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A link between topological entropy and Lyapunov exponents

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Abstract. We show that a C^1-generic non-partially hyperbolic symplectic diffeomorphism f has topological entropy equal to the supremum of the sum of the positive Lyapunov exponents of its hyperbolic periodic points. Moreover, we also prove that f has topological entropy approximated by the topological entropy of f restricted to basic hyperbolic sets. In particular, the topological entropy map is lower semicontinuous in a C^1-generic set of symplectic diffeomorphisms far from partial hyperbolicity.

1. *Introduction*

An important problem in the theory of dynamical systems is how to measure/compare the complexity of different dynamics. The most famous tool, in this sense, is the topological entropy. This invariant quantity (which means that it is invariant under conjugacy), roughly, measures the rate of exponential growth of the number of distinct orbits when we compare orbits with finite length. Formally, if $f : X \to X$ is a continuous map defined in a compact metric space (X, d), then we define the *topological entropy* of f by

$$h_{\text{top}}(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log r(n, \varepsilon),$$

where $r(n, \varepsilon)$ is the maximum number of ε-distinct orbits of length n. Two points x and y have ε-distinct orbits of length n if there is $0 \leq j \leq n$ such that $d(f^j(x), f^j(y)) > \varepsilon$.

Another way to measure the complexity of dynamics is using Lyapunov exponents. Informally, these numbers tell us the rate of divergence of nearby trajectories. Lyapunov exponents measure the asymptotic behavior of dynamics in the tangent-space level. Given a diffeomorphism f of a manifold M, we say that a real number $\chi(x, f)$ is a *Lyapunov exponent* of $x \in M$ for f if there exists a non-zero vector $v \in T_x M$ such that

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|Df^n(x)\, v\| = \chi(x, f).$$

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The main result of this paper links topological entropy and Lyapunov exponents in a particular setting. Namely, we prove that a C^1-generic non-partially hyperbolic symplectic diffeomorphism f has topological entropy equal to the supremum of the sum of the positive Lyapunov exponents of its hyperbolic periodic points. This was already proved in dimension two by the author together with Tahzibi in [**CT**]. Also, it is worth pointing out that our result improves [**CH**, Theorem B], which gives a lower bound for topological entropy by means of Lyapunov exponents of periodic points. See also [**Ne1**].

Let us make precise the statement.

Here we will consider M^{2n} as a compact, connected and boundaryless Riemannian manifold of dimension $2n$. Recall that M^{2n} is a symplectic manifold if M^{2n} admits a *symplectic form*, i.e., if the manifold has a non-degenerate, closed, differential 2-form ω. We denote a symplectic manifold by (M^{2n}, ω). We say that a diffeomorphism f acting on M^{2n} is symplectic if it preserves the symplectic form ω, i.e., $f^*\omega = \omega$. The set of C^1 symplectic diffeomorphisms is denoted by $\mathrm{Diff}^1_\omega(M^{2n})$.

Let $f\colon M^{2n} \to M^{2n}$ be a C^1-diffeomorphism and Λ be a closed, f-invariant subset of M. We say that Λ is a *partially hyperbolic set* if there exists a continuous Df-invariant splitting $T_\Lambda M = E^s \oplus E^c \oplus E^u$ with non-trivial extremal subbundles E^s and E^u such that for every $x \in \Lambda$ and every m large enough:

- the splitting is *dominated*:
$$\|Df^m|E^i(x)\|\,\|Df^{-m}|E^j(f^m(x))\| \le \tfrac{1}{2} \quad \text{for any } (i,j) = (s,c),\ (s,u),\ (c,u);$$
and
- the extremal subbundles are hyperbolic:
$$\|Df^m|E^s(x)\| \le \tfrac{1}{2} \quad \text{and} \quad \|Df^{-m}|E^u(x)\| \le \tfrac{1}{2}.$$

If E^c is trivial, then the set Λ is called *hyperbolic.* When M is a partially hyperbolic set of f, we say that f *is partially hyperbolic* and, when M is a hyperbolic set of f, we say that f is an *Anosov diffeomorphism.* We will denote by $\mathcal{PH}$ the set of C^1 partially hyperbolic diffeomorphisms. It is worth pointing out that decompositions satisfying the first item in the definition of partial hyperbolicity are called *dominated splittings*.

Now, if p is a periodic point of a C^1-diffeomorphism f, and if λ is a complex eigenvalue of $Df^{\tau(p,f)}$, then $\chi = (1/\tau(p,f))\log|\lambda|$ is a Lyapunov exponent of p for f. In this paper, $\tau(p,f)$ denotes *the minimal period* of the periodic point p. We say that χ has multiplicity m if λ is an eigenvalue of $Df^{\tau(p,f)}$ with multiplicity m. Hence, we define

$$S(p,f) = \sum_{\chi_i^+(p,f)>0} \chi_i^+(p,f),$$

where the sum is over all positive Lyapunov exponents of p for f, counting multiplicity. Using this notation, we also define

$$S(f) = \sup\{S(p,f),\ \text{where } p \text{ is a hyperbolic periodic point of } f\}.$$

We say that a *periodic point is hyperbolic* if the orbit of p is a hyperbolic set of f. Since these points are robust (they are preserved after perturbations of the diffeomorphism), it is possible to conclude that the functional $S(\cdot)$ is a lower semicontinuous map for C^1-diffeomorphisms.

Finally, we can state the main result of the paper.

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THEOREM A. *There exists a residual subset* $\mathcal{R} \subset \mathrm{Diff}^1_\omega(M^{2n})$ *such that if* $f \in \mathcal{R}$ *is not partially hyperbolic, then*

$$h_{\mathrm{top}}(f) = S(f).$$

It is important to remark that this equality does not happen for Anosov diffeomorphisms, in general. In fact, there is an open and dense subset of Anosov symplectic diffeomorphisms f where $h_{\mathrm{top}}(f) < S(f)$. Hence, we put a question: what happens for partially hyperbolic diffeomorphisms with non-trivial center bundle? In this sense, it is worth pointing out that in [**CH**] it was proved that the topological entropy is lower bounded by means of the largest Lyapunov exponent among the Lyapunov exponents of the hyperbolic periodic points restricted to the central direction of generic partially hyperbolic symplectic diffeomorphisms.

Another important problem in smooth theory is about the regularity of the topological entropy. Using Theorem A, it will be possible to prove something in this direction. For instance, Newhouse has proved in [**Ne2**] that the topological entropy map varies upper semicontinuously for C^∞-diffeomorphisms (C^∞ topology), which is not true in the C^1 topology. See [**CT**, Theorem C]. On the other hand, Katok showed that topological entropy varies lower semicontinuously for C^2-diffeomorphisms (C^2 topology) of surfaces. See [**K**] and also [**KH**]. In particular, the topological entropy is continuous for C^∞-surface diffeomorphisms (C^∞ topology). To prove the lower semicontinuity, Katok proved that the topological entropy of a C^2-diffeomorphism f is approximated by topological entropy of f restricted to hyperbolic sets. In the lack of partial hyperbolicity we can find a similar result for C^1-generic symplectic diffeomorphisms.

Given a diffeomorphism f of M^{2n}, we define

$$H(f) = \sup\{h_{\mathrm{top}}(f|\Lambda), \text{ where } \Lambda \text{ is a basic hyperbolic set of } f\}.$$

By a *basic hyperbolic set* we mean a hyperbolic set which is isolated, i.e., there is a neighborhood U of Λ such that $\Lambda = \bigcap_{j\in\mathbb{Z}} f^j(U)$.

THEOREM B. *There is a residual subset* $\mathcal{R} \subset \mathrm{Diff}^1_\omega(M^{2n})$ *such that if* $f \in \mathcal{R}$ *is not partially hyperbolic, then*

$$h_{\mathrm{top}}(f) = H(f).$$

Therefore, by invariance of the topological entropy and robustness of basic hyperbolic sets, the previous theorem implies the existence of points where the topological entropy varies semicontinuously. More precisely, we have the following result.

COROLLARY C. *There is a non-empty residual subset* $\mathcal{R} \subset \mathrm{Diff}^1_\omega(M^{2n}) - \mathrm{cl}(\mathcal{PH})$ *such that if* $f \in \mathcal{R}$, *then the topological entropy map,* $h_{\mathrm{top}} : \mathrm{Diff}^1(M) \to \mathbb{R}^+$, *is lower semicontinuous at* f.

Recall that the topological entropy map is locally constant for Anosov diffeomorphisms. However, we do not know a similar result to Corollary C for partially hyperbolic diffeomorphisms with non-trivial center bundle.

This paper is organized in the following way: in §2, we prove Theorem A using Propositions 2.7, 2.8 and 2.9. In §3, we prove Theorem B and Corollary C. Finally, in §§4, 5 and 6, we prove Propositions 2.7, 2.8 and 2.9, respectively.

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2. *Proof of Theorem A*

We divide the proof of Theorem A into two results. The first one shows that the functional $S(\cdot)$ is generically an upper bound for the topological entropy. And, in the second one, we prove that generically any symplectic non-partially hyperbolic diffeomorphism f has topological entropy lower bounded by $S(f)$. More precisely, Theorem A follows directly from the following two results.

THEOREM 2.1. *There exists a residual subset* $\mathcal{R} \subset \mathrm{Diff}^1_\omega(M^{2n})$ *such that if* $f \in \mathcal{R}$, *then*

$$h_{\mathrm{top}}(f) \leq S(f).$$

THEOREM 2.2. *There exists a residual subset* $\mathcal{R} \subset \mathrm{Diff}^1_\omega(M^{2n})$ *such that if* $f \in \mathcal{R}$ *is not partially hyperbolic, then*

$$h_{\mathrm{top}}(f) \geq S(f).$$

Before we start the proofs, we recall some perturbative results. First, we state a symplectic version of Franks' lemma [**F**].

LEMMA 2.3. (Lemma 5.1 (symplectic Franks' lemma) in [**HT**]) *Let* $f \in \mathrm{Diff}^1_\omega(M^{2n})$ *and let* E *be a finite* f*-invariant set. Assume that* B *is a small symplectic perturbation of* Df *along* E. *Then for every neighborhood* $V \subset M$ *of* E *there is a symplectic diffeomorphism* g *arbitrarily* C^1*-close to* f *coinciding with* f *on* E *and outside of* V, *and such that* Dg *is equal to* B *on* E.

We then present a pasting lemma of Arbieto and Matheus [**AM**].

THEOREM 2.4. (Pasting lemma) *If* f *is a* C^k*-symplectic diffeomorphism of* M $(k \geq 1)$, *and* $x \in M$, *then for every* $\varepsilon > 0$ *there exists a* C^1 *symplectic diffeomorphism* g $\varepsilon - C^1$ *close to* f *such that for small neighborhoods of* x, $V \subset U$, $g|U^c = f$ *and* $g|V = Df(x)$ *(in local coordinates).*

2.1. *Proof of Theorem 2.1.*

We remark that Theorem 2.1 is a consequence of Ruelle's inequality and a symplectic version of a result of Abdenur *et al* [**ABC**, Theorem 3.8].

Given an ergodic invariant measure μ of a diffeomorphism f on M^{2n}, there are real numbers (not necessarily distinct) $\chi_1 \leq \chi_2 \leq \cdots \leq \chi_{2n}$, which are the Lyapunov exponents for μ-almost every point $x \in M$. We define the *Lyapunov vector* of μ by $L(\mu) = (\chi_1, \ldots, \chi_{2n})$. Also, given a periodic point p of a diffeomorphism f, we denote by μ_p *the Dirac measure defined on the orbit of* p, and then we define $L(p) = L(\mu_p)$.

THEOREM 2.5. (Symplectic version of [**ABC**, Theorem 3.8]) *There exists a residual subset* $\mathcal{R} \subset \mathrm{Diff}^1_\omega(M^{2n})$ *such that if* $f \in \mathcal{R}$, *then for any ergodic measure* μ *of* f *there is a sequence of periodic points* p_n *of* f *such that* μ_{p_n} *converge to* μ *in the weak topology, and moreover the Lyapunov vectors* $L(\mu_{p_n})$ *also converge to the Lyapunov vector* $L(\mu)$.

Proof. The original proof given in [**ABC**] also works here, since there exist symplectic versions of the perturbative results used there. □

To prove Theorem 2.1, we also need the following technical lemma.

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LEMMA 2.6. *There exists a residual subset* $\mathcal{R} \subset \mathrm{Diff}^1_\omega(M^{2n})$ *such that if* $f \in \mathcal{R}$, *then*

$$S(f) = \sup_{p \in \mathrm{Per}(f)} S(p, f).$$

Proof. By the Kupka–Smale theorem, generically a symplectic diffeomorphism has only hyperbolic or m-elliptic periodic points. Recall that p is a m-elliptic periodic point, $0 < m \leq n$, if $Df^{\tau(p,f)}$ has exactly $2m$ non-real and simple eigenvalues of modulus one, and all other eigenvalues have modulus different from one. These periodic points are also robust. By the Kupka–Smale theorem, we have, in fact, that if we fix a period, there exists an open and dense subset of symplectic diffeomorphisms for which there are only finitely many periodic points (hyperbolic or m-elliptic) of such period. Hence, by standard generic arguments, the proof of the lemma follows naturally from the following claim.

Claim. For any $\varepsilon > 0$ and any m-elliptic periodic point p of a symplectic diffeomorphism f, there exist a symplectic diffeomorphism $\tilde{f}$ arbitrarily close to f and a hyperbolic periodic point $\tilde{p}$ of $\tilde{f}$ such that $L(\tilde{p})$ is ε-close to $L(p)$.

In fact, after a perturbation of f, we can suppose that the arguments of all non-real eigenvalues with modulus equal to one of $Df^{\tau(p)}(p, f)$ are rational. Now, using the pasting lemma, Lemma 2.4, to perform a local perturbation of f, we can also suppose that $f^{\tau(p,f)} = Df^{\tau(p,f)}(p)$ in a small neighborhood of p (in local coordinates). Hence, we can conclude the existence of a non-hyperbolic periodic point $\tilde{p}$ close to p having only real eigenvalues, such that $Df^{\tau(\tilde{p},f)}(\tilde{p}) = (Df^{\tau(p),f}(p))^k$, for some $k \in \mathbb{N}$. In particular, the Lyapunov vector of $\tilde{p}$ is the same of the Lyapunov vector of p. Moreover, we can choose linear maps A_i arbitrarily close to $Df^i(\tilde{p})$, $1 \leq i \leq \tau(\tilde{p}, f)$, such that $\prod A_i$ has no eigenvalue with absolute value equal to one. Thus, we can use Franks' lemma to find $\tilde{f}$ arbitrarily close to f such that $\tilde{p}$ still is a periodic point of $\tilde{f}$, and $D\tilde{f}^i(\tilde{p}) = A_i$, for $1 \leq i \leq \tau(\tilde{p}, \tilde{f})$, which implies that $\tilde{p}$ is hyperbolic for $\tilde{f}$. By continuity of the Lyapunov vector, the proof of the claim is finished and thus the proof of the lemma. □

Proof of Theorem 2.1. Let $\mathcal{R}$ be a residual subset for which Theorem 2.5 and Lemma 2.6 are true. Consider now $f \in \mathcal{R}$. Then, for any $\mu \in \mathcal{M}(f)$, there exists a sequence of periodic points p_n^μ such that the Dirac measures on the orbit of p_n^μ converge to μ and the Lyapunov exponents of p_n^μ converge to the Lyapunov exponents of μ. By the variational principle, $h_{\mathrm{top}}(f) = \sup_\mu h_\mu(f)$, where the supremum is over all f-invariant ergodic probability measures. By Ruelle's inequality, $h_\mu(f) \leq \sum \chi_i^+(\mu)$, where the sum is over all positive Lyapunov exponents of μ, counting multiplicity. Then we have that

$$h_{\mathrm{top}}(f) \leq \sup_{p \in \mathrm{Per}(f)} S(p, f).$$

Since $f \in \mathcal{R}$, by Lemma 2.6, we have $h_{\mathrm{top}}(f) \leq S(f)$. □

2.2. *Proof of Theorem 2.2.* Theorem 2.2 will be a consequence of some propositions. We prefer to conclude the proof of the theorem and postpone the proofs of the propositions for other sections.

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Proof of Theorem 2.2. Given a positive integer $k > 0$, we define $\mathcal{B}_k \subset \mathrm{Diff}^1_\omega(M^{2n}) - \mathrm{cl}(\mathcal{PH})$ as the subset of diffeomorphisms g such that

$$h_{\mathrm{top}}(g) > S(g) - \frac{1}{k}.$$

The proof of the theorem follows directly from the following claim, by considering

$$\mathcal{R} = \bigcap_{k\in\mathbb{N}} (\mathcal{B}_k \cup \mathcal{PH}).$$

Claim. For every $k \in \mathbb{N}$, $\mathcal{B}_k$ is an open and dense subset in $\mathrm{Diff}^1_\omega(M^{2n}) - \mathrm{cl}(\mathcal{PH})$.

Proof of the claim. First, since $S(\cdot)$ is a lower semicontinuous map defined on $\mathrm{Diff}^1_\omega(M^{2n})$, there is a residual subset $\mathcal{R} \subset \mathrm{Diff}^1_\omega(M^{2n})$ where $S(\cdot)$ is continuous.

Now, let $f \in \mathcal{R} \cap (\mathrm{Diff}^1_\omega(M^{2n}) - \mathrm{cl}(\mathcal{PH}))$ and $\mathcal{V} \subset \mathrm{Diff}^1_\omega(M^{2n}) - \mathrm{cl}(\mathcal{PH})$ be a small neighborhood of f such that

$$S(f) > S(g) - \frac{1}{5k} \quad \text{for every } g \in \mathcal{V}. \tag{1}$$

By definition of $S(f)$, there exists a hyperbolic periodic point p of f such that

$$S(f) > S(p, f) - \frac{1}{5k}. \tag{2}$$

We need now the following result.

PROPOSITION 2.7. *Let $f \in \mathrm{Diff}^1_\omega(M^{2n})$ be not approximated by partially hyperbolic symplectic diffeomorphisms and p be a hyperbolic periodic point of f. For any C^1-neighborhood $\mathcal{U} \subset \mathrm{Diff}^1_\omega(M^{2n})$ of f and $\varepsilon > 0$, there exist a diffeomorphism $g \in \mathcal{U}$ and a hyperbolic periodic point $\tilde{p}$ of g such that*

$$|S(p, f) - n\chi^+_{\min}(\tilde{p}, g)| < \varepsilon,$$

where $\chi^+_{\min}(\tilde{p}, g)$ is the smallest positive Lyapunov exponent of $\tilde{p}$ for g.

Hence, by Proposition 2.7, we can perturb f to $f_1 \in \mathcal{V}$ such that there is a hyperbolic periodic point $\tilde{p}$ of f_1 satisfying

$$|S(p, f) - n\chi^+_{\min}(\tilde{p}, f_1)| < \frac{1}{5k}. \tag{3}$$

Also, since f_1 still is not approximated by partially hyperbolic diffeomorphisms, it is possible to find a full homoclinic tangency for $\tilde{p}$ after perturbing f_1. See the next result. We say that a periodic point $\tilde{p}$ of a symplectic diffeomorphism f_1 *exhibits a full homoclinic tangency* if there exists a point q (*homoclinic point*) which is a point of intersection of stable and unstable manifolds of $\tilde{p}$, $W^s(\tilde{p}, f_1)$ and $W^u(\tilde{p}, f_1)$, respectively, such that $T_q W^s(\tilde{p}) = T_q W^u(\tilde{p})$. Recall that hyperbolic periodic points of symplectic diffeomorphisms of $2n$-manifolds have index equal to n, i.e., the dimension of the stable manifold is n.

PROPOSITION 2.8. *Let $f \in \mathrm{Diff}^1_\omega(M^{2n})$ be a symplectic diffeomorphism which is not C^1 approximated by partially hyperbolic symplectic diffeomorphisms. If p is a hyperbolic periodic point of f, then, for any C^1 small neighborhood $\mathcal{U} \subset \mathrm{Diff}^1_\omega(M^{2n})$ of f, there exists a diffeomorphism $g \in \mathcal{U}$ exhibiting a full homoclinic tangency for $p(g)$, where $p(g)$ is the analytic continuation of p for g.*

Recall that a hyperbolic periodic point p is robust, i.e., after any perturbation of the diffeomorphism there always exists a hyperbolic periodic point close to p, and moreover

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with the same period. Such a hyperbolic periodic point is called the *analytic continuation of p*.

Thus, using Proposition 2.8, we can find $f_2 \in \mathcal{V}$ arbitrarily close to f_1 such that $\tilde{p}(f_2)$ has a full homoclinic tangency for f_2. For convenience, we still denote $\tilde{p}(f_2)$ by $\tilde{p}$.

The following result is the key technical point in this proof.

PROPOSITION 2.9. *Let $f \in \mathrm{Diff}^1_\omega(M^{2n})$ and let p be a hyperbolic periodic point of f. If f exhibits a full homoclinic tangency q for p, then, for any neighborhood $\mathcal{U}$ of f and any $\varepsilon > 0$, there exists a diffeomorphism $g \in \mathcal{U}$ such that p still is a hyperbolic periodic point of g and moreover there exists a basic hyperbolic set Λ_p of g such that*

$$h(g|\Lambda_p) \geq n\ \chi^+_{\min}(p,\, g) - \varepsilon.$$

By Proposition 2.9, there exist a diffeomorphism $f_3 \in \mathcal{V}$ arbitrarily close to f_2 and a hyperbolic set $\Lambda_{\tilde{p}}$ of f_3 such that

$$h(f_3|\Lambda_{\tilde{p}}) \geq n\ \chi^+_{\min}(\tilde{p},\, f_3) - \frac{1}{5k}. \tag{4}$$

Therefore, if $\mathcal{U} \subset \mathcal{V}$ is a small enough neighborhood of f_3, then for any $g \in \mathcal{U}$ there is a continuation of the hyperbolic basic set $\Lambda_{\tilde{p}}$, which we denote by $\Lambda(\tilde{p},\, g)$. Recall, as a hyperbolic periodic point, a hyperbolic basic set also is robust, and then it has a continuation. Thus, using, respectively, properties of entropy, equation (4), continuity of the Lyapunov exponents, equations (3), (2) and (1), we have

$$\begin{aligned} h_{\mathrm{top}}(g) \geq h_{\mathrm{top}}(g|\Lambda(\tilde{p},\, g)) = h_{\mathrm{top}}(f_3|\Lambda_{\tilde{p}}) & \\ \geq n\ \chi^+_{\min}(\tilde{p},\, f_3) - \frac{1}{5k} & \\ \geq n\ \chi^+_{\min}(\tilde{p},\, f_1) - \frac{2}{5k} & \\ \geq S(p,\, f) - \frac{3}{5k} & \\ \geq S(f) - \frac{4}{5k} & \\ \geq S(g) - \frac{1}{k}. & \end{aligned} \tag{5}$$

Hence, every $g \in \mathcal{U}$ belongs to $\mathcal{B}_k$. Since $\mathcal{R}$ is dense in $\mathrm{Diff}^1_\omega(M^{2n}) - \mathrm{cl}(\mathcal{PH})$, we can conclude that $\mathcal{B}_k$ contains an open and dense subset in $\mathrm{Diff}^1_\omega(M^{2n}) - \mathrm{cl}(\mathcal{PH})$. This completes the proof of the claim. □

And thus, the proof of Theorem 2.2 is also completed. □

3. *Regularity of topological entropy*

First we prove Theorem B.

Proof of Theorem B. Clearly, $h_{\mathrm{top}}(f) \geq H(f)$. Now we prove that $H(f) \geq h_{\mathrm{top}}(f)$ for generic symplectic diffeomorphisms in the lack of partial hyperbolicity. For that, we define $\mathcal{C}_k$ as the set formed by symplectic diffeomorphisms such that $H(g) \geq S(g) - 1/k$

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for any $k \in \mathbb{N}$. Analogously to the proof of Theorem 2.2, we can use equation (5) to conclude that $\mathcal{C}_k \cap (\mathrm{Diff}^1_\omega(M^{2n}) - \mathrm{cl}(\mathcal{PH}))$ contains an open and dense subset. Hence,

$$\mathcal{R}_1 = \bigcap_{k\in\mathbb{N}} \mathcal{C}_k \cup \mathcal{PH} \text{ is a residual subset of } \mathrm{Diff}^1_\omega(M^{2n}).$$

Let $\mathcal{R}_2 \subset \mathrm{Diff}^1_\omega(M^{2n})$ be the residual subset given by Theorem A. Now, if $f \in \mathcal{R} = \mathcal{R}_1 \cap \mathcal{R}_2$ is not partially hyperbolic, then $h_{\mathrm{top}}(f) = S(f)$ and $H(f) \geq S(f)$, by choice of $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively, which implies that $H(f) \geq h_{\mathrm{top}}(f)$. And, thus, $H(f) = h_{\mathrm{top}}(f)$ for any $f \in \mathcal{R}$. □

Proof of Corollary C. Let $\mathcal{R}$ be the residual subset given by Theorem B and let $f \in \mathcal{R}$. By definition of $H(f)$, given any $\varepsilon > 0$ there exists a basic hyperbolic set Λ of f such that $h_{\mathrm{top}}(f|\Lambda) > h_{\mathrm{top}}(f) - \varepsilon$. As a consequence of hyperbolicity properties, for any C^1-diffeomorphism g, not necessarily symplectic, close to f there is a basic hyperbolic set $\Lambda(g)$ which is close and topologically conjugated to Λ. Thus, we have

$$\begin{aligned} h_{\mathrm{top}}(g) &\geq h_{\mathrm{top}}(g|\Lambda(g)) \\ &= h_{\mathrm{top}}(f|\Lambda) \\ &\geq h_{\mathrm{top}}(f) - \varepsilon \end{aligned}$$

for any g C^1-close to f. That is, f is a point of lower semicontinuity of the topological entropy map in $\mathrm{Diff}^1(M)$. □

4. *Proof of Proposition 2.7*

This proposition is a consequence of some results about *periodic linear systems with transitions*. This kind of system was introduced by Bonatti *et al* in [**BDP**]. Many of the results in [**BDP**] are also true in the symplectic setting, proved by Horita and Tahzibi in [**HT**]. In this section, we will show how to use their results and arguments to prove Proposition 2.7. We also use some arguments developed in [**CH**].

We remark that since the theory of periodic linear systems is well known nowadays, we will give only a sketch of the proof.

Let we recall some definitions and some basic results used in this section.

Given a diffeomorphism f of M^{2n}, we say that two hyperbolic periodic points p_1 and p_2 are *homoclinically related* if $W^s(p_1, f)$ and $W^u(p_2, f)$ intersect transversally $W^u(p_2, f)$ and $W^s(p_1, f)$, respectively. That is, the stable and unstable manifolds of p_1 and p_2 have non-empty transversal intersections. We denote by $H(p_1, f)$ the closure of all hyperbolic periodic points of f homoclinically related to p_1. This set is called the *homoclinic class of* p_1. It is worth pointing out that f restricted to a homoclinic class is *topologically transitive*, i.e., this set contains a forward dense orbit of f.

The following result due to Arnaud *et al* says that any homoclinic class of a generic symplectic diffeomorphism coincides with the ambient manifold. This is a symplectic version of the main result in [**BC**].

THEOREM 4.1. [**ArBC**, Theorem 1] *There is a residual subset $\mathcal{R} \subset \mathrm{Diff}^1_\omega(M^{2n})$ such that if $f \in \mathcal{R}$ and p is a hyperbolic periodic point of f, then $H(p, f) = M$.*

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4.1. *Periodic linear systems with symplectic transitions.* In this subsection we will recall some basic definitions. A *periodic symplectic linear system (or linear symplectic cocycle over f)* is a 4-tuple $\mathcal{P} = (\Sigma, f, \mathcal{E}, A)$, where f is a homeomorphism, Σ is an infinite set of periodic points of f, $\mathcal{E}$ is a symplectic vector bundle defined on Σ (i.e., each fiber $\mathcal{E}(x)$ is a symplectic vector space), and $A : \mathcal{E} \to \mathcal{E}$ is such that $A(x) : \mathcal{E}_x \to \mathcal{E}_{f(x)}$ is a symplectic linear isomorphism for each x. Also, a system $(\Sigma, f, \mathcal{E}, A)$ can be considered as a system of matrices (Σ, f, A). In particular, we denote

$$M_A(x) = A(f^{n-1}(x))A(f^{n-2}(x)) \cdots A(x)$$

if $x \in \Sigma$ and n is the period of x. A periodic linear system is *diagonalizable* at the point $x \in \Sigma$ if $M_A(x)$ has only real eigenvalues of multiplicity one.

Given a set $\mathcal{B}$, a *word* with letters in $\mathcal{B}$ is a finite sequence of elements of $\mathcal{B}$. The product of the word $[a] = (a_1, \ldots, a_n)$ by the word $[b] = (b_1, \ldots, b_m)$ is the word $(a_1, \ldots, a_n, b_1, \ldots, b_m)$. We say that a word is *not a power* if $[a] \neq [b]^k$ for every word $[b]$ and $k > 1$, and we say that two words $[a]$ and $[b]$ are ε-close if they have the same length, and the correspondents letters are ε-close. With this notation, for a periodic symplectic linear system $(\Sigma, f, \mathcal{E}, A)$, if we consider the word $[M_A(x)] = (A(f^{n-1}(x)), \ldots, A(x))$, where n is the period of $x \in \Sigma$, then the matrix $M_A(x)$ is the product of the letters of the word $[M_A(x)]$.

Definition 4.2. [**BDP**, Definition 1.6] Given $\varepsilon > 0$, a periodic symplectic linear system $(\Sigma, f, \mathcal{E}, A)$ *admits ε-transitions* if for every finite family of points $x_1, \ldots, x_n = x_1 \in \Sigma$ and for any $(i, j) \in \{1, \ldots, n\}^2$ there exist $k(i, j) \in \mathbb{N}$ and a finite word $[t^{i,j}] = (t_1^{i,j}, \ldots, t_{k(i,j)}^{i,j})$ of symplectic matrices, satisfying the following properties.

(1) For every $m \in \mathbb{N}$, $\iota = (i_1, \ldots, i_m) \in \{1, \ldots, n\}^m$, and $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, consider the word

$$[W(\iota, \alpha)] = [t^{i_1, i_m}][M_A(x_{i_m})]^{\alpha_m}[t^{i_m, i_{m-1}}][M_A(x_{i_{m-1}})]^{\alpha_{m-1}} \cdots \cdots [t^{i_2, i_1}][M_A(x_{i_1})]^{\alpha_1},$$

where the word $[W(\iota, \alpha)]$ is not a power. Then there is $x(\iota, \alpha) \in \Sigma$ such that:
- the length of $[W(\iota, \alpha)]$ is the period of $x(\iota, \alpha)$;
- the word $[M_A(x(\iota, \alpha))]$ is ε-close to $[W(\iota, \alpha)]$ and there is an ε-symplectic perturbation $\tilde{A}$ of A such that the word $[M_{\tilde{A}}(x(\iota, \alpha))]$ is $[W(\iota, \alpha)]$.

(2) One can choose $x(\iota, \alpha)$ such that the distance between the orbit of $x(\iota, \alpha)$ and any point x_{i_k} is bounded by some function of α_k which tends to zero as k goes to infinity.

In our setting, results about periodic linear systems are useful because of the following result, which is a symplectic version of [**BDP**, Lemma 1.9].

LEMMA 4.3. [**HT**, Lemma 4.5] *Let f be a symplectic diffeomorphism and let p be a hyperbolic periodic point of f. The derivative Df induces a continuous periodic symplectic linear system with transitions on the set Σ formed by the hyperbolic periodic points homoclinically related to p.*

A nice property of periodic linear systems with transitions is the fact that we have a diagonalizable arbitrarily small perturbation of A defined on a dense subset of Σ, as we

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can see in [**BDP**, Lemma 4.16] (see [**HT**, Lemma 4.7] for a symplectic version). Recall that we say that a point $x \in \Sigma$ is *diagonalizable for some linear periodic system* if the matrix $M_A(x)$ is diagonalizable. However, their proof could be improved. That is, we can proceed in the same way as in the proof of [**CH**, Lemma 4.3] (see also [**CH**, Remark 4.4]) to prove that these diagonalizable perturbations keep, in some sense, the Lyapunov exponents. More precisely, the following result holds.

LEMMA 4.4. *Let $(\Sigma, f, \mathcal{E}, A)$ be a periodic symplectic linear system. For any $\varepsilon > 0$ and $x \in \Sigma$, there exist $y \in \Sigma$ and a symplectic perturbation $\tilde{A}$ of A defined on the orbit of y such that $M_{\tilde{A}}(y)$ is diagonalizable and moreover*

$$|S(x, A) - S(y, \tilde{A})| < \varepsilon.$$

We refer the reader to [**CH**] to see details about the proof of the previous lemma.

Now, given a diagonalizable periodic point x of a periodic symplectic linear system $(\Sigma, f, \mathcal{E}, A)$, let $\lambda_1 \leq \cdots \leq \lambda_{2n}$ be the $2n$ distinct eigenvalues of $M_A(x)$. Hence, we will consider the following decomposition: $\mathcal{E}(x) = E_1 \oplus \cdots \oplus E_{2n}(x)$, where E_i is the one-dimensional eigenspace with respect to λ_i for $1 \leq i \leq 2n$.

We will see now how to use the existence of specific periodic points, namely periodic points having complex eigenvalue of rank $(i,\ i+1)$, to mix the Lyapunov exponents of a diagonalizable periodic point x.

Before that, we point out that as done for diffeomorphisms, we can define similar dominated splittings and partial hyperbolicity on $\mathcal{E}$ for periodic linear systems $(\Sigma, f, \mathcal{E}, A)$.

Definition 4.5. (see [**BDP**, Definition 2.2]) We say that a periodic linear system $(\Sigma, f, \mathcal{E}, A)$ has *a complex eigenvalue of rank* $(i, i+1)$ if there is $x \in \Sigma$ such that the matrix $M_A(x)$ has a complex eigenvalue of rank $(i, i+1)$. That is, if there is an $M_A(x)$-invariant dominated splitting of $\mathbb{R}^{2n}$, $F \oplus G \oplus H$, such that:

- every eigenvalue σ of $M_A(x)|F$ (respectively $M_A(x)|H$) has modulus $|\sigma| < |\lambda|$ (respectively $|\sigma| > |\lambda|$);
- $\dim(F) = i - 1$ and $\dim(H) = 2n - i - 1$;
- the plane G is the eigenspace of λ.

The following result is a new statement of [**HT**, Proposition 7.1].

PROPOSITION 4.6. *Let $(\Sigma, f, \mathcal{E}, A)$ be a periodic symplectic linear system with symplectic transitions. Fix $\varepsilon_0 > 0$ and assume that a symplectic ε_0-perturbation of A has a complex eigenvalue of rank $(i, i+1)$ for some $i \in \{1, \ldots, 2N-1\}$. Then, for every $0 < \varepsilon_1 < \varepsilon_0$, if $x \in \Sigma$ is a diagonalizable periodic point, then there is an ε_1-transition $[t^i]$ from x to itself such that:*

- *there is a symplectic $(\varepsilon_0 + \varepsilon_1)$-perturbation $[\tilde{t}^i]$ of the transition $[t^i]$ such that the corresponding matrix $\tilde{T}^i$ satisfies:*
  - *$\tilde{T}^i(E_j(x)) = E_j(x)$ if $j \notin [i,\ i+1,\ i^*, (i+1)^*]$;*
  - *$\tilde{T}^i(E_i(x)) = E_{i+1}(x)$ and $\tilde{T}^i(E_{i+1}(x)) = E_i(x)$.*

Using this proposition, we can proceed in the same way as in the proof of [**HT**, Proposition 5.6] to conclude the following result.

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PROPOSITION 4.7. *Let $(\Sigma, f, \mathcal{E}, A)$ be a periodic symplectic linear system with symplectic transitions. Given $\varepsilon > 0$, if there exists $\delta > 0$ such that for any $i \in \{1, \dots, 2n-1\}$ there is a symplectic δ-perturbation of A having a complex eigenvalue of rank $(i, i+1)$, then for every diagonalizable point $x \in \Sigma$ there are a point $y \in \Sigma$ and an ε-perturbation $\tilde{A}$ of A defined in the orbit of y such that*

$$|n\chi^+_{\min}(y, \tilde{A}) - S(x, A)| < \varepsilon.$$

Since the previous proposition is the main technical result in this subsection, we decided to include a proof.

Proof of Proposition 4.7. Fix $\delta = \varepsilon/3$. Considering $\varepsilon_0 = \delta$ and $\varepsilon_1 = \varepsilon/3$, let $[t^i]$ be the ε_1-transition from x to itself and $[\tilde{t}^i]$ the $(\varepsilon_0 + \varepsilon_1)$-perturbation of $[t^i]$ given by Proposition 4.6 for any $0 < i < 2n$. For convenience, we suppose that $[t^0] = [t^{2n}] = \mathrm{Id}$.

Note that the maps $[\tilde{t}^i]$ are transpositions acting on the finite set $\{E_j\}_{1 \le j \le 2n}$ of eigenspaces of $M_A(x)$ which interchange E_i and E_{i+1}. Hence, as any permutation is a composition of transpositions, for every $0 \le k \le n$ there exists a transition map $[S_k]$, obtained by composing some of the $[t^i]$, such that if we consider $[\tilde{S}_k]$ as the similar word formed by the perturbations $[\tilde{t}^i]$ instead of $[t^i]$, then $\tilde{S}_k(E_{j+n}) = E_{(j+k)+n}$, for any $1 \le j \le n$, and considering $(j+k)$ in the cyclic group $\mathbb{Z}/n\mathbb{Z}$. Also, recall that $\tilde{S}_k$ is the matrix corresponding to $[\tilde{S}_k]$.

Now, we consider $[W_{k,m}] = [S_{n-k}][M_A(x)]^m[S_k]$, and $[\tilde{W}_{k,m}] = [\tilde{S}_{n-k}][M_A(x)]^m[\tilde{S}_k]$, for any $0 \le k \le n$. And, using such words, we define

$$[M_m] = [W_{n-1,m}] \circ \cdots \circ [W_{1,m}] \circ [W_{0,m}] \quad \text{and}$$
$$[\tilde{M}_m] = [\tilde{W}_{n-1,m}] \circ \cdots \circ [\tilde{W}_{1,m}] \circ [\tilde{W}_{0,m}].$$

By definition of periodic linear systems with transitions, there exists a point $y_m \in \Sigma$ such that $[M_A(y_m)]$ is $\varepsilon/3$-close to $[M_m]$; in particular, ε-close to $\tilde{M}_m$. Note that $\tilde{S}_{n-k}\tilde{S}_k$ and $\tilde{M}_m$ keep invariant any E_j, $n < j \le 2n$. We denote by $\mu_{j,k}$ and $\tilde{\lambda}_j$ the eigenvalues of $\tilde{S}_{n-k}\tilde{S}_k|E_j$ and $\tilde{M}_m|E_j$, respectively. With this notation, we have the following:

$$\tilde{\lambda}_j = \sum_{j=n+1}^{2n} \lambda_j^m + \sum_{k=0}^{n-1} \mu_{j,k} \quad \text{for any } n+1 \le j \le 2n.$$

Note now that the period of y is $\tau(y) = nm\tau(x) + K$, where K is a constant which depends only on the length of the words $[S_k]$. Thus, if $\tilde{A}_m$ is defined in the orbit of y_m such that $[M_{\tilde{A}}] = [\tilde{M}_m]$, we have that

$$\chi_j(y_m, \tilde{A}_m) = \frac{1}{n} \sum_{i=n+1}^{2n} \chi_i(x, A) + \frac{1}{\tau(y_m)} \sum_{k=0}^{n-1} \log \mu_{j,k} \quad \text{for any } n+1 \le j \le 2n.$$

Since $\tau(y_m)$ goes to infinity when m does, the proof is finished by taking m large enough and considering $y = y_m$ and $\tilde{A} = \tilde{A}_m$. □

Finally, to finish this subsection we state a result which is a direct consequence of [**HT**, Proposition 5.5].

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PROPOSITION 4.8. *Let $(\Sigma, f, \mathcal{E}, A)$ be a continuous periodic $2n$-dimensional linear system with symplectic transitions such that its norm $\|A\|$ is bounded by a positive integer K. Given $\varepsilon > 0$, if $(\Sigma, f, \mathcal{E}, A)$ admits no dominated splitting, then for any $1 \le i \le 2n-1$ there is an ε-perturbation $\tilde{A}$ of A having a complex eigenvalue of rank $(i, i+1)$.*

4.2. *Proof of Proposition 2.7.* Let $\varepsilon > 0$ be arbitrarily small and $\mathcal{U}$ be a small enough neighborhood of f such that any diffeomorphism inside $\mathcal{U}$ is not partially hyperbolic. In particular, every diffeomorphism in this neighborhood admits no dominated splitting. Recall that for symplectic diffeomorphisms, partial hyperbolicity is equivalent to the existence of a dominated splitting on M. After a perturbation of f, using Theorem 4.1, we can also suppose that $H(p, f) = M$. Using Lemma 4.3, we can consider the following periodic linear system with symplectic transitions: (Σ, f, TM, Df), where Σ is the set formed by all the hyperbolic periodic points of f homoclinically related to p. By Lemma 4.4, there exist a periodic point $p_1 \in \Sigma$ and a diagonalizable symplectic perturbation A of Df along the orbit of p_1, such that

$$|S(p, f) - S(p_1, A)| < \varepsilon/2. \tag{6}$$

Since this perturbation is arbitrarily close to Df, we can use Franks' lemma (Lemma 2.3) to perform a nonlinear perturbation of f to $f_1 \in \mathcal{U}$ such that p_1 is a periodic point of f_1, and moreover Df_1 along the orbit of p_1 coincides with A. In particular, p_1 is a diagonalizable periodic point of f_1, and $S(p_1, f_1) = S(p_1, A)$. As before, after a perturbation we can suppose that $H(p_1, f_1)$ coincides with M. And, thus, we can consider the following periodic linear system: $(\Sigma_1, f_1, TM, Df_1)$, where Σ_1 is now the set formed by all the hyperbolic periodic points of f_1 homoclinically related to p_1.

Since $f_1 \in \mathcal{U}$, it has no dominated splitting. In particular, the above periodic linear system has no dominated splitting either, which implies, by Proposition 4.8, that for any $1 \le i \le 2n-1$ there is an arbitrary linear perturbation of Df_1 having complex eigenvalue of rank $(i, i+1)$. Therefore, this linear periodic system is in the hypothesis of Proposition 4.7, which implies the existence of a periodic point $\tilde{p} \in \Sigma_1$ and a symplectic perturbation $\tilde{A}$ of Df_1 along the orbit of $\tilde{p}$ such that

$$|n\chi^+_{\min}(\tilde{p}, \tilde{A}) - S(p_1, f_1)| < \varepsilon/2. \tag{7}$$

Once again, we can use Franks' lemma (Lemma 2.3) to perform a local perturbation of f_1 to $g \in \mathcal{U}$ along the orbit of $\tilde{p}$ such that $\tilde{p}$ still is a periodic point of g and Dg coincides with $\tilde{A}$ in the orbit of $\tilde{p}$. In particular, $\tilde{p}$ is a hyperbolic periodic point of g and, moreover, from equations (6) and (7) we have $|n\chi^+_{\min}(\tilde{p}, g) - S(p, f)| < \varepsilon$. □

5. *Abundance of full homoclinic tangencies: proof of Proposition 2.8*

In this section we will prove Proposition 2.8. That is, we prove that in the lack of partial hyperbolicity any hyperbolic periodic point exhibits a full homoclinic tangency, after a perturbation. Recall that a hyperbolic periodic point p of a C^1-diffeomorphism f exhibits a homoclinic tangency if there is a non-transversal intersection q between its stable and unstable manifolds, $W^s(p, f)$ and $W^u(p, f)$. We define $c_T(q) = \dim(T_q W^u(p, f) \cap T_q W^s(p, f))$ as the *codimension of the tangency at q*. According with such definition,

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we can say that we have a *full homoclinic tangency* when we have a tangency having the maximal possible codimension. In particular, if $f \in \mathrm{Diff}^1_\omega(M^{2n})$, then a hyperbolic periodic point p has a full homoclinic tangency if it exhibits a homoclinic tangency with codimension n.

We would like to point out that recently Barrientos and Raibekas proved in [**BR**] the existence of robust homoclinic tangencies with large codimension. Their result also works for symplectic diffeomorphisms.

However, to prove Proposition 2.8 we will use some ideas developed in [**CH**]. But, it is worth pointing out that in our setting, we do not get robustness.

Before we start the proof, we need some definitions and results.

Considering (M^{2n}, ω) a symplectic manifold, we say that a submanifold $N \subset M$ is Lagrangian if $\dim N = n$ and moreover if $\omega(x)|T_xN \times T_xN = 0$ for any $x \in N$. In particular, if p is a hyperbolic periodic point of $f \in \mathrm{Diff}^1_\omega(M^{2n})$, then its stable and unstable manifolds are Lagrangian submanifolds of M.

In the proof below we will need to connect stable and unstable manifolds of periodic points. To do this we will use the famous connecting lemma of Hayashi [**H**]. The next result is a symplectic version of it, which was proved by Wen and Xia [**XW**].

THEOREM 5.1. (C^1-connecting lemma) *Let $f \in \mathrm{Diff}^1_\omega(M^{2n})$ and let p_1, p_2 be hyperbolic periodic points of f such that there exist sequences $y_n \in M$ and positive integers k_n such that:*

- $y_n \to y \in W^u_{\mathrm{loc}}(p_1, f)$, $y \neq p_1$; *and*
- $f^{k_n}(y_n) \to x \in W^s_{\mathrm{loc}}(p_2, f)$, $x \neq p_2$.

Then there exists a C^1 symplectic diffeomorphism g, C^1-close to f, such that $W^u(p_1, g)$ and $W^s(p_2, g)$ have a non-empty intersection close to y.

Proof of Proposition 2.8. Let $\mathcal{U} \subset \mathrm{Diff}^1_\omega(M^{2n})$ be a small neighborhood of f and p be a hyperbolic periodic point of f, as given in the proposition. Now, taking $\mathcal{U}$ smaller, if necessary, we can assume that for any $g \in \mathcal{U}$:

- the analytic continuation of p is defined for g; and
- g is not partially hyperbolic.

Now, we can use [**HT**, Proposition 5.3] to find a diffeomorphism $f_1 \in \mathcal{U}$ having a periodic point $\tilde{p}$ such that $Df_1^{\tau(\tilde{p}, f_1)} = \mathrm{Id}$.

Let $\gamma > 0$ be an arbitrarily small positive real number. By the same arguments as in the proof of Theorem A in [**CH**], we can perform a local perturbation of f_1 to $f_2 \in \mathcal{U}$ such that $\tilde{p}$ is now a hyperbolic periodic point of f_2 and moreover $W^s_{\mathrm{loc}}(\tilde{p}, f_2)$ is a Lagrangian submanifold of M $\gamma - C^1$ close to the Lagrangian submanifold $W^u_{\mathrm{loc}}(\tilde{p}, f_2)$. Let $\tilde{\mathcal{U}} \subset \mathcal{U}$ be a small neighborhood of f_2 such that the analytic continuation of $\tilde{p}$ is defined in $\tilde{\mathcal{U}}$ and moreover the local stable and unstable manifolds of $\tilde{p}(g)$ still are submanifolds $\gamma - C^1$ close, for any $g \in \tilde{\mathcal{U}}$.

The next step of the proof is similar to the proof of [**CH**, Lemma 5.2]. In words, we want to perturb f_2 in order to find a transversal homoclinic point q of p near $\tilde{p}$, such that the stable and unstable manifolds of p at q are locally C^1-close to the local stable and unstable manifolds of $\tilde{p}$, respectively.

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First, we use Theorem 4.1 to perturb f_2 to a transitive symplectic diffeomorphism $f_3 \in \tilde{\mathcal{U}}$. Hence, we can use Theorem 5.1 to perturb f_3 to f_4 such that there is an intersection between $W^s(p(f_4))$ and $W^u(\tilde{p}(f_4))$. After a perturbation, if necessary, we can suppose that this intersection is transversal. By transversality and continuity of stable and unstable manifolds of hyperbolic periodic points, this intersection is robust. Hence, we can use again Theorems 4.1 and 5.1 to perturb f_4 to a diffeomorphism $\tilde{g} \in \tilde{\mathcal{U}}$ such that for $\tilde{g}$ there is also an intersection between $W^u(p(\tilde{g}))$ and $W^s(\tilde{p}(\tilde{g}))$. Now, since $\tilde{p}(\tilde{g})$ and $p(\tilde{g})$ have the same indices (i.e., their stable manifolds have the same dimension), the existence of such transversal intersections between the stable and unstable manifolds of $\tilde{p}(\tilde{g})$ and $p(\tilde{g})$, together with the famous lambda lemma, allow us to find a point $q \in W^s(p(\tilde{g})) \cap W^u(p(\tilde{g}))$, such that:

- $T_q W^s(p(\tilde{g}))$ is γ-close to $E^s(\tilde{p}(\tilde{g})) = T_{\tilde{p}} W^s(\tilde{p}(\tilde{g}))$; and
- $T_q W^u(p(\tilde{g}))$ is γ-close to $E^u(\tilde{p}(\tilde{g})) = T_{\tilde{p}} W^u(\tilde{p}(\tilde{g}))$.

And, thus, $T_q W^s(p(\tilde{g}))$ is 2γ-close to $T_q W^u(p(\tilde{g}))$.

Now, we have the following lemma.

LEMMA 5.2. *Suppose that we have a symplectic vector space $V = E \oplus F$, where E is a Lagrangian subspace. For any $\delta > 0$, there exists $\gamma > 0$ such that if $W \subset V$ is a Lagrangian subspace γ-close to E, then there exists a symplectic linear map B on V δ-close to* Id *such that $B(W) = E$ and $B|F = \mathrm{Id}_F$.*

There is a proof of such lemma in [**CH**, §2].

Therefore, given $\delta > 0$, we can choose $\gamma > 0$ as in Lemma 5.2 and, thus, since the stable and unstable manifolds of p are Lagrangian manifolds, there exists a symplectic linear map B in $T_q M$, $\delta - C^1$ close to the identity, such that $B(T_q W^u(p(\tilde{g}))) = T_q W^s(p(\tilde{g}))$.

Finally, we can use Franks' lemma, Lemma 2.3, to perform a local perturbation in a small neighborhood of $\tilde{g}^{-1}(q)$, and find $g \in \tilde{\mathcal{U}}$ such that q still is a homoclinic point of p and moreover $Dg(\tilde{g}^{-1}(q)) = B \circ D\tilde{g}(\tilde{g}^{-1}(q))$. This implies that $T_q W^s(p(g)) = T_q W^u(p(g))$, as we wanted. □

6. *Snake perturbation product and proof of Proposition 2.9*

In this section we prove the key technical result (Proposition 2.9) in the proof of Theorem 2.2. To do that we improve the snake perturbation created by Newhouse [**Ne1**], and used by many authors since then. For instance, we found a way to make snake perturbations simultaneously in many directions.

In this section we will need to perform local C^1-perturbations of symplectic diffeomorphisms. More precisely, we need to paste distinct maps, locally. For general diffeomorphisms this is quite easy to do by using bump functions. However, since we are in the symplectic setting, we need to keep the symplectic structure after we paste the maps. Fortunately, we can do this, locally, by using generating functions. More precisely, we have the following result.

LEMMA 6.1. [**AM**, Lemma 3.9] *If f is a C^k-symplectic diffeomorphism on M^{2n} ($k \geq 1$), then for any $x \in M$ and any local symplectic diffeomorphism g (C^k-close to f) defined in*

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a small neighborhood U of x, there exist a C^k-symplectic diffeomorphism h (C^k-close to f) and some neighborhood $V \subset U$ of x satisfying $h|V = g$ and $g|U^c = f$.

For more details about generating functions, see also [**AM**].

Proof of Proposition 2.9. Let $f \in \mathrm{Diff}^1_\omega(M^{2n})$ and p be a hyperbolic periodic point of f exhibiting a full homoclinic tangency at q, as in the hypothesis of the proposition.

We can take a C^1 neighborhood $\mathcal{U} \subset \mathrm{Diff}^1_\omega(M^{2n})$ of f small enough, such that there exists a common Lipschitz constant, say $\lambda > 0$, for every diffeomorphism in $\mathcal{U}$. To simplify notation, we will use in this proof $\tau = \tau(p, f)$.

Now, we will linearize the diffeomorphism in a small neighborhood of p. After a local perturbation, by transversality, we can assume that q is a transversal homoclinic point but having the spaces $T_qW^s(p)$ and $T_qW^u(p)$ arbitrarily close to each other. We use now the pasting lemma, Lemma 2.4, to perform a C^1-perturbation of f to $f_1 \in \mathcal{U}$ such that p still is a hyperbolic periodic point of f_1 and there is a small neighborhood V of p such that $f_1^\tau|V = Df^\tau(p)$ (in local coordinates) and $q \notin V$. By continuity of the stable and unstable manifolds of p, f_1 still exhibits a transversal homoclinic point $\tilde{q}$ of p close to q, where the tangent spaces to $W^s(p)$ and $W^u(p)$ at $\tilde{q}$ still are near. Hence, after another local C^1-perturbation of f_1 in a small neighborhood of $\tilde{q}$, as done in the proof of Proposition 2.8, we can recover the homoclinic tangency to f_1 at $\tilde{q}$ as in the hypothesis of the proposition.

Replacing $\tilde{q}$ by an iterate, we can suppose that $\tilde{q} \in V$ and $f_1^{-1}(\tilde{q}) \notin V$. Thus, we can choose a small open set $U \subset V$ containing $\tilde{q}$ such that $f_1^{-1}(U) \cap V = \emptyset$. Since f_1^τ is linear on V (in local coordinates), we can choose a local coordinate ψ of U such that $\psi(W^s_{\mathrm{loc}}(p) \cap U) \subset R^n \times \{0\}^n$ and such that $\psi(\tilde{q}) = 0$. Let Ξ^u be the connected component of $W^u(p) \cap U$ containing $\tilde{q}$. After shrinking U, if necessary, we have that $\psi(\Xi^u)$ is a graph of a map $\xi : \mathbb{R}^n \to \mathbb{R}^n$, with zero derivative at the origin since $\tilde{q}$ is a homoclinic tangency. Now, since the stable and unstable manifolds of p are Lagrangian submanifolds of M, the map $\pi(x, y) = (x, y - \xi(x))$ $(x, y \in \mathbb{R}^n)$ defined in $\psi(U)$ preserves the symplectic structure and moreover is arbitrarily C^1-close to the identity in a small neighborhood of the origin. Thus, we can use Lemma 6.1 to perform a local C^1-perturbation of f_1 to find a diffeomorphism $f_2 \in \mathcal{U}$ such that f_2 is equal to $\psi^{-1} \circ \pi \circ \psi \circ f_1$ in a small neighborhood of $f_1^{-1}(\tilde{q})$. In particular, f_2 satisfies:

(1) $f_2(x) = f_1(x)$ for any x in the complement of $f_1^{-1}(U)$;
(2) there are $a > 0$ and a disc $\tilde{D} = [-a, a]^n \times \{0\}^n \subset \mathbb{R}^{2n}$ such that $D^s = \psi^{-1}(\tilde{D}) \subset W^s_{\mathrm{loc}}(p) \cap W^u(p)$.

In words, item (2) says that f_2 has a disc D^s of homoclinic points of p.

Since D^s is a small disc of homoclinic points of p around $\tilde{q}$, let T be a positive integer such that $D^u = f_2^{-T}(D^s) \subset V$. Note that D^u is a disc inside $W^u_{\mathrm{loc}}(p)$.

The next step of the proof is to perform a local C^1-perturbation of f_2 in order to create arbitrarily finitely many transversal homoclinic points of p inside U. This kind of perturbation is known as a snake perturbation and was first introduced by Newhouse [**Ne1**]. However, here we repeat his idea in many directions, which we call the *snake perturbation product.*

For that, we fix $\delta > 0$ arbitrarily small such that any diffeomorphism C^1 δ-close to f must be in $\mathcal{U}$. Let N be a large odd natural number and $\tilde{A} > 0$ be a small real number.

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Then we consider the following symplectic map defined in $\mathbb{R}^{2n}$:

$$\Theta_{\tilde{A},N}(x_1,\ldots,x_n,y_1,\ldots,y_n) = \left(x_1,\ldots,x_n,y_1+\tilde{A}\sin\frac{x_1\pi N}{2a},\ldots,y_n+\tilde{A}\sin\frac{x_n\pi N}{2a}\right).$$

It is not difficult to see that $\Theta(\tilde{D})\cap\tilde{D}$ is a finite set containing N^n points. Moreover, note that $\Theta_{\tilde{A},N}$ is C^1 $(\tilde{A}\pi N/2a)$-close to the identity map. Hence, considering $A = 2a\alpha\delta/\pi N$, where α is a constant depending on the local coordinate ψ, we have that the map $\psi^{-1}\circ\Theta_{A,N}\circ\psi$ defined in U is C^1 δ-close to the identity map restricted to U. Thus, we can use Lemma 6.1 to make a local perturbation of f_2 to $f_{3,N}\in\mathcal{U}$ such that:

- $f_{3,N}(x) = f_2(x)$ for any x in the complement of $f_2^{-1}(U)$;
- $f_{3,N}$ is equal to $\psi^{-1}\circ\Theta_{A,N}\circ\psi\circ f_2$ in an open set $\tilde{U}\subset U$ which contains the disc D^s.

Hence, by construction of $\Theta_{A,N}$, we remark that this C^1 local perturbation destroys the disc of homoclinic points D^s of p by creating N^n transversal homoclinic points. We will denote $J = \psi^{-1}\circ\Theta_{A,N}\circ\psi(D^s)$, which is a disc in $W^u(p)$ containing these N^n homoclinic points. It is also important to remark that by properties of the unstable and stable manifolds it is true that $f_{3,N}^{-T}(J) = D^u\subset W^u_{\mathrm{loc}}(p)$, and D^s still belongs to $W^s_{\mathrm{loc}}(p)$, since $f_{3,N}$ was obtained by a C^1 local perturbation.

We will use these transversal homoclinic points to find a hyperbolic set Λ_p satisfying the thesis of the proposition. For that, we consider a local coordinate map ϕ on V such that $\phi(p) = 0$ and $\phi\circ f_{3,N}^{\tau}\circ\phi^{-1}$ is linear. Also, we consider a decomposition in $\mathbb{R}^{2n}$ induced by the hyperbolic splitting $E^s(p)\oplus E^u(p)$ existent in T_pM for $Df_{3,N}^{\tau}(p)$, using the local coordinate ϕ. To simplify notation, we continue denoting this splitting by $E^s(p)\oplus E^u(p)$. Using such decomposition, we define $\Gamma_l = B_l^s\times D^u$ and $b_l^s(x) = B_l^s\times\{x\}$ for any $x\in\phi(D^u)$. Here B_l^s is the closed ball with center zero and radius l inside $E^s(p)$. We will fix l_D such that $B_{l_D}^s$ contains the disc D^s.

Since $f_{3,N}\in\mathcal{U}$, λ is a Lipschitz constant to $f_{3,N}$. Hence, if $l_A = A\beta/4\lambda^T$, then $f_{3,N}^T(\phi^{-1}(\Gamma_{l_A}))$ is a neighborhood of J contained in a $A/4-C^1$ neighborhood of J. Here β is a constant which depends only on ϕ.

We choose now the smallest positive integer t such that

$$f_{3,N}^{-\tau t}(\phi^{-1}(\Gamma_{l_A}))\cap V \text{ is } A/4-C^1 \text{ close to } W^s_{\mathrm{loc}}(p); \text{ and} \tag{8}$$

$$\text{the projection of } \phi\circ f_{3,N}^{-\tau t}\circ\phi^{-1}(b^s_{l_A}(x)) \text{ in } E^s(p) \text{ contains the subset } B^s_{l_D}\cap\phi(U). \tag{9}$$

The next lemma says that there is a relation between t and A.

LEMMA 6.2. *For A and t defined as before, there exists a positive integer K, which is independent of A (in particular, of N), such that*

$$K^{-1}\min\{\|Df^{t\tau}(p)|E^u\|^{-1},\|Df^{-t\tau}(p)|E^s\|^{-1}\}\le A\le K\max\{\|Df^{-t\tau}(p)|E^u\|,\|Df^{t\tau}(p)|E^s\|\}.$$

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Proof. By construction of $f_{3,N}$, note that $\phi \circ f_{3,N}^{\tau} \circ \phi^{-1}$ is linear and equal to $D\phi(p)\ Df^{\tau}(p)\ D\phi^{-1}(0)$ in $\phi(V)$. In particular, the action of $f_{3,N}^{\tau}$ on V does not depend on N. Also, recall that D^u and l_{D^s} do not depend on N, either.

Hence, by the above remarks and the choice of l_A, there exist constants K_1 and K_2, which do not depend on N (in particular, do not depend on A) such that for any $0 \leq i \leq t$ and $x \in f_{3,N}^{-i\tau}(f_{3,N}^{i\tau}(\phi^{-1}(\Gamma_{l_A})) \cap V)$:

$$K_1^{-1} \|Df^{i\tau}(p)|E^u\|^{-1} \leq d(f_{3,N}^{-i\tau}(x),\ W^s_{\mathrm{loc}}(p)) \leq K_1 \|Df^{-i\tau}(p)|E^u\|; \quad \text{and} \quad (10)$$

$$K_2^{-1}\ A\ \|Df^{i\tau}(p)|E^s\|^{-1} \leq d(f_{3,N}^{-i\tau}(x),\ W^u_{\mathrm{loc}}(p)) \leq K_2\ A\ \|Df^{-i\tau}(p)|E^s\|. \quad (11)$$

Thus, using the above inequalities for a point $x \in f_{3,N}^{t\tau}(f_{3,N}^{-\tau t}(\phi(\Gamma_{l_A})) \cap V)$ together with (8) and (9), we obtain

$$A \geq \min\{4K_1^{-1}\|Df^{t\tau}(p)|E^u\|^{-1},\ l_{D^s}K_2^{-1}\|Df^{-t\tau}(p)|E^s\|^{-1}\}.$$

On the other hand, by choice of t, there exists $x \in \phi(\Gamma_{l_A})$ such that $f_{3,N}^{i\tau}(x) \in V$ and:

- either $d(f_{3,N}^{-(t+1)\tau}(x),\ W^s_{\mathrm{loc}}(p)) \geq A/4$;
- or $d(f_{3,N}^{-(t+1)\tau}(x),\ W^u_{\mathrm{loc}}(p)) \leq l_{D^s}$.

Now, the two above inequalities together with the inequalities (10) and (11) imply that

$$A \leq \max\{4K_1\|Df^{-(t-1)\tau}(p)|E^u\|,\ l_{D^s}K_2\|Df^{(t-1)\tau}(p)|E^s\|\}.$$

Therefore, the proof is finished by taking

$$K = \max\{4K_1\|Df^{\tau}(p)|E^u\|,\ l_{D^s}K_2\|Df^{-\tau}(p)|E^s\|,\ K_1/4,\ K_2/l_{D^s}\}. \qquad \square$$

Finally, using t we will construct a hyperbolic set close to the disc D^s and using the previous lemma we will estimate the topological entropy of such hyperbolic set by means of the Lyapunov exponents of p.

For that, we define $\tilde{R} = \phi(D^s) \times \phi(f_{3,N}^{-\tau t}(D^u))$ and $R = \phi^{-1}(\tilde{R})$. Thus, the choice of t implies that $f_{3,N}^{T+\tau t}(R)$ intersects transversally R in N^n disjoint connected components. Then it is known that the maximal invariant set in R is a basic hyperbolic set $\tilde{\Lambda}_{p,N}$ which is conjugated to the shift acting in the space of sequences of N^n symbols. In particular, $h_{\mathrm{top}}(f_{3,N}^{-T-\tau t}|\tilde{\Lambda}_{p,N}) = \log N^n$. Moreover, if we define $\Lambda_{p,N} = \bigcup_{i=0}^{T+t\tau-1} f_{3,N}^{i}(\tilde{\Lambda}_{p,N})$, we have that

$$h_{\mathrm{top}}(f_{3,N}|\Lambda_{p,N}) = \frac{1}{T+\tau t}\log N^n. \quad (12)$$

It is important to remark now that a direct consequence of Lemma 6.2 is that t goes to infinity when N does, since $A = 2a\alpha\delta/\pi N$. Hence, taking $\varepsilon > 0$ arbitrarily small, we can use this information together with Lemma 6.2 to choose a large positive integer N such that

$$\frac{1}{T+\tau t}\log N > \min\left\{\frac{1}{T+\tau t}\log\|Df^{-\tau t}(p)|E^u\|^{-1},\ \frac{1}{T+\tau t}\log\|Df^{\tau t}(p)|E^s\|^{-1}\right\} - \frac{\varepsilon}{2}.$$

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Then, when t goes to infinity (which happens when N does), the minimum of the right-hand side of the above inequality converges to $\chi^+_{\min}(p)$ (the smallest positive Lyapunov exponent of p), by definition. Thus, there exists a large positive integer N_0 such that

$$\frac{1}{T+\tau t}\log N_0 > n\ \chi^+_{\min}(p) - \varepsilon. \tag{13}$$

Therefore, taking $g = f_{3,N_0}$ and $\Lambda_p = \Lambda_{p,N_0}$, we can use (12) and (13) to conclude that

$$h_{\text{top}}(g|\Lambda_p) > n\ \chi^+_{\min}(p, g) - \varepsilon,$$

which finishes the proof. □

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References

[**ABC**] F. Abdenur, C. Bonatti and S. Crovisier. Nonuniform hyperbolicity for C^1-generic diffeomorphisms. *Israel J. Math.* **183** (2011), 1–60.

[**AM**] A. Arbieto and C. Matheus. A pasting lemma and some applications for conservative systems. *Ergod. Th. & Dynam. Sys.* **27**(5) (2007), 1399–1417.

[**ArBC**] M.-C. Arnaud, C. Bonatti and S. Crovisier. Dynamiques symplectiques génériques. *Ergod. Th. & Dynam. Sys.* **25** (2005), 1401–1436.

[**BC**] C. Bonatti and S. Crovisier. Récurrence et généricité. *Invent. Math.* **158** (2004), 33–104.

[**BDP**] C. Bonatti, L. Diaz and E. Pujals. A C^1-generic dichotomy for diffeomorphisms: weak forms of hyperbolicity or infinitely many sinks or sources. *Ann. of Math.* **158** (2003), 355–418.

[**BR**] P. Barrientos and A. Raibekas. Robust cycles and tangencies of large codimension. *Preprint*, 2015, arXiv:1509.05325.

[**CH**] T. Catalan and V. Horita. C^1-genericity of symplectic diffeomorphisms and lower bounds for topological entropy. *Dyn. Syst.* (2017), http://dx.doi.org/10.1080/14689367.2017.1278744.

[**CT**] T. Catalan and A. Tahzibi. A lower bound for topological entropy of generic non Anosov symplectic diffeomorphisms. *Ergod. Th. & Dynam. Sys.* **34** (2014), 1503–1524.

[**F**] J. Franks. Necessary conditions for stability of diffeomorphisms. *Trans. Amer. Math. Soc.* **158** (1971), 301–308.

[**H**] S. Hayashi. Connecting invariant manifolds and the solution of the C^1 stability conjectures for flows. *Ann. of Math. (2)* **145**(1) (1997), 81–137.

[**HT**] V. Horita and A. Tahzibi. Partial hyperbolicity for symplectic diffeomorphisms. *Ann. Inst. H. Poincaré* **23** (2006), 641–661.

[**K**] A. Katok. Lyapunov exponents, entropy and periodic orbits for diffeomorphisms. *Publ. Math. Inst. Hautes Études Sci.* **51** (1980), 137–173.

[**KH**] A. Katok and B. Hasselblat. *Introduction to the Modern Theory of Dynamical Systems (Encyclopedia of Mathematics and its Applications, 54)*. Cambridge University Press, Cambridge, 1995.

[**Ne1**] S. E. Newhouse. Topological entropy and Hausdorff dimension for area preserving diffeomorphisms of surfaces. *Astérisque* **51** (1978), 323–334.

[**Ne2**] S. E. Newhouse. Continuity properties of entropy. *Ann. of Math.* **129**(2) (1989), 215–235.

[**XW**] L. Wen and Z. Xia. C^1 connecting lemmas. *Trans. Amer. Math. Soc.* **352**(11) (2000), 5213–5230.

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