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The shear dynamo problem for small magnetic Reynolds numbers

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We study large-scale kinematic dynamo action due to turbulence in the presence of a linear shear flow in the low-conductivity limit. Our treatment is non-perturbative in the shear strength and makes systematic use of both the shearing coordinate transformation and the Galilean invariance of the linear shear flow. The velocity fluctuations are assumed to have low magnetic Reynolds number (Re_m), but could have arbitrary fluid Reynolds number. The equation for the magnetic fluctuations is expanded perturbatively in the small quantity, Re_m. Our principal results are as follows: (i) the magnetic fluctuations are determined to the lowest order in Re_m by explicit calculation of the resistive Green's function for the linear shear flow; (ii) the mean electromotive force is then calculated and an integro-differential equation is derived for the time evolution of the mean magnetic field. In this equation, velocity fluctuations contribute to two different kinds of terms, the 'C' and 'D' terms, respectively, in which first and second spatial derivatives of the mean magnetic field, respectively, appear inside the space–time integrals; (iii) the contribution of the D term is such that its contribution to the time evolution of the cross-shear components of the mean field does not depend on any other components except itself. Therefore, to the lowest order in Re_m, but to all orders in the shear strength, the D term cannot give rise to a shear-current-assisted dynamo effect; (iv) casting the integro-differential equation in Fourier space, we show that the normal modes of the theory are a set of shearing waves, labelled by their sheared wavevectors; (v) the integral kernels are expressed in terms of the velocity-spectrum tensor, which is the fundamental dynamical quantity that needs to be specified to complete the integro-differential equation description of the time evolution of the mean magnetic field; (vi) the C term couples different components of the mean magnetic field, so they can, in principle, give rise to a shear-current-type effect. We discuss the application to a slowly varying magnetic field, where it can be shown that forced non-helical velocity dynamics at low fluid Reynolds number does not result in a shear-current-assisted dynamo effect.

Key words: dynamo theory, MHD and electrohydrodynamics

1. Introduction

Large-scale magnetic fields in many astrophysical systems, from planets to clusters of galaxies, are thought to originate from dynamo action in the electrically conducting fluids in these systems. The standard paradigm involves amplification of seed magnetic

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fields, due to non-mirror-symmetric (i.e. helical) turbulent flows, through the α-effect (Moffatt 1978; Parker 1979). It is only relatively recently that the role of the mean shear in the turbulent flows is beginning to be appreciated. Dynamo action due to shear and turbulence has received some attention in the astrophysical contexts of accretion disks (Vishniac & Brandenburg 1997) and galactic disks (Blackman 1998). Also it has been demonstrated that shear, in conjunction with rotating turbulent convection, can drive a large-scale dynamo (Käpylä, Korpi & Brandenberg 2008; Hughes & Proctor 2009). We are interested in the more specific problem of large-scale-dynamo action due to non-helical turbulence with mean shear. Direct numerical simulations now provide strong support for such a *shear dynamo*. Yousef *et al.* (2008*a*) demonstrated that forced small-scale non-helical turbulence in non-rotating linear shear flows leads to the exponential growth of large-scale magnetic fields. These findings were later generalized by Yousef *et al.* (2008*b*) to a shearing-sheet model of a differentially rotating disk with a Keplerian rotation profile. The investigations of Brandenburg *et al.* (2008) demonstrated the shear dynamo effect for a range of values of the Reynolds numbers and the shear parameter, and measured the tensorial magnetic diffusivity tensor. While the shear dynamo has been conclusively demonstrated to function, it is not yet clear what makes it to work. This outstanding unsolved problem is the focus of the present investigation.

One possibility that has been suggested is dynamo action due to a 'fluctuating α-effect' in turbulent flows, which have zero mean helicities. In this approach, large-scale dynamo action derives from the interaction of mean shear with fluctuations of helicity (Sokolov 1997; Vishniac & Brandenburg 1997; Proctor 2007; Brandenburg *et al.* 2008; Rogachevskii & Kleeorin 2008; Schekochihin *et al.* 2008). Another approach is that, even if transient growth makes non-axisymmetric mean magnetic fields strong enough, they themselves might drive motions which could lead to subcritical dynamo action (Rincon *et al.* 2008). Yet another possibility that has been suggested is the shear-current effect (Rogachevskii & Kleeorin 2003, 2004, 2008). In this mechanism, it is thought that the mean shear gives rise to anisotropic turbulence, which causes an extra component of the mean electromotive force (EMF), leading to the generation of the cross-shear component of the mean magnetic field from the component which is parallel to the shear flow. However, there is no agreement yet on whether the sign of such a coupling is favourable to the operation of a dynamo. Some analytical calculations (Rädler & Stepanov 2006; Rüdiger & Kitchatinov 2006) and numerical experiments (Brandenburg *et al.* 2008) find that the sign of the shear-current term is unfavourable for dynamo action. A quasi-linear theory of dynamo action in a linear shear flow of an incompressible fluid which has random velocity fluctuations was presented in Sridhar & Subramanian (2009*a*,*b*). Unlike earlier analytical studies, which treated shear as a small perturbation, this study did not place any restriction on the strength of the shear. The authors proposed an integro-differential equation for the evolution of the mean magnetic field and argued that the shear-current-assisted dynamo is essentially absent. It should be noted that the quasi-linear theory of Sridhar & Subramanian (2009*a*,*b*) assumes zero resistivity, and is valid in the limit of small velocity correlation times when the 'first-order-smoothing approximation' holds.

In this paper, we present a kinematic theory of the shear dynamo, which is non-perturbative in the shear strength, but perturbative in Re_m; this may be thought of as a first-order-smoothing approximation with finite resistivity. Thus, we are not limited to the quasi-linear limit of small velocity correlation times, and our conclusions are rigorously valid for velocity fluctuations which have small Re_m but arbitrary fluid Reynolds number. In § 2, we formulate the shear dynamo problem for small Re_m.

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Using Reynolds averaging, we split the magnetic field into mean and fluctuating components. The equation for the fluctuations is expanded perturbatively in the small parameter, Re_m. Using the shearing-coordinate transformation, we make an explicit calculation of the resistive Green's function for the linear shear flow. In § 3, the magnetic fluctuations and the mean EMF are determined to the lowest order in Re_m. The transport coefficients are given in general form in terms of the two-point correlators of the velocity fluctuations. Galilean invariance (GI) is a basic symmetry in the problem and is the focus of § 4. For Galilean invariant (G-invariant) velocity fluctuations, it is proved that the transport coefficients, although space-dependent, possess the property of translational invariance in sheared coordinate space. An explicit expression for the G-invariant mean EMF is derived. We put together all the results in § 5 by deriving the integro-differential equation governing the time evolution of the mean magnetic field. Some important properties of this equation are also discussed. In particular, it is shown that, in the formal limit of zero resistivity, the quasi-linear results of Sridhar & Subramanian (2009*a*,*b*) are recovered. We also show that the natural setting for the integro-differential equation governing mean-field evolution is in sheared Fourier space. We prove a theorem on the form of the two-point velocity correlator in Fourier space; the velocity spectrum tensor and its general properties are discussed. We then express all the integral kernels in terms of the velocity spectrum tensor, which is the fundamental dynamical quantity that needs to be specified. Summary and conclusions are presented in § 6.

2. The shear dynamo problem

2.1. *The small Re_m limit*

Let us consider a Cartesian coordinate system with unit vectors $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ constructed on a co-moving patch of a differentially rotating disk. Henceforth this will be referred to as the lab frame and we will use notation $\boldsymbol{X} = (X_1, X_2, X_3)$ for the position vector and τ for time. The fluid velocity is given by $(SX_1\boldsymbol{e}_2 + \boldsymbol{v})$, where S is the rate of shear parameter and $\boldsymbol{v}(\boldsymbol{X}, \tau)$ is a randomly fluctuating velocity field. The total magnetic field, $\boldsymbol{B}^{tot}(\boldsymbol{X}, \tau)$, obeys the induction equation

$$\left(\frac{\partial}{\partial\tau} + SX_1\frac{\partial}{\partial X_2}\right)\boldsymbol{B}^{tot} - SB_1^{tot}\boldsymbol{e}_2 = \boldsymbol{\nabla}\times\left(\boldsymbol{v}\times\boldsymbol{B}^{tot}\right) + \eta\nabla^2\boldsymbol{B}^{tot}. \tag{2.1}$$

It is useful to note that the induction equation is unaffected by a uniform rotation of the frame of reference. Therefore, our coordinate system can refer to an inertial frame, or to a co-moving patch of a differentially rotating disk. We study a kinematic problem in this paper, so will assume that the velocity field is prescribed. We also assume that the velocity fluctuations have zero mean ($\langle\boldsymbol{v}\rangle = \boldsymbol{0}$), with root-mean-squared amplitude, v_{rms}, on some typical spatial scale, ℓ. The *magnetic Reynolds number* may be defined as $Re_m = (v_{rms}\ell/\eta)$; note that Re_m has been defined with respect to the fluctuation velocity field, not the background shear velocity field. To address the dynamo problem, we will use the approach of the theory of mean-field electrodynamics (Moffatt 1978; Krause & Rädler 1980; Brandenburg & Subramanian 2005). Here, the action of the velocity fluctuations on some seed magnetic field is assumed to produce a total magnetic field with a well-defined *mean field* ($\boldsymbol{B}$) and a *fluctuating field* ($\boldsymbol{b}$)

$$\boldsymbol{B}^{tot} = \boldsymbol{B} + \boldsymbol{b}, \quad \langle\boldsymbol{B}^{tot}\rangle = \boldsymbol{B}, \quad \langle\boldsymbol{b}\rangle = \boldsymbol{0}, \tag{2.2}$$

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where $\langle\ \rangle$ denotes ensemble averaging in the sense of Reynolds. Applying Reynolds averaging to the induction equation (2.1), we obtain the following equations governing the dynamics of the mean and fluctuating magnetic fields:

$$\left(\frac{\partial}{\partial\tau}+SX_1\frac{\partial}{\partial X_2}\right)\boldsymbol{B}-SB_1\boldsymbol{e}_2=\boldsymbol{\nabla}\times\boldsymbol{\mathscr{E}}+\eta\nabla^2\boldsymbol{B}, \tag{2.3}$$

$$\left(\frac{\partial}{\partial\tau}+SX_1\frac{\partial}{\partial X_2}\right)\boldsymbol{b}-Sb_1\boldsymbol{e}_2=\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B})+\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{b}-\langle\boldsymbol{v}\times\boldsymbol{b}\rangle)+\eta\nabla^2\boldsymbol{b}, \tag{2.4}$$

where $\boldsymbol{\mathscr{E}}=\langle\boldsymbol{v}\times\boldsymbol{b}\rangle$ is the mean EMF. The first step towards solving the problem is to solve (2.4) for $\boldsymbol{b}$, then calculate $\boldsymbol{\mathscr{E}}$ and obtain a closed equation for the mean field $\boldsymbol{B}(\boldsymbol{X},\tau)$. In the framework of the above mean-field theory, the *shear dynamo problem* may be proposed as follows: under what conditions does the equation for $\boldsymbol{B}(\boldsymbol{X},\tau)$ admit growing solutions? In particular, are growing solutions possible when the velocity field is non-helical (i.e. when the velocity field is mirror symmetric)?

