SHEARING DISPLACEMENT OF A RECTANGULAR PLATE

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Communicated by W. R. Dean

Received 20 May 1948

1. *Introduction.* This problem arose in connexion with the determination of the shearing stress under which a pair of dislocations is formed in a raft of bubbles, the model used by Prof. Bragg (1) to represent a crystalline material. It is applicable also to cases in which a flat rectangular plate of isotropic material is stressed by uniform displacements along two opposite edges with the other pair of sides left free. It may therefore be of interest to designers of stiffened thin shell constructions, where load is transmitted from one stiffener to the next in a similar manner.

The problem has already been solved by Prof. Inglis (3), but it was considered worth while to publish this new method of solution, which is believed more rapid and accurate.

2. *Statement of the problem.* The problem is to find the stresses in a thin flat rectangular plate of elastic material, when two opposite edges are displaced a given distance relative and parallel to one another, the other two opposite edges being left free.

In the usual notation for problems of elasticity, u, v are the components of displacement in the directions x, y of Cartesian axes in the plane of the plate. If the z-axis is normal to the plate, the stress components $\widehat{xz}$, $\widehat{yz}$, $\widehat{zz}$ are assumed zero. The differential equations to be satisfied by the three remaining stress components are then

$$\frac{\partial \widehat{xx}}{\partial x}+\frac{\partial \widehat{xy}}{\partial y}=0, \quad \frac{\partial \widehat{xy}}{\partial x}+\frac{\partial \widehat{yy}}{\partial y}=0, \tag{1}$$

subject to the boundary conditions, and the requirement that components of displacement should exist such that

$$\left.\begin{aligned} \widehat{xx} &= \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\frac{\partial u}{\partial x}+\frac{2\lambda\mu}{\lambda+2\mu}\frac{\partial v}{\partial y}, \\ \widehat{yy} &= \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}\frac{\partial v}{\partial y}+\frac{2\lambda\mu}{\lambda+2\mu}\frac{\partial u}{\partial x}, \\ \widehat{xy} &= \mu\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right), \end{aligned}\right\} \tag{2}$$

where λ, μ are the usual elastic constants.

If the axes x, y are so chosen that the boundary of the plate coincides with the lines $x = \pm\pi$, $y = \pm h\pi$ (h some constant), the boundary conditions may be written

$$\widehat{xy} = 0 = \widehat{xx} \quad \text{on} \quad x = +\pi,$$

$$u = \text{constant} \begin{cases} = U, \text{ say,} & \text{on} \quad y = h\pi, \\ = -U & \text{on} \quad y = -h\pi, \end{cases}$$

$$v = 0 \quad \text{on} \quad y = \pm h\pi.$$

https://doi.org/10.1017/S0305004100024816 Published online by Cambridge University Press

Boundary conditions such as these, which are expressed partly in terms of displacements and partly in terms of stresses, are known as 'mixed boundary conditions' and often cause difficulty. It would be possible to express everything in terms of the displacements by substituting (2) for the stresses, but the first boundary condition would then involve derivatives of both u and v, so that little would be gained in simplicity.

The method used here will be to build up a solution in series, each term of which satisfies the differential equation and one or more of the boundary conditions, and finally to adjust the coefficients so that the remaining conditions are satisfied.

3. *Solution of the problem.* The form of the solution is similar to that used in a recent paper by J. S. Pickett (4). Let

$$\left.\begin{aligned} u &= \sum_{n=1}^{\infty} A_n \sin\frac{ny}{h}\left[x\sinh\frac{nx}{h} - \left(\pi\coth\frac{n\pi}{h} + \frac{h}{n}\frac{\lambda+2\mu}{3\lambda+2\mu}\right)\cosh\frac{nx}{h}\right] \\ &\quad + B_0 y + \sum_{n=1}^{\infty} B_n \cos nx[y\cosh ny - h\pi\coth nh\pi\sinh ny], \\ v &= A_0 x + \sum_{n=1}^{\infty} A_n \cos\frac{ny}{h}\left[x\cosh\frac{nx}{h} - \left(\pi\coth\frac{n\pi}{h} - \frac{4h}{n}\frac{\lambda+\mu}{3\lambda+2\mu}\right)\sinh\frac{nx}{h}\right] \\ &\quad + \sum_{n=1}^{\infty} B_n \sin nx\left[y\sinh ny - \left(h\pi\coth nh\pi + \frac{5\lambda+6\mu}{n(3\lambda+2\mu)}\right)\cosh ny\right], \end{aligned}\right\} \tag{3}$$

