}}$ for all $z \in \overline{\mathbb{D}}$, it follows that for any $(z_1, \ldots, z_n) \in \sigma(\Phi_1(z), \ldots, \Phi_n(z))$,

$$\underline{\mathbf{z}} := z_1 \cdots z_n = z. \tag{3.5}$$

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Therefore,

$$\begin{aligned}
&\sup_{z\in\mathbb{T}} \|p(\Phi_1(z),\dots,\Phi_n(z))\|_{\mathcal{B}(\mathcal{E})}\\
&\quad= \sup\{|p(z_1,\dots,z_n)| : (z_1,\dots,z_n)\in\sigma(\Phi_1(z),\dots,\Phi_n(z)), z\in\mathbb{T}\}\\
&\quad= \sup\{|p(\mathbf{z})| : \mathbf{z}=(z_1,\dots,z_n)\in\mathbb{T}^n\cap\sigma(\Phi_1(\underline{z}),\dots,\Phi_n(\underline{z}))\}\ \text{(by (3.5))}\\
&\quad= \sup_{\mathbf{z}\in W\cap\mathbb{T}^n}|p(\mathbf{z})|\\
&\quad\le \sup_{\mathbf{z}\in W\cap\overline{\mathbb{D}}^n}|p(\mathbf{z})|,
\end{aligned}$$

where W is the one-dimensional symmetric algebraic variety as in (3.4). The above computation implies that $\sup_{\theta\in[0,2\pi]}\|p(\Phi_1(e^{i\theta}),\dots,\Phi_n(e^{i\theta}))\|_{\mathcal{B}(\mathcal{E})}\le\sup_{\mathbf{z}\in W\cap\overline{\mathbb{D}}^n}|p(\mathbf{z})|$. This completes the proof. ∎

4 Commutant lifting theorem for $\mathfrak{B}_0^n(\mathcal{H})$

For $T=(T_1,\dots,T_n)\in\mathcal{T}^n(\mathcal{H})$, we denote by $\{T\}'$ the set of all bounded operators on $\mathcal{H}$ which commute with each T_i $(i=1,\dots,n)$; that is,

$$\{T\}' := \{X\in\mathcal{B}(\mathcal{H}) : XT_i = T_iX, i=1,\dots,n\}.$$

The foundation of the commutant lifting theorem for $\mathfrak{B}_0^n(\mathcal{H})$ is the explicit isometric dilation of an element $T\in\mathfrak{B}_0^n(\mathcal{H})$ to a BCL n-tuple $M\in\mathfrak{B}_0^n(H^2_{\mathcal{E}}(\mathbb{D}))$ which we obtained in the previous section. As a result, our proof of the commutant lifting theorem hinges on the detailed analysis of the structure of $\{T\}'$ as well as of $\{M\}'$, where $T\in\mathfrak{B}_0^n(\mathcal{H})$ and $M\in\mathfrak{B}_0^n(H^2_{\mathcal{E}}(\mathbb{D}))$ is a BCL n-tuple. To begin with, we consider the case of BCL n-tuples.

Proposition 4.1 *Let $M=(M_{\Phi_1},\dots,M_{\Phi_n})\in\mathfrak{B}_0^n(H^2_{\mathcal{E}}(\mathbb{D}))$ be a BCL n-tuple corresponding to $(\mathcal{E},\underset{\sim}{U},\underset{\sim}{P})$. Suppose that $\mathcal{E}_i=\operatorname{ran}P_i$ for all $i=1,\dots,n$. If X is a contraction in $\{M\}'$, then X has the following block matrix form with respect to decomposition $H^2_{\mathcal{E}}(\mathbb{D})=H^2_{\mathcal{E}_1}(\mathbb{D})\oplus\cdots\oplus H^2_{\mathcal{E}_n}(\mathbb{D})$:*

$$X=\begin{bmatrix} M_{\Theta_1} & 0 & \cdots & 0\\ 0 & M_{\Theta_2} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & M_{\Theta_n}\end{bmatrix}, \tag{4.1}$$

where for all $i=1,\dots,n$, Θ_i is a contractive $\mathcal{B}(\mathcal{E}_i)$-valued analytic function on $\mathbb{D}$.

Proof For $X\in\{M\}'$, since $XM_z=M_zX$, then there exists $\Theta\in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ such that $X=M_\Theta$. Let $\Theta(z)=\sum_{k\ge0}A_kz^k$, where $A_k\in\mathcal{B}(\mathcal{E})$ $(k\ge0)$. We fix $i\in\{1,\dots,n\}$. Since $XM_{\Phi_i}=M_{\Phi_i}X$, then the identity $\Phi_i(z)\Theta(z)=\Theta(z)\Phi_i(z)$ implies that for all $z\in\mathbb{D}$,

$$U_iP_i^\perp A_0+\sum_{k\ge1}(U_iP_i^\perp A_k+U_iP_iA_{k-1})z^k = A_0U_iP_i^\perp+\sum_{k\ge1}(A_kU_iP_i^\perp+A_{k-1}U_iP_i)z^k.$$

Comparing the coefficients, we obtain

$$U_iP_i^\perp A_0=A_0U_iP_i^\perp \text{ and } U_iP_i^\perp A_k+U_iP_iA_{k-1}=A_kU_iP_i^\perp+A_{k-1}U_iP_i\ (k\ge1). \tag{4.2}$$

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Note that as $M \in \mathfrak{B}_0^n(H^2_{\mathcal{E}}(\mathbb{D}))$, we have $U_iP_j = P_jU_i$ for all $i, j = 1, \ldots, n$. Then the identity $U_iP_i^{\perp}A_0 = A_0U_iP_i^{\perp}$ together with $A_0^*P_i^{\perp}U_i^* = P_i^{\perp}U_i^*A_0^*$, obtained by taking adjoint, imply that $\mathcal{E}_i^{\perp}$ is a reducing subspace for A_0. That is, $A_0P_i^{\perp} = P_i^{\perp}A_0$. Letting $k = 1$ in 4.2, we get

$$U_iP_i^{\perp}A_1 + U_iP_iA_0 = A_1U_iP_i^{\perp} + A_0U_iP_i.$$

Consequently,

$$U_iP_i^{\perp}A_1P_i^{\perp} = (U_iP_i^{\perp}A_1 + U_iP_iA_0)P_i^{\perp} = (A_1U_iP_i^{\perp} + A_0U_iP_i)P_i^{\perp} = A_1U_iP_i^{\perp}.$$

By the same argument as before, the above identity together with $P_i^{\perp}A_1^*P_i^{\perp}U_i^* = P_i^{\perp}U_i^*A_1^*$ imply that $\mathcal{E}_i^{\perp}$ is a reducing subspace for A_1. Proceeding in the same way, we can show that $\mathcal{E}_i^{\perp}$ is a reducing subspace for A_k for all $k \geq 2$. We set $A_k^{(i)} := A_k|_{\mathcal{E}_i}$ for all $k \geq 0$, and $\Theta_i(z) := \sum_{k\geq 0} A_k^{(i)}z^k$. Then $\Theta_i \in H^{\infty}_{\mathcal{B}(\mathcal{E}_i)}(\mathbb{D})$. Since i was arbitrarily chosen, we have $\Theta_i \in H^{\infty}_{\mathcal{B}(\mathcal{E}_i)}(\mathbb{D})$ for all $i = 1, \ldots, n$, and it is now easy to see that

$$\Theta(z) = \oplus_{i=1}^n \Theta_i(z), \text{ and therefore } M_{\Theta} = \oplus_{i=1}^n M_{\Theta_i}.$$

This completes the proof. ∎

As a consequence of the above result, we find certain decomposition of the defect operator next.

Proposition 4.2 *Let $M = (M_{\Phi_1}, \ldots, M_{\Phi_n}) \in \mathfrak{B}_0^n(H^2_{\mathcal{E}}(\mathbb{D}))$ be a BCL n-tuple corresponding to $(\mathcal{E}, \underset{\sim}{U}, \underset{\sim}{P})$. If X is a contraction in $\{M\}'$, then there exist positive operators $G_1, \ldots, G_n$ on $H^2_{\mathcal{E}}(\mathbb{D})$ such that*

$$I - XX^* = G_1 + \cdots + G_n \quad \text{and} \quad (I - C_{M_{\Phi_j}})(G_i) = G_i - M_{\Phi_j}G_iM_{\Phi_j}^* = 0 \quad (i \neq j).$$

Proof Since X is a contraction in $\{M\}'$, using Proposition 4.1, X has the block matrix from as in 4.1 for some $\Theta_i \in H^{\infty}_{\mathcal{B}(\mathcal{E}_i)}(\mathbb{D})(i = 1, \ldots, n)$. We consider

$$G_i = \begin{bmatrix} 0 & \ldots & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \ldots & I_{H^2_{\mathcal{E}_i}(\mathbb{D})} - M_{\Theta_i}M_{\Theta_i}^* & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \ldots & 0 \end{bmatrix}, \quad (i = 1, \ldots, n)$$

where the block matrix representation is taken with respect to the decomposition $H^2_{\mathcal{E}}(\mathbb{D}) = H^2_{\mathcal{E}_1}(\mathbb{D}) \oplus \cdots \oplus H^2_{\mathcal{E}_n}(\mathbb{D})$. Since M_{Θ_i} is a contraction, it follows that $G_i \geq 0$ and $I - XX^* = G_1 + \cdots + G_n$. Now for $i \neq j$, using the block matrix form 2.3 of M_{Φ_j} and $M_{\Phi_j}X = XM_{\Phi_j}$, it can be easily verified that $M_{\Phi_j}G_iM_{\Phi_j}^* = G_i$. The proof now follows. ∎