The problem is, in general, a difficult one, but it can be approached perturbatively in the limit of small Re_m. When $Re_m\ll 1$, we can expand $\boldsymbol{b}$ in a series

$$\boldsymbol{b}=\boldsymbol{b}^{(0)}+\boldsymbol{b}^{(1)}+\boldsymbol{b}^{(2)}+\cdots, \tag{2.5}$$

where $\boldsymbol{b}^{(n)}$ is of the order of $\boldsymbol{b}^{(n-1)}$ multiplied by the small quantity Re_m. The equations governing the time evolution of these quantities are

$$\left(\frac{\partial}{\partial\tau}+SX_1\frac{\partial}{\partial X_2}\right)\boldsymbol{b}^{(0)}-Sb_1^{(0)}\boldsymbol{e}_2=\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B})+\eta\nabla^2\boldsymbol{b}^{(0)}, \tag{2.6}$$

$$\left(\frac{\partial}{\partial\tau}+SX_1\frac{\partial}{\partial X_2}\right)\boldsymbol{b}^{(n)}-Sb_1^{(n)}\boldsymbol{e}_2=\boldsymbol{\nabla}\times\left(\boldsymbol{v}\times\boldsymbol{b}^{(n-1)}-\left\langle\boldsymbol{v}\times\boldsymbol{b}^{(n-1)}\right\rangle\right)+\eta\nabla^2\boldsymbol{b}^{(n)},\quad \text{for } n=1,2,\ldots. \tag{2.7}$$

Note that $\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{B})$ acts as a source term for $\boldsymbol{b}^{(0)}$, whereas the source term for $\boldsymbol{b}^{(n)}$ is $\boldsymbol{\nabla}\times(\boldsymbol{v}\times\boldsymbol{b}^{(n-1)}-\langle\boldsymbol{v}\times\boldsymbol{b}^{(n-1)}\rangle)$. Once the $\boldsymbol{b}^{(n)}$ have been determined, the mean EMF can be calculated directly by

$$\boldsymbol{\mathscr{E}}=\langle\boldsymbol{v}\times\boldsymbol{b}\rangle=\left\langle\boldsymbol{v}\times\left(\boldsymbol{b}^{(0)}+\boldsymbol{b}^{(1)}+\boldsymbol{b}^{(2)}+\cdots\right)\right\rangle. \tag{2.8}$$

In this paper, we focus on the lowest order in Re_m, therefore we need to work out only $\boldsymbol{b}^{(0)}$; (2.7) will not be used.

2.2. *The shearing coordinate transformation*

In this paper, we will focus on the determination of the lowest order term $\boldsymbol{b}^{(0)}$. We also assume that the fluctuating velocity field is incompressible; i.e. $\boldsymbol{\nabla}\cdot\boldsymbol{v}=0$. Then the evolution of $\boldsymbol{b}^{(0)}$ is governed by

$$\left(\frac{\partial}{\partial\tau}+SX_1\frac{\partial}{\partial X_2}\right)\boldsymbol{b}^{(0)}-Sb_1^{(0)}\boldsymbol{e}_2=(\boldsymbol{B}\cdot\boldsymbol{\nabla})\,\boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{\nabla})\,\boldsymbol{B}+\eta\nabla^2\boldsymbol{b}^{(0)}. \tag{2.9}$$

We will now solve (2.9) for $\boldsymbol{b}^{(0)}$ and determine the mean EMF. General methods of solving equations such as (2.9) are presented in Krause & Rädler (1980), but we prefer to employ the shearing coordinate transformation because it is directly adapted to the problem at hand and greatly simplifies the task of writing down the Green's function solution. The term $(X_1\partial/\partial X_2)$ makes (2.9) inhomogeneous in the coordinate X_1. This term can be eliminated through a shearing transformation to new space–time variables

$$x_1=X_1,\quad x_2=X_2-S\tau X_1,\quad x_3=X_3,\quad t=\tau. \tag{2.10}$$

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Partial derivatives transform as

$$\frac{\partial}{\partial \tau} = \frac{\partial}{\partial t} - Sx_1 \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial X_1} = \frac{\partial}{\partial x_1} - St \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial X_2} = \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial X_3} = \frac{\partial}{\partial x_3}. \tag{2.11}$$

Therefore,

$$\nabla^2 \equiv \frac{\partial^2}{\partial X_p \partial X_p} = \left(\frac{\partial}{\partial x_p} - St\delta_{p1} \frac{\partial}{\partial x_2} \right)^2 = \frac{\partial^2}{\partial x_p \partial x_p} - 2St \frac{\partial^2}{\partial x_1 \partial x_2} + S^2 t^2 \frac{\partial^2}{\partial x_2^2}. \tag{2.12}$$

We also define new variables, which are componentwise equal to the old variables

$$\boldsymbol{H}(\boldsymbol{x}, t) = \boldsymbol{B}(\boldsymbol{X}, \tau), \quad \boldsymbol{h}(\boldsymbol{x}, t) = \boldsymbol{b}^{(0)}(\boldsymbol{X}, \tau), \quad \boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{v}(\boldsymbol{X}, \tau). \tag{2.13}$$

Note that, just like the old variables, the new variables are expanded in the fixed Cartesian basis of the lab frame. For example, $\boldsymbol{H} = H_1 \boldsymbol{e}_1 + H_2 \boldsymbol{e}_2 + H_3 \boldsymbol{e}_3$, where $H_i(\boldsymbol{x}, t) = B_i(\boldsymbol{X}, \tau)$ and similarly for the other variables. With the new variables, (2.9) becomes

$$\frac{\partial \boldsymbol{h}}{\partial t} - Sh_1 \boldsymbol{e}_2 = \left(\boldsymbol{H} \cdot \frac{\partial}{\partial \boldsymbol{x}} - St H_1 \frac{\partial}{\partial x_2} \right) \boldsymbol{u} - \left(\boldsymbol{u} \cdot \frac{\partial}{\partial \boldsymbol{x}} - St u_1 \frac{\partial}{\partial x_2} \right) \boldsymbol{H} + \eta \nabla^2 \boldsymbol{h}, \tag{2.14}$$

which can be expressed in component form as

$$\left(\frac{\partial}{\partial t} - \eta \nabla^2 \right) h_m(\boldsymbol{x}, t) = q_m(\boldsymbol{x}, t), \tag{2.15}$$

where ∇^2 is given by (2.12), and

$$q_m(\boldsymbol{x}, t) = [H_l - St\delta_{l2} H_1]\, u_{ml} - [u_l - St\delta_{l2} u_1]\, H_{ml} + S\delta_{m2} h_1. \tag{2.16}$$

We have used notation $u_{ml} = (\partial u_m / \partial x_l)$ and $H_{ml} = (\partial H_m / \partial x_l)$. Below we construct the Green's function for (2.15).

2.3. *The resistive Green's function for a linear shear flow*

Equation (2.15) is linear, homogeneous in $\boldsymbol{x}$ and inhomogeneous in t. Therefore, the general solution can be written in the form

$$h_m(\boldsymbol{x}, t) = \int \mathrm{d}^3 x' G_\eta(\boldsymbol{x} - \boldsymbol{x}', t, s) h_m(\boldsymbol{x}', s) + \int_s^t \mathrm{d}t' \int \mathrm{d}^3 x' G_\eta(\boldsymbol{x} - \boldsymbol{x}', t, t') q_m(\boldsymbol{x}', t'), \quad \text{for any } s < t, \tag{2.17}$$

where $G_\eta(\boldsymbol{x}, t, t')$ is the *resistive Green's function* for the linear shear flow, which satisfies

$$\left(\frac{\partial}{\partial t} - \eta \nabla^2 \right) G_\eta(\boldsymbol{x}, t, t') = 0, \tag{2.18a}$$

$$\lim_{t' \to t_-} G_\eta(\boldsymbol{x}, t, t') = \delta^3(\boldsymbol{x}), \tag{2.18b}$$

$$G_\eta(\boldsymbol{x}, t, t') \quad \text{is non-zero only when} \quad 0 \leqslant t' < t, \tag{2.18c}$$

$$G_\eta(\boldsymbol{x} - \boldsymbol{x}', t, t_0) = \int \mathrm{d}^3 x'' G_\eta(\boldsymbol{x} - \boldsymbol{x}'', t, s) G_\eta(\boldsymbol{x}'' - \boldsymbol{x}', s, t_0), \quad \text{for } t_0 < s < t. \tag{2.18d}$$

Let us define the spatial Fourier transform of the Green's function as

$$\widetilde{G}_\eta(\boldsymbol{k}, t, t') = \int \mathrm{d}^3 x \exp(-\mathrm{i}\, \boldsymbol{k} \cdot \boldsymbol{x}) G_\eta(\boldsymbol{x}, t, t'), \tag{2.19}$$

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where $\boldsymbol{k}$ being conjugate to the sheared coordinate vector $\boldsymbol{x}$, can be regarded as a *sheared wavevector*. Then

$$\frac{\partial \widetilde{G}_\eta}{\partial t} + \eta K^2(\boldsymbol{k}, t)\widetilde{G}_\eta = 0, \tag{2.20}$$

where $K^2(\boldsymbol{k}, t) = (k_1 - Stk_2)^2 + k_2^2 + k_3^2$. It is now straightforward to write down the solution

$$\begin{aligned}\widetilde{G}_\eta(\boldsymbol{k}, t, t') &= \exp\left[-\eta \int_{t'}^{t} \mathrm{d}s\, K^2(\boldsymbol{k}, s)\right] \\ &= \exp\left[-\eta\left(k^2(t-t') - Sk_1k_2(t^2 - t'^2) + \frac{S^2}{3}k_2^2(t^3 - t'^3)\right)\right],\end{aligned} \tag{2.21}$$

where, as per (2.18*c*) above, $t > t'$. Also note that $\widetilde{G}_\eta(\boldsymbol{k}, t, t')$ is a positive quantity which takes values between 0 and 1, and that it is an even function of $\boldsymbol{k}$ and k_3.

Now we take the inverse Fourier transform of (2.21) to get $G_\eta(\boldsymbol{x}, t, t')$. It is convenient to write this as

$$G_\eta(\boldsymbol{x}, t, t') = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \exp\left[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x} - \eta(t-t')\{k^2 + T_{ij}k_ik_j\}\right], \tag{2.22}$$

where T_{ij} is a 2×2 symmetric matrix whose elements are given by

$$T_{11} = 0, \quad T_{12} = T_{21} = -\frac{S}{2}(t + t'), \quad T_{22} = \frac{S^2}{3}(t^2 + tt' + t'^2). \tag{2.23}$$

The integral in (2.22) can be evaluated by diagonalizing the matrix T_{ij}. It proves useful to express $G_\eta(\boldsymbol{x}, t, t')$ in terms of the principal-axes coordinates, $\overline{\boldsymbol{x}} = (\overline{x}_1, \overline{x}_2, \overline{x}_3)$. These are defined by the orthogonal transformation

$$\begin{pmatrix} \overline{x}_1 \\ \overline{x}_2 \\ \overline{x}_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \tag{2.24}$$

which is a time-dependent rotation of the coordinate axes in the x_1–x_2 plane. The angle of rotation, θ, is determined by

$$\tan\theta = f + \sqrt{1 + f^2}, \tag{2.25}$$

$$f = -\frac{1}{3}\frac{S(t^2 + tt' + t'^2)}{(t + t')}. \tag{2.26}$$

Note that θ depends on the shear parameter, S, and on times, t and t'. Let us define the dimensionless quantities

$$\sigma_1 = \left[1 - \frac{S}{2}(t + t')\tan\theta\right]^{1/2}, \tag{2.27}$$

$$\sigma_2 = \left[1 + \frac{S}{2}(t + t')\cot\theta\right]^{1/2}. \tag{2.28}$$

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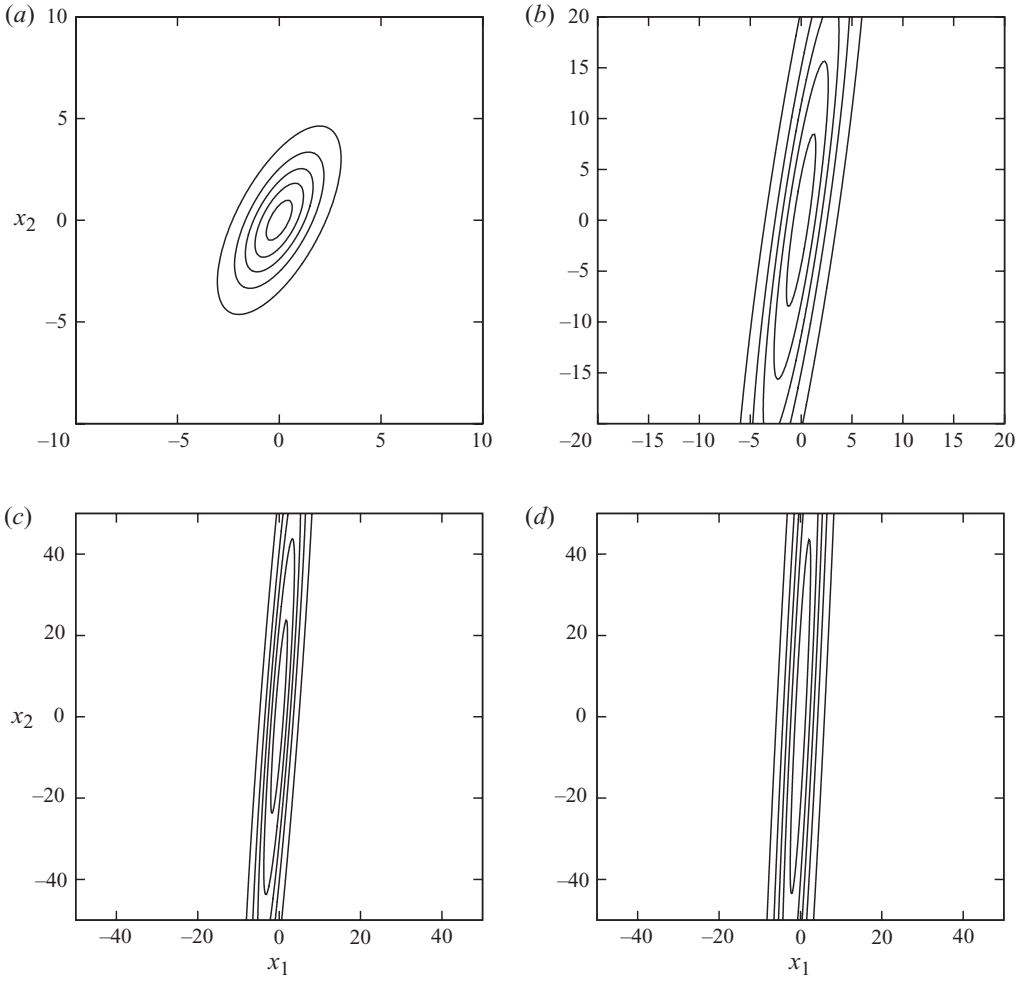


FIGURE 1. Isocontours of the resistive Green's function $G_\eta(\boldsymbol{x}, t, t')$ plotted in the x_1–x_2 plane of the shearing coordinate system, for $t' = 0$ at four different values of t. Units are such that $S = -2; \eta = 1$. Five isocontours at 90 %, 70 %, 50 %, 30 % and 10 % of the maximum value are displayed. (*a–d*) correspond to times $t = 1$, 5, 10 and 15, respectively.

Now we can write the Green's function as a *sheared heat kernel*

$$G_\eta(\boldsymbol{x}, t, t') = \left[4\pi\eta(t-t')\right]^{-3/2} \left[1 + \frac{S^2}{12}(t-t')^2\right]^{-1/2} \times \exp\left[-\frac{1}{4\eta(t-t')}\left(\frac{\overline{x}_1^2}{\sigma_1^2} + \frac{\overline{x}_2^2}{\sigma_2^2} + \overline{x}_3^2\right)\right], \quad (2.29)$$

which is equivalent to the equation that was first derived by Krause & Rädler (1971).

Now we note some properties of the Green's function. For convenience, we choose the shear parameter, S, to be negative: then the quantities, $f \geqslant 0, 0 \leqslant \theta \leqslant \pi/2, \sigma_1 \geqslant 1$ and $0 \leqslant \sigma_2 \leqslant 1$. At fixed t and t', the Green's function is a Gaussian function with *long axis* along $\overline{x}_1$, *short axis* along $\overline{x}_2$ and the *intermediate axis* along $\overline{x}_3$. To obtain some idea of the behaviour of the Green's function, it is useful to plot isocontours in the sheared coordinate space (x_1, x_2, x_3) at different values of t and t'. Figure 1

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displays isocontours in the x_1–x_2 plane at four different values of t for $t' = 0$; we have chosen $x_3 = 0$ and $t' = 0$ in the interests of brevity of presentation. The figure is plotted in shearing coordinates with respect to which diffusion is anisotropic and there is no advection. It may be noted that the Green's function shows a shearing motion *against* the direction of the actual shear. As t increases from zero to infinity, θ (which is the angle the long axis makes with the x_1-axis) increases from 45° to 90°, and all the principal axes increase without bound.

3. Magnetic fluctuations and mean electromotive force at small Re_m

3.1. *Explicit solution for* $\boldsymbol{h}(\boldsymbol{x}, t)$

We are interested in the particular solution to (2.15) (i.e. the *forced solution*), which vanishes at $t = 0$. This can be written as

$$h_m(\boldsymbol{x}, t) = \int_0^t \mathrm{d}t' \int \mathrm{d}^3x' G_\eta(\boldsymbol{x} - \boldsymbol{x}', t, t') q_m(\boldsymbol{x}', t'). \tag{3.1}$$

Substituting the expression for q_m from (2.16) in (3.1), we have

$$\begin{aligned} h_m(\boldsymbol{x}, t) = \int_0^t \mathrm{d}t' \int \mathrm{d}^3x' G_\eta(\boldsymbol{x} - \boldsymbol{x}', t, t') \\ \times \left\{ \left[H_l' - St' \delta_{l2} H_1' \right] u_{ml}' - \left[u_l' - St' \delta_{l2} u_1' \right] H_{ml}' \right\} \\ + S\delta_{m2} \int_0^t \mathrm{d}t' \int \mathrm{d}^3x' G_\eta(\boldsymbol{x} - \boldsymbol{x}', t, t') h_1(\boldsymbol{x}', t'), \end{aligned} \tag{3.2}$$

where primes denote evaluation at space–time point $(\boldsymbol{x}', t')$. The solution is not yet in explicit form because the last term on the right-hand side contains the unknown quantity, $h_1(\boldsymbol{x}', t')$. Thus, we need to define the integral

$$\begin{aligned} \int_0^t \mathrm{d}t' \int \mathrm{d}^3x' G_\eta(\boldsymbol{x} - \boldsymbol{x}', t, t') h_1(\boldsymbol{x}', t') = \int_0^t \mathrm{d}t' \int \mathrm{d}^3x' G_\eta(\boldsymbol{x} - \boldsymbol{x}', t, t') \\ \times \int_0^{t'} \mathrm{d}t'' \int \mathrm{d}^3x'' G_\eta(\boldsymbol{x}' - \boldsymbol{x}'', t', t'') \\ \times \left\{ \left[H_l'' - St'' \delta_{l2} H_1'' \right] u_{1l}'' - \left[u_l'' - St'' \delta_{l2} u_1'' \right] H_{1l}'' \right\}, \end{aligned} \tag{3.3}$$

where $''$ means evaluation at space–time point $(\boldsymbol{x}'', t'')$. Note that, on the right-hand side of (3.3), $\boldsymbol{x}'$ occurs only in the Green's functions. Therefore, by using the property given in (2.18*d*), the integral over $\boldsymbol{x}'$ can be performed. Then

$$\begin{aligned} \int_0^t \mathrm{d}t' \int \mathrm{d}^3x' G_\eta(\boldsymbol{x} - \boldsymbol{x}', t, t') h_1(\boldsymbol{x}', t') = \int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \int \mathrm{d}^3x'' G_\eta(\boldsymbol{x} - \boldsymbol{x}'', t, t'') \\ \times \left\{ \left[H_l'' - St'' \delta_{l2} H_1'' \right] u_{1l}'' - \left[u_l'' - St'' \delta_{l2} u_1'' \right] H_{1l}'' \right\}. \end{aligned} \tag{3.4}$$

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The double time integrals can be reduced to single time integrals because of the following simple identity. For any function $f(\boldsymbol{x}, t)$, we have

$$\begin{aligned}\int_0^t \mathrm{d}t' \int_0^{t'} \mathrm{d}t'' \int \mathrm{d}^3x'' f(\boldsymbol{x}'', t'') &= \int_0^t \mathrm{d}t'' \int \mathrm{d}^3x'' f(\boldsymbol{x}'', t'') \int_{t''}^t \mathrm{d}t' \\ &= \int_0^t \mathrm{d}t''(t - t'') \int \mathrm{d}^3x'' f(\boldsymbol{x}'', t'') \\ &= \int_0^t \mathrm{d}t'(t - t') \int \mathrm{d}^3x' f(\boldsymbol{x}', t'), \end{aligned} \tag{3.5}$$

where in the last equality we have merely replaced the dummy integration variables $(\boldsymbol{x}'', t'')$ by $(\boldsymbol{x}', t')$. Then we have

$$\begin{aligned}\int_0^t \mathrm{d}t' \int \mathrm{d}^3x' G_\eta(\boldsymbol{x} - \boldsymbol{x}', t, t') h_1(\boldsymbol{x}', t') = \int_0^t \mathrm{d}t'(t - t') \int \mathrm{d}^3x' G_\eta(\boldsymbol{x} - \boldsymbol{x}'', t, t'') \\ \times \left\{ \left[H_l' - St'\delta_{l2} H_1' \right] u_{1l}' - \left[u_l' - St'\delta_{l2} u_1' \right] H_{1l}' \right\}. \end{aligned} \tag{3.6}$$

Therefore, the forced solution to (2.15) can finally be written in explicit form as

$$\begin{aligned} h_m(\boldsymbol{x}, t) = &\int_0^t \mathrm{d}t' \int \mathrm{d}^3x' G_\eta(\boldsymbol{x} - \boldsymbol{x}', t, t') \left[u_{ml}' + S(t - t')\delta_{m2} u_{1l}' \right] \left[H_l' - St'\delta_{l2} H_1' \right] \\ &- \int_0^t \mathrm{d}t' \int \mathrm{d}^3x' G_\eta(\boldsymbol{x} - \boldsymbol{x}', t, t') \left[H_{ml}' + S(t - t')\delta_{m2} H_{1l}' \right] \left[u_l' - St'\delta_{l2} u_1' \right]. \end{aligned} \tag{3.7}$$

This gives the magnetic fluctuation to the lowest order in Re_m.

3.2. *Explicit expression for the mean electromotive force*

For the lowest order in Re_m, the mean EMF is given by $\mathscr{E} = \langle \boldsymbol{v} \times \boldsymbol{b}^{(0)} \rangle = \langle \boldsymbol{u} \times \boldsymbol{h} \rangle$, where $\boldsymbol{h}$ should be substituted from (3.7). The averaging, $\langle\ \rangle$, acts only on the velocity variables but not the mean field; i.e. $\langle \boldsymbol{uuH} \rangle = \langle \boldsymbol{uu} \rangle \boldsymbol{H}$, etc. After interchanging the dummy indices (l, m) in the last term, the mean EMF is given in the component form as

$$\begin{aligned} \mathscr{E}_i(\boldsymbol{x}, t) &= \epsilon_{ijm} \langle u_j h_m \rangle \\ &= \int_0^t \mathrm{d}t' \int \mathrm{d}^3x' G_\eta(\boldsymbol{x} - \boldsymbol{x}', t, t') \left[\widehat{\alpha}_{il}(\boldsymbol{x}, t, \boldsymbol{x}', t') + S(t - t')\widehat{\beta}_{il}(\boldsymbol{x}, t, \boldsymbol{x}', t') \right] \\ &\quad \times \left[H_l' - St'\delta_{l2} H_1' \right] \\ &\quad - \int_0^t \mathrm{d}t' \int \mathrm{d}^3x' G_\eta(\boldsymbol{x} - \boldsymbol{x}', t, t') \left[\widehat{\eta}_{iml}(\boldsymbol{x}, t, \boldsymbol{x}', t') - St'\delta_{m2} \widehat{\eta}_{i1l}(\boldsymbol{x}, t, \boldsymbol{x}', t') \right] \\ &\quad \times \left[H_{lm}' + S(t - t')\delta_{l2} H_{1m}' \right]. \end{aligned} \tag{3.8}$$

Here, $(\widehat{\alpha}, \widehat{\beta}, \widehat{\eta})$ are *transport coefficients*, which are defined in terms of the $\boldsymbol{uu}$ velocity correlators by

$$\widehat{\alpha}_{il}(\boldsymbol{x}, t, \boldsymbol{x}', t') = \epsilon_{ijm} \langle u_j(\boldsymbol{x}, t) u_{ml}(\boldsymbol{x}', t') \rangle, \tag{3.9}$$

$$\widehat{\beta}_{il}(\boldsymbol{x}, t, \boldsymbol{x}', t') = \epsilon_{ij2} \langle u_j(\boldsymbol{x}, t) u_{1l}(\boldsymbol{x}', t') \rangle, \tag{3.10}$$

$$\widehat{\eta}_{iml}(\boldsymbol{x}, t, \boldsymbol{x}', t') = \epsilon_{ijl} \langle u_j(\boldsymbol{x}, t) u_m(\boldsymbol{x}', t') \rangle. \tag{3.11}$$

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It is also useful to consider velocity statistics in terms of $\boldsymbol{vv}$ velocity correlators, because this is referred to the lab frame. By definition, from (2.13)

$$u_m(\boldsymbol{x}, t) = v_m(\boldsymbol{X}(\boldsymbol{x}, t), t), \tag{3.12}$$

where

$$X_1 = x_1, \quad X_2 = x_2 + St x_1, \quad X_3 = x_3, \quad \tau = t \tag{3.13}$$

is the inverse of the shearing transformation given in (2.10). Using

$$\frac{\partial}{\partial x_l} = \frac{\partial}{\partial X_l} + S\tau\delta_{l1}\frac{\partial}{\partial X_2} \tag{3.14}$$

the velocity gradient u_{ml} can be written as

$$u_{ml} \equiv \frac{\partial u_m}{\partial x_l} = \left(\frac{\partial}{\partial X_l} + S\tau\delta_{l1}\frac{\partial}{\partial X_2}\right) v_m = v_{ml} + S\tau\delta_{l1}v_{m2}, \tag{3.15}$$

where $v_{ml} = (\partial v_m/\partial X_l)$. Then the transport coefficients are given in terms of the $\boldsymbol{vv}$ velocity correlators by

$$\widehat{\alpha}_{il}(\boldsymbol{x}, t, \boldsymbol{x}', t') = \epsilon_{ijm}\left[\langle v_j(\boldsymbol{X}, t)v_{ml}(\boldsymbol{X}', t')\rangle + St'\delta_{l1}\langle v_j(\boldsymbol{X}, t)v_{m2}(\boldsymbol{X}', t')\rangle\right], \tag{3.16}$$

$$\widehat{\beta}_{il}(\boldsymbol{x}, t, \boldsymbol{x}', t') = \epsilon_{ij2}\left[\langle v_j(\boldsymbol{X}, t)v_{1l}(\boldsymbol{X}', t')\rangle + St'\delta_{l1}\langle v_j(\boldsymbol{X}, t)v_{12}(\boldsymbol{X}', t')\rangle\right], \tag{3.17}$$

$$\widehat{\eta}_{iml}(\boldsymbol{x}, t, \boldsymbol{x}', t') = \epsilon_{ijl}\langle v_j(\boldsymbol{X}, t)v_m(\boldsymbol{X}', t')\rangle, \tag{3.18}$$

where $\boldsymbol{X}$ and $\boldsymbol{X}'$ are shorthand for

$$\boldsymbol{X} = (x_1, x_2 + St x_1, x_3), \quad \boldsymbol{X}' = (x_1', x_2' + St' x_1', x_3'). \tag{3.19}$$

Equation (3.8), together with equations (3.9)–(3.11) or (3.16)–(3.18), gives the mean EMF in general form. $\boldsymbol{X}$ can be thought of as the coordinates of the origin at time t of an observer *co-moving* with the background shear flow, who was at $\boldsymbol{x}$ at time equal to zero. Similarly, $\boldsymbol{X}'$ can be thought of as the coordinates of the origin at time t' of an observer *co-moving* with the background shear flow, who was at $\boldsymbol{x}'$ at time equal to zero.

4. Galilean-invariant velocity statistics

4.1. *Galilean invariance of the induction equation*

The induction equation (2.1) for the total magnetic field – and also (2.3) and (2.4) for the mean and fluctuating components – have a fundamental invariance property relating to measurements made by a special subset of all observers called *co-moving observers* in Sridhar & Subramanian (2009*a*,*b*). A co-moving observer translates with the velocity of the background shear flow, and such an observer can be labelled by the coordinates, $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$, of their origin at time $\tau = 0$. At any time τ, the origin is at position (figure 2)

$$\boldsymbol{X}_c(\boldsymbol{\xi}, \tau) = (\xi_1, \xi_2 + S\tau\xi_1, \xi_3). \tag{4.1}$$

An event with space–time coordinates $(\boldsymbol{X}, \tau)$ in the lab frame has space–time coordinates $(\tilde{\boldsymbol{X}}, \tilde{\tau})$ with respect to the co-moving observer, given by

$$\tilde{\boldsymbol{X}} = \boldsymbol{X} - \boldsymbol{X}_c(\boldsymbol{\xi}, \tau), \quad \tilde{\tau} = \tau - \tau_0, \tag{4.2}$$

where the arbitrary constant τ_0 allows for translation in time as well.

Let $[\tilde{\boldsymbol{B}}^{tot}(\tilde{\boldsymbol{X}}, \tilde{\tau}), \tilde{\boldsymbol{B}}(\tilde{\boldsymbol{X}}, \tilde{\tau}), \tilde{\boldsymbol{b}}(\tilde{\boldsymbol{X}}, \tilde{\tau}), \tilde{\boldsymbol{v}}(\tilde{\boldsymbol{X}}, \tilde{\tau})]$ denote the total, the mean and the fluctuating magnetic fields and the fluctuating velocity field, respectively, as measured by the

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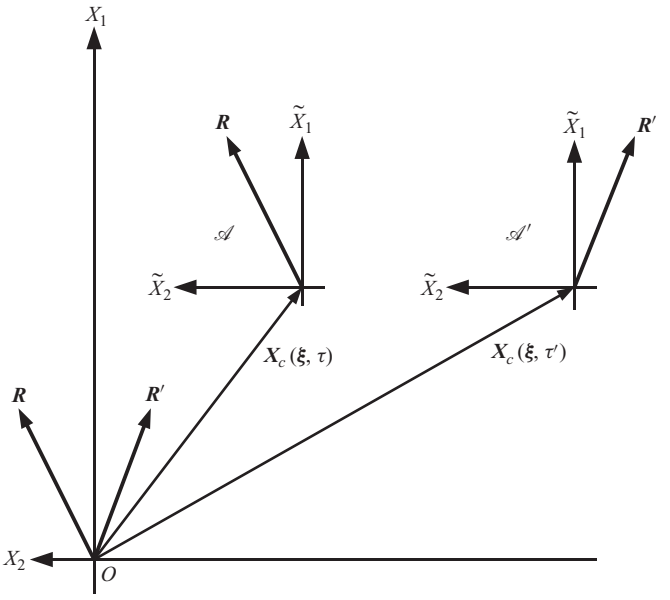


FIGURE 2. Galilean invariance of the two-point velocity correlator given in (4.5). 'O' labels the observer in the laboratory frame who correlates the velocity fluctuation at location $\boldsymbol{R}$ at time τ with the velocity fluctuation at location $\boldsymbol{R}'$ at a later time τ'. $\mathscr{A}$ and $\mathscr{A}'$ label a co-moving observer the origin of whose cooordinate axes is at $\boldsymbol{\xi}$ at the initial time and who makes an equivalent measurement at the times τ and τ'.

co-moving observer. As explained in Sridhar & Subramanian (2009*a*,*b*), these are all equal to the respective quantities measured in the lab frame

$$[\tilde{\boldsymbol{B}}^{tot}(\tilde{\boldsymbol{X}}, \tilde{\tau}), \tilde{\boldsymbol{B}}(\tilde{\boldsymbol{X}}, \tilde{\tau}), \tilde{\boldsymbol{b}}(\tilde{\boldsymbol{X}}, \tilde{\tau}), \tilde{\boldsymbol{v}}(\tilde{\boldsymbol{X}}, \tilde{\tau})] = [\boldsymbol{B}^{tot}(\boldsymbol{X}, \tau), \boldsymbol{B}(\boldsymbol{X}, \tau), \boldsymbol{b}(\boldsymbol{X}, \tau), \boldsymbol{v}(\boldsymbol{X}, \tau)]. \quad (4.3)$$

It is proved in Sridhar & Subramanian (2009*a*,*b*) that (2.1), (2.3) and (2.4) are invariant under the simultaneous transformations given in (4.2) and (4.3). This symmetry property is actually invariance under a subset of the full ten parameter Galilean group, parametrized by the five quantities $(\xi_1, \xi_2, \xi_3, \tau_0, S)$; for brevity we will refer to this restricted symmetry as GI.

It is important to note that the lab and co-moving frames need not constitute inertial coordinate systems. One of the main applications of our theory is to the *shearing sheet*, which is a local description of a differentially rotating disk. In this case, the velocity field will be affected by Coriolis forces. The only requirement is that the magnetic field satisfies the induction equation (2.1).

4.2. *Galilean-invariant velocity correlators*

Now we explore the consequences of the requirement that the statistics of the velocity fluctuations be Galilean-invariant. We consider the n-point velocity correlator measured by the observer in the lab frame. Let this observer correlate v_{j_1} at space–time location $(\boldsymbol{R}_1, \tau_1)$, with v_{j_2} at space–time location $(\boldsymbol{R}_2, \tau_2)$ and so on up to v_{j_n} at space–time location $(\boldsymbol{R}_n, \tau_n)$. Now let us consider a co-moving observer, the position vector of whose origin is given by $\boldsymbol{X}_c(\boldsymbol{\xi}, \tau)$ of (4.1). An identical experiment performed by this observer must yield the same results, the measurements now made at the space–time points denoted by $(\boldsymbol{R}_1 + \boldsymbol{X}_c(\boldsymbol{\xi}, \tau_1), \tau_1), (\boldsymbol{R}_2 + \boldsymbol{X}_c(\boldsymbol{\xi}, \tau_2), \tau_2), \ldots, (\boldsymbol{R}_n + \boldsymbol{X}_c(\boldsymbol{\xi}, \tau_n), \tau_n)$. If

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the velocity statistics is GI, the n-point velocity correlator must satisfy the condition

$$\langle v_{j_1}(\boldsymbol{R}_1, \tau_1) \dots v_{j_n}(\boldsymbol{R}_n, \tau_n)\rangle = \langle v_{j_1}(\boldsymbol{R}_1 + \boldsymbol{X}_c(\boldsymbol{\xi}, \tau_1), \tau_1) \dots v_{j_n}(\boldsymbol{R}_n + \boldsymbol{X}_c(\boldsymbol{\xi}, \tau_n), \tau_n)\rangle, \quad (4.4)$$

for all $(\boldsymbol{R}_1, \dots, \boldsymbol{R}_n, \tau_1, \dots, \tau_n, \boldsymbol{\xi})$.

In the low-Re_m limit, we require only the two-point velocity correlators, for which

$$\langle v_i(\boldsymbol{R}, \tau) v_j(\boldsymbol{R}', \tau')\rangle = \langle v_i(\boldsymbol{R} + \boldsymbol{X}_c(\boldsymbol{\xi}, \tau), \tau) v_j(\boldsymbol{R}' + \boldsymbol{X}_c(\boldsymbol{\xi}, \tau'), \tau')\rangle, \quad (4.5)$$

for all $(\boldsymbol{R}, \boldsymbol{R}', \tau, \tau', \boldsymbol{\xi})$. We also need to define the correlation between velocities and their gradients

$$\begin{aligned}\langle v_i(\boldsymbol{R}, \tau) v_{jl}(\boldsymbol{R}', \tau')\rangle &= \frac{\partial}{\partial R'_l} \langle v_i(\boldsymbol{R}, \tau) v_j(\boldsymbol{R}', \tau')\rangle \\ &= \frac{\partial}{\partial R'_l} \langle v_i(\boldsymbol{R} + \boldsymbol{X}_c(\boldsymbol{\xi}, \tau), \tau) v_j(\boldsymbol{R}' + \boldsymbol{X}_c(\boldsymbol{\xi}, \tau'), \tau')\rangle \\ &= \langle v_i(\boldsymbol{R} + \boldsymbol{X}_c(\boldsymbol{\xi}, \tau), \tau) v_{jl}(\boldsymbol{R}' + \boldsymbol{X}_c(\boldsymbol{\xi}, \tau'), \tau')\rangle. \quad (4.6)\end{aligned}$$

We want to choose $(\boldsymbol{R}, \boldsymbol{R}', \tau, \tau', \boldsymbol{\xi})$ as functions of $(\boldsymbol{x}, \boldsymbol{x}', t, t')$ such that we can use (4.5) and (4.6) to simplify the velocity correlators in equations (3.16)–(3.18). We note that (3.19) and (4.1) give

$$\boldsymbol{X} = \boldsymbol{X}_c(\boldsymbol{x}, t), \quad \boldsymbol{X}' = \boldsymbol{X}_c(\boldsymbol{x}', t'). \quad (4.7)$$

It is, therefore, natural to choose

$$\tau = t, \quad \tau' = t'. \quad (4.8)$$

Thus the required velocity correlators can now be written as

$$\langle v_i(\boldsymbol{X}, t) v_j(\boldsymbol{X}', t')\rangle = \langle v_i(\boldsymbol{X}_c(\boldsymbol{x}, t), t) v_j(\boldsymbol{X}_c(\boldsymbol{x}', t'), t')\rangle, \quad (4.9)$$

$$\langle v_i(\boldsymbol{X}, t) v_{jl}(\boldsymbol{X}', t')\rangle = \langle v_i(\boldsymbol{X}_c(\boldsymbol{x}, t), t) v_{jl}(\boldsymbol{X}_c(\boldsymbol{x}', t'), t')\rangle. \quad (4.10)$$

Comparing (4.9) and (4.10) with (4.5) and (4.6), we see that if we choose

$$\boldsymbol{R} = \boldsymbol{X}_c(\boldsymbol{x}, t), \quad \boldsymbol{R}' = \boldsymbol{X}_c(\boldsymbol{x}', t'), \quad (4.11)$$

then (4.9) and (4.10) together with (4.1), (4.5) and (4.6) imply that

$$\begin{aligned}\langle v_i(\boldsymbol{X}, t) v_j(\boldsymbol{X}', t')\rangle &= \langle v_i(\boldsymbol{R}, \tau) v_j(\boldsymbol{R}', \tau')\rangle \\ &= \langle v_i(\boldsymbol{R} + \boldsymbol{X}_c(\boldsymbol{\xi}, \tau), \tau) v_j(\boldsymbol{R}' + \boldsymbol{X}_c(\boldsymbol{\xi}, \tau'), \tau')\rangle \\ &= \langle v_i(\boldsymbol{X}_c(\boldsymbol{x} + \boldsymbol{\xi}, t), t) v_j(\boldsymbol{X}_c(\boldsymbol{x}' + \boldsymbol{\xi}, t'), t')\rangle. \quad (4.12)\end{aligned}$$

Similarly,

$$\langle v_i(\boldsymbol{X}, t) v_{jl}(\boldsymbol{X}', t')\rangle = \langle v_i(\boldsymbol{X}_c(\boldsymbol{x} + \boldsymbol{\xi}, t), t) v_{jl}(\boldsymbol{X}_c(\boldsymbol{x}' + \boldsymbol{\xi}, t'), t')\rangle. \quad (4.13)$$

Now it is natural to choose

$$\boldsymbol{\xi} = -\tfrac{1}{2}(\boldsymbol{x} + \boldsymbol{x}'). \quad (4.14)$$

Then

$$\begin{aligned}\langle v_i(\boldsymbol{X}, \tau) v_j(\boldsymbol{X}', \tau')\rangle &= \left\langle v_i\left(\boldsymbol{X}_c\left(\frac{\boldsymbol{x} - \boldsymbol{x}'}{2}, t\right), t\right) v_j\left(\boldsymbol{X}_c\left(\frac{\boldsymbol{x}' - \boldsymbol{x}}{2}, t'\right), t'\right)\right\rangle \\ &= R_{ij}(\boldsymbol{x} - \boldsymbol{x}', t, t'). \quad (4.15)\end{aligned}$$

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Similarly,

$$\langle v_i(\boldsymbol{X},\tau)v_{jl}(\boldsymbol{X}',\tau')\rangle = \left\langle v_i\left(\boldsymbol{X}_c\left(\frac{\boldsymbol{x}-\boldsymbol{x}'}{2},t\right),t\right)v_{jl}\left(\boldsymbol{X}_c\left(\frac{\boldsymbol{x}'-\boldsymbol{x}}{2},t'\right),t'\right)\right\rangle$$
$$= Q_{ijl}(\boldsymbol{x}-\boldsymbol{x}',t,t'). \tag{4.16}$$

We note that symmetry and incompressibility imply that

$$R_{ij}(\boldsymbol{r},t,t') = R_{ji}(-\boldsymbol{r},t',t), \tag{4.17}$$
$$Q_{ijj}(\boldsymbol{r},t,t') = 0. \tag{4.18}$$

4.3. *Galilean-invariant mean electromotive force*

The transport coefficients are completely determined by the form of the velocity correlator. Using (4.15) and (4.16) in equations (3.16)–(3.18) and noting the fact that the velocity correlators defined above are functions of only $(\boldsymbol{x}-\boldsymbol{x}')$, t and t', we can see that the GI transport coefficients

$$\widehat{\alpha}_{il}(\boldsymbol{x},t,\boldsymbol{x}',t') = \epsilon_{ijm}\left[Q_{jml}(\boldsymbol{x}-\boldsymbol{x}',t,t') + St'\delta_{l1}Q_{jm2}(\boldsymbol{x}-\boldsymbol{x}',t,t')\right], \tag{4.19}$$
$$\widehat{\beta}_{il}(\boldsymbol{x},t,\boldsymbol{x}',t') = \epsilon_{ij2}\left[Q_{j1l}(\boldsymbol{x}-\boldsymbol{x}',t,t') + St'\delta_{l1}Q_{j12}(\boldsymbol{x}-\boldsymbol{x}',t,t')\right], \tag{4.20}$$
$$\widehat{\eta}_{iml}(\boldsymbol{x},t,\boldsymbol{x}',t') = \epsilon_{ijl}R_{jm}(\boldsymbol{x}-\boldsymbol{x}',t,t'). \tag{4.21}$$

The transport coefficients depend on $\boldsymbol{x}$ and $\boldsymbol{x}'$ only through the combination, $(\boldsymbol{x}-\boldsymbol{x}')$, which arises because of GI. We can derive an expression for the Galilean-invariant mean EMF by using equations (4.19)–(4.21) for the transport coefficients in (3.8). We also change the integration variable in (3.8) to $\boldsymbol{r} = \boldsymbol{x}-\boldsymbol{x}'$. The integrands can be simplified as follows:

$$\widehat{\alpha}_{il}(\boldsymbol{x},t,\boldsymbol{x}',t')\left[H_l' - St'\delta_{l2}H_1'\right] = \epsilon_{ijm}\left[Q_{jml} + St'\delta_{l1}Q_{jm2}\right]\left[H_l' - St'\delta_{l2}H_1'\right]$$
$$= \epsilon_{ijm}Q_{jml}(\boldsymbol{r},t,t')H_l(\boldsymbol{x}-\boldsymbol{r},t'), \tag{4.22}$$
$$\widehat{\beta}_{il}(\boldsymbol{x},t,\boldsymbol{x}',t')\left[H_l' - St'\delta_{l2}H_1'\right] = \epsilon_{ij2}\left[Q_{j1l} + St'\delta_{l1}Q_{j12}\right]\left[H_l' - St'\delta_{l2}H_1'\right]$$
$$= \epsilon_{ij2}Q_{j1l}(\boldsymbol{r},t,t')H_l(\boldsymbol{x}-\boldsymbol{r},t'), \tag{4.23}$$
$$\left[\widehat{\eta}_{iml} - St'\delta_{m2}\widehat{\eta}_{i1l}\right]H_{lm}' = \epsilon_{ijl}\left[R_{jm}(\boldsymbol{r},t,t') - St'\delta_{m2}R_{j1}(\boldsymbol{r},t,t')\right]H_{lm}(\boldsymbol{x}-\boldsymbol{r},t'), \tag{4.24}$$
$$\left[\widehat{\eta}_{im2} - St'\delta_{m2}\widehat{\eta}_{i12}\right]H_{1m}' = \epsilon_{ij2}\delta_{l1}\left[R_{jm}(\boldsymbol{r},t,t') - St'\delta_{m2}R_{j1}(\boldsymbol{r},t,t')\right]H_{lm}(\boldsymbol{x}-\boldsymbol{r},t'). \tag{4.25}$$

Let us define

$$C_{jml}(\boldsymbol{r},t,t') = Q_{jml}(\boldsymbol{r},t,t') + S(t-t')\delta_{m2}Q_{j1l}(\boldsymbol{r},t,t'), \tag{4.26}$$
$$D_{jm}(\boldsymbol{r},t,t') = R_{jm}(\boldsymbol{r},t,t') - St'\delta_{m2}R_{j1}(\boldsymbol{r},t,t'). \tag{4.27}$$

The mean EMF can now be written compactly as

$$\mathscr{E}_i(\boldsymbol{x},t) = \epsilon_{ijm}\int_0^t \mathrm{d}t'\int \mathrm{d}^3r\, G_\eta(\boldsymbol{r},t,t')C_{jml}(\boldsymbol{r},t,t')H_l(\boldsymbol{x}-\boldsymbol{r},t')$$
$$-\int_0^t \mathrm{d}t'\int \mathrm{d}^3r\, G_\eta(\boldsymbol{r},t,t')\left[\epsilon_{ijl} + S(t-t')\delta_{l1}\epsilon_{ij2}\right]$$
$$\times D_{jm}(\boldsymbol{r},t,t')H_{lm}(\boldsymbol{x}-\boldsymbol{r},t'). \tag{4.28}$$

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5. Mean-field induction equation

5.1. *Mean-field induction equation in sheared coordinate space*

Applying the shearing transformation given in (2.10) and (2.11) to the mean field equation (2.3), we see that the mean field, $\boldsymbol{H}(\boldsymbol{x}, t)$, obeys

$$\frac{\partial H_i}{\partial t} - S\delta_{i2}H_1 = (\nabla \times \mathscr{E})_i + \eta\nabla^2 H_i, \tag{5.1}$$

where

$$(\nabla)_p \equiv \frac{\partial}{\partial X_p} = \frac{\partial}{\partial x_p} - St\delta_{p1}\frac{\partial}{\partial x_2}. \tag{5.2}$$

We note that the divergence condition on the mean magnetic field can be written as

$$\nabla \cdot \boldsymbol{H} \equiv \frac{\partial H_p}{\partial X_p} = H_{pp} - StH_{12} = 0. \tag{5.3}$$

It may be verified that (5.1) preserves the condition $\nabla \cdot \boldsymbol{H} = 0$. We now use (4.28) and (5.2) to evaluate $\nabla \times \mathscr{E}$

$$\begin{aligned}(\nabla \times \mathscr{E})_i &= \epsilon_{ipq}\frac{\partial \mathscr{E}_q}{\partial X_p} = \epsilon_{ipq}\left(\frac{\partial}{\partial x_p} - St\delta_{p1}\frac{\partial}{\partial x_2}\right)\mathscr{E}_q \\ &= \epsilon_{ipq}\epsilon_{qjm}\int_0^t \mathrm{d}t' \int \mathrm{d}^3 r G_\eta(\boldsymbol{r}, t, t')C_{jml}(\boldsymbol{r}, t, t')\left[H'_{lp} - St\delta_{p1}H'_{l2}\right] \\ &\quad - \int_0^t \mathrm{d}t' \int \mathrm{d}^3 r G_\eta(\boldsymbol{r}, t, t')D_{jm}(\boldsymbol{r}, t, t')\left[\epsilon_{ipq}\epsilon_{qjl} + S(t-t')\delta_{l1}\epsilon_{ipq}\epsilon_{qj2}\right] \\ &\quad \times \left[H'_{lmp} - St\delta_{p1}H'_{lm2}\right],\end{aligned} \tag{5.4}$$

where $H'_i = H_i(\boldsymbol{x} - \boldsymbol{r}, t')$. Expanding $\epsilon_{ipq}\epsilon_{qjm} = (\delta_{ij}\delta_{mp} - \delta_{im}\delta_{jp})$, the contribution from the C term is

$$(\nabla \times \mathscr{E})_i^C = \int_0^t \mathrm{d}t' \int \mathrm{d}^3 r G_\eta(\boldsymbol{r}, t, t')\left[C_{ipl}(\boldsymbol{r}, t, t') - C_{pil}(\boldsymbol{r}, t, t')\right]\left[H'_{lp} - St\delta_{p1}H'_{l2}\right]. \tag{5.5}$$

Evaluating the D term is a bit more involved. Again, we begin by expanding $\epsilon_{ipq}\epsilon_{qjl} = (\delta_{ij}\delta_{lp} - \delta_{il}\delta_{jp})$. Then we get

$$\begin{aligned}(\nabla \times \mathscr{E})_i^D &= \int_0^t \mathrm{d}t' \int \mathrm{d}^3 r G_\eta(\boldsymbol{r}, t, t')D_{pm}(\boldsymbol{r}, t, t') \\ &\quad \times \left\{H'_{ipm} - St\delta_{p1}H'_{i2m} + S(t-t')\delta_{i2}\left[H'_{1pm} - St\delta_{p1}H'_{12m}\right]\right\} \\ &\quad - \int_0^t \mathrm{d}t' \int \mathrm{d}^3 r G_\eta(\boldsymbol{r}, t, t')D_{im}(\boldsymbol{r}, t, t')\left[H'_{ppm} - St'H'_{12m}\right].\end{aligned} \tag{5.6}$$

The second integral vanishes because the factor in [ ] multiplying D_{im} is zero: to see this, differentiate the divergence-free condition of (5.3) with respect to x_m. Now we can use (5.5) and (5.6) to write $(\nabla \times \mathscr{E}) = (\nabla \times \mathscr{E})^C + (\nabla \times \mathscr{E})^D$. Substituting this expression in (5.1), we obtain a set of integro-differential equations governing the

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dynamics of the mean field, $\boldsymbol{H}(\boldsymbol{x},t)$, valid for arbitrary values of the shear strength, S

$$\begin{aligned}\frac{\partial H_i}{\partial t} - S\delta_{i2}H_1 = \eta\nabla^2 H_i &+ \int_0^t \mathrm{d}t' \int \mathrm{d}^3 r G_\eta(\boldsymbol{r},t,t')\left[C_{iml}(\boldsymbol{r},t,t') - C_{mil}(\boldsymbol{r},t,t')\right] \\ &\times \left[H_{lm}(\boldsymbol{x}-\boldsymbol{r},t') - St\delta_{m1}H_{l2}(\boldsymbol{x}-\boldsymbol{r},t')\right] \\ &+ \int_0^t \mathrm{d}t' \int \mathrm{d}^3 r G_\eta(\boldsymbol{r},t,t') D_{jm}(\boldsymbol{r},t,t') \\ &\times \left[H_{ijm}(\boldsymbol{x}-\boldsymbol{r},t') - St\delta_{j1}H_{i2m}(\boldsymbol{x}-\boldsymbol{r},t')\right. \\ &+ \left. S(t-t')\delta_{i2}\left\{H_{1jm}(\boldsymbol{x}-\boldsymbol{r},t') - St\delta_{j1}H_{12m}(\boldsymbol{x}-\boldsymbol{r},t')\right\}\right]. \end{aligned} \tag{5.7}$$

We note some important properties of the mean-field induction equation (5.7) as follows:

(*a*) The $D_{jm}(\boldsymbol{r},t,t')$ terms are such that $(\nabla\times\mathscr{E})_i$ involves only H_i for $i=1$ and $i=3$, whereas $(\nabla\times\mathscr{E})_2$ depends on both H_2 and H_1. The implications for the original field, $\boldsymbol{B}(\boldsymbol{X},\tau)$, can be read off, because it is equal to $\boldsymbol{H}(\boldsymbol{x},t)$ component-wise (i.e. $B_i(\boldsymbol{X},\tau) = H_i(\boldsymbol{x},t)$). Therefore, in the mean-field induction equation, the D term is of such a form that (i) the equations for B_1 or B_3 involve only B_1 or B_3, respectively; (ii) the equation for B_2 involves both B_1 and B_2.

(*b*) Only the part of $C_{iml}(\boldsymbol{r},t,t')$ that is antisymmetric in the indices (i,m) contributes. We note that it is possible that the C term can lead to a coupling of different components of the mean magnetic field. To investigate this, it is necessary to specify the statistics of the velocity fluctuations.

(*c*) In the formal limit of zero resistivity, $\eta\to 0$, the resistive Green's function, $G(\boldsymbol{x},t,t')\to\delta(\boldsymbol{x})$. Then the mean-field induction equation simplifies to

$$\begin{aligned}\frac{\partial H_i}{\partial t} - S\delta_{i2}H_1 = &\int_0^t \mathrm{d}t' \left[C_{iml}(\boldsymbol{0},t,t') - C_{mil}(\boldsymbol{0},t,t')\right] \\ &\times \left[H_{lm}(\boldsymbol{x},t') - St\delta_{m1}H_{l2}(\boldsymbol{x},t')\right] \\ &+ \int_0^t \mathrm{d}t' D_{jm}(\boldsymbol{0},t,t')\left[H_{ijm}(\boldsymbol{x},t') - St\delta_{j1}H_{i2m}(\boldsymbol{x},t')\right. \\ &+ \left. S(t-t')\delta_{i2}\left\{H_{1jm}(\boldsymbol{x},t') - St\delta_{j1}H_{12m}(\boldsymbol{x},t')\right\}\right], \end{aligned} \tag{5.8}$$

which is identical to the equation derived by Sridhar & Subramanian (2009*a*,*b*).

5.2. *Mean-field induction equation in sheared Fourier space*

Equation (5.7) governing the time evolution of the mean field may be simplified further by taking a spatial Fourier transformation. Let us define

$$\tilde{\boldsymbol{H}}(\boldsymbol{k},t) = \int \mathrm{d}^3 x \boldsymbol{H}(\boldsymbol{x},t)\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}) \tag{5.9}$$

and the quantities

$$\tilde{I}_{iml}(\boldsymbol{k},t,t') = \int \mathrm{d}^3 r G_\eta(\boldsymbol{r},t,t') C_{iml}(\boldsymbol{r},t,t')\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}), \tag{5.10}$$

$$\tilde{J}_{jm}(\boldsymbol{k},t,t') = \int \mathrm{d}^3 r G_\eta(\boldsymbol{r},t,t') D_{jm}(\boldsymbol{r},t,t')\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}). \tag{5.11}$$

Both $\tilde{I}_{iml}(\boldsymbol{k},t,t')$ and $\tilde{J}_{jm}(\boldsymbol{k},t,t')$ are to be regarded as given quantities, because they are known once the velocity correlators have been specified. Taking the spatial

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Fourier transform of (5.7), we obtain

$$\frac{\partial \tilde{H}_i}{\partial t} - S\delta_{i2}\tilde{H}_1 = -\eta K^2 \tilde{H}_i + \mathrm{i}K_m \int_0^t \mathrm{d}t' \left[\tilde{I}_{iml}(\boldsymbol{k}, t, t') - \tilde{I}_{mil}(\boldsymbol{k}, t, t')\right] \tilde{H}_l(\boldsymbol{k}, t') - k_m K_j \int_0^t \mathrm{d}t' \tilde{J}_{jm}(\boldsymbol{k}, t, t') \left[\tilde{H}_i(\boldsymbol{k}, t') + S(t - t')\delta_{i2}\tilde{H}_1(\boldsymbol{k}, t')\right], \quad (5.12)$$

where $\boldsymbol{K}(\boldsymbol{k}, t) = (k_1 - Stk_2, k_2, k_3)$ and $K^2 = |\boldsymbol{K}|^2 = (k_1 - Stk_2)^2 + k_2^2 + k_3^2$. Once the initial data, $\tilde{\boldsymbol{H}}(\boldsymbol{k}, 0)$, have been specified, (5.12) can be integrated in time to determine $\tilde{\boldsymbol{H}}(\boldsymbol{k}, t)$. Whereas these equations are not easy to solve, we note some of their important properties as follows:

(*a*) Only the part of $\tilde{I}_{iml}(\boldsymbol{k}, t, t')$ that is antisymmetric in the indices (i, m) contributes.

(*b*) The time evolution of $\tilde{\boldsymbol{H}}(\boldsymbol{k}, t)$ depends only on $\tilde{\boldsymbol{H}}(\boldsymbol{k}, t')$ for $0 \leqslant t' < t$, not on the values of $\tilde{\boldsymbol{H}}$ at other values of $\boldsymbol{k}$. Thus each $\boldsymbol{k}$ labels a *normal mode* whose *amplitude* and *polarization* are given by $\tilde{\boldsymbol{H}}(\boldsymbol{k}, t)$, the time evolution of which is independent of all the other normal modes.

(*c*) When we have determined $\tilde{\boldsymbol{H}}(\boldsymbol{k}, t)$, the magnetic field in the original variables, $\boldsymbol{B}(\boldsymbol{X}, \tau)$, can be recovered by using the shearing transformation, (2.10), to write $(\boldsymbol{x}, t)$ in terms of the lab frame coordinates $(\boldsymbol{X}, \tau)$

$$\boldsymbol{B}(\boldsymbol{X}, \tau) = \boldsymbol{H}(\boldsymbol{x}, t) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \tilde{\boldsymbol{H}}(\boldsymbol{k}, t) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \tilde{\boldsymbol{H}}(\boldsymbol{k}, \tau) \exp[\mathrm{i}\boldsymbol{K}(\boldsymbol{k}, \tau) \cdot \boldsymbol{X}], \quad (5.13)$$

where we have used $\boldsymbol{K} \cdot \boldsymbol{X} = \boldsymbol{k} \cdot \boldsymbol{x}$. Thus $\boldsymbol{B}(\boldsymbol{X}, \tau)$ has been expressed as a superposition of the normal modes, each of which is a *shearing wave*, whose spatial structure is given by

$$\exp[\mathrm{i}\boldsymbol{K}(\boldsymbol{k}, \tau) \cdot \boldsymbol{X}] = \exp[\mathrm{i}\{(k_1 - Stk_2)X_1 + k_2 X_2 + k_3 X_3\}]. \quad (5.14)$$

For non-axisymmetric waves, $k_2 \neq 0$ and, as time progresses, the shearing wave develops fine structure along the X_1-direction with a time-dependent spatial frequency equal to $(k_1 - Stk_2)$.

5.3. *The integral kernels expressed in terms of the velocity spectrum tensor*

We have derived the integral equation satisfied by the mean magnetic field, to the lowest order in Re_m; in sheared coordinate space it is given by (5.7) and in sheared Fourier space it is given by (5.12). One can proceed to look for solutions if the integral kernels are known. This means that either the pair $[C_{iml}(\boldsymbol{r}, t, t'), D_{jm}(\boldsymbol{r}, t, t')]$ or the pair $[\tilde{I}_{iml}(\boldsymbol{k}, t, t'), \tilde{J}_{jm}(\boldsymbol{k}, t, t')]$ needs to be specified. Here, we show that all these integral kernels can be expressed in terms of a single entity, which is the velocity spectrum tensor, $\Pi_{ij}(\boldsymbol{k}, t, t')$. The GI of velocity correlators stated in (4.5) is most compactly expressed in Fourier space; this is stated in the theorem given in the Appendix. Let $\tilde{\boldsymbol{v}}(\boldsymbol{K}, \tau)$ be the spatial Fourier transform of $\boldsymbol{v}(\boldsymbol{X}, \tau)$, defined by

$$\tilde{\boldsymbol{v}}(\boldsymbol{K}, \tau) = \int \mathrm{d}^3 X \exp(-\mathrm{i}\boldsymbol{K} \cdot \boldsymbol{X}) \boldsymbol{v}(\boldsymbol{X}, \tau), \quad [\boldsymbol{K} \cdot \tilde{\boldsymbol{v}}(\boldsymbol{K}, \tau)] = 0. \quad (5.15)$$

It is proved in the Appendix that a GI Fourier-space two-point velocity correlator must be of the form

$$\left\langle \tilde{v}_i(\boldsymbol{K}, \tau) \tilde{v}_j^*(\boldsymbol{K}', \tau') \right\rangle = (2\pi)^6 \delta(\boldsymbol{k} - \boldsymbol{k}') \Pi_{ij}(\boldsymbol{k}, t, t'). \quad (5.16)$$

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We first define $R_{jm}(\boldsymbol{r},t,t')$ and $Q_{jml}(\boldsymbol{r},t,t')$ in terms of $\Pi_{jm}(\boldsymbol{k},t,t')$. From (4.15) and (4.16), we have

$$\begin{aligned} R_{jm}(\boldsymbol{r},t,t') &= \left\langle v_j\left(\boldsymbol{X}_c\left(\frac{\boldsymbol{r}}{2},t\right),t\right) v_m\left(\boldsymbol{X}_c\left(-\frac{\boldsymbol{r}}{2},t'\right),t'\right)\right\rangle \\ &= \int \frac{\mathrm{d}^3K}{(2\pi)^3}\frac{\mathrm{d}^3K'}{(2\pi)^3}\left\langle v_j(\boldsymbol{K},t)\, v_m^*(\boldsymbol{K}',t')\right\rangle \\ &\quad\times \exp\left[\mathrm{i}\left(\boldsymbol{K}\cdot\boldsymbol{X}_c\left(\frac{\boldsymbol{r}}{2},t\right)-\boldsymbol{K}'\cdot\boldsymbol{X}_c\left(-\frac{\boldsymbol{r}}{2},t'\right)\right)\right], \qquad (5.17) \\ Q_{jml}(\boldsymbol{r},t,t') &= \left\langle v_j\left(\boldsymbol{X}_c\left(\frac{\boldsymbol{r}}{2},t\right),t\right) v_{ml}\left(\boldsymbol{X}_c\left(-\frac{\boldsymbol{r}}{2},t'\right),t'\right)\right\rangle \\ &= \int \frac{\mathrm{d}^3K}{(2\pi)^3}\frac{\mathrm{d}^3K'}{(2\pi)^3}\left(-\mathrm{i}K_l'\right)\left\langle v_j(\boldsymbol{K},t)\, v_m^*(\boldsymbol{K}',t')\right\rangle \\ &\quad\times \exp\left[\mathrm{i}\left(\boldsymbol{K}\cdot\boldsymbol{X}_c\left(\frac{\boldsymbol{r}}{2},t\right)-\boldsymbol{K}'\cdot\boldsymbol{X}_c\left(-\frac{\boldsymbol{r}}{2},t'\right)\right)\right]. \qquad (5.18) \end{aligned}$$

Substituting for $\boldsymbol{X}_c$ from (4.1), we can write the phase

$$\boldsymbol{K}\cdot\boldsymbol{X}_c\left(\frac{\boldsymbol{r}}{2},t\right)-\boldsymbol{K}'\cdot\boldsymbol{X}_c\left(-\frac{\boldsymbol{r}}{2},t'\right)=\left(\boldsymbol{k}+\boldsymbol{k}'\right)\cdot\frac{\boldsymbol{r}}{2}, \qquad (5.19)$$

where $\boldsymbol{k}$ and $\boldsymbol{k}'$ are defined in (A 8) and (A 9), respectively. Therefore,

$$R_{jm}(\boldsymbol{r},t,t')=\int \mathrm{d}^3k\Pi_{jm}(\boldsymbol{k},t,t')\exp[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}], \qquad (5.20)$$

$$Q_{jml}(\boldsymbol{r},t,t')=-\mathrm{i}\int \mathrm{d}^3k\left[k_l-St'\delta_{l1}k_2\right]\Pi_{jm}(\boldsymbol{k},t,t')\exp[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}]. \qquad (5.21)$$

Using (4.26) and (4.27), we can write the real-space integral kernels, $C_{jml}(\boldsymbol{r},t,t')$ and $D_{jm}(\boldsymbol{r},t,t')$ as

$$\begin{aligned} D_{jm}(\boldsymbol{r},t,t') &= R_{jm}(\boldsymbol{r},t,t')-St'\delta_{m2}R_{j1}(\boldsymbol{r},t,t') \\ &= \int \mathrm{d}^3k\left[\Pi_{jm}(\boldsymbol{k},t,t')-St'\delta_{m2}\Pi_{j1}(\boldsymbol{k},t,t')\right]\exp[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}], \qquad (5.22) \\ C_{jml}(\boldsymbol{r},t,t') &= Q_{jml}(\boldsymbol{r},t,t')+S(t-t')\delta_{m2}Q_{j1l}(\boldsymbol{r},t,t') \\ &= -\mathrm{i}\int \mathrm{d}^3k\left[k_l-St'\delta_{l1}k_2\right]\left[\Pi_{jm}(\boldsymbol{k},t,t')+S(t-t')\delta_{m2}\Pi_{j1}(\boldsymbol{k},t,t')\right] \\ &\quad\times\exp\left[\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}\right]. \qquad (5.23) \end{aligned}$$

Using (5.10) and (5.11), we can express the Fourier-space integral kernels, $\tilde{I}_{jml}(\boldsymbol{k},t,t')$ and $\tilde{J}_{jm}(\boldsymbol{k},t,t')$ as

$$\tilde{J}_{jm}(\boldsymbol{k},t,t')=\int \mathrm{d}^3k'\tilde{G}_\eta(\boldsymbol{k}-\boldsymbol{k}',t,t')\left[\Pi_{jm}(\boldsymbol{k}',t,t')-St'\delta_{m2}\Pi_{j1}(\boldsymbol{k}',t,t')\right], \qquad (5.24)$$

$$\begin{aligned} \tilde{I}_{jml}(\boldsymbol{k},t,t') &= -\mathrm{i}\int \mathrm{d}^3k'\tilde{G}_\eta(\boldsymbol{k}-\boldsymbol{k}',t,t')\left[k_l'-St'\delta_{l1}k_2'\right] \\ &\quad\times\left[\Pi_{jm}(\boldsymbol{k}',t,t')+S(t-t')\delta_{m2}\Pi_{j1}(\boldsymbol{k}',t,t')\right]. \qquad (5.25) \end{aligned}$$

Thus, we have expressed the integral kernels in terms of the velocity spectrum tensor, $\Pi_{jm}(\boldsymbol{k},t,t')$, which is the fundamental dynamical quantity that needs to be

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calculated before the integro-differential equation for the mean magnetic field can be solved.

6. Conclusions

We have formulated the problem of large-scale kinematic dynamo action due to turbulence in the presence of a linear shear flow, in the limit of small Re_m, but arbitrary fluid Reynolds number. The mean-field theory we present is non-perturbative in the shear parameter, and makes systematic use of the shearing coordinate transformation and the GI of the linear shear flow. Using Reynolds averaging, we split the magnetic field into mean and fluctuating components. The mean magnetic field is driven by the curl of the mean EMF, which in turn must be determined in terms of the statistics of the velocity fluctuations. In order to do this, it is necessary to determine the magnetic fluctuations in terms of the mean magnetic field and the velocity fluctuations. Therefore, we develop the equation for the fluctuations perturbatively in the small parameter, Re_m. Using the shearing coordinate transformation, we make an explicit calculation of the resistive Green's function for the linear shear flow. From the perturbative scheme, it is clear that the fluctuations can be determined to any order in Re_m. Here we determine the magnetic fluctuations and the mean EMF to the lowest order in Re_m. The transport coefficients are given in general form in terms of the two-point correlators of the velocity fluctuations. At this point, we make use of GI, which is a fundamental symmetry of the problem. For GI velocity statistics, we prove that the transport coefficients, although space-dependent, possess the property of translational invariance in sheared coordinate space. An explicit expression for the GI mean EMF is derived.

We put together all the results in §5 by deriving the integro-differential equation governing the time evolution of the mean magnetic field. Some important properties of this equation are given in the following.

(*a*) Velocity fluctuations contribute to two different kinds of terms, the C and D terms, in which first and second spatial derivatives of the mean magnetic field, respectively, appear inside the space–time integrals.

(*b*) The C term is a generalization to the case of shear, of the 'α' term familiar from mean-field electrodynamics in the absence of shear. However, they can also contribute to 'magnetic diffusion' (see below discussion). Similarly, the D term is a generalization to the case of shear, of the 'magnetic diffusion' term familiar from mean-field electrodynamics in the absence of shear. It must be noted that the generalization is non-perturbative in the shear strength.

(*c*) In the mean-field induction equation, the D term is of such a form that (i) the equations for B_1 or B_3 involve only B_1 or B_3, respectively; (ii) the equation for B_2 involves both B_1 and B_2. Therefore, to the lowest order in Re_m but to all orders in the shear strength, the D term cannot give rise to a shear-current-assisted dynamo effect.

(*d*) In the formal limit of zero resistivity, the quasi-linear theory of Sridhar & Subramanian (2009*a*,*b*) is recovered. In this case, the C term vanishes when the velocity field is non-helical. However, this may not be the case when the resistivity is non-zero. Whether the C term gives rise to such a shear-current-type effect depends on the form of the velocity correlators, which will be strongly affected by shear and highly anisotropic; hence it is difficult to guess their tensorial forms *a priori*, and it is necessary to develop a dynamical theory of velocity correlators – see below for further discussion.

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(*e*) Sheared Fourier space is the natural setting for the mean magnetic field; the normal modes of the theory are a set of shearing waves, labelled by their sheared wavevectors.

(*f*) We prove a theorem (in the Appendix) on the form of the two-point velocity correlator in Fourier space; the velocity spectrum tensor and its general properties are discussed. The integral kernels are expressed in terms of the velocity spectrum tensor, which is the fundamental dynamical quantity that needs to be specified to complete the integro-differential equation description of the time evolution of the mean magnetic field.

The physical meaning of the C and D terms becomes clear in the limit of a slowly varying magnetic field, when the integro-differential equation reduces to a partial differential equation (Singh & Sridhar 2010). Then we encounter the well-known α-effect and turbulent magnetic diffusion (η), *albeit* in tensorial form. The C term alone contributes to α, whereas both the C and D terms contribute to magnetic diffusion. When the velocity field is non-helical, the velocity spectrum tensor is real and the tensorial α coefficient vanishes; this result is true for arbitrary values of the shear parameter. The C term can, in principle, contribute to a shear-current effect, through the off-diagonal components of the diffusivity tensor (which couple the streamwise component of the mean magnetic field with the cross-stream components). It turns out that these off-diagonal components depend on the microscopic resistivity in such a manner that they vanish when the microscopic resistivity vanishes. This result is consistent with the results of Sridhar & Subramanian (2009*a*,*b*). To deal with the case when the microscopic resistivity does not vanish, it is necessary to provide our kinematic development with a dynamical model for the velocity field. Singh & Sridhar (2010) show that, for forced non-helical driving at low fluid Reynolds number, the sign of the off-diagonal terms of the diffusivity tensor does not favour the shear-current effect. This conclusion agrees with those reported by Rädler & Stepanov (2006), Rüdiger & Kitchatinov (2006) and Brandenburg *et al.* (2008), even though our results are limited to low Reynolds numbers. If we seek a different explanation for the dynamo action seen in numerical simulations, the 'fluctuating α-effect' still remains a promising candidate. α itself is described by second-order velocity correlators, so to describe fluctuatons of α, it is necessary to deal with either fourth-order velocity correlators or products of two second-order velocity correlators. This requires extending our perturbative calculations by at least two higher orders, a task which, while tractable, is beyond the scope of the present investigation.

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Appendix. A theorem on Galilean invariant velocity correlators

THEOREM 1. *A GI Fourier-space two-point velocity correlator must be of the form*

$$\left\langle \tilde{v}_i(\boldsymbol{K},\tau)\,\tilde{v}_j^*(\boldsymbol{K}',\tau')\right\rangle = (2\pi)^6\delta(\boldsymbol{k}-\boldsymbol{k}')\Pi_{ij}(\boldsymbol{k},t,t'), \tag{A 1}$$

where Π_{ij} is the velocity spectrum tensor, which must possess the following properties:

$$\Pi_{ij}(\boldsymbol{k},t,t') = \Pi_{ij}^*(-\boldsymbol{k},t,t') = \Pi_{ji}(-\boldsymbol{k},t',t), \tag{A 2}$$

$$K_i\Pi_{ij}(\boldsymbol{k},t,t') = (k_i - St\delta_{i1}k_2)\,\Pi_{ij}(\boldsymbol{k},t,t') = 0, \tag{A 3}$$

$$K_j'\Pi_{ij}(\boldsymbol{k},t,t') = \left(k_j - St'\delta_{j1}k_2\right)\Pi_{ij}(\boldsymbol{k},t,t') = 0. \tag{A 4}$$

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Proof. The velocity correlator in Fourier space is

$$\begin{aligned}\left\langle \tilde{v}_i(\boldsymbol{K},\tau)\tilde{v}_j^*(\boldsymbol{K}',\tau')\right\rangle &= \int \mathrm{d}^3X\mathrm{d}^3X' \exp\left[\mathrm{i}\left(\boldsymbol{K}'\cdot\boldsymbol{X}'-\boldsymbol{K}\cdot\boldsymbol{X}\right)\right]\left\langle v_i(\boldsymbol{X},\tau)v_j(\boldsymbol{X}',\tau')\right\rangle \\ &= \int \mathrm{d}^3X\mathrm{d}^3X' \exp\left[\mathrm{i}\left(\boldsymbol{K}'\cdot\boldsymbol{X}'-\boldsymbol{K}\cdot\boldsymbol{X}\right)\right] \\ &\quad\times\left\langle v_i(\boldsymbol{X}+\boldsymbol{X}_c(\boldsymbol{\xi},\tau),\tau)v_j(\boldsymbol{X}'+\boldsymbol{X}_c(\boldsymbol{\xi},\tau'),\tau')\right\rangle, \end{aligned} \tag{A 5}$$

where (4.5) has been used. Using new dummy variables of integration, $\boldsymbol{X}\rightarrow\boldsymbol{X}-\boldsymbol{X}_c(\boldsymbol{\xi},\tau)$ and $\boldsymbol{X}'\rightarrow\boldsymbol{X}'-\boldsymbol{X}_c(\boldsymbol{\xi},\tau')$, we write

$$\begin{aligned}\left\langle \tilde{v}_i(\boldsymbol{K},\tau)\tilde{v}_j^*(\boldsymbol{K}',\tau')\right\rangle &= \exp\left[\mathrm{i}\left(\boldsymbol{K}\cdot\boldsymbol{X}_c(\boldsymbol{\xi},\tau)-\boldsymbol{K}'\cdot\boldsymbol{X}_c(\boldsymbol{\xi},\tau')\right)\right] \\ &\quad\times\int \mathrm{d}^3X\mathrm{d}^3X' \exp\left[\mathrm{i}\left(\boldsymbol{K}'\cdot\boldsymbol{X}'-\boldsymbol{K}\cdot\boldsymbol{X}\right)\right]\left\langle v_i(\boldsymbol{X},\tau)v_j(\boldsymbol{X}',\tau')\right\rangle \\ &= \exp\left[\mathrm{i}\left(\boldsymbol{K}\cdot\boldsymbol{X}_c(\boldsymbol{\xi},\tau)-\boldsymbol{K}'\cdot\boldsymbol{X}_c(\boldsymbol{\xi},\tau')\right)\right]\times\left\langle \tilde{v}_i(\boldsymbol{K},\tau)\tilde{v}_j^*(\boldsymbol{K}',\tau')\right\rangle. \end{aligned} \tag{A 6}$$

Comparing the left-hand side and the right-hand side, we conclude that the phase, $[\boldsymbol{K}\cdot\boldsymbol{X}_c(\boldsymbol{\xi},\tau)-\boldsymbol{K}'\cdot\boldsymbol{X}_c(\boldsymbol{\xi},\tau')]$, must vanish. Substituting for $\boldsymbol{X}_c$ from (4.1), the condition of zero phase implies that

$$(k_1-k_1')\xi_1+(k_2-k_2')\xi_2+(k_3-k_3')\xi_3=0, \tag{A 7}$$

where $\boldsymbol{k}\equiv(k_1,k_2,k_3)$ and $\boldsymbol{k}'\equiv(k_1',k_2',k_3')$ are sheared wavevectors which are related to $\boldsymbol{K}$ and $\boldsymbol{K}'$ through the Fourier-space shearing transformation

$$k_1=K_1+S\tau K_2,\quad k_2=K_2,\quad k_3=K_3,\quad t=\tau, \tag{A 8}$$

$$k_1'=K_1'+S\tau' K_2',\quad k_2'=K_2',\quad k_3'=K_3',\quad t'=\tau'. \tag{A 9}$$

Since (A 7) must be valid for arbitrary (ξ_1,ξ_2,ξ_3), we must have $\boldsymbol{k}=\boldsymbol{k}'$. In other words, the Galilean-invariant Fourier-space velocity correlator must be of the general form stated in (A 1). Moreover, the listed properties of the velocity spectrum tensor, Π_{ij}, given in equations (A 2)–(A 4) follow from the reality of $\boldsymbol{v}(\boldsymbol{X},\tau)$, symmetry with respect to simultaneous interchange of (i,j) and (t,t') and incompressibility. □

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