where the A_n, B_n are constants as yet undetermined. Substituting these series in (2) gives the stress components

$$\begin{aligned} \widehat{xx} &= 2\mu\left\{\sum_{n=1}^{\infty} A_n \sin\frac{ny}{h}\left[\frac{nx}{h}\cosh\frac{nx}{h} - \frac{n\pi}{h}\coth\frac{n\pi}{h}\sinh\frac{nx}{h}\right]\right. \\ &\quad \left. + \sum_{n=1}^{\infty} B_n \sin nx\left[-ny\cosh ny + \left(nh\pi\coth nh\pi - \frac{2\lambda}{3\lambda+2\mu}\right)\sinh ny\right]\right\}, \\ \widehat{yy} &= 2\mu\left\{\sum_{n=1}^{\infty} A_n \sin\frac{ny}{h}\left[-\frac{nx}{h}\cosh\frac{nx}{h} + \left(\frac{n\pi}{h}\coth\frac{n\pi}{h} - 2\right)\sinh\frac{nx}{h}\right]\right. \\ &\quad \left. + \sum_{n=1}^{\infty} B_n \sin nx\left[ny\cosh ny - \left(nh\pi\coth nh\pi + \frac{4(\lambda+\mu)}{3\lambda+2\mu}\right)\sinh ny\right]\right\}, \\ \widehat{xy} &= 2\mu\left\{\frac{A_0+B_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{ny}{h}\left[\frac{nx}{h}\sinh\frac{nx}{h} - \left(\frac{n\pi}{h}\coth\frac{n\pi}{h} - 1\right)\cosh\frac{nx}{h}\right]\right. \\ &\quad \left. + \sum_{n=1}^{\infty} B_n \cos nx\left[ny\sinh ny - \left(nh\pi\coth nh\pi + \frac{\lambda+2\mu}{3\lambda+2\mu}\right)\cosh ny\right]\right\}. \end{aligned}$$

These series have the correct type of symmetry for the problem, for each displacement is a function $f(x,y)$ such that $f(-x,-y) = -f(x,y)$ and each stress a function $g(x,y) = g(-x,-y)$. They satisfy the equations of equilibrium (1) for all values of A_n, B_n, and also two of the boundary conditions

$$\widehat{xx} = 0 \quad \text{at} \quad x = \pm\pi,$$

$$u = \text{constant} \begin{cases} = B_0 h\pi & \text{at} \quad y = h\pi, \\ = -B_0 h\pi & \text{at} \quad y = -h\pi. \end{cases}$$

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B_0 is therefore chosen equal to $U/h\pi$. The remaining boundary conditions

$$\widehat{xy} = 0 \quad \text{at} \quad x = \pm\pi, \qquad v = 0 \quad \text{at} \quad y = \pm h\pi,$$

may now be used to determine the constants A_n, B_n.

These conditions give two equations:

$$\tfrac{1}{2}(A_0+B_0)+\sum_{m=1}^{\infty} A_m \cos\frac{my}{h}\left[\cosh\frac{m\pi}{h}-\frac{m\pi}{h\sinh\frac{m\pi}{h}}\right]$$

$$+\sum_{n=1}^{\infty}(-1)^n B_n\left[ny\sinh ny-\left(nh\pi\coth nh\pi+\frac{\lambda+2\mu}{3\lambda+2\mu}\right)\cosh ny\right]=0, \tag{4}$$