Let $T = (T_1, \ldots, T_d) \in \mathfrak{B}_0^n(\mathcal{H})$ and $M = (M_{\Phi_1}, \ldots, M_{\Phi_n}) \in \mathfrak{B}_0^n(H^2_{\mathcal{E}}(\mathbb{D}))$ be an isometric dilation of T. That is, there exists an isometry $\Pi : \mathcal{H} \to H^2_{\mathcal{E}}(\mathbb{D})$ such that

$$\Pi T_i^* = M_{\Phi_i}^*\Pi \quad (i = 1, \ldots, n).$$

For a contractive $X \in \{T\}'$, if there exists a contractive multiplier $\Theta \in H^{\infty}_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ such that $M_{\Theta} \in \{M\}'$ and $\Pi X^* = M_{\Theta}^*\Pi$, then by the above proposition, $I - M_{\Theta}M_{\Theta}^* =$

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$G_1 + \cdots + G_n$ for some positive operators G_i on $H^2_{\mathcal{E}}(\mathbb{D})$ such that $(I - C_{M_{\Phi_j}})(G_i) = G_i - M_{\Phi_j} G_i M^*_{\Phi_j} = 0 \quad (i \neq j)$. Then observe that

$$I - XX^* = \Pi^*(I - M_\Theta M^*_\Theta)\Pi = \Pi^* G_1 \Pi + \cdots + \Pi^* G_n \Pi$$

and

$$(I - C_{T_j})(\Pi^* G_i \Pi) = \Pi^*(I - C_{M_{\Phi_j}})(G_i)\Pi = 0 \quad (i \neq j).$$

Thus, the condition that there should exist positive operators $\tilde{G}_1, \ldots \tilde{G}_n$ on $\mathcal{H}$ for which

$$I - XX^* = \tilde{G}_1 + \cdots + \tilde{G}_n, \text{ and } (I - C_{T_j})(\tilde{G}_i) = 0, \quad (i \neq j)$$

is necessary for the commutant lifting theorem to hold. In fact, we show below that every commutant of $T \in \mathfrak{B}^n_0(\mathcal{H})$ indeed satisfies the necessary condition for which we need the following lemma.

Lemma 4.3 *Let $T = (T_1, \ldots, T_n) \in \mathcal{T}^n(\mathcal{H})$ be such that $\Delta_T^{e_i + e_j} = 0 (i \neq j)$. Then $I - P_T P_T^* = \sum_{i=1}^n D^2_{T_i}$, where $P_T = T_1 \cdots T_n$ and D_{T_i} is the defect operator of T_i.*

Proof Using Agler's hereditary functional calculus ([1]) to the following identity, for $(z_1, \ldots, z_n)$ and $(w_1, \ldots, w_n) \in \mathbb{D}^n$,

$$\begin{aligned}(1 - z_1\bar{w}_1 \cdots z_n\bar{w}_n) &= \sum_{i=1}^n (1 - z_i\bar{w}_i) - \sum_{\substack{i,j=1 \\ i<j}}^n (1 - z_i\bar{w}_i)(1 - z_j\bar{w}_j) + \cdots + (-1)^{n-1}(1 - z_1\bar{w}_1)\cdots(1 - z_n\bar{w}_n),\end{aligned}$$

we get

$$I - P_T P_T^* = \sum_{i=1}^n D^2_{T_i} - \sum_{\substack{i,j=1 \\ i<j}}^n \Delta_T^{\mathbf{e}_i + \mathbf{e}_j} + \sum_{\substack{i,j,k=1 \\ i<j<k}}^n \Delta_T^{\mathbf{e}_i + \mathbf{e}_j + \mathbf{e}_k} - \cdots + (-1)^{n-1} \Delta_T^{\mathbf{e}_i + \cdots + \mathbf{e}_n}.$$

Hence, by Lemma 2.1, $I - P_T P_T^* = \sum_{j=1}^n D^2_{T_j}$. ∎

For $T = (T_1, \ldots, T_n) \in \mathcal{T}^n(\mathcal{H})$, we denote by $\tilde{T}_j$ the $(n-1)$-tuple obtained from T by deleting T_j; that is,

$$\tilde{T}_j := (T_1, \ldots, T_{j-1}, T_{j+1}, \ldots, T_n).$$

Theorem 4.4 *Let $T = (T_1, \ldots, T_n) \in \mathfrak{B}^n_0(\mathcal{H})$. If X is a contraction in $\{T\}'$, then there exist positive operators $G_1, \ldots, G_n$ on $\mathcal{H}$ such that*

$$I - XX^* = G_1 + \cdots + G_n \text{ and } (I - C_{T_i})(G_j) = G_j - T_i G_j T_i^* = 0 \quad (i \neq j). \tag{4.3}$$

Proof First, we define

$$G_j = \text{SOT} - \lim_{\alpha = (\alpha_1, \ldots, \alpha_{n-1}) \to \infty} \tilde{T}_j^\alpha (I - XX^*) \tilde{T}_j^{*\alpha} \quad (j = 1, \ldots, n).$$

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Clearly, the above SOT limit exists as T_i's are contractions. It then follows from the definition that $(I - C_{T_i})(G_j) = G_j - T_i G_j T_i^* = 0$ for all $i \neq j$. Now,

$$\sum_{j=1}^{n} G_j = \text{SOT} - \lim_{\alpha\to\infty} \sum_{j=1}^{n} \tilde{T}_j^{\alpha}(I - XX^*)\tilde{T}_j^{*\alpha} = \text{SOT} - \lim_{\alpha\to\infty} \Big(\sum_{j=1}^{n} \tilde{T}_j^{\alpha}\tilde{T}_j^{*\alpha} - X\big(\sum_{j=1}^{n} \tilde{T}_j^{\alpha}\tilde{T}_j^{*\alpha}\big)X^* \Big).$$

To complete the proof, it is enough to show that $\text{SOT} - \lim_{\alpha\to\infty} \sum_{j=1}^{n} \tilde{T}_j^{\alpha}\tilde{T}_j^{*\alpha} = I_{\mathcal{H}}$. Since $\Delta_T^{\mathbf{e}_i+\mathbf{e}_j} = 0$ (that is, $D_{T_j}^2 = T_i D_{T_j}^2 T_i^*$ $(i \neq j)$), then

$$D_{T_j}^2 = \tilde{T}_j^{\alpha} D_{T_j}^2 \tilde{T}_j^{*\alpha} \quad (j = 1, \ldots, n, \alpha = (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{Z}_+^{n-1}).$$

Taking sum over j in both sides to the above identity, we get for all $\alpha \in \mathbb{Z}_+^{n-1}$,

$$\begin{aligned}\sum_{j=1}^{n} \tilde{T}_j^{\alpha}\tilde{T}_j^{*\alpha} &= \sum_{j=1}^{n} D_{T_j}^2 + \sum_{j=1}^{n} \tilde{T}_j^{\alpha} T_j T_j^* \tilde{T}_j^{*\alpha} \\ &= I_{\mathcal{H}} - P_T P_T^* + \sum_{j=1}^{n} \tilde{T}_j^{\alpha} T_j T_j^* \tilde{T}_j^{*\alpha} \quad \text{(by Lemma 4.3)}.\end{aligned}$$

Consequently, for $\alpha \geq \mathbf{e} = (1, \ldots, 1)$,

$$I_{\mathcal{H}} - \sum_{j=1}^{n} \tilde{T}_j^{\alpha}\tilde{T}_j^{*\alpha} = P_T\big(I_{\mathcal{H}} - \sum_{j=1}^{n} \tilde{T}_j^{\alpha-e}\tilde{T}_j^{*\alpha-e}\big)P_T^*.$$

Set $Y_\alpha := I_{\mathcal{H}} - \sum_{j=1}^{n} \tilde{T}_j^{\alpha}\tilde{T}_j^{*\alpha}$, $(\alpha \in \mathbb{Z}_+^{n-1})$. By the above identity, for $\alpha \geq \mathbf{e}$, if $k = \min\{\alpha_1, \ldots, \alpha_{n-1}\}$, then

$$Y_\alpha = P_T Y_{\alpha-\mathbf{e}} P_T^* = \cdots = P_T^k Y_{\alpha-k\mathbf{e}} P_T^{*k}.$$

Thus, for $h \in \mathcal{H}$,

$$\|Y_\alpha h\| = \|P_T^k Y_{\alpha-k\mathbf{e}} P_T^{*k} h\| \leq \|Y_{\alpha-k\mathbf{e}}\| \|P_T^{*k} h\| \leq (n+1)\|P_T^{*k} h\|.$$

Since P_T is pure, it then follows that $\text{SOT} - \lim_{\alpha\to\infty} Y_\alpha = 0$. This completes the proof. ∎

The identity (4.3), in the above theorem, is crucial for commutant lifting theorem to hold, which is one of the main results of this section. The reader familiar with the work of [9] could easily recognize the identity (4.3) and must be tempted to use [9] to prove the commutant lifting theorem. Despite the similarity, one cannot use [9] directly, as the isometric dilation of T as well as the dilation map are completely different. As a result, our proof is entirely different compared to that of [9]. Before going into the commutant lifting theorem, we need a couple of preparatory results. The first one of them is of independent interest. It is well-known that given any contractive $\mathcal{B}(\mathcal{E})$-valued analytic function Θ, there exist an auxiliary Hilbert space $\mathcal{F}$ and a contraction

$$W = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{E} \oplus \mathcal{F} \to \mathcal{E} \oplus \mathcal{F}$$

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such that $\Theta(z) = \tau_W(z)$, where τ_W is the transfer function corresponding to W; that is,

$$\tau_W(z) = A + zB(I_{\mathcal{F}} - zD)^{-1}C \quad (z \in \mathbb{D}).$$

Moreover, the contraction W can be taken to be a unitary if necessary. This result is known as transfer function realizations for operator valued bounded analytic functions on the unit disc. Recall that a contraction T on $\mathcal{H}$ is said to be a *completely non-coisometric* if there is no nontrivial invariant subspace of T^* on which T^* is an isometry. A contraction T is said to be a *completely nonunitary* (cnu) if there is no nonzero T-reducing subspace $\mathcal{S}$ of $\mathcal{H}$ such that $T|_{\mathcal{S}}$ is a unitary operator. For a contraction T on $\mathcal{H}$, there exist unique T-reducing subspaces $\mathcal{H}_{cnu}$ and $\mathcal{H}_u$ such that $\mathcal{H} = \mathcal{H}_{cnu} \oplus \mathcal{H}_u$, $T_{cnu} := T|_{\mathcal{H}_{cnu}}$ is a cnu, and $T_u := T|_{\mathcal{H}_u}$ is a unitary (see [31, Theorem 3.2]). Thus, T has the following block matrix representation of T with respect to the decomposition $\mathcal{H}_{cnu} \oplus \mathcal{H}_u$:

$$T = \begin{bmatrix} T_{cnu} & 0 \\ 0 & T_u \end{bmatrix}.$$

Such a decomposition is commonly referred to as canonical decomposition of T. If $U \in \mathcal{B}(\mathcal{E})$ is an isometry, in the result below, we find a necessary and sufficient condition for the commutativity of $\tau_W(z)$ and U for all $z \in \mathbb{D}$ in terms of W and U.

Theorem 4.5 *Let $\tau_W(z) = A + zB(I - zD)^{-1}C$ be the transfer function corresponding to a unitary $W = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ on $\mathcal{E} \oplus \mathcal{F}$ for some Hilbert spaces $\mathcal{E}$ and $\mathcal{F}$, and let U be an isometry on $\mathcal{E}$. Let*

$$D = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix}$$

be the canonical decomposition of D into the cnu part D_1 and the unitary part D_2 with respect to the decomposition $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$. Consider the following statements.

(1) $M_{\tau_W}(I_{H^2(\mathbb{D})} \otimes U) = (I_{H^2(\mathbb{D})} \otimes U)M_{\tau_W}$.

(2) *There exists a contraction Y on $\mathcal{F}$ such that $W\tilde{U} = \tilde{U}W$, where $\tilde{U} = \begin{bmatrix} U & 0 \\ 0 & Y \end{bmatrix}$.*

Then (2) $\Longrightarrow$ (1). *Moreover, if D_1 is completely non-coisometric, then* (1) $\Longrightarrow$ (2).

Proof Using the power series expansion of $\tau_W(z)$, it is easy to see that the identity in (1) is equivalent to

$$AU = UA \text{ and } (BD^{k-1}C)U = U(BD^{k-1}C) \quad (k \geq 1). \tag{4.4}$$

Now we assume that (2) holds. Then by a routine block matrix multiplication, $AU = UA$, $BY = UB$, $CU = YC$, and $DY = YD$. Thus, $AU = UA$ and

$$BD^{k-1}CU = BD^{k-1}YC = BYD^{k-1}C = UBD^{k-1}C \quad (k \geq 1).$$

Hence, by the observation we made earlier, $M_{\tau_W}(I_{H^2(\mathbb{D})} \otimes U) = (I_{H^2(\mathbb{D})} \otimes U)M_{\tau_W}$. This proves (2) $\Longrightarrow$ (1).

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For the proof of the moreover part, note that the block matrix representation of W with respect to the decomposition $\mathcal{E} \oplus \mathcal{F}_1 \oplus \mathcal{F}_2$ is

$$W = \begin{bmatrix} A & B|_{\mathcal{F}_1} & 0 \\ P_{\mathcal{F}_1} C & D_1 & 0 \\ 0 & 0 & D_2 \end{bmatrix}.$$

Since W is a unitary,

$$A^*A + C^*P_{\mathcal{F}_1}C = I_{\mathcal{E}}, \quad P_{\mathcal{F}_1}B^*B|_{\mathcal{F}_1} + D_1^*D_1 = I_{\mathcal{F}_1},$$

$$P_{\mathcal{F}_1}B^*A + D_1^*P_{\mathcal{F}_1}C = 0, \text{ and } P_{\mathcal{F}_1}CC^*|_{\mathcal{F}_1} + D_1D_1^* = I_{\mathcal{F}_1}.$$

Claim:

$$\bigvee_{k\geq 0} D_1^k P_{\mathcal{F}_1} C(\mathcal{E}) = \mathcal{F}_1.$$

The proof of the claim follows from [8, Theorem 4.2]. However, we supply a proof for the completeness. To this end, it is enough to show that $\mathcal{S} := (\mathcal{F}_1 \ominus \bigvee_{k\geq 0} D_1^k P_{\mathcal{F}_1} C(\mathcal{E})) = \{0\}$. *Let* $h \in \mathcal{S}$; *that is,* $h \in \mathcal{F}_1$ *and* $h \perp \bigvee_{k\geq 0} D_1^k P_{\mathcal{F}_1} C(\mathcal{E})$. *Therefore,* $h \perp D_1^k P_{\mathcal{F}_1} C\eta$ $(k \geq 0,$ $\eta \in \mathcal{E})$. *In particular,* $h \perp P_{\mathcal{F}_1} C\eta$ $(\eta \in \mathcal{E})$; *that is,*

$$\langle h, P_{\mathcal{F}_1} C\eta \rangle = 0 \quad (\eta \in \mathcal{E}).$$

This implies $h \in \ker C^*$. *In other words,* $\mathcal{S} \subseteq \ker C^*$. *Now it is easy to see that* $\mathcal{S}$ *is a* D_1^**-invariant subspace and the identity* $P_{\mathcal{F}_1}CC^* + D_1D_1^* = I_{\mathcal{F}}$ *restricted on* $\mathcal{S}$ *becomes* $D_1D_1^*|_{\mathcal{S}} = I_{\mathcal{S}}$, *which contradicts that* D_1 *is completely non-coisometric. Hence, the claim is proved.* ∎

We now proceed to construct a contraction Y on $\mathcal{F}$ such that $W\tilde{U} = \tilde{U}W$; that is,

$$BY = UB, CU = YC, \text{ and } DY = YD.$$

Since $\mathcal{F}_1 = \bigvee_{k\geq 0} D_1^k P_{\mathcal{F}_1} C(\mathcal{E})$, we define

$$Y : \mathcal{F}_1 \oplus \mathcal{F}_2 \to \mathcal{F}_1 \oplus \mathcal{F}_2, \text{ by } (D_1^k P_{\mathcal{F}_1} C\eta, f) \mapsto (D_1^k P_{\mathcal{F}_1} CU\eta, 0) \quad (\eta \in \mathcal{E}, f \in \mathcal{F}_2, k \geq 0).$$

It remains to show that Y is a well-defined contraction on $\mathcal{F}$ and $BY = UB$ as the identities $CU = YC$ and $DY = YD$ are immediate from the definition of Y. In fact, the reader must have noticed that it is the only way one could define Y with required properties, and therefore, Y is unique in this case. We show below that $Y|_{\mathcal{F}_1}$ is a unitary. To this end, it is enough to show that

$$\langle D_1^k P_{\mathcal{F}_1} C\eta, D_1^l P_{\mathcal{F}_1} C\eta' \rangle = \langle D_1^k P_{\mathcal{F}_1} CU\eta, D_1^l P_{\mathcal{F}_1} CU\eta' \rangle$$

for all $k, l \in \mathbb{Z}_+$ and $\eta, \eta' \in \mathcal{E}$. When $l = k = 0$, note that

$$\begin{aligned} \langle P_{\mathcal{F}_1} C\eta, P_{\mathcal{F}_1} C\eta' \rangle &= \langle \eta, (I_{\mathcal{E}} - A^*A)\eta' \rangle \quad (\text{as } A^*A + C^*P_{\mathcal{F}_1}C = I_{\mathcal{E}}) \\ &= \langle U\eta, U\eta' \rangle - \langle UA\eta, UA\eta' \rangle \quad (\text{as } U \text{ is an isometry}) \\ &= \langle U\eta, U\eta' \rangle - \langle AU\eta, AU\eta' \rangle \quad (\text{as } AU = UA) \\ &= \langle P_{\mathcal{F}_1} CU\eta, P_{\mathcal{F}_1} CU\eta' \rangle. \end{aligned}$$

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For the general case, we use the following identity repeatedly:

$$\begin{aligned}\langle D_1^m P_{\mathcal{F}_1} C\eta, D_1^n P_{\mathcal{F}_1} C\eta'\rangle &= \langle D_1^{m-1} P_{\mathcal{F}_1} C\eta, D_1^* D_1 D_1^{n-1} P_{\mathcal{F}_1} C\eta'\rangle \\ &= \langle D_1^{m-1} P_{\mathcal{F}_1} C\eta, D_1^{n-1} P_{\mathcal{F}_1} C\eta'\rangle - \langle BD_1^{m-1} P_{\mathcal{F}_1} C\eta, BD_1^{n-1} P_{\mathcal{F}_1} C\eta'\rangle, \end{aligned} \tag{4.5}$$

as $P_{\mathcal{F}_1} B^*B|_{\mathcal{F}_1} + D_1^* D_1 = I_{\mathcal{F}_1}$, where $m, n \geq 1$. Now consider the case when at least one of l and k is nonzero. We assume, without any loss of generality, that $k \geq l$. Then for any $\eta, \eta' \in \mathcal{E}$, by applying (4.5) l times, we have

$$\begin{aligned}\langle D_1^k P_{\mathcal{F}_1} C\eta, D_1^l P_{\mathcal{F}_1} C\eta'\rangle &= \langle D_1^{k-l} P_{\mathcal{F}_1} C\eta, P_{\mathcal{F}_1} C\eta'\rangle - \sum_{j=1}^{l} \langle BD_1^{k-j} P_{\mathcal{F}_1} C\eta, BD_1^{l-j} P_{\mathcal{F}_1} C\eta'\rangle \\ &= \langle D_1^{k-l-1} P_{\mathcal{F}_1} C\eta, D_1^* P_{\mathcal{F}_1} C\eta'\rangle - \sum_{j=1}^{l} \langle BD_1^{k-j} P_{\mathcal{F}_1} C\eta, BD_1^{l-j} P_{\mathcal{F}_1} C\eta'\rangle \\ &= -\langle D_1^{k-l-1} P_{\mathcal{F}_1} C\eta, B^* A\eta'\rangle - \sum_{j=1}^{l} \langle BD_1^{k-j} P_{\mathcal{F}_1} C\eta, BD_1^{l-j} P_{\mathcal{F}_1} C\eta'\rangle \\ &= -\langle BD_1^{k-l-1} P_{\mathcal{F}_1} C\eta, A\eta'\rangle - \sum_{j=1}^{l} \langle BD_1^{k-j} P_{\mathcal{F}_1} C\eta, BD_1^{l-j} P_{\mathcal{F}_1} C\eta'\rangle. \end{aligned}$$

In the above identity, replacing η and η' by $U\eta$ and $U\eta'$, respectively, and using (4.4), we get

$$\begin{aligned}&\langle D_1^k P_{\mathcal{F}_1} CU\eta, D_1^l P_{\mathcal{F}_1} CU\eta'\rangle \\ &= -\langle BD_1^{k-l-1} P_{\mathcal{F}_1} CU\eta, AU\eta'\rangle - \sum_{j=1}^{l} \langle BD_1^{k-j} P_{\mathcal{F}_1} CU\eta, BD_1^{l-j} P_{\mathcal{F}_1} CU\eta'\rangle \\ &= -\langle UBD_1^{k-l-1} P_{\mathcal{F}_1} C\eta, UA\eta'\rangle - \sum_{j=1}^{l} \langle UBD_1^{k-j} P_{\mathcal{F}_1} C\eta, UBD_1^{l-j} P_{\mathcal{F}_1} C\eta'\rangle \\ &= -\langle BD_1^{k-l-1} P_{\mathcal{F}_1} C\eta, A\eta'\rangle - \sum_{j=1}^{l} \langle BD_1^{k-j} P_{\mathcal{F}_1} C\eta, BD_1^{l-j} P_{\mathcal{F}_1} C\eta'\rangle \\ &= \langle D_1^k P_{\mathcal{F}_1} C\eta, D_1^l P_{\mathcal{F}_1} C\eta'\rangle. \end{aligned}$$

Thus, Y is a unitary on $\mathcal{F}_1$. From the definition of Y, it follows that Y is a contraction. Now for $g = \sum_{k=0}^{l} D_1^k P_{\mathcal{F}_1} C\eta_k$ ($\eta_k \in \mathcal{E}$) and $f \in \mathcal{F}_2$,

$$\begin{aligned} BY(g, f) = BY\Big(\sum_{k=0}^{l} D_1^k P_{\mathcal{F}_1} C\eta_k, f\Big) &= \Big(\sum_{k=0}^{l} BD_1^k P_{\mathcal{F}_1} CU\eta_k, 0\Big) \\ &= UB\Big(\sum_{k=0}^{l} D_1^k P_{\mathcal{F}_1} C\eta_k, f\Big) \\ &= UB(g, f). \end{aligned}$$

Therefore, $BY = UB$, and this completes the proof.

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The other result that we need appeared in different contexts before (see [17, Lemma 4.1] and [23, Lemma 2.1]). According to our need, we present a slightly modified version.

Lemma 4.6 *Let (S, T, X) be a triple of commuting contractions on $\mathcal{H}$ such that T is pure, and let G and Q be two contractions on $\mathcal{H}$ such that $\overline{ran}\, G \subseteq \mathcal{G}$ and $\overline{ran}\, Q \subseteq \mathcal{K}$ for some Hilbert spaces $\mathcal{G}$ and $\mathcal{K}$. Assume that $\Pi : \mathcal{H} \to H^2_{\mathcal{K}}(\mathbb{D})$, defined by $h \mapsto \sum_{k\geq 0} z^k QS^* T^{*k} h$, is a bounded operator and*

$$W = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{K} \oplus \mathcal{G} \to \mathcal{K} \oplus \mathcal{G}$$

is a contraction satisfying

$$W(Qh, GT^* h) = (QX^* h, Gh) \quad (h \in \mathcal{H}).$$

Then the transfer function $\Theta(z) = \tau_{W^*}(z) = A^* + C^* z(I - zD^*)^{-1} B^* (z \in \mathbb{D})$, corresponding to the contraction W^*, satisfies

$$\Pi X^* = M^*_\Theta \Pi.$$

Proof Since

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} Qh \\ GT^* h \end{bmatrix} = \begin{bmatrix} QX^* h \\ Gh \end{bmatrix},$$

we have

$$AQh + BGT^* h = QX^* h, \tag{4.6}$$

and

$$CQh + DGT^* h = Gh. \tag{4.7}$$

Now, replacing h by $T^* h$ in 4.7, we have

$$GT^* h = CQT^* h + DGT^{*2} h.$$

Then, substituting $GT^* h$ by $CQT^* h + DGT^{*2} h$ in 4.6, we obtain

$$QX^* h = AQh + BCQT^* h + BDGT^{*2} h.$$

By repeating the procedure m times, we get

$$QX^* h = AQh + \sum_{k=0}^{m-1} BD^k CQT^{*k+1} h + BD^m GT^{*m+1} h.$$

Since T is pure, for all $h \in \mathcal{H}$, $\|BD^m GT^{*m+1} h\| \leq \|T^{*m+1} h\| \to 0$ as $m \to \infty$. Therefore,

$$QX^* = AQ + \sum_{k=0}^{\infty} BD^k CQT^{*k+1}. \tag{4.8}$$

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Now, for $k \in \mathbb{Z}_+, \eta \in \mathcal{E}, h \in \mathcal{H}$,

$$\begin{aligned}\langle M_\Theta^* \Pi h, z^k \eta\rangle &= \langle \sum_{m\geq 0} z^m Q S^{*m} T^{*m} h, (A^* + C^* z(I - zD^*)^{-1} B^*) z^k \eta\rangle \\ &= \langle \sum_{m\geq 0} z^m Q S^{*m} T^{*m} h, (A^* + \sum_{m\geq 0} z^{m+1} C^* D^{*m} B^*) z^k \eta\rangle \\ &= \langle AQS^* T^{*k} h, \eta\rangle + \sum_{m\geq 0} \langle BD^m CQS^* T^{*m+k+1} h, \eta\rangle \\ &= \langle \big(AQ + \sum_{m\geq 0} BD^m CQT^{*m+1}\big) S^* T^{*k} h, \eta\rangle \\ &= \langle QX^* S^* T^{*k} h, \eta\rangle \quad \text{(by the identity 4.8)},\end{aligned}$$

and on the other hand,

$$\langle \Pi X^* h, z^k \eta\rangle = \langle \sum_{m\geq 0} z^m Q S^{*m} T^{*m} X^* h, z^k \eta\rangle = \langle QX^* S^* T^{*k} h, \eta\rangle.$$

Hence, $\Pi X^* = M_\Theta^* \Pi$. This completes the proof. ∎

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, we define $\alpha_+ := (\alpha_1^+, \ldots, \alpha_n^+) \in \mathbb{Z}_+^n$ and $\alpha_- := (\alpha_1^-, \ldots, \alpha_n^-) \in \mathbb{Z}_+^n$, where $\alpha_i^+ = \max\{\alpha_i, 0\}$ and $\alpha_i^- = \max\{-\alpha_i, 0\}$ $(i = 1, \ldots, n)$. We are now ready to prove the commutant lifting theorem.

Theorem 4.7 *Let $T = (T_1, \ldots, T_n) \in \mathfrak{B}_0^n(\mathcal{H})$. Suppose that the isometric dilation of T is given by the BCL n-tuple $M = (M_{\Phi_1}, \ldots, M_{\Phi_n}) \in \mathfrak{B}_0^n(H_\mathcal{E}^2(\mathbb{D}))$ via the dilation map $\Pi : \mathcal{H} \to H_\mathcal{E}^2(\mathbb{D})$ for some Hilbert space $\mathcal{E}$, as in Theorem 3.2. If X is a contraction in $\{T\}'$, then there exists a contractive multiplier $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ such that $M_\Theta \in \{M\}'$ and $\Pi X^* = M_\Theta^* \Pi$.*

Proof Since the proof uses the explicit isometric dilation of $T \in \mathfrak{B}_0^n(\mathcal{H})$ as constructed in Theorem 3.2, we first recall some of the necessary ingredients from the proof of Theorem 3.2. By Theorem 3.2, the Hilbert space $\mathcal{E}$ has the decomposition $\mathcal{E} = \oplus_{j=1}^n \mathcal{K}_j$, where $\mathcal{D}_{T_j} \subseteq \mathcal{K}_j$ for all $j = 1, \ldots, n$. The dilation map $\Pi : \mathcal{H} \to H_\mathcal{E}^2(\mathbb{D})$ is an isometry given by

$$\Pi h = \sum_{k=0}^\infty z^k (D_{T_1} P_T^{*k} h, D_{T_2} T_1^* P_T^{*k} h, \ldots, D_{T_n} T_1^* \cdots T_{n-1}^* P_T^{*k} h) \quad (h \in \mathcal{H}).$$

The dilating BCL n-tuple M corresponds to the triple $(\mathcal{E}, \underset{\sim}{U}, \underset{\sim}{P})$, where $\underset{\sim}{U} = (U_1, \ldots, U_n)$ and $\underset{\sim}{P} = (P_1, \ldots, P_n)$ are defined as

(4.9)

$$U_i^* = \bigoplus_{j=1}^n W_j^{(i)} \text{ and } P_i(x_1, \ldots, x_n) = (0, \ldots 0, x_i, 0, \ldots 0) \quad (x_l \in \mathcal{K}_l, l = 1, \ldots, n),$$

for some unitaries $W_j^{(i)} : \mathcal{K}_j \to \mathcal{K}_j$ satisfying

$$W_j^{(i)}(D_{T_j} h) = D_{T_j} T_i^* h \quad (i \neq j) \text{ and } W_j^{(j)}(D_{T_j} P_T^* h) = D_{T_j} T_j^* h \tag{4.10}$$

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for all $h \in \mathcal{H}$. Moreover, for all $j = 1, \ldots, n$,

$$W_j^{(j)} = W_j^{(1)*} \cdots W_j^{(j-1)*} W_j^{(j+1)*} \cdots W_j^{(n)*}. \tag{4.11}$$

For all $j = 1, \ldots, n$, we define

$$\Pi_j : \mathcal{H} \to H^2_{\mathcal{K}_j}(\mathbb{D}) \text{ by } h \mapsto \sum_{k=0}^{\infty} z^k D_{T_j} T_1^* \cdots T_{j-1}^* P_T^{*k} h.$$

It is now clear that for all $h \in \mathcal{H}$, $\Pi h = \oplus_{j=1}^n \Pi_j h$. Suppose any contractive multiplier $\Theta \in H^\infty_{\mathcal{B}(\mathcal{E})}(\mathbb{D})$ is in $\{M\}'$. Then, by Proposition 4.1, M_Θ has the form $M_\Theta = \oplus_{j=1}^n M_{\Theta_j}$, where $\Theta_j \in H^\infty_{\mathcal{B}(\mathcal{K}_j)}(\mathbb{D})$ for all $j = 1, \ldots, n$. By the explicit form of M_{Φ_i}, as in 2.3, M_Θ commutes with M_{Φ_i} for all $i = 1, \ldots, n$ if and only if

$$\Theta_j(z) W_j^{(i)} = W_j^{(i)} \Theta_j(z) \quad (z \in \mathbb{D}, i, j = 1, \ldots, n). \tag{4.12}$$

Moreover, $\Pi X^* = M_\Theta^* \Pi$ if and only if for all $h \in \mathcal{H}$,

$$\bigoplus_{j=1}^n \Pi_j X^* h = \bigoplus_{j=1}^n M_{\Theta_j}^* \Pi_j h, \text{ equivalently } \Pi_j X^* = M_{\Theta_j}^* \Pi_j \quad (j = 1, \ldots, n). \tag{4.13}$$

Thus, by the above discussion, to complete the proof, we need to construct $\Theta_j \in H^\infty_{\mathcal{B}(\mathcal{K}_j)}(\mathbb{D})$ for all $j = 1, \ldots, n$ such that (4.13) and (4.12) holds. To this end, for the rest of the proof, we fix j and find an auxiliary Hilbert space $\mathcal{G}$ and a contraction R on $\mathcal{K}_j \oplus \mathcal{G}$ such that $\Theta = \tau_{R^*}$, the transfer function corresponding to R^*. Also note that for τ_{R^*} to satisfy equation (4.12), by Theorem 4.5, it is sufficient to find contractions Y_i on $\mathcal{G}$ such that

$$R^* \begin{bmatrix} W_j^{(i)} & 0 \\ 0 & Y_i \end{bmatrix} = \begin{bmatrix} W_j^{(i)} & 0 \\ 0 & Y_i \end{bmatrix} R^* \quad (i = 1, \ldots, n). \tag{4.14}$$

This is the point of departure to the construction of the space $\mathcal{G}$ and contractions R and Y_i with required properties. Since X is a contraction in $\{T\}'$, by Theorem 4.4, there exist positive operators $G_1, \ldots, G_n$ on $\mathcal{H}$ such that

$$I - XX^* = G_1 + \cdots + G_n \text{ and } (I - C_{T_i})(G_m) = G_m - T_i G_m T_i^* = 0 \quad (i \neq m). \tag{4.15}$$

Repeated application of the second identity above yields $T_j G_j T_j^* = P_T G_j P_T^*$. Now,

$$\begin{aligned} D^2_{T_j} - X D^2_{T_j} X^* &= I - T_j T_j^* - X(I - T_j T_j^*) X^* \\ &= (I - C_{T_j})(I - XX^*) \\ &= (I - C_{T_j})(G_1 + \cdots + G_n) \\ &= G_j - T_j G_j T_j^* \quad (\text{by } 4.15). \end{aligned}$$

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A rearrangement of the above identity is

$$D_{T_j}^2 + P_T G_j P_T^* = X D_{T_j}^2 X^* + G_j. \tag{4.16}$$

We set $\tilde{\mathcal{G}} := \overline{\operatorname{ran}}\, G_j^{1/2}$,

$$\mathcal{M} := \{(D_{T_j} h, G_j^{1/2} P_T^* h) : h \in \mathcal{H}\} \subseteq \mathcal{K}_j \oplus \tilde{\mathcal{G}}, \text{ and } \mathcal{N} := \{(D_{T_j} X^* h, G_j^{1/2} h) : h \in \mathcal{H}\} \subseteq \mathcal{K}_j \oplus \tilde{\mathcal{G}}.$$

Therefore, the identity 4.16 induces an isometry $\tilde{R} : \mathcal{M} \to \mathcal{N}$, defined by

$$\tilde{R}(D_{T_j} h, G_j^{1/2} P_T^* h) = (D_{T_j} X^* h, G_j^{1/2} h) \quad (h \in \mathcal{H}). \tag{4.17}$$

Again, using the identity $G_j = T_i G_j T_i^*$ $(i \neq j)$, we define an isometry for each $i \neq j$,

$$\tilde{Y}_i : \tilde{\mathcal{G}} \to \tilde{\mathcal{G}}, \text{ by } G_j^{1/2} h \mapsto G_j^{1/2} T_i^* h \quad (h \in \mathcal{H}). \tag{4.18}$$

Since $(\tilde{Y}_1, \ldots, \tilde{Y}_{j-1}, \tilde{Y}_{j+1}, \ldots, \tilde{Y}_n)$ is an $(n-1)$-tuple of commuting isometries on $\tilde{\mathcal{G}}$, it has minimal unitary extension $(Y_1, \ldots, Y_{j-1}, Y_{j+1}, \ldots, Y_n)$ on $\mathcal{G}$, say. We set

$$Y_j := Y_1^* \cdots Y_{j-1}^* Y_{j+1}^* \cdots Y_n^*. \tag{4.19}$$

For the simplicity of notations, we set

$$V_i := \begin{bmatrix} W_j^{(i)} & 0 \\ 0 & Y_i \end{bmatrix} \quad (i = 1, \ldots, n).$$

The auxiliary space $\mathcal{G}$ and the required contractions Y_i's on $\mathcal{G}$ are already constructed. To construct the contraction R, we take extension of $\tilde{R}$, as in (4.17), in certain way so that (4.14) holds. It follows from (4.10) and (4.18) that $\mathcal{M}$ and $\mathcal{N}$ are invariant subspaces of each of the unitary V_i on $\mathcal{K}_j \oplus \mathcal{G}$ with $i \neq j$. Also note that for $h \in \mathcal{H}$ and $i \neq j$,

$$\tilde{R} V_i (D_{T_j} h, G_j^{1/2} P_T^* h) = \tilde{R}(D_{T_j} T_i^* h, G_j^{1/2} T_i^* P_T^* h) = (D_{T_j} T_i^* X^* h, G_j^{1/2} T_i^* h),$$

and on the other hand,

$$V_i \tilde{R}(D_{T_j} h, G_j^{1/2} P_T^* h) = V_i (D_{T_j} X^* h, G_j^{1/2} h) = (D_{T_j} T_i^* X^* h, G_j^{1/2} T_i^* h).$$

Thus, $\tilde{R} V_i|_{\mathcal{M}} = V_i \tilde{R}|_{\mathcal{M}}$ for all $i \neq j$. We consider

$$\tilde{\mathcal{M}} := \bigvee_{\alpha \in \mathbb{Z}_+^{n-1}} \tilde{V}^{*\alpha}(\mathcal{M}) \text{ and } \tilde{\mathcal{N}} := \bigvee_{\alpha \in \mathbb{Z}_+^{n-1}} \tilde{V}^{*\alpha}(\mathcal{N}),$$

where $\tilde{V} := (V_1, \ldots, V_{j-1}, V_{j+1}, \ldots, V_n)$, and extend $\tilde{R}$ to get an operator from $\tilde{\mathcal{M}}$ onto $\tilde{\mathcal{N}}$ by the following prescription:

$$\sum_{0 \leq \alpha \leq \gamma} \tilde{V}^{*\alpha} x_\alpha \mapsto \sum_{0 \leq \alpha \leq \gamma} \tilde{V}^{*\alpha} \tilde{R} x_\alpha \quad (x_\alpha \in \mathcal{M}).$$

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The extension is well-defined as

$$\begin{aligned}
\Big\| \sum_{0\le\alpha\le\gamma} \tilde{V}^{*\alpha}\tilde{R}x_\alpha \Big\|^2 &= \sum_{0\le\alpha,\beta\le\gamma} \langle \tilde{V}^{*\alpha}\tilde{R}x_\alpha, \tilde{V}^{*\beta}\tilde{R}x_\beta\rangle \\
&= \sum_{0\le\alpha,\beta\le\gamma} \langle \tilde{V}^{(\beta-\alpha)_+}\tilde{R}x_\alpha, \tilde{V}^{(\beta-\alpha)_-}\tilde{R}x_\beta\rangle \\
&= \sum_{0\le\alpha,\beta\le\gamma} \langle \tilde{R}\tilde{V}^{(\beta-\alpha)_+}x_\alpha, \tilde{R}\tilde{V}^{(\beta-\alpha)_-}x_\beta\rangle \quad (\text{as } \tilde{R}V_i|_{\mathcal{M}} = V_i\tilde{R}|_{\mathcal{M}}, i\neq j) \\
&= \sum_{0\le\alpha,\beta\le\gamma} \langle \tilde{V}^{(\beta-\alpha)_+}x_\alpha, \tilde{V}^{(\beta-\alpha)_-}x_\beta\rangle \\
&= \Big\| \sum_{0\le\alpha\le\gamma} \tilde{V}^{*\alpha}x_\alpha \Big\|^2.
\end{aligned}$$

We denoted the extended operator again by $\tilde{R}:\tilde{\mathcal{M}}\to\tilde{\mathcal{N}}$, and it follows from the definition that

$$\tilde{R}V_i|_{\tilde{\mathcal{M}}} = V_i\tilde{R}|_{\tilde{\mathcal{M}}} \quad (i\neq j).$$

Finally, we define the contraction R on $\mathcal{K}_j\oplus\mathcal{G}$ by setting $R=\tilde{R}$ on $\tilde{\mathcal{M}}$ and $R\equiv 0$ on $\tilde{\mathcal{M}}^\perp$. Since $\tilde{\mathcal{M}}$ is a reducing subspace for $V_i(i\neq j)$, we have $RV_i = V_iR$ for all $i\neq j$. Also since $V_j = V_1^*\cdots V_{j-1}^*V_{j+1}^*\cdots V_n^*$, which can be read off from (4.11) and (4.19), it follows that $RV_j = V_jR$. Then by Fuglede-Putnam, we have (4.14). It only remains to prove that

$$\Pi_j X^* = M_{\Theta_j}^*\Pi_j,$$

where $\Theta_j(z) = \tau_{R^*}(z) = A^* + C^*z(I-zD^*)^{-1}B^*$, transfer function of R^* and

$$R = \begin{bmatrix} A & B \\ C & D \end{bmatrix} : \mathcal{K}_j\oplus\mathcal{G}\to\mathcal{K}_j\oplus\mathcal{G}.$$

This is obtained by applying Lemma 4.6 for $S := T_1\cdots T_{j-1}$, $T := P_T$, $G := G_j^{1/2}$, $Q := D_{T_j}$, $\Pi := \Pi_j$, $W := R$, and using 4.17. Hence, the theorem is proved. ∎

Remark 4.8 For $T=(T_1,\dots,T_n)\in\mathfrak{B}_0^n(\mathcal{H})$ and contractive X in $\{T\}'$, the above theorem shows that X has a contractive lifting in the isometric dilation space of T. This in particular implies that one can find a norm-preserving lifting of X. Indeed, if $\|X\|=1$, then there is nothing to prove. If $0<\|X\|<1$, and if Y is a contractive lifting of $\frac{X}{\|X\|}$, then $\|Y\|=1$, and clearly, $\|X\|Y$ is a norm-preserving lifting of X.

The following result is a particular case of [31, Proposition 9.2]; we include a proof for the sake of completeness.

Proposition 4.9 *Let $(V_1,\dots,V_n)$ be an n-tuple of commuting isometries on $\mathcal{H}$, and let X be a contraction on $\mathcal{H}$ such that $XV_i = V_iX$ for all $i=1,\dots,n$. Then the tuple $(X,V_1,\dots,V_n)$ has an isometric dilation on the minimal isometric dilation space of X.*

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Proof Since X is a contraction, suppose Y on $\mathcal{K}$ is the minimal isometric dilation of X. That is,

$$X^k = P_{\mathcal{H}} Y^k|_{\mathcal{H}} \ (k \in \mathbb{Z}_+) \text{ and } \mathcal{K} = \bigvee_{k\in\mathbb{Z}_+} Y^k(\mathcal{H}).$$

We define, for each $i = 1, \ldots, n$, $\tilde{V}_i : \mathcal{K} \to \mathcal{K}$ by

$$\tilde{V}_i(\sum_{k=0}^{m} Y^k h_k) = \sum_{k=0}^{m} Y^k V_i h_k \quad (h_k \in \mathcal{H}).$$

It is now enough to prove that each $\tilde{V}_i$ is well-defined, which readily follows from the following computation. For any $h, h' \in \mathcal{H}$ and $k, l \in \mathbb{Z}_+$ with $k \geq l$,

$$\langle Y^k V_i h, Y^l V_i h' \rangle = \langle Y^{k-l} V_i h, V_i h' \rangle = \langle X^{k-l} V_i h, V_i h' \rangle = \langle V_i X^{k-l} h, V_i h' \rangle = \langle X^{k-l} h, h' \rangle.$$

This completes the proof. ∎

As a consequence of Theorem 4.7 and Proposition 4.9, we have the following dilation result.

Theorem 4.10 *Let $(T_1, \ldots, T_n, T_{n+1})$ be an $(n+1)$-tuple of commuting contractions on $\mathcal{H}$ such that $(T_1, \ldots, T_n) \in \mathfrak{B}_0^n(\mathcal{H})$. Then $(T_1, \ldots, T_n, T_{n+1})$ has an isometric dilation and, consequently, satisfies the von Neumann inequality.*

5 Examples

In this section, we exhibit structures of tuples in $\mathfrak{B}_0^n(\mathcal{H})$ when $\mathcal{H}$ is finite dimensional. Also, we find an example to demonstrate that such a structure theorem does not hold for the general case of infinite-dimensional Hilbert space. We have pointed out earlier that if (T_1, T_2) is a pair of commuting contractions on $\mathcal{H}$ such that T_1 is a co-isometry and T_2 is a pure contraction, then $T_1 T_2$ is a pure contraction and $\Delta^{(1,1)}_{(T_1,T_2)} = 0$, and therefore, $(T_1, T_2) \in \mathfrak{B}_0^2(\mathcal{H})$. In fact, in the next proposition, we show that pureness of one of the contractions forces the other operators to be co-isometries and vice versa.

Proposition 5.1 *Let $T = (T_1, \ldots, T_n) \in \mathfrak{B}_0^n(\mathcal{H})$. Then T_i is a pure contraction if and only if T_j is a co-isometry for all $j \neq i$.*

Proof Without of loss generality, let T_1 be a pure contraction. Since $(T_1, \ldots, T_n) \in \mathfrak{B}_0^n(\mathcal{H})$, then, for $i = 2, \ldots, n$, $\Delta^{(1,1)}_{(T_1,T_i)} = D^2_{T_i} - T_1 D^2_{T_i} T_1^* = 0$, where $D^2_{T_i} = I - T_i T_i^*$. Iterating the identity k-times, we have

$$D^2_{T_i} = T_1^k D^2_{T_i} T_1^{*k} \quad (k \in \mathbb{Z}_+).$$

Using pureness of T_1, we obtain $D^2_{T_i} = 0$, and therefore, T_i is a co-isometry. Conversely, let $T_2, \ldots, T_n$ be co-isometries. Since P_T is a pure contraction, for $h \in \mathcal{H}$,

$$\|T_1^{*k} h\| = \|T_2^k \cdots T_n^k T_n^{*k} \cdots T_2^{*k} T_1^{*k} h\| \leq \|P_T^{*k} h\| \to 0 \text{ as } n \to \infty.$$

This completes the proof. ∎

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It turns out that if $\mathcal{H}$ is finite dimensional, then any n-tuple $(T_1, \ldots, T_n)$ in $\mathfrak{B}_0^n(\mathcal{H})$ arises in this way. We first prove it for $n = 2$, as it is much simpler, and subsequently, we treat the general case. We denote by $\mathbb{M}_m(\mathbb{C})$ the space of all $m \times m$ complex matrices.

Proposition 5.2 *Let $(A, B) \in \mathfrak{B}_0^2(\mathbb{C}^m)$. Then there exist commuting pairs (U_1, T_1) in $\mathbb{M}_k(\mathbb{C})$ and (T_2, U_2) in $\mathbb{M}_{m-k}(\mathbb{C})$, where $k \leq m$, U_1 and U_2 are unitary, and T_1 and T_2 are matrices with no unimodular eigenvalues such that*

$$(A, B) \cong \left(\begin{bmatrix} U_1 & 0 \\ 0 & T_2 \end{bmatrix}, \begin{bmatrix} T_1 & 0 \\ 0 & U_2 \end{bmatrix} \right).$$

Proof Since A and B are commuting matrices, then the pair (A^*, B^*) can be simultaneously upper tringularizable. Suppose that

$$(A^*, B^*) \cong \left(\begin{bmatrix} a_{11} & \cdots & \cdots & a_{1m} \\ 0 & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{mm} \end{bmatrix}, \begin{bmatrix} b_{11} & \cdots & \cdots & b_{1m} \\ 0 & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{mm} \end{bmatrix} \right).$$

Then the joint eigenvalues of A^* and B^* are given by $\{(a_{ii}, b_{ii}) : i = 1, \ldots, m\}$. Let v_i be a joint eigenvector of norm one corresponding to the joint eigenvalues (a_{ii}, b_{ii}). Then

$$0 = \langle \Delta_{(A,B)}^{(1,1)} v_i, v_i \rangle = \langle (I - AA^* - BB^* + ABA^*B^*)v_i, v_i \rangle = (1 - |a_{ii}|^2)(1 - |b_{ii}|^2).$$

This shows that either $|a_{ii}| = 1$ or $|b_{ii}| = 1$. Also note that both a_{ii} and b_{ii} cannot have modulus one, as $A^{*n}B^{*n} \to 0$ in SOT as $n \to \infty$. Now the result follows by rearranging all the unimodular eigenvalues of A^* and B^*. ∎

Now we prove the result for any n-tuple ($n \geq 3$) of matrices in $\mathfrak{B}_0^n(\mathbb{C}^m)$. Let $A = (A_1, \ldots, A_n) \in \mathfrak{B}_0^n(\mathbb{C}^m)$. Since $(A_1, \ldots, A_n)$ is a tuple of commuting matrices, the tuple $(A_1^*, \ldots, A_n^*)$ is simultaneously upper tringularizable. Let us assume that

$$(A_1^*, \ldots, A_n^*) \cong \left(\begin{bmatrix} a_{11}^{(1)} & \cdots & \cdots & a_{1m}^{(1)} \\ 0 & a_{22}^{(1)} & \cdots & a_{2m}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{mm}^{(1)} \end{bmatrix}, \ldots, \begin{bmatrix} a_{11}^{(n)} & \cdots & \cdots & a_{1m}^{(n)} \\ 0 & a_{22}^{(n)} & \cdots & a_{2m}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{mm}^{(n)} \end{bmatrix} \right).$$

Let us fix i ($1 \leq i \leq m$). Since $A_1^{*k} \cdots A_n^{*k} \to 0$ in SOT as $k \to \infty$ and $(a_{ii}^{(1)}, \ldots, a_{ii}^{(n)})$ is a joint eigenvalues of $(A_1^*, \ldots, A_n^*)$, $a_{ii}^{(k)} \in \mathbb{D}$ for some $1 \leq k \leq n$.

On the other hand, for $l \neq j$, a similar computation as in the proof of Proposition 5.2 reveals that

$$0 = \langle \Delta_A^{\mathbf{e}_l + \mathbf{e}_j} v_i, v_i \rangle = (1 - |a_{ii}^{(l)}|^2)(1 - |a_{ii}^{(j)}|^2) \quad (l \neq j),$$

where v_i is a joint eigenvector of norm one corresponding to the joint eigenvalue $(a_{ii}^{(1)}, \ldots, a_{ii}^{(n)})$ of the tuple $(A_1^*, \ldots, A_n^*)$. The above identity implies that $a_{ii}^{(k)} \in \mathbb{D}$ for exactly one $k \in \{1, \ldots, n\}$, and the rest are unimodular. Since this is true for all $i = 1, \ldots, m$, it leads to the following result.

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Proposition 5.3 *Let $A = (A_1, \dots, A_n) \in \mathfrak{B}_0^n(\mathbb{C}^m)$. Set $k = \min\{n, m\}$. Then a rearrangement of the tuple A is unitarily equivalent to a commuting tuple $(B_1, \dots, B_k, W_1, \dots, W_{n-k})$, where B_j's and W_i's are block diagonal matrices corresponding to the decomposition $\mathbb{C}^m = \mathbb{C}^{m_1} \oplus \cdots \oplus \mathbb{C}^{m_k}$ and have the form*

$$B_j = \begin{bmatrix} U_j^{(1)} & & & & \\ & \ddots & & & \\ & & T_j & & \\ & & & \ddots & \\ & & & & U_j^{(k)} \end{bmatrix} \quad \text{and} \quad W_i = \begin{bmatrix} V_i^{(1)} & & & & \\ & \ddots & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & V_i^{(k)} \end{bmatrix},$$

for some matrices T_j's in $\mathbb{M}_{m_j}(\mathbb{C})$ with no unimodular eigenvalues and for unitaries $U_j^{(l)} (l \neq j)$ and $V_i^{(l)}$ in $\mathbb{M}_{m_l}(\mathbb{C})$.

It is worth mentioning here that a pair of commuting isometries $(V_1, V_2) \in \mathfrak{B}_0^2(\mathcal{H})$ also has the form as in Proposition 5.2. Indeed, since $V_1 V_2$ is pure, then (V_1, V_2) is unitarily equivalent to a BCL pair $(M_{\Phi_1}, M_{\Phi_2}) \in \mathfrak{B}_0^2(H_{\mathcal{E}}^2(\mathbb{D}))$ for some co-efficient Hilbert space $\mathcal{E}$. Then it follows from Proposition 2.7 and (2.3) that (M_{Φ_1}, M_{Φ_2}) has the required form. However, Proposition 5.2 is not true for arbitrary elements in $\mathfrak{B}_0^2(\mathcal{H})$ when $\mathcal{H}$ is infinite-dimensional. We demonstrate this in the following example.

Example Let B be the bilateral shift on $l^2(\mathbb{Z})$, defined by

$$B(\{\dots, a_{-2}, a_{-1}, \mathbf{a_0}, a_1, a_2, \dots\}) = \{\dots, a_{-3}, a_{-2}, \mathbf{a_{-1}}, a_0, a_1, \dots\}.$$

Set $\mathcal{E} := l^2(\mathbb{Z}) \oplus l^2(\mathbb{Z})$. Consider the unitary $U := B \oplus B$ on $\mathcal{E}$ and define a projection

$$P : \mathcal{E} \to \mathcal{E}, \text{ by } P((x, y)) = (x, 0) \quad (x, y \in l^2(\mathbb{Z})).$$

Let (M_Φ, M_Ψ) on $H_{\mathcal{E}}^2(\mathbb{D})$ be the BCL pair corresponding to U and P; that is,

$$\Phi(z) = (P + zP^\perp)U^* \quad \text{and} \quad \Psi(z) = U(P^\perp + zP) \quad (z \in \mathbb{D}).$$

Since $UP = PU$, it follows from Proposition 2.7 that $(M_\Phi, M_\Psi) \in \mathfrak{B}_0^2(H_{\mathcal{E}}^2(\mathbb{D}))$. Moreover, with respect to the decomposition $H^2_{l^2(\mathbb{Z})}(\mathbb{D}) \oplus H^2_{l^2(\mathbb{Z})}(\mathbb{D})$,

$$M_\Phi = \begin{bmatrix} I \otimes B^* & 0 \\ 0 & M_z \otimes B^* \end{bmatrix} \text{ and } M_\Psi = \begin{bmatrix} M_z \otimes B & 0 \\ 0 & I \otimes B \end{bmatrix}. \tag{5.1}$$

For a fixed $\lambda_0 \in \mathbb{D}$, let us define a subspace $\mathcal{Q}$ of $H_{\mathcal{E}}^2(\mathbb{D})$ as follows:

$$\mathcal{Q} := \bigvee_{k,l \geq 0} M_\Phi^{*k} M_\Psi^{*l} (K_{\lambda_0} \otimes (e_0, e_0)),$$

where $e_0 = \{\dots, 0, \mathbf{1}, 0, \dots\} \in l^2(\mathbb{Z})$ and K_{λ_0} is the kernel function at λ_0; that is,

$$K_{\lambda_0}(z) = \frac{1}{(1 - \bar{\lambda}_0 z)} \quad (z \in \mathbb{D}).$$

It is easy to see that $\mathcal{Q}$ is a joint (M_Φ^*, M_Ψ^*) invariant subspace. We define

$$T_1 = P_{\mathcal{Q}} M_\Phi|_{\mathcal{Q}} \text{ and } T_2 = P_{\mathcal{Q}} M_\Psi|_{\mathcal{Q}}.$$

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Then (T_1, T_2) is a pair of commuting contractions on $\mathcal{Q}$ with $T_1T_2 = P_{\mathcal{Q}}M_z|_{\mathcal{Q}}$. Moreover, since $\Delta^{(1,1)}_{(M_\phi . M_\psi)} = 0$, we have $\Delta^{(1,1)}_{(T_1 . T_2)} = 0$. Therefore, $(T_1, T_2) \in \mathfrak{B}^2_0(\mathcal{Q})$. We show below that the pair (T_1, T_2) does not have the form as in Proposition 5.2. We divide the arguments in two cases.

Case I: We claim that neither T_1 nor T_2 is a co-isometry; equivalently, by Proposition 5.1 neither T_1 nor T_2 is a pure contraction. Without any loss of generality, suppose that T_1 is a pure contraction. Since $K_{\lambda_0} \otimes (e_0, e_0) \in \mathcal{Q}$, for $n \in \mathbb{N}$,

$$\begin{aligned}\|T_1^{*n}\big(K_{\lambda_0} \otimes (e_0, e_0)\big)\|^2 &= \|M_\Phi^{*n}\big(K_{\lambda_0} \otimes (e_0, e_0)\big)\|^2 \\ &= \left\|\begin{bmatrix} I \otimes B^n & 0 \\ 0 & M_z^{*n} \otimes B^n \end{bmatrix}\begin{pmatrix} K_{\lambda_0} \otimes e_0 \\ K_{\lambda_0} \otimes e_0 \end{pmatrix}\right\|^2 \\ &= \|K_{\lambda_0} \otimes B^n e_0\|^2 + \|M_z^{*n} K_{\lambda_0} \otimes B^n e_0\|^2.\end{aligned}$$

This shows that $\|T_1^{*n}\big(K_{\lambda_0} \otimes (e_0, e_0)\big)\|$ does not converge to 0 as $n \to \infty$, which is a contradiction.

Case II: We show that there is no joint (T_1, T_2)-reducing subspace $\mathcal{Q}_1$ of $\mathcal{Q}$ such that both $T_1|_{\mathcal{Q}_1}$ and $T_2|_{\mathcal{Q}_2}$ are co-isometry, where $\mathcal{Q} = \mathcal{Q}_1 \oplus \mathcal{Q}_2$. Suppose on the contrary that there exists a joint (T_1, T_2)-reducing subspace $\mathcal{Q}_1$ of $\mathcal{Q}$ so that both $T_1|_{\mathcal{Q}_1}$ and $T_2|_{\mathcal{Q}_2}$ are co-isometry. Since an arbitrary element of $\mathcal{Q}$ is of the form $K_{\lambda_0} \otimes (x', y')$ for some $x', y' \in l^2(\mathbb{Z})$ and $\mathcal{Q}_1 \subseteq \mathcal{Q}$, suppose that $K_{\lambda_0} \otimes (x, y) \in \mathcal{Q}_1$. We claim that $y = 0$. Indeed, for any $n \in \mathbb{N}$,

$$\begin{aligned}\|K_{\lambda_0} \otimes (x, y)\|^2 &= \|T_1^{*n}\big(K_{\lambda_0} \otimes (x, y)\big)\|^2 \\ &= \|M_\Phi^{*n}\big(K_{\lambda_0} \otimes (x, y)\big)\|^2 \\ &= \left\|\begin{bmatrix} I \otimes B^n & 0 \\ 0 & M_z^{*n} \otimes B^n \end{bmatrix}\begin{pmatrix} K_{\lambda_0} \otimes x \\ K_{\lambda_0} \otimes y \end{pmatrix}\right\|^2 \\ &= \|K_{\lambda_0} \otimes B^n x\|^2 + \|M_z^{*n} K_{\lambda_0} \otimes B^n y\|^2.\end{aligned}$$

Since M_z is a pure isometry, by taking $n \to \infty$ in the above identity, we have $\|(x, y)\|^2 = \|x\|^2$, and therefore $y = 0$. Thus we have shown that $\mathcal{Q}_1 \subseteq H^2_{l^2(\mathbb{Z})\oplus\{0\}}$. By a similar set of arguments, along with the fact that $T_2|_{\mathcal{Q}_2}$ is a co-isometry, we also have $\mathcal{Q}_2 \subseteq H^2_{\{0\}\oplus l^2(\mathbb{Z})}$. Since $\mathcal{Q} = \mathcal{Q}_1 \oplus \mathcal{Q}_2$ and $K_{\lambda_0} \otimes (e_0, e_0) \in \mathcal{Q}$, then $K_{\lambda_0} \otimes (e_0, 0) \in \mathcal{Q}_1$. Next, we claim that $\mathcal{Q}$ has the following orthogonal decomposition:

$$\mathcal{Q} = \text{Span}\,\{K_{\lambda_0} \otimes (e_0, e_0)\} \oplus \bigvee_{k \neq l} M_\Phi^{*k} M_\Psi^{*l} K_{\lambda_0} \otimes (e_0, e_0).$$

Indeed, note that for $k = l$, $M_\Phi^{*k} M_\Psi^{*l} K_{\lambda_0} \otimes (e_0, e_0) = M_z^* K_{\lambda_0} \otimes (e_0, e_0) = K_{\lambda_0} \otimes \tilde{\lambda}_0(e_0, e_0)$. Thus, we only need to show the orthogonality. Without any loss of generality, suppose that $k > l$. Then

$$\begin{aligned}\langle M_\Phi^{*k} M_\Psi^{*l} K_{\lambda_0} \otimes (e_0, e_0), K_{\lambda_0} \otimes (e_0, e_0)\rangle &= \langle M_\Phi^{*(k-l)} M_z^{*l} K_{\lambda_0} \otimes (e_0, e_0), K_{\lambda_0} \otimes (e_0, e_0)\rangle \\ &= \langle K_{\lambda_0} \otimes (B^{(k-l)} e_0, \tilde{\lambda}_0^{(k-l)} B^{(k-l)} e_0), K_{\lambda_0} \otimes \lambda_0^l (e_0, e_0)\rangle \\ &= 0.\end{aligned}$$

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Here, we have used the fact that for $k \in \mathbb{Z}_+$, $M_\Phi^{*k} K_{\lambda_0} \otimes (e_0, e_0) = K_{\lambda_0} \otimes (B^k e_0, \bar{\lambda}_0^k B^k e_0)$, which can be computed using 5.1. This proves the claim. Also, by the same computation as we have done above, for $l \neq k$, $\langle M_\Phi^{*k} M_\Psi^{*l} K_{\lambda_0} \otimes (e_0, e_0), K_{\lambda_0} \otimes (e_0, 0) \rangle = 0$. Therefore, $K_{\lambda_0} \otimes (e_0, 0) \in \text{Span}\{K_{\lambda_0} \otimes (e_0, e_0)\}$, which is a contradiction.

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