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Instantaneous kinematics and singularity analysis of three-legged parallel manipulators

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SUMMARY
Instantaneous kinematics and singularity analysis of a class of three-legged, 6-DOF parallel manipulators are addressed in this paper. A generic method of derivation of reciprocal screw and consequently, the instantaneous kinematics model is presented. The advantage of this formulation is that the instantaneous kinematics model possesses well-defined geometric meaning and algebraic structure. Singularity analysis is performed under three categories, namely forward, inverse and combined singularities. A new concept of Passive Joint Plane is introduced to correlate the physical structure of the manipulator and these geometric conditions. In the inverse kinematic analysis, a new approach is introduced. At each leg end point a characteristic parallelepiped is defined whose sides are the linear velocity components from three main joints of the leg. An inverse singularity occurs when the volume of this parallelepiped becomes zero. Examples are demonstrated using RRRS and RPRS-type parallel manipulators.

KEYWORDS: Instantaneous kinematics; Singularity analysis; Parallel manipulators.

1. INTRODUCTION

Three-legged parallel manipulators unify the advantages of fully parallel manipulators and serial manipulators. Fully parallel manipulators (of the type Stewart-Gough platform mechanisms) have a high rigidity and accuracy, but have a small workspace and suffer from interference of the six legs. Serial manipulators, on the other hand, have a large workspace though it is less rigid and accurate. Three-legged parallel manipulators, combining the features of serial and fully parallel manipulators, have a fairly large workspace, symmetric actuation scheme, sufficient rigidity and reduced branch interference. However, the design, trajectory planning and application development of these manipulators are challenging because of the closed-loop nature of the mechanism. Therefore, they have received attention from many researchers[1–4].

The class of three-legged parallel manipulators, investigated here, comprises of a base and a mobile platform connected by three kinematic chains or legs. Each leg has three 1-degrees of freedom (DOF) joints and a spherical joint at the leg-end connecting to the mobile platform. Hence, each leg has 6 DOFs. To have a symmetric actuation scheme, two joints in each leg are actuated. Two prototypes of this class of manipulators, designed and constructed in our laboratory using modularity concept are shown in Figure 1.

One of the main concerns in the design of parallel manipulators is the occurrence of kinematic singularities. Singular configurations are particular poses of the end-effector, in which parallel robots lose their inherent rigidity and the end-effector gains or loses degrees of freedom. In other words, the parallel manipulator becomes uncontrollable upon encountering a singularity pose. Thus, singularity analysis plays an important role in the design of any parallel manipulator. Singularity analysis is performed by analyzing the two matrices relating the instantaneous velocities of the actuators and the end-effector. Derivation of this instantaneous kinematic relationship is the key to determination of singularity poses. These two important issues, namely study of instantaneous kinematics and singularity analysis of this class of parallel manipulators are the objective of this article.

Ma and Angeles[5] classified the kinematic singularities of a closed-loop mechanism into three categories: *architecture*, *configuration* and *formulation* singularities. While architecture singularity is related to a particular architecture of a parallel mechanism, formulation singularity is due to a bad scheme of modeling. Configuration singularity is further classified into three different categories namely forward, inverse and combined singularities.[6] To identify configuration singularities, the instantaneous relationship between the vectors of the mobile platform velocity (v) and the active joint velocity ($\dot{q}$) is established as:

$$A\boldsymbol{v} = B\dot{q}. \tag{1}$$

The first, second and third kind of configuration singularities occur respectively when matrix A is singular, matrix B is singular and both matrices are simultaneously singular.

Derivation of the relationship as described in Equation (1) is the stepping stone of any singularity analysis. A number of approaches based on the screw theory and line geometry have been proposed to establish this relationship. Mohamed and Duffy[7] introduced a concept of partial twists for instantaneous kinematic analysis. They proposed that for any parallel device, the twist representing the instantaneous motion of the end-effector is equal to the sum of its partial

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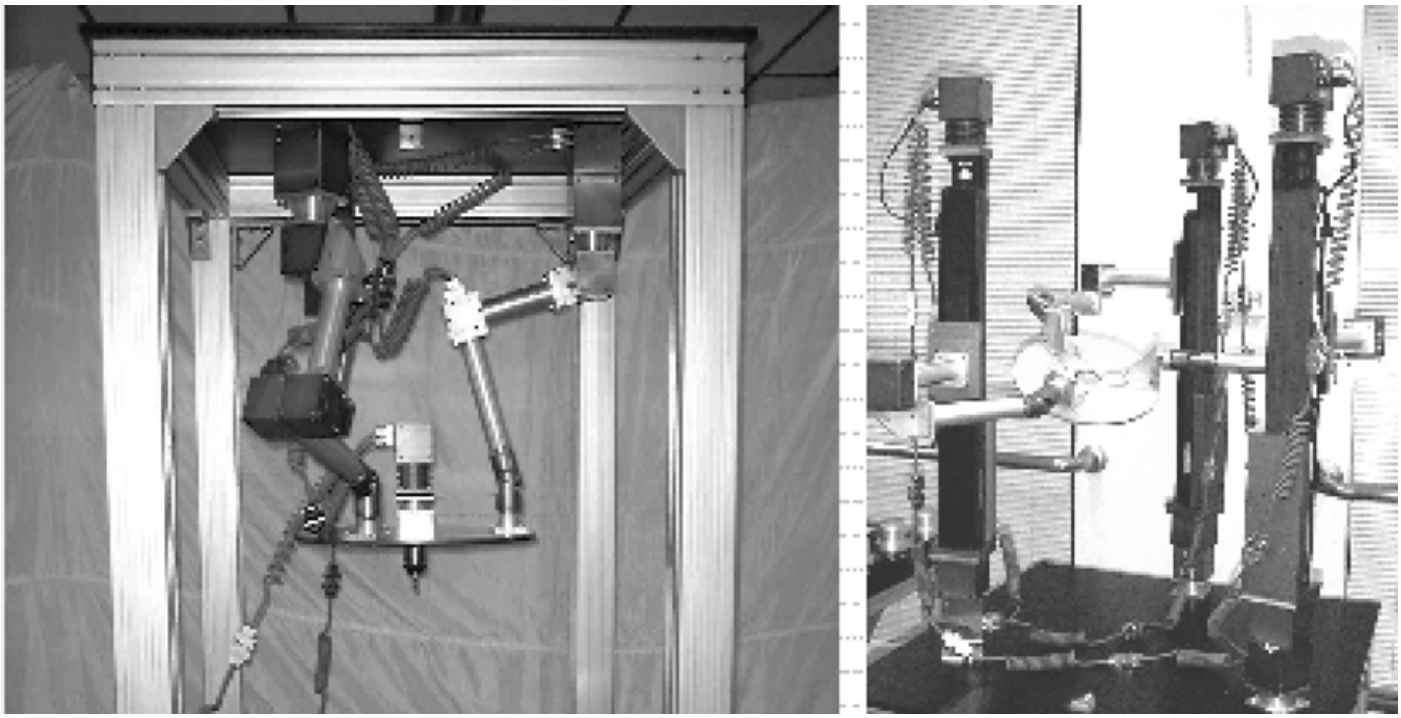


Fig. 1. RRRS and RPRS-type three legged parallel manipulators.

twists. Kumar[8] derived the instantaneous kinematic relationship using the reciprocal screw and provided a detailed singularity analysis by using concept of screw reciprocity. Kim *et al*.[9] extended on this concept to develop a general formulation to obtain an analytic Jacobian matrix for in-parallel manipulators. Angeles[10] introduced the concept of screw system annihilators (SSA) to reduce the number of unknowns in the derivation of instantaneous kinematics relations systematically to a minimum. Merlet[11] showed that by applying line geometry, it is possible to enumerate all the singularities of the Stewart-Gough platform. Leading from this, Notash *et al*.[2] identified the uncertainty configurations of three-branch parallel manipulators and proposed methods to eliminate them. Park and Kim[12] proposed a coordinate-invariant differential geometric analysis of singularities of closed kinematic chains. Shi and Fenton[13,14] developed a variation of the formulation of the forward velocity and acceleration kinematics based on the motion of three platform points. Yang *et al*.[4] proposed a new technique of deriving the instantaneous kinematic relationship and singularity analysis based on passive joint velocities of three-legged parallel manipulators. Other recent approaches to singularity analysis of closed-loop parallel mechanisms can be found in Agarwal,[15] Zlatonov *et al*.,[16] Xu *et al*.[17]

Unlike the above works, we have followed an approach based on *Product-of-Exponentials* (POE) formula. The advantage of POE representation method is that it describes the joint axes as twists based on line geometry.[18,19] As a result, it is equally applicable to the modeling of robot manipulators with both revolute and prismatic joints. Most important of all, the kinematic equations derived using POE has well defined algebraic structure so that they can be readily differentiated with respect to time. These features make the POE formula a very suitable tool for modeling and singularity analysis of the parallel manipulators.

In this paper, a very intuitive and easy method of construction of reciprocal screw is demonstrated while analyzing the instantaneous kinematic relationship. In the derivation of instantaneous kinematics model, each leg of this class of three-legged parallel manipulators is modeled as a serial 6-DOF chain. The end-effector twist is expressed in terms of 6 joint rates of a leg. Then five joint rates are systematically eliminated by introducing twist annihilators (discussed in Section 2.2). The result is an expression relating an actuated joint rate to the end-effector twist. The important feature of this expression is that it contains the reciprocal screw associated with an actuator joint preserving the geometry of the mechanism. This procedure is repeated for the remaining five actuators of the manipulator to finally obtain the overall kinematic relations between the six actuator joint rates and the end-effector twist. The notable advantage of this method is that reciprocal screws can be constructed with equal ease for all combinations of revolute and prismatic joints of a leg of this class of parallel manipulators. Also, the reciprocal screws are geometrically meaningful unlike the case in $D-H$ notation method.

To determine the singularity configurations, the classical way is to find the roots of the determinant of manipulator Jacobian matrices. However, in case of a parallel manipulator, the forward matrix, A, becomes complex and it is difficult to find its roots. Based on reciprocal screw and Grassmann line geometry, we propose geometric conditions to establish linear dependency among wrenches to identify forward singularity configurations. As these geometric conditions sometimes become complex and difficult to visualize, a new concept of *Passive joint plane* is introduced for easy detection of singularity postures satisfying these geometric conditions. Each case of linear dependency among wrenches is examined, DOF gained or lost, therein, is found and specific robot configurations are presented to

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illustrate the singularity postures. A geometrical characteristic of these wrenches is that they are of zero pitch. It is demonstrated that even for parallel structures having prismatic joints in legs, we can very easily identify these wrenches and consequently the singularity poses. The example configurations are demonstrated with RRRS-type and RPRS-type parallel manipulators.

In finding inverse singularity configurations, a new approach is followed. A characteristic parallelepiped is defined at spherical joint of each leg. The sides of this parallelepiped are the linear velocity components from three one-DOF joints of that leg. It is shown that an inverse singularity occurs when the volume of this parallelepiped becomes zero. The advantage of this method is that given any configuration of a parallel manipulator, we can physically view this parallelepiped and determine whether the parallel manipulator is in inverse singularity or not. We can also determine which joint(s) of which leg(s) are responsible for vanishing of the volume of this parallelepiped.

The remaining sections of this paper is organized as follows. In Section 2, the Product-of-exponentials (POE) formula and concept of twist annihilators are briefly discussed. Instantaneous kinematics model of three-legged parallel manipulators is derived in Section 3. In Section 4, singularity analysis is presented. The paper is summarized in Section 5.

2. Background

2.1. Product of exponentials (POE) formula

Brockett[18] showed that the forward kinematic equation of an open-chain robot containing either revolute or prismatic joints can be uniformly expressed as a product of matrix exponentials. Because of its compact representation and its connection with Lie groups and Lie algebras, the POE formula has proven to be a useful modeling tool in kinematics.[20–23]

In Figure 2(a), A is inertial reference frame and B is body reference frame. Relative pose (position and orientation) of frame B with respect to frame A, under a joint displacement q can be described by a 4×4 homogeneous matrix which is an element of $SE(3)$:

$$T_{ab}(q)=e^{\hat{s}q}T_{ab}(0), \tag{2}$$

where $T_{ab}(0)\in SE(3)$ is the initial pose of frame B relative to frame A. $T_{ab}(0)\in SE(3)$ and can be written as

$$T_{ab}(0)=\begin{bmatrix} R_{ab}(0) & \boldsymbol{p}_{ab}(0) \\ 0 & 1 \end{bmatrix}, \tag{3}$$

where $R_{ab}(0)\in SO(3)$ and $\boldsymbol{p}_{ab}(0)\in\mathbb{R}^{3\times1}$ are the initial orientation and position of frame B relative to frame A respectively. In Equation (2), $\hat{s}\in se(3)$ is the twist associated with joint B expressed in frame A. The twist, $\hat{s}$, can be written as

$$\hat{s}=\begin{bmatrix} \hat{\boldsymbol{w}} & \boldsymbol{v} \\ 0 & 0 \end{bmatrix}, \tag{4}$$

where $\boldsymbol{v}=[v_x, v_y, v_z]^T\in\mathbb{R}^{3\times1}$, and $\hat{\boldsymbol{w}}$ is the cross-product matrix (a skew symmetric matrix) of $\boldsymbol{w}=[w_x, w_y, w_z]^T\in\mathbb{R}^{3\times1}$. $\hat{\boldsymbol{w}}$ is given by

$$\hat{\boldsymbol{w}}=\begin{bmatrix} 0 & -w_z & w_y \\ w_z & 0 & -w_x \\ -w_y & w_x & 0 \end{bmatrix}. \tag{5}$$

The twist, $\hat{s}$, can also be expressed as a 6-dimensional vector through a mapping: $\hat{s}\mapsto s=(\boldsymbol{v}, \boldsymbol{w})\in\mathbb{R}^{6\times1}$, termed the twist coordinates. They represent the line coordinates of the screw axis at joint B which generates the rigid body motion. Given any screw, there exists a twist such that the rigid motion associated with the screw is generated by the twist (refer to[22] for more details). Here we define two operators: $\vee$ (vee) and $\wedge$ (wedge). A '$\vee$' operator is defined to extract the 6-dimensional vector which parameterizes a twist, i.e.,

$$\begin{bmatrix} \hat{\boldsymbol{w}} & \boldsymbol{v} \\ 0 & 0 \end{bmatrix}^{\vee}=\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix}. \tag{6}$$

The inverse operator, '$\wedge$', forms a matrix in $se(3)$ out of a given vector in $\mathbb{R}^6$, i.e.,

$$\begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{bmatrix}^{\wedge}=\begin{bmatrix} \hat{\boldsymbol{w}} & \boldsymbol{v} \\ 0 & 0 \end{bmatrix}. \tag{7}$$

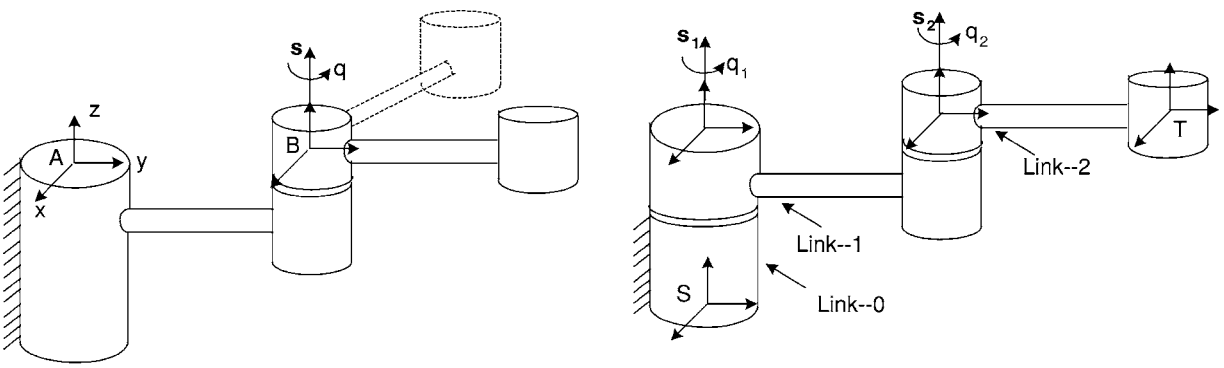


Fig. 2. Rigid motion.

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An explicit formula for the computation of $e^{\hat{s}q}$ is given as

$$e^{\hat{s}q} = \begin{bmatrix} e^{\hat{w}_q} & (I - e^{\hat{w}_q})(\boldsymbol{w} \times \boldsymbol{v}) + \boldsymbol{v}q \\ 0 & 1 \end{bmatrix}, \tag{8}$$

where

$$e^{\hat{w}q} = I + \hat{\boldsymbol{w}}\ sin\ q + \hat{\boldsymbol{w}}^2(1 - \cos\ q) \tag{9}$$

is the rotation matrix about an axis passing through the origin of the local frame and parallel to the joint axis $\boldsymbol{w}$ with an angle q.

Now we derive the forward kinematic transformation of a two degrees of freedom manipulator as shown in Figure 2(b). There are two frames: the inertial reference frame S and the body frame T. The two twists associated with two joints are s_1 and s_2. The respective displacements are q_1 and q_2. In case of a prismatic joint, $q \in \mathbb{R}$ is the amount of translation. To find the pose of body frame T with respect to inertial frame S, denoted as $T_{st}(q)$, let us assume that the first joint is fixed. As a result, the tool frame is a function of q_2 only, i.e.

$$T_{st}(q_2) = e^{\hat{s}_2 q_2} T_{st}(0). \tag{10}$$

Now fix the second joint and move the first joint. The end-effector configuration then becomes,

$$T_{st}(q_1, q_2) = e^{\hat{s}_1 q_1} T_{st}(q_2) = e^{\hat{s}_1 q_1} e^{\hat{s}_2 q_2} T_{st}(0). \tag{11}$$

Note that s_1 and s_2 are constant twists obtained by evaluating the screw motion for each joint at $q_1 = q_2 = 0$ configuration of the manipulator. Also, this representation does not depend upon the order in which we move the joints.

This procedure can be generalized to find the forward kinematic transformation of an open kinematic chain. Consider an open kinematic chain with $n+1$ links, sequentially numbered as $0, 1, \ldots, n$ (from the base link 0 to the end link n). The forward kinematic transformation, thus can be given by

$$T_{0,n} = T_{0,1}(q_1) T_{1,2}(q_2) \ldots T_{(n-1),n}(q_n) = e^{\hat{s}_1 q_1} e^{\hat{s}_2 q_2} \ldots e^{\hat{s}_n q_n} T_{0,n}(0). \tag{12}$$

Rigid body velocity. Consider $T_{ab}(t)$ to be a one-parameter curve representing trajectory of frame B (Figure 2(a)) relative to frame A. The spatial velocity of frame B[22] relative to frame A, denoted as $\hat{V}^s_{ab} \in se(3)$, is defined as

$$\hat{V}^s_{ab} = \dot{T}_{ab} T^{-1}_{ab}. \tag{13}$$

If $T_{ab}(t)$ is defined as

$$T_{ab}(t) = \begin{bmatrix} R_{ab}(t) & \boldsymbol{p}_{ab}(t) \\ 0 & 1 \end{bmatrix},$$

we have

$$\hat{V}^s_{ab} = \begin{bmatrix} \dot{R}_{ab} R^T_{ab} & -\dot{R}_{ab} R^T_{ab} \boldsymbol{p}_{ab} + \dot{p}_{ab} \\ 0 & 0 \end{bmatrix} \tag{14}$$

which is in the form of a twist. Consequently,

$$V^s_{ab} = (\dot{T}_{ab} T^{-1}_{ab})^\vee = \begin{bmatrix} \boldsymbol{v}^s_{ab} \\ \boldsymbol{w}^s_{ab} \end{bmatrix} = \begin{bmatrix} -\dot{R}_{ab} R^T_{ab} \boldsymbol{p}_{ab} + \dot{\boldsymbol{p}}_{ab} \\ (\dot{R}_{ab} R^T_{ab})^\vee \end{bmatrix}. \tag{15}$$

Twist coordinates of this spatial velocity is expressed as $V^s_{ab} = (\boldsymbol{v}^s_{ab}, \boldsymbol{w}^s_{ab})^T \in \mathbb{R}^{6\times 1}$. Here, the angular component $\boldsymbol{w}^s_{ab}$, is the instantaneous angular velocity of frame B as viewed in frame A. The linear component, $\boldsymbol{v}^s_{ab}$, is the velocity of a point on the rigid body travelling through the origin of the frame A. For an open kinematic chain of $n+1$ links, the forward kinematic transformation is given in Equation (12). The spatial velocity of the end-link frame relative to the base link frame, $\hat{V}^s_{0,n}$, can be expressed as:

$$\begin{aligned} \hat{V}^s_{0n} &= \dot{T}_{0n} T^{-1}_{0n} \\ &= \hat{s}_1 \dot{q}_1 + e^{\hat{s}_1 q_1} \hat{s}_2 (e^{\hat{s}_1 q_1})^{-1} \dot{q}_2 + (e^{\hat{s}_1 q_1} e^{\hat{s}_2 q_2}) \hat{s}_3 (e^{\hat{s}_1 q_1} e^{\hat{s}_2 q_2})^{-1} \dot{q}_3 + \ldots \\ &\quad + (e^{\hat{s}_1 q_1} \ldots e^{\hat{s}_{n-1} q_{n-1}}) \hat{s}_n (e^{\hat{s}_1 q_1} \ldots e^{\hat{s}_{n-1} q_{n-1}})^{-1} \dot{q}_n \\ &= (s_1)^\wedge \dot{q}_1 + (Ad_{e^{\hat{s}_1 q_1} s_2})^\wedge \dot{q}_2 + \ldots + (Ad_{(e^{\hat{s}_1 q_1} \ldots e^{\hat{s}_{n-1} q_{n-1}}) s_n})^\wedge \dot{q}_n \\ &= (s_1)^\wedge \dot{q}_1 + (s'_2)^\wedge \dot{q}_2 + (s'_3)^\wedge \dot{q}_3 + \ldots + (s'_n)^\wedge \dot{q}_n, \end{aligned} \tag{16}$$

where $Ad_{T_{i-1,i}}$ is the *adjoint transformation* associated with a rigid body motion $T_{i-1,i}$ transforming twists from frame $i-1$ to frame i.

$$Ad_{T_{i-1,i}} = \begin{bmatrix} R_{i-1,i} & \hat{\boldsymbol{p}}_{i-1,i} R_{i-1,i} \\ 0 & R_{i-1,i} \end{bmatrix}. \tag{15}$$

In Equation (16), $Ad_{T_{i-1,i}} s$ is the product of two matrices, $Ad_{T_{i-1,i}}$ and s. More clearly, if $\hat{s} \in se(3)$ is a twist with twist coordinates $s \in \mathbb{R}^6$, then for any rigid body motion $T \in SE(3)$, $T\hat{s}T^{-1}$ is a twist with twist coordinates $Ad_T s \in \mathbb{R}^6$ and mathematically,

$$(Ad_T s)^\wedge = T\hat{s}T^{-1}. \tag{18}$$

Writing Equation (16) in twist coordinates,

$$\begin{aligned} V^s_{0n} &= s_1 \dot{q}_1 + (Ad_{e^{\hat{s}_1 q_1} s_2}) \dot{q}_2 + \ldots + (Ad_{(e^{\hat{s}_1 q_1} \ldots e^{\hat{s}_{n-1} q_{n-1}}) s_n}) \dot{q}_n \\ &= [s_1 \quad s'_2 \quad \ldots \quad s'_n] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}. \end{aligned} \tag{19}$$

where

$$s'_i = Ad_{e^{\hat{s}_1 q_1} \ldots e^{\hat{s}_{i-1} q_{i-1}}} s_i. \tag{20}$$

Note that s'_i, the i^{th} column of the Jacobian, depends only on $q_1, q_2, \ldots, q_{i-1}$. In other words, the contribution on the i^{th} joint velocity is independent of the configuration of later joints in the chain. Furthermore, s'_i corresponds to the i^{th} joint twist, s_i, transformed to the current manipulator configuration by the rigid transformation given by $exp(\hat{s}_1 q_1) \ldots exp(\hat{s}_{i-1} q_{i-1})$. This gives us a powerful structural property that the manipulator Jacobian can be calculated even by inspection. Thus, *Product-of-Exponential* formula leads to a natural and explicit description of the manipulator Jacobian, which highlights the geometry of the mechanism.

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2.2. Twist annihilator

The concept of twist generators and twist annihilators was introduced by Angeles.[10] The twist generator of a kinematic pair is a screw which generates the twist of that kinematic pair when it is multiplied with the amplitude of the motion. Five lower kinematic pairs connecting link 1 and link 2 are shown in Figure 3. In this figure, $\boldsymbol{e}$ is the unit vector representing the direction of movement of the lower pair; $\boldsymbol{p}$ is the vector joining point o, on axis of the joint and a point P on link 2. The twist generators for these commonly used five lower kinematic pairs are:

Twist generator for a *revolute* pair, $s_r = \begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{e} \times \boldsymbol{p} \end{bmatrix}$.

Twist generator for a *prismatic* pair, $s_p = \begin{bmatrix} 0 \\ \boldsymbol{e} \end{bmatrix}$.

Twist generator for a *screw* pair, $s_h = \begin{bmatrix} \boldsymbol{e} \\ \boldsymbol{e} \times \boldsymbol{p} + h\boldsymbol{e} \end{bmatrix}$, where h is the pitch.

Twist generator for a *cylindrical* pair, $s_c = \begin{bmatrix} \boldsymbol{e} & \boldsymbol{0} \\ \boldsymbol{e} \times \boldsymbol{p} & \boldsymbol{e} \end{bmatrix}$.

Twist generator for a *spherical* pair,

$$s_s = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \boldsymbol{e}_1 \times \boldsymbol{p} & \boldsymbol{e}_2 \times \boldsymbol{p} & \boldsymbol{e}_3 \times \boldsymbol{p} \end{bmatrix}.$$

Here, a twist annihilator is defined as a 3×6 matrix which maps any of the above mentioned twist generators into the three-dimension zero array (in reference [10], it is defined as a 6×6 matrix whose last three rows are always zero). That is, the columns of the twist generator lie in the null space of the corresponding twist annihilator. Twist annihilators of the above mentioned twist generators are given as follows.

- Twist annihilator of the *revolute pair,* $A_r = [\hat{\boldsymbol{p}} \quad I]$, where $\hat{\boldsymbol{p}}$ is the skew-symmetric matrix made out of vector $\boldsymbol{p}$ and I is the identity matrix.
- Twist annihilator of the *prismatic* pair, $A_p = [I \quad 0]$, where 0 is a 3×3 zero matrix.
- Twist annihilator of the *screw* pair, $A_h = [\hat{\boldsymbol{p}} - hI \quad I]$.
- Twist annihilator of the *cylindrical* pair, $A_c = [\hat{\boldsymbol{p}} - I \quad I]$.
- Twist annihilator of the *spherical* pair, $A_s = [\hat{\boldsymbol{p}} \quad I]$.

Usually the leg of a parallel manipulator is configured using some of the above mentioned kinematic pairs. Inclusion of these commonly used lower kinematic pairs introduces idle variables. Twist annihilators allow the elimination of these idle variables in analysis of kinematic chains with multiple loops. Thus it eases the formulation of underlying kinematic

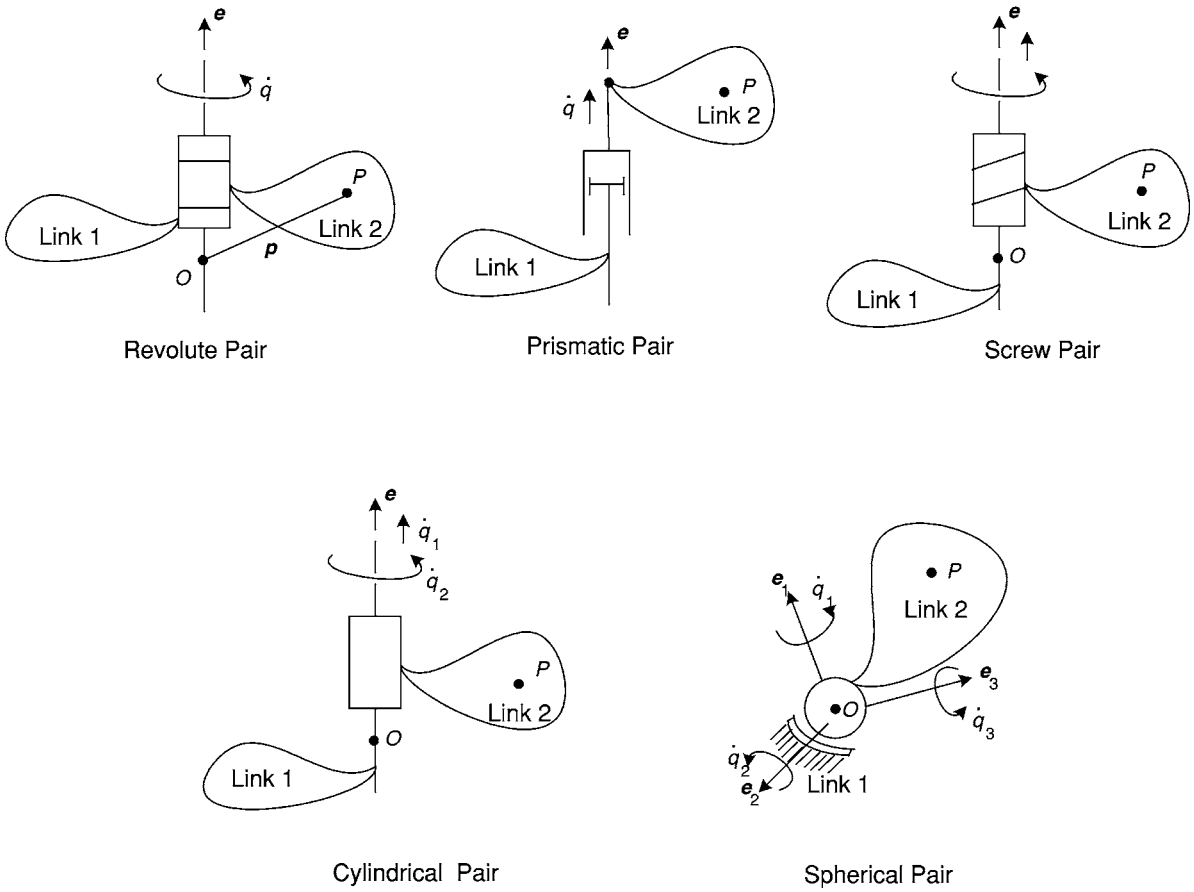


Fig. 3. Lower kinematic pairs.

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relations. In the following formulation, we will see how the twist annihilators eliminate the unwanted joint twists.

3. INSTANTANEOUS KINEMATICS

Shown in Figure 4 is the i^{th} leg of the three-legged parallel manipulator under study, where $i=1, 2, 3$. In each leg of the parallel manipulator, there are three 1-DOF joints ($i1$, $i2$ and $i3$) and a 3-DOF spherical joint. Among these three 1-DOF joints, often mentioned as *main* joints of the leg, two are active joints and one is passive joint. The spherical joint at the end of the leg is modeled as three revolute joints with their axes intersecting at one point. Without loss of generality, we assumed that the first two joints from the base ($i1$ and $i2$) as active joints and the third joint from the base ($i3$) as a passive joint. Twists of these three joints are shown as s_{i1}, s'_{i2} and s'_{i3} and those of the three revolute joints representing the spherical joint are shown as s'_{i4}, s'_{i5} and s'_{i6}. We define frame O as the local frame attached to the center of the mobile platform and frame B as the base frame. Each leg can be considered as a 6-DOF serial chain consisting of six 1-DOF joints. Using Equation (19), we express the end-effector twist (V^s_{bo}) as a linear combination of the six joint rates:

$$s_{i1}\dot{q}_{i1}+s'_{i2}\dot{q}_{i2}+s'_{i3}\dot{q}_{i3}+s'_{i4}\dot{q}_{i4}+s'_{i5}\dot{q}_{i5}+s'_{i6}\dot{q}_{i6}=V^s_{bo}, \qquad (21)$$

where

$$s'_{i2}=Ad_{e^{\hat{s}_1}q_1}s_{i2}, \quad s'_{i3}=Ad_{e^{\hat{s}_1}q_1e^{\hat{s}_2}q_2}s_{i3},$$
$$s'_{i4}=Ad_{e^{\hat{s}_1}q_1e^{\hat{s}_2}q_2e^{\hat{s}_3}q_3}s_{i4}, \quad s'_{i5}=Ad_{e^{\hat{s}_1}q_1e^{\hat{s}_2}q_2e^{\hat{s}_3}q_3e^{\hat{s}_4}q_4}s_{i5},$$
$$s'_{i6}=Ad_{e^{\hat{s}_1}q_1e^{\hat{s}_2}q_2e^{\hat{s}_3}q_3e^{\hat{s}_4}q_4e^{\hat{s}_5}q_5}s_{i6}.$$

Regardless of the type of joints, we can express the twist coordinates of the first three joints, s_{i1}, s'_{i2}, s'_{i3} in Equation (21) as:

$$s_{i1}=\begin{bmatrix}\boldsymbol{v}_{i1}\\ \boldsymbol{w}_{i1}\end{bmatrix}, \quad s'_{i2}=\begin{bmatrix}\boldsymbol{v}'_{i2}\\ \boldsymbol{w}'_{i2}\end{bmatrix}, \quad s'_{i3}=\begin{bmatrix}\boldsymbol{v}'_{i3}\\ \boldsymbol{w}'_{i3}\end{bmatrix}.$$

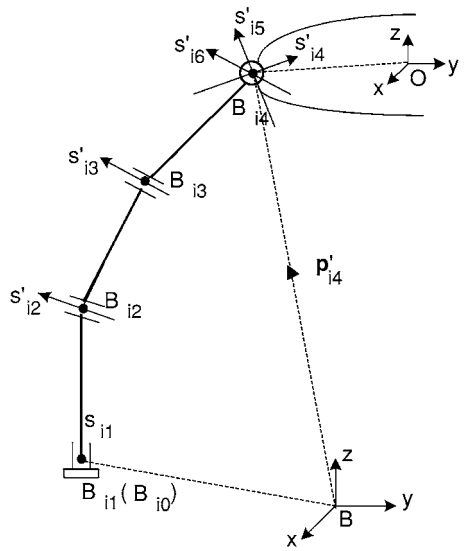


Fig. 4. i^{th} leg of the parallel manipulator.

Since the spherical joint is modeled as three revolute joints, we can write

$$s'_{i4}=\begin{bmatrix}-\boldsymbol{w}'_{i4}\times\boldsymbol{p}'_{i4}\\ \boldsymbol{w}'_{i4}\end{bmatrix}, s'_{i5}=\begin{bmatrix}-\boldsymbol{w}'_{i5}\times\boldsymbol{p}'_{i4}\\ \boldsymbol{w}'_{i5}\end{bmatrix}, s'_{i6}=\begin{bmatrix}-\boldsymbol{w}'_{i6}\times\boldsymbol{p}'_{i4}\\ \boldsymbol{w}'_{i6}\end{bmatrix}.$$

Note that $\boldsymbol{p}'_{i4}$ is the vector joining the origin of base frame B to the spherical joint centre of leg i, B_{i4}. Putting above expressions of $s_{i1}, \ldots, s'_{i6}$ into Equation (21), we obtain

$$\begin{bmatrix}\boldsymbol{v}_{i1} & \boldsymbol{v}'_{i2} & \boldsymbol{v}'_{i3} & -\boldsymbol{w}'_{i4}\times\boldsymbol{p}'_{i4} & -\boldsymbol{w}'_{i5}\times\boldsymbol{p}'_{i4} & -\boldsymbol{w}'_{i6}\times\boldsymbol{p}'_{i4}\\ \boldsymbol{w}_{i1} & \boldsymbol{w}'_{i2} & \boldsymbol{w}'_{i3} & \boldsymbol{w}'_{i4} & \boldsymbol{w}'_{i5} & \boldsymbol{w}'_{i6}\end{bmatrix}$$
$$\times\begin{bmatrix}\dot{q}_{i1}\\ \dot{q}_{i2}\\ \dot{q}_{i3}\\ \dot{q}_{i4}\\ \dot{q}_{i5}\\ \dot{q}_{i6}\end{bmatrix}=V^s_{bo} \qquad (22)$$

The twist annihilator to annihilate the three twists representing the spherical joint, namely the 4^{th}, 5^{th} and 6^{th} columns, is given by $[I \quad -\hat{\boldsymbol{p}}'_{i4}]$ (a 3×6 matrix). Here $\hat{\boldsymbol{p}}'_{i4}$ is the skew-symmetric matrix of the vector $\boldsymbol{p}'_{i4}$, and I is a 3×3 identity matrix. Pre-multiplying it to both sides of Equation (22), we obtain

$$[\boldsymbol{v}_{i1}-\boldsymbol{p}'_{i4}\times\boldsymbol{w}_{i1} \quad \boldsymbol{v}'_{i2}-\boldsymbol{p}'_{i4}\times\boldsymbol{w}'_{i2} \quad \boldsymbol{v}'_{i3}-\boldsymbol{p}'_{i4}\times\boldsymbol{w}'_{i3}]$$
$$\times\begin{bmatrix}\dot{q}_{i1}\\ \dot{q}_{i2}\\ \dot{q}_{i3}\end{bmatrix}=[I \quad -\hat{\boldsymbol{p}}'_{i4}]V^s_{bo}. \qquad (23)$$

Since $\boldsymbol{v}'_{ij}-\boldsymbol{p}'_{i4}\times\boldsymbol{w}'_{ij}=[I \quad -\hat{\boldsymbol{p}}'_{i4}]s'_{ij}$ (for $j=2, 3$), Equation (23) can be written as

$$[[I \quad -\hat{\boldsymbol{p}}'_{i4}]s_{i1} \quad [I \quad -\hat{\boldsymbol{p}}'_{i4}]s'_{i2} \quad [I \quad -\hat{\boldsymbol{p}}'_{i4}]s'_{i3}]$$
$$\times\begin{bmatrix}\dot{q}_{i1}\\ \dot{q}_{i2}\\ \dot{q}_{i3}\end{bmatrix}=[I \quad -\hat{\boldsymbol{p}}'_{i4}]V^s_{bo}. \qquad (24)$$

where $[I \quad -\hat{\boldsymbol{p}}'_{i4}]s_{i1}$, $[I \quad -\hat{\boldsymbol{p}}'_{i4}]s'_{i2}$ and $[I \quad -\hat{\boldsymbol{p}}'_{i4}]s'_{i3}$ are 3×1 vectors representing the linear velocity contributions of joint $i1$, $i2$ and $i3$ at the spherical joint center of the i^{th} leg respectively. The annihilator to eliminate the second and third actuator joint rates, $\dot{q}_{i2}$ and $\dot{q}_{i3}$, from Equation (24) is given by $[([I \quad -\hat{\boldsymbol{p}}'_{i4}]s'_{i2})\times([I \quad -\hat{\boldsymbol{p}}'_{i4}]s'_{i3})]^T$. Note that this is the transpose of the cross product of the two vectors in column-2 and column-3 of the matrix at left of left hand side of Equation (24). Pre-multiplying it to both sides of Equation (24), we obtain

$$b_{i1}\dot{q}_{i1}=W_{i1}V^s_{bo}, \qquad (25)$$

where

$$b_{i1}=([I \quad -\hat{\boldsymbol{p}}'_{i4}]s'_{i2})\times([I \quad -\hat{\boldsymbol{p}}'_{i4}]s'_{i3})\cdot([I \quad -\hat{\boldsymbol{p}}'_{i4}]s_{i1}), \qquad (26)$$

is a scalar and

$$W_{i1}=\begin{bmatrix}([I \quad -\hat{\boldsymbol{p}}'_{i4}]s'_{i2})\times([I \quad -\hat{\boldsymbol{p}}'_{i4}]s'_{i3})\\ -\{([I \quad -\hat{\boldsymbol{p}}'_{i4}]s'_{i2})\times([I \quad -\hat{\boldsymbol{p}}'_{i4}]s'_{i3})\}\times\boldsymbol{p}'_{i4}\end{bmatrix}^T, \qquad (27)$$

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is a 1×6 vector. The important thing on the derivation of Equation (25) from Equation (21) is that W_{i1} is reciprocal to all the screws of i^{th} leg except to s_{i1}, i.e.

$$W_{i1}\,.\,s'_{ij}=0 \quad \text{for} \quad i=1,\,2,\,3 \quad \text{and} \quad j=2,\,3,\,4,\,5,\,6$$

$$W_{i1}\,.\,s_{ij}=b_{i1} \quad \text{for} \quad i=1,\,2,\,3 \quad \text{and} \quad j=1.$$

The essential point about this derivation of W_{i1} (Equation (27) is that W_{i1} is a zero-pitch screw and no matter what may be the combination of the joint types in the leg this form is retained. Using vector formulation method, it would not have been possible to give a generic reciprocal screw for this class of parallel manipulators containing revolute and prismatic joints. In other words, W_{i1} is a zero-pitch reciprocal screw (not in normalized screw coordinates) associated with the first joint of the i^{th} leg. It is also called the wrench associated with the first actuator joint acting on the platform. As it can be seen, this method of constructing the reciprocal screw is simple and systematic and can be uniformly applied to any combinations of revolute and prismatic joints for the first three joints of a leg in this class of parallel manipulators. Adopting a similar method, we eliminate the first and third actuator joint rates, $\dot{q}_{i1}$ and $\dot{q}_{i3}$, from Equation (24) by pre-multiplying $[([I \quad -\hat{\boldsymbol{p}}'_{i4}]s_{i1}) \times ([I \quad -\hat{\boldsymbol{p}}'_{i4}]s'_{i3})]^T$ to both sides of Eq. (4). Note that this is the transpose of the cross product of the two vectors in column-1 and column-3 of the left matrix at the LHS of Equation (24). The result is a relationship between the second actuator joint rate and the end-effector twist

$$b_{i2}\dot{q}_{i2}=W_{i2}V^s_{bo}. \tag{28}$$

where

$$b_{i2}=([I \quad -\hat{\boldsymbol{p}}'_{i4}]s_{i1})\times([I \quad -\hat{\boldsymbol{p}}'_{i4}]s'_{i3})\cdot([I \quad -\hat{\boldsymbol{p}}'_{i4}]s'_{i2}). \tag{29}$$

Deriving such relationships for the remaining four actuators in the other two legs and arranging them in matrix form, we obtain

$$Diag\,.\,\{b_{11},\,b_{12},\,b_{21},\,b_{22},\,b_{31},\,b_{32}\}\begin{bmatrix}\dot{q}_{11}\\ \dot{q}_{12}\\ \dot{q}_{21}\\ \dot{q}_{22}\\ \dot{q}_{31}\\ \dot{q}_{32}\end{bmatrix}=\begin{bmatrix}W_{11}\\ W_{12}\\ W_{21}\\ W_{22}\\ W_{31}\\ W_{32}\end{bmatrix}V^s_{bo} \tag{30}$$

or,

$$B\dot{q}_a=AV^s_{bo}, \tag{31}$$

where B, called an *inverse Jacobian matrix*, is a 6×6 diagonal matrix and b_{ij} is a scalar element and A, called a *forward Jacobian matrix*, is a 6×6 square matrix, where W_{ij} is the wrench associated with the j^{th} joint of the i^{th} leg. In Equation (30), whether the subscript j in W_{ij}, b_{ij} and $\dot{q}_{ij}$ is 1, 2 or 3, depends upon the labels of the actuated joints in the i^{th} leg. Equation 31 is derived when the first two joints of each leg are active, i.e., $j=1,\,2$. When the actuators are placed at first and third joints or at second and third joints of a leg then similar relationship as given in Equation 31 can be derived. In these cases, the entries of matrices A, B and $\dot{q}_a$ will refer to the labels of the active joints.

4. Singularity Analysis

Based on the kinematic relationship derived in Equation (31), singularity analysis can be carried out. Three kinds of singularities are identified in the configuration singularity category, namely forward, inverse and combined singularities.

4.1. Forward Singularity

A forward singularity occurs when matrix A becomes singular. Matrix A consists of six zero-pitch wrenches associated with six actuators. Hence, the singularity condition of matrix A is the same as the condition of linear dependency of two or more such wrenches. A zero-pitch wrench can be represented as a line and thus, linear dependency among the wrenches becomes equivalent to the linear dependency among the lines they represent. Line varieties from rank one to five in *projective space*, $P5$[11] are shown in Figure 5. While discussing a line variety of rank n ($n=1,\,2,\,\ldots,\,5$), we find the condition in which $n+1$ lines constitute the corresponding line system of rank n. This will be the case in which matrix A becomes singular. These conditions are uniform to any combination of revolute and prismatic joints in the leg of a parallel manipulator. All the cases of singularities are illustrated with two example configurations: a 6-DOF, RRRS-type parallel manipulator whose base and distal joints are active joints and a 6-DOF, RPRS-type parallel manipulator whose base and middle joints are active joints. A leg of this RRRS-type manipulator is shown in Figure 6. The following discussion in this paragraph shows how to find the wrench associated with the second actuator joint (third joint from the base) of this manipulator geometrically.

The wrench associated with the third joint will be reciprocal to the first two joints (revolute joints) and to the spherical joint. A reciprocal screw to a revolute joint intersects the revolute joint axis and that to a spherical joint passes through the center of the spherical joint. Therefore, the wrench should intersect s_{11} and s'_{12} and pass through B_{14}. The line joining B_{12} and B_{14} then represents the wrench W_{13}.

Passive joint plane. In general, a three-legged 6-DOF, nonredundant parallel manipulator having symmetric actuation scheme and passive spherical end-joints has the following features:

(i) Each leg has two zero-pitch wrenches.
(ii) Each of these wrenches passes through the center of the spherical joint.
(iii) If the passive joint is a revolute joint, these two wrenches lie on a plane formed by the passive revolute joint axis and the passive spherical joint center. If it is a prismatic joint, these wrenches lie on a plane which passes through the spherical joint center and normal to the passive prismatic joint axis. We denote this plane as *Passive joint plane*. The concept of Passive joint plane becomes very useful to locate the wrenches of a parallel manipulator. This plays an important role in finding conditions for linear dependency in line systems of different ranks.

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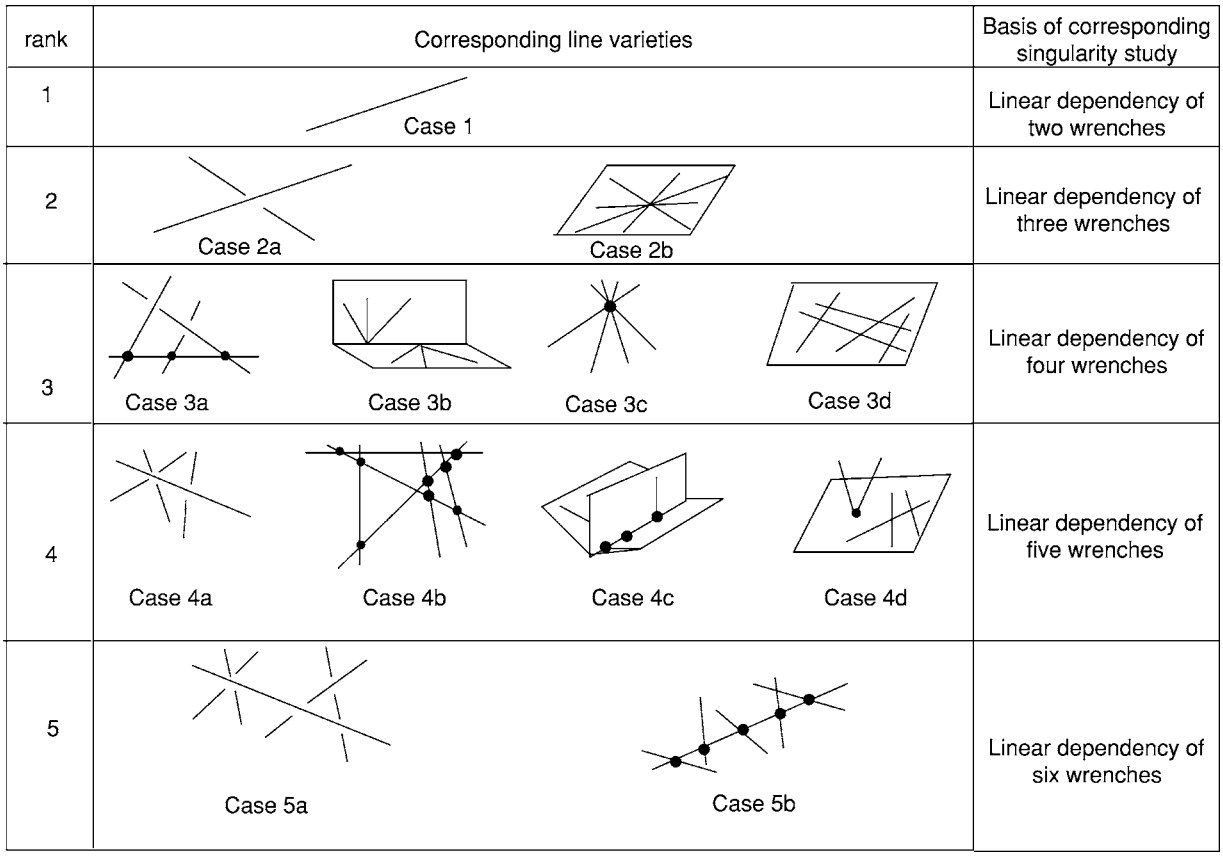


Fig. 5. Line varieties of ranks 1, 2, 3, 4 and 5 [11].

As shown in Figure 5, there can be five cases of linear dependency among lines/zero-pitch wrenches and each case represents a forward singularity configuration. In the following discussion each of these cases is analyzed.

4.1.1. Linear dependency of two wrenches.
Case 1: Two zero-pitch wrenches, hence lines are linearly dependent if they lie on the same axis (Figure 5).

Condition: A wrench of one leg is co-linear with a wrench of either of the two other legs.

Example 1: For an RPRS-type three-legged parallel manipulator with the first and second joints as actuator joints, this case occurs when B_{13}, B_{14}, B_{24} and B_{23} fall on a line (Figure 7). As a result, wrenches associated with the first joints (the first active joint in the leg) of leg-1 and leg-2 become co-linear. Therefore, W_{11} and W_{21} in matrix A become linearly dependent, and matrix A becomes singular.

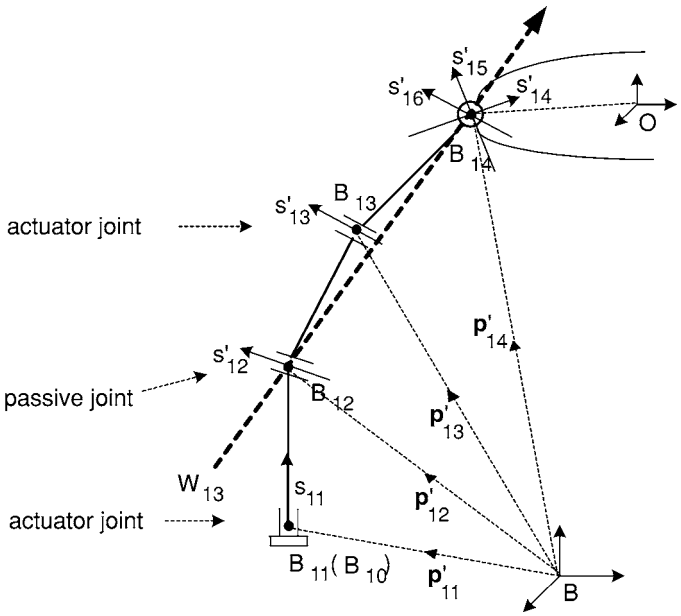


Fig. 6. Construction of wrench of third joint of RRRS leg.

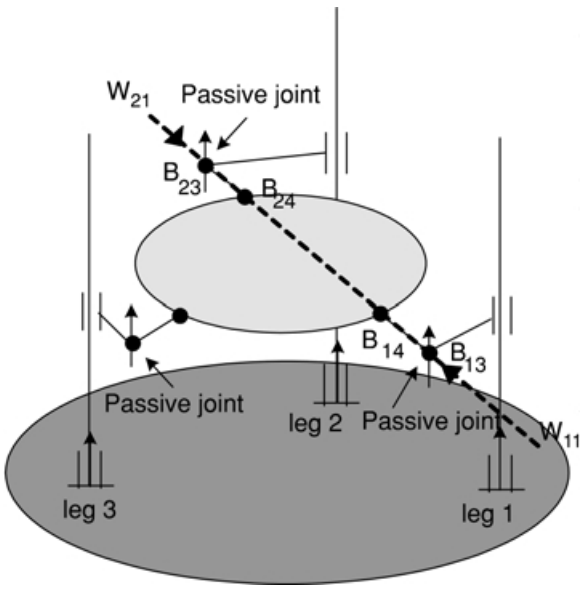


Fig. 7. **Case 1** singularity of RPRS-type parallel manipulator.

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4.1.2. Linear dependency of three wrenches.
Case 2a: Two skew lines constitute a line system of rank two.

Condition: This case can be eliminated because two skew zero-pitch lines are only dependent on themselves.

Case 2b: A minimum of two co-planar and concurrent lines constitute a line system of rank two.

Condition: The conditions for linear dependency are: (1) one wrench of any leg passes through the center of the spherical joint of another leg and also lies in the passive joint plane of that leg; (2) three wrenches, each from a different leg, lie on the plane of the platform and intersect at a common point. This condition is discussed in Example 2.

Example 2: For the RPRS-type three legged parallel manipulators having base and middle joints as active joints, such a case is shown in Figure 8. In this configuration, passive joint centers and the spherical joint centers of all the three legs lie on a plane such that three lines, each connecting these two joints of a leg, are concurrent. Hence, the three wrenches, W_{11}, W_{21} and W_{31} become coplanar and intersect at a common point. They constrain the motion of the platform coincident with the common intersecting point along two translational DOFs on the plane of the platform. Assuming that the remaining three wrenches are linearly independent of this line system, all the six wrenches constrain 5 DOFs leaving one DOF unconstrained. Thus the platform gains one DOF.

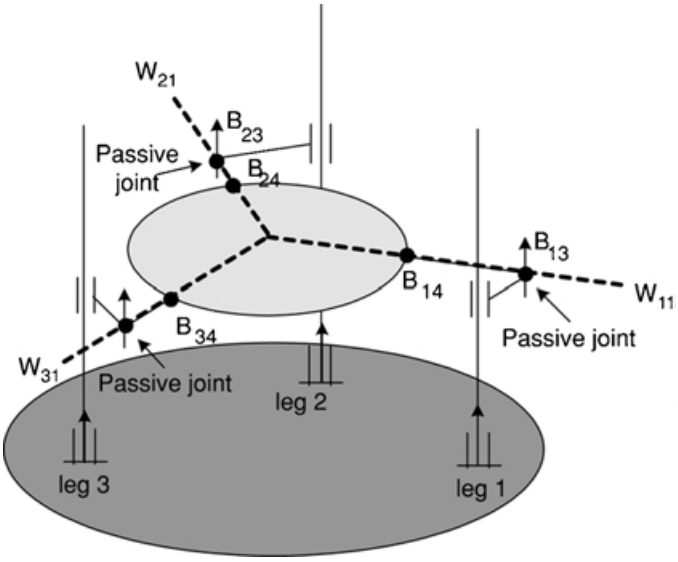


Fig. 8. **Case 2b** singularity for RPRS-type parallel manipulator.

4.1.3. Linear dependency of four wrenches.
Case 3a: Three independent lines belonging to a regulus constitute a line system of rank three.

Condition: This case of line dependency does not exist because it is impossible to find four lines belonging to a regulus constituting a line system of rank three.[24]

Case 3b: Two flat pencils (Flat pencil refers to lines lying on a plane and passing through some point on that plane) having a line in common and lying in distinct planes and with distinct centers constitute a line system of rank three.

Condition: The condition for linear dependency is that the passive joint planes of any two legs, contain each other's spherical joint centers. If, for example, these two legs are the first and second legs and their passive joint planes contain B_{24} and B_{14}, respectively, then the four wrenches of these two legs constrain the translation of line $B_{14}B_{24}$ on the mobile platform in three directions. The three linearly independent unconstrained DOFs are: (1) a twist about $B_{14}B_{24}$, (2) twist about a line normal to passive joint plane of leg 1 and passing through B_{14}, (3) twist about a line normal to passive joint plane of leg 2 and passing through B_{24}. The two wrenches of the third leg, if linearly independent, constrain two DOFs out of these three and the mobile platform is left with one unconstrained DOF.

Example 3: For the RRRS-type three legged parallel manipulators having base and distal joints as active joints, such a case is shown in Figure 9. In this configuration, the passive joint axes of leg 1 and leg 2 are both parallel to the line $B_{14}B_{24}$. As a result, both wrenches of leg 1 and leg 2 lie on respective passive joint planes.

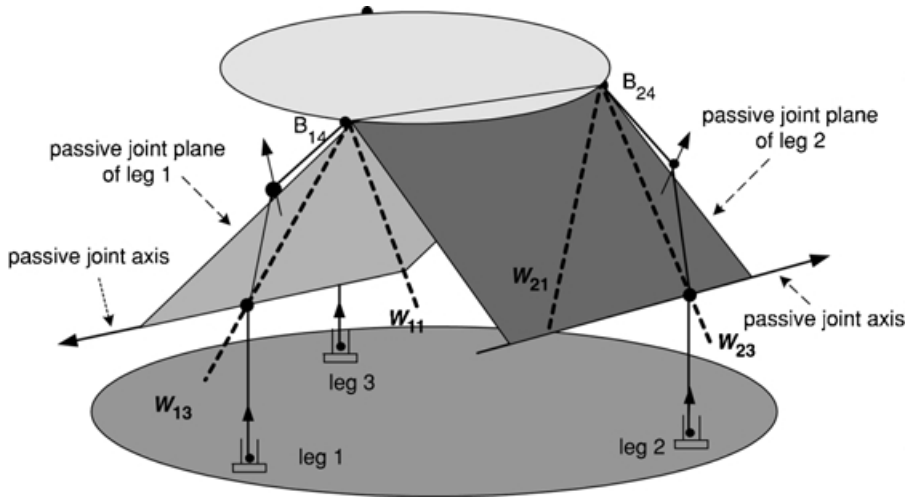


Fig. 9. **Case 3b** singularity for RRRS-type parallel manipulator.

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Case 3c: Three or more lines passing through a point constitute a line system of rank three. This common point is called the bundle center.

Condition: If two wrenches, each from a different leg, pass through the spherical joint center of the third leg, then there are four non-coplanar wrenches passing through the spherical joint center (bundle center) of the third leg. This constitutes a line system of rank three.

In this configuration, four concurrent wrenches will constrain the mobile platform point coincident with the bundle center (the spherical joint center of the third leg) along three translational DOFs. The remaining two wrenches can constrain the bundle center along two rotational DOFs. The unconstrained rotational DOF corresponds to a zero-pitch instantaneous screw passing through the bundle center and intersecting the remaining two wrenches.

Example 4: As shown in Figure 10, wrench W_{13} of leg 1 and wrench W_{23} of leg 2 pass through B_{34}, the spherical joint centre of the third leg. Through B_{34} also pass the two wrenches of the third leg. Now four wrenches pass through the bundle center, B_{34}, constituting this case of dependency. The unconstrained DOF will be a rotation about a line passing through B_{34} and intersecting the two remaining wrenches, W_{11} and W_{21} (not shown in the figure).

Case 3d: A minimum of three coplanar and non-concurrent lines constitute a line system of rank three.

Condition: Four wrenches out of the six wrenches are coplanar. This happens when

- passive joint planes of any two legs are coplanar (discussed in Example-5)
- two wrenches, each from a different leg, fall on the passive joint plane of the third leg.

The three constrained DOFs are: two translational DOFs in the passive joint plane and one rotational DOF about a line normal to this plane. The three unconstrained DOFs are: one translational DOF in the direction normal to the passive joint plane and two rotational DOFs about lines lying on this plane with the direction perpendicular to the normal of the plane. Depending upon the DOFs constrained by the two remaining wrenches, the mobile platform gains one or more DOFs from the above three mentioned unconstrained DOFs.

Example 5: For the RRRS-type parallel manipulator having base and distal joints as active joints, s_{12}, s_{22} and $B_{14}B_{24}$ lie on a single plane when s_{12} and s_{22} are co-linear and parallel to $B_{14}B_{24}$. Hence, all the four wrenches of the two legs lie on this plane. Since two wrenches of each leg pass through the spherical joint center, the four wrenches are non-concurrent (Figure 11) and constitute the line dependency.

4.1.4. Linear dependency of five wrenches.

Case 4a: Four independent skew lines constitute a line system of rank four.

Condition: This case of line dependency does not exist because three wrenches of this class of parallel manipulators intersect the other three wrenches.

Case 4b: If four lines are concurrent with two skew lines, these four lines constitute a line system of rank four.

Condition: In this case, five wrenches have to intersect two skew lines. Lines to which five wrenches can intersect are,

- a line joining the spherical joint centers of two legs if any wrench of the third leg lies on the plane of the platform
- a common line between the passive joint planes of two legs if it also intersects one wrench of the third leg
- a line passing through following two points; first point: spherical joint of a leg, second point: intersection point of a wrench of another leg with the passive joint plane of the third leg.

This case of dependency arises if any two of these lines are skew. These five wrenches constrain 4 DOFs. The two unconstrained DOFs are the twists along the two skew lines. If the sixth wrench does not intersect with these two skew

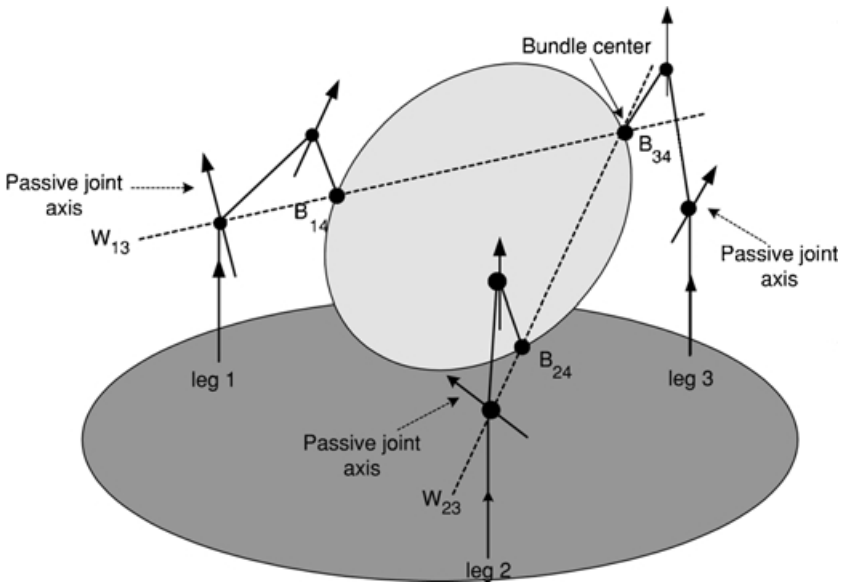


Fig. 10. **Case 3c** singularity for RRRS-type parallel manipulator.

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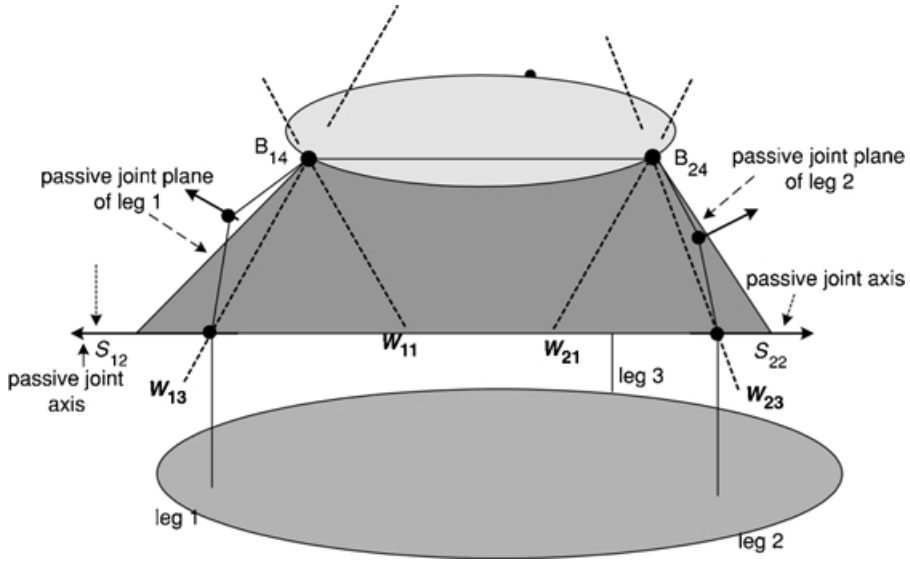


Fig. 11. **Case 3d** singularity for RRRS-type parallel manipulator.

lines, it will constrain one more DOF and leave the unconstrained DOF to one.

Example 6: Figure 12 shows this case of singularity for RRRS-type parallel manipulator. Line $B_{24}B_{34}$ is parallel to the base. A wrench of leg 1, W_{13}, lies in the plane of the platform, so line $B_{24}B_{34}$ intersects with five wrenches (two of leg-2, two of leg-3 and W_{13}). Line L_1 is the common line of the passive joint planes of leg 2 and leg 3. It is skew to line $B_{24}B_{34}$ because screws s_{22} and s_{32} are parallel to the base and are not coincident. Also line $B_{24}B_{34}$ is not coplanar with either of these two joint axes (s_{22} and s_{32}). Thus line L_1 intersects with the same five wrenches as line $B_{24}B_{34}$ does. These five wrenches constrain four DOFs; the two unconstrained DOFs are twists about line L_1 and line $B_{24}B_{34}$. The sixth wrench can constrain only one of these two, leaving behind minimum one unconstrained DOF.

Case 4c: When four lines form among themselves three planes, as shown in Figure 5, they constitute a line system of rank four.

Condition: This case of line dependency can arise only if the three spherical joint centers become co-linear.

Case 4d: If a minimum of two out of five lines are coplanar and the remaining three lines intersect at a point on the plane, then all these lines constitute a line system of rank four.

Condition: There are two possibilities:

- A wrench of the first leg lies in the passive joint plane of the second leg and a wrench of the third leg passes through the spherical joint center of the first leg.
- Two wrenches of a leg and one wrench of another leg pass through the spherical joint center of the third leg.

The five wrenches constrain only 4 DOFs. The two linearly independent unconstrained DOFs are the twists about two orthogonal axes lying on the plane of wrenches and passing through the spherical joint center. In the following example, the first possibility is described.

Example 7: For the RRRS-type three-legged parallel manipulator, this case arises when the passive joint axis of leg 2 and one wrench of leg 1, W_{13}, pass through B_{34} (Figure 13). As a result, two wrenches of leg 2 lie on its passive joint plane and three other wrenches (two of leg 3 and W_{13} of leg 1) intersect at B_{34}, a point lying on the passive joint plane of leg-2.

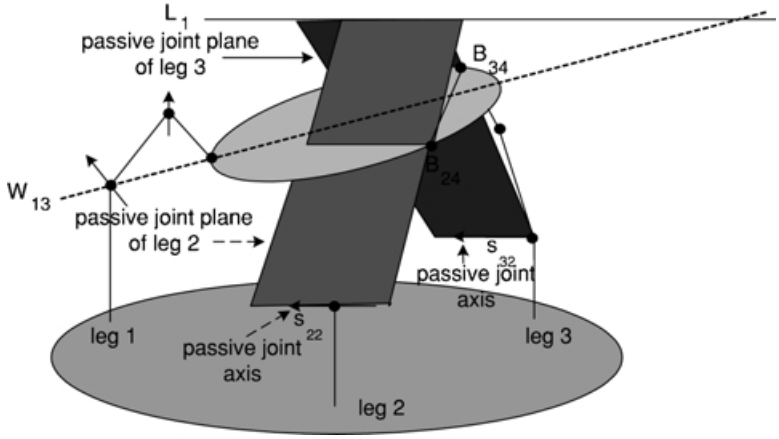


Fig. 12. **Case 4b** singularity for RRRS-type parallel manipulator.

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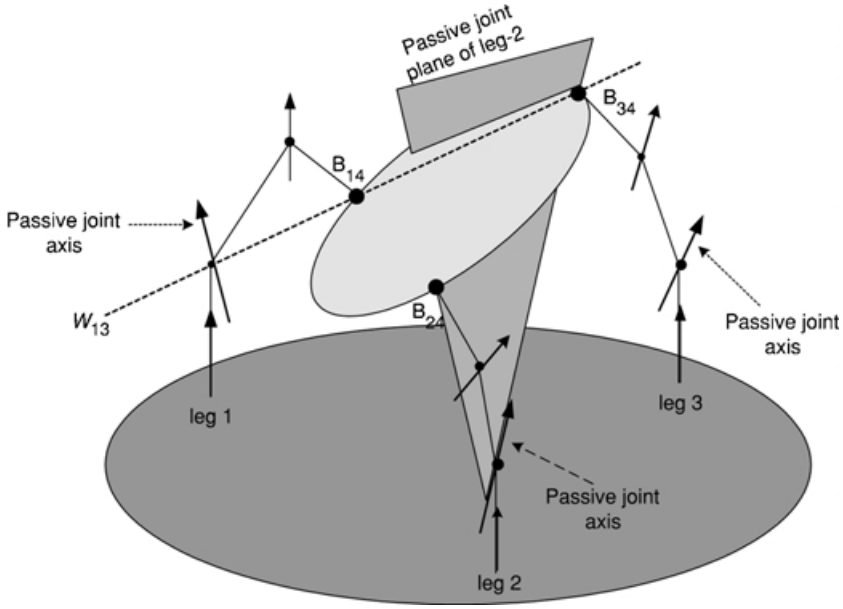


Fig. 13. **Case 4d** singularity of RRRS-type parallel manipulator.

4.1.5. Linear dependency of six wrenches.
Case 5a: Five independent skew lines constitute a line system of rank five.

Condition: This case does not exist since there cannot be more than three skew wrenches.

Case 5b: When six lines intersect with a line, these six lines constitute a line system of rank five.

Condition: All the six wrenches of the parallel manipulator must intersect a line. This can happen in following ways:

- When two wrenches of a leg lie on the plane of the mobile platform, the line joining spherical joint centers of other two legs intersects with six wrenches.
- If the common line of the two passive joint planes passes through the spherical joint center of the third leg, this line intersects six wrenches.
- A common line of all the three passive joint planes, if it exists, intersects six wrenches.
- If the passive joint plane of a leg passes through the spherical joint center of another leg and the two intersection points of the two wrenches of the third leg are co-linear with this spherical joint center, the line joining these three points (the spherical joint center and the two intersection points) will intersect six wrenches.
- When the passive joint axis of a leg passes through the spherical joint centers of the other two legs. This is shown in the following example.

In this case, the platform gains a DOF along the direction of this common line which intersects all the six wrenches.

Example 8: As shown in Figure 14, the passive joint axis of leg 3 is coincident with line $B_{14}B_{24}$. In this case, the line $B_{14}B_{24}$ intersects with all the six wrenches and the platform gains a rotational DOF about $B_{14}B_{24}$.

4.2. Inverse singularity
An inverse singularity occurs when matrix B in Equation (31) becomes singular. Matrix B is a diagonal matrix and each of its entry is a scalar. For element b_{i1} of matrix B, we have

$$b_{i1} = ([I \quad -\hat{\boldsymbol{p}}'_{i4}]s'_{i2}) \times ([I \quad -\hat{\boldsymbol{p}}'_{i4}]s'_{i3}) \cdot ([I \quad -\hat{\boldsymbol{p}}'_{i4}]s_{i1})$$
$$= (\boldsymbol{v}'_{i2} - \boldsymbol{p}'_{i4} \times \boldsymbol{w}'_{i2}) \times (\boldsymbol{v}'_{i3} - \boldsymbol{p}'_{i4} \times \boldsymbol{w}'_{i3}) \cdot (\boldsymbol{v}_{i1} - \boldsymbol{p}'_{i4} \times \boldsymbol{w}_{i1}) \quad (32)$$

It is easy to observe that the first, second and third vectors, at the right hand side of the above expression of b_{i1}, are, respectively, the linear velocity components from the second, third and first joints at the spherical joint center of i^{th} leg. Considering the vector triple product, b_{i1} becomes the volume of a parallelepiped with sides as the linear velocity components from the three main joints of the i^{th} leg at its

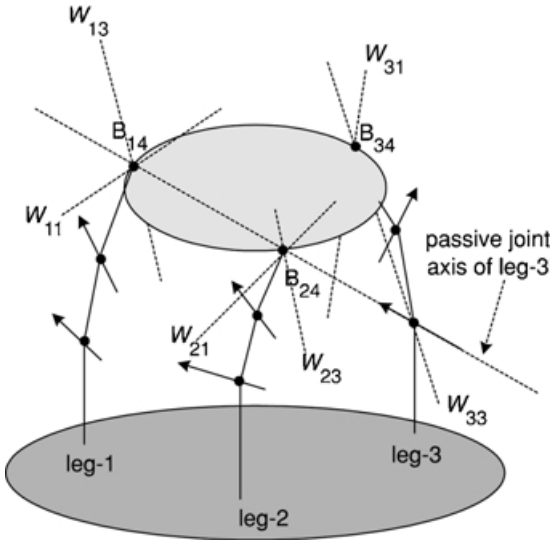


Fig. 14. **Case 5b** singularity for RRRS-type parallel manipulator.

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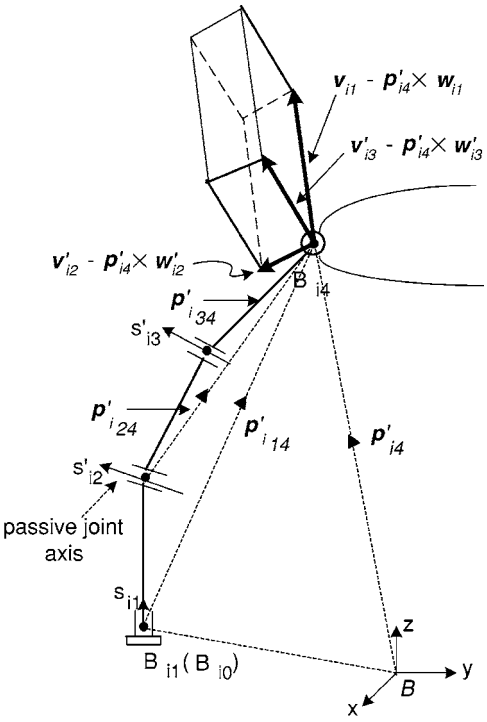


Fig. 15. Parallelepiped at the spherical joint center of i^{th} leg.

spherical joint center (Figure 15). Observing Equation (29), the next entry of the diagonal matrix B, $b_{i2}=-b_{i1}$. Thus, the six diagonal elements of matrix B correspond to three such parallelepipeds. For the RRRS-type parallel manipulators (Figure 15), entries of matrix B corresponding to the i^{th} leg are

$$b_{i1}=(\boldsymbol{w}'_{i3}\times\boldsymbol{p}'_{i_{34}})\times(\boldsymbol{w}'_{i2}\times\boldsymbol{p}'_{i_{24}})\cdot(\boldsymbol{w}_{i1}\times\boldsymbol{p}'_{i_{14}}) \quad (33)$$

$$b_{i3}=(\boldsymbol{w}_{i1}\times\boldsymbol{p}'_{i_{14}})\times(\boldsymbol{w}'_{i2}\times\boldsymbol{p}'_{i_{24}})\cdot(\boldsymbol{w}'_{i3}\times\boldsymbol{p}'_{i_{34}}) \quad (34)$$

It can be observed that $(\boldsymbol{w}_{i1}\times\boldsymbol{p}'_{i_{14}})\times(\boldsymbol{w}'_{i2}\times\boldsymbol{p}'_{i_{24}})$ in the above expression of b_{i3} is equal to a vector represented by $\boldsymbol{p}'_{i_{24}}$.

The occurrence of inverse singularity can thus be stated as, "An inverse singularity occurs when the volume of such a parallelepiped of any leg becomes zero". This condition is satisfied in three cases.

Case I: Magnitude of any vector representing the side of the parallelepiped is zero.

Condition: Any joint axis (revolute) passes through the corresponding spherical joint center of a leg. This kind of singularity does not exist for a prismatic joint in a chain.

Example 9: Figure 16(a) shows a configuration in which the joint axis $\boldsymbol{w}_{11}$ passes through the spherical joint center B_{14}. As a result, $b_{11}=b_{13}=0$ and matrix B becomes singular.

Case II: Any two non-zero vectors are parallel or perpendicular to each other.

Condition: They are parallel (1.) when any two joint axes (revolute) and corresponding lines joining the respective joint centers with the spherical joint center of a leg are coplanar (shown in Example 10), (2.) when a prismatic joint axis becomes normal to the plane formed by any revolute joint axis and and the spherical joint center. They are perpendicular to each other when any two planes, each being formed by a joint axis (revolute) and the spherical joint center are perpendicular to each other (shown in Example 11).

Example 10: Figure 16(b) shows a configuration in which $\boldsymbol{w}'_{12}$, $\boldsymbol{w}'_{13}$, $\boldsymbol{p}'_{1_{24}}$ and $\boldsymbol{p}'_{1_{34}}$ are coplanar. As a result, the cross product of the linear velocity components from the second and third joints becomes a null vector. Consequently the volume of the parallelopiped becomes zero and causes inverse singularity.

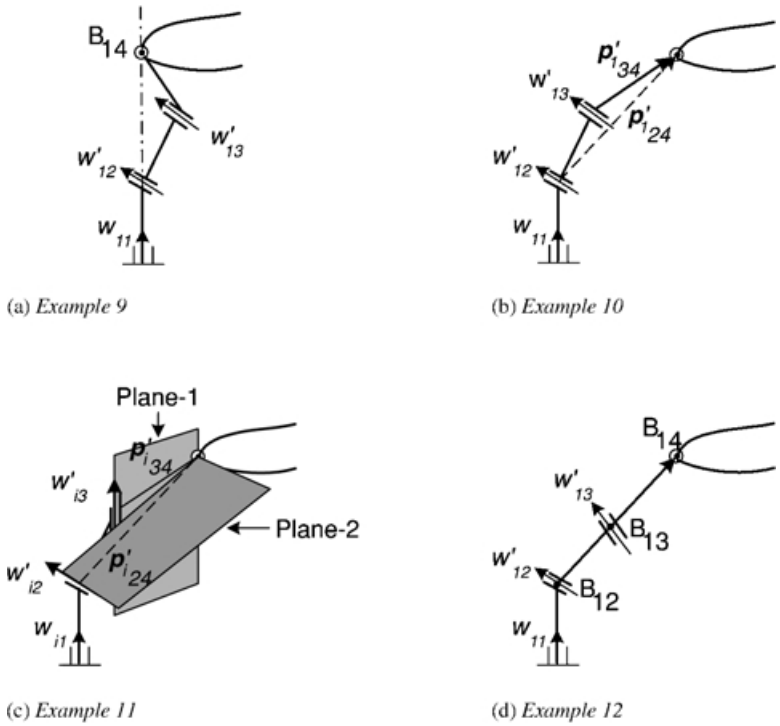


Fig. 16. Inverse singularity.

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Example 11: Figure 16(c) shows a configuration in which the plane formed by $\boldsymbol{w}'_{12}$ and $\boldsymbol{p}'_{1_{24}}$ is perpendicular to that formed by $\boldsymbol{w}'_{13}$ and $\boldsymbol{p}'_{1_{34}}$. As a result, the inner product of linear velocity components from second and third joints and consequently the volume of the parallelepiped becomes zero. So this causes an inverse singularity.

Case III: All the three non-zero vectors are coplanar.

Condition: This happens in following ways: when positions of any two joints (revolute) and the spherical joint are co-linear (discussed in Example 12), when three joint axes (revolute) are coplanar and when any joint axis (revolute), the spherical joint center and any other joint center (revolute) become coplanar.

Example 12: Figure 16(d) shows a configuration in which B_{12}, B_{13} and B_{14} are co-linear. As a result $b_{13}=0$ and matrix B becomes singular. Similar thing also happens when $\boldsymbol{w}_{13}$ becomes colinear with line $B_{12}B_{14}$.

4.3. Combined singularity

A combined singularity occurs when both matrices, A and B, in Equation (31) become singular. Example 10 and Example 11 are the configurations in which combined singularities occur. In these cases, there are a maximum of five linearly independent wrenches (out of six wrenches in total) to constrain the mobile platform, thus matrix A becomes singular. Also in these cases, the volume of the parallelepiped at the leg end becomes zero. Thus they cause combined singularities.

5. CONCLUSION

The class of parallel manipulators, investigated here, comprises of a base and a mobile platform connected by three kinematic chains or legs having a spherical joint at the end of each leg. For this class of parallel manipulators, instantaneous kinematics and singularity analysis are presented in this paper. In the derivation of instantaneous kinematic relationship, it is shown that reciprocal screws can be constructed easily and intuitively adopting twist annihilator method. The instantaneous kinematics model formulated using POE scheme treats any combination of revolute and prismatic joints in the leg uniformly. In other words, a generic method for derivation of expressions of reciprocal screws and consequently, the instantaneous kinematics model is presented. Taking two different configurations (RRRS-type and RPRS-type) as examples, the generic nature of the instantaneous kinematics model is demonstrated. Using the concepts of reciprocal screw and Grassman Line geometry, simple geometric conditions are proposed to identify the configurations in forward singularity. A concept of Passive joint plane is introduced to correlate the physical structure of the manipulator and these geometric conditions. This is very helpful for easy identification of singularity configurations, specifically when the geometric conditions become complicated. Each case of linear dependency of line systems is identified, DOF gained/lost therein is found and an example configuration is presented. It is shown that the conditions for forward singularity are equally applicable to all kinds of joint combinations between revolute and prismatic joints. Knowledge of these cofigurations can be employed at the design stage to eliminate the singularities by choosing appropriate redundancy in actuation. In the inverse singularity analysis, a characteristic parallelepiped is defined at the spherical joint center of each leg. Three sides of this parallelepiped are the linear velocity components from the three joints at the spherical joint center of that leg. It is shown that an inverse singularity occurs when the volume of the parallelepiped is zero. This method gives a physical view of the parallelepiped from any configuration of parallel manipulator. As a result, it becomes easy to determine inverse singularity poses.

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