and

$$A_0 x+\sum_{m=1}^{\infty}(-1)^m A_m\left[x\cosh\frac{mx}{h}-\left(\pi\coth\frac{m\pi}{h}-\frac{4h}{m}\frac{\lambda+\mu}{3\lambda+2\mu}\right)\sinh\frac{mx}{h}\right]$$

$$-\sum_{n=1}^{\infty} B_n \sin nx\left[\frac{h\pi}{\sinh nh\pi}+\frac{5\lambda+6\mu}{n(3\lambda+2\mu)}\cosh nh\pi\right], \tag{5}$$

where, for a material of Poisson's ratio $\frac{1}{3}$,

$$\frac{\lambda+2\mu}{3\lambda+2\mu}=\frac{1}{2}, \quad \frac{4(\lambda+\mu)}{3\lambda+2\mu}=\frac{3}{2}, \quad \frac{5\lambda+6\mu}{3\lambda+2\mu}=2.$$

Equation (4) may be solved for each A_m in terms of all the B_n by multiplying by $\cos my/h$ $(m=0,1,2,\ldots)$ and integrating from $-h\pi$ to $h\pi$:

$$\left.\begin{aligned} A_0 h\pi &= -B_0 h\pi+3\sum_{n=1}^{\infty}(-1)^n\frac{B_n}{n}\sinh nh\pi,\\ A_m h\pi\left[\cosh\frac{m\pi}{h}-\frac{m\pi}{h\sinh(m\pi/h)}\right] &= \sum_{n=1}^{\infty}(-1)^{n+m+1}B_n\frac{nh^2(m^2-3n^2h^2)}{(m^2+n^2h^2)^2}\sinh nh\pi. \end{aligned}\right\} \tag{6}$$

In other words, $A_m\{\cosh(m\pi/h)-m\pi/h\sinh(m\pi/h)\}$ is the mth Fourier coefficient of the function

$$\sum_{n=1}^{\infty}(-1)^{n+1}B_n[ny\sinh ny-(nh\pi\coth nh\pi+\tfrac{1}{2})\cosh ny]$$

in the range $-h\pi\leqslant y\leqslant h\pi$.

The solution of (5) is less direct, since multiplying by $\sin nx$ and integrating from $-\pi$ to π gives each B_n in terms of A_0 and all the A_m, which depend on all the B_n by (6):

$$B_n\pi\left[\frac{h\pi}{\sinh nh\pi}+\frac{2}{n}\cosh nh\pi\right]=-\frac{2\pi}{n}A_0(-1)^n+\sum_{m=1}^{\infty}(-1)^{m+n}A_m\frac{nh^3(m^2-3n^2h^2)}{m(m^2+n^2h^2)^2}\sinh\frac{m\pi}{h}. \tag{7}$$

However, (7) is in fact easily solved, since the A_m terms are in most cases very small compared with those in B_n. The B_n series is practically the sine series for $A_0 x$, slightly adjusted to allow for the A_m series.

The method of solution is to take, as a first approximation, values of B_n given by

$$B_n\left[\frac{h\pi}{\sinh nh\pi}+\frac{2}{n}\cosh nh\pi\right]=-\frac{2A_0(-1)^n}{n},$$

and substitute these values in (6) for a convenient number of B_n, to give the first approximation to A_m. These are all obtained as multiples of A_0, which is related to B_0 and hence to U by the first equation (6). These values substituted in (7) give the

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second approximation to B_n, and corresponding values of A_m, which here agreed so closely with the first that no further repetitions were necessary.

For example, with $h = \frac{1}{2}$, and with only four terms of each series the values were:

	1st approx.	2nd approx.
B_1	$0{\cdot}112\pi A_0$	$0{\cdot}112\pi A_0$
B_2	$-0{\cdot}0272\pi A_0$	$-0{\cdot}0265\pi A_0$
B_3	$0{\cdot}00572\pi A_0$	$0{\cdot}00585\pi A_0$
B_4	$-0{\cdot}00119\pi A_0$	$-0{\cdot}00108\pi A_0$

with $$\pi A_0 = -0{\cdot}446U.$$

4. *Results and discussion.* To compare this solution with Inglis's (3) theory and Coker's (2) experimental results, the case $h = \frac{1}{2}$ has been investigated and the shear stress calculated along the centre line $y = 0$. Only four terms were sufficient to give a result in close agreement with that obtained by Inglis with twenty-two terms of a different Fourier series. Both methods give a flat-topped curve for $\widehat{xy}$ with a maximum at the centre and steep slope to zero at the ends. The present solution lies between the curves of Inglis and Coker, in accordance with the different values of Poisson's ratio, which were $\frac{1}{3}$, 0·3 and 0·4 respectively. The maximum shear stresses (when Inglis's solution is expressed in terms of the displacement) are $0{\cdot}657\mu U$ (Inglis) and $0{\cdot}655\mu U$ (this solution).

When four terms only are used in (3), so that the summation is $\sum_{n=1}^{4}$ instead of $\sum_{n=1}^{\infty}$, the solution is not exact. The error occurs in two of the boundary conditions, namely v on $y = \pm h\pi$ and $\widehat{xy}$ on $x = \pm\pi$ as follows:

at $y = \frac{1}{2}\pi$,	$x = 0$	$0{\cdot}2\pi$	$0{\cdot}4\pi$	$0{\cdot}6\pi$	$0{\cdot}8\pi$	π
	$v/U = 0$	$-0{\cdot}007$	$0{\cdot}02$	$-0{\cdot}033$	$0{\cdot}067$	$-0{\cdot}38$

at $x = \pi$,	$y = 0$	$0{\cdot}1\pi$	$0{\cdot}2\pi$	$0{\cdot}3\pi$	$0{\cdot}4\pi$	$0{\cdot}5\pi$
	$\widehat{xy}/\mu U = -0{\cdot}004$	$0{\cdot}0071$	$-0{\cdot}0054$	$0{\cdot}0098$	$-0{\cdot}014$	$0{\cdot}086$

The large error in v at $x = \pm\pi$ is the usual boundary error in attempting to represent x by a series of sines, and, as Inglis points out, has little or no effect on the stresses elsewhere.

Values of the shear stress on the centre line $y = 0$ are:

$x = 0$	$0{\cdot}2\pi$	$0{\cdot}4\pi$	$0{\cdot}6\pi$	$0{\cdot}8\pi$	π
$\widehat{xy} = 0{\cdot}655$	$0{\cdot}655$	$0{\cdot}623$	$0{\cdot}538$	$0{\cdot}318$	$-0{\cdot}004 \times \mu U$

differing from Inglis's results by $0{\cdot}03\mu U$ at most.

Although Inglis gives values of the shear stress on the centre line to six figures, the boundary conditions are not satisfied to the same accuracy. As in the present solution, there are errors at the corners of the plate, and the harmonic approximation gives $\widehat{xy}$ varying between $\pm 0{\cdot}02\mu U$ all along the 'free' edges.

Calculations have also been made for the case $h = \frac{1}{4}$, i.e. a plate of length four times its breadth. In this case also four terms were sufficient to give good agreement with Inglis's more lengthy solution, namely, a maximum shear stress of $1{\cdot}34\mu U$ as compared with $1{\cdot}31\mu U$ (Inglis). This maximum occurs at two humps near the ends of the plate,

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as shown by Inglis, not at the centre. Although four terms sufficed to give the maximum value, it was necessary in this case to take six terms to eliminate a second pair of smaller humps due to the Fourier approximation.

This solution confirms the results of Inglis and Coker, which show that for purposes of design a plate loaded in this manner will be subjected to a shear stress only slightly above that expected for the corresponding case of uniform shear, the maximum being approximately 1·03 times the uniform value.

The justification of this paper lies not in the results but in the ease with which they were obtained. This is due to the form of the series chosen in (3) which have the property of simplifying to series of sines or cosines when substituted in the boundary conditions. There are no simultaneous equations to be solved, the coefficients B_n being multiples of the Fourier coefficients for x with a small correction, and the A_n known exact functions of the B_n.

Inglis's solution on the other hand involves two approximations by means of cosine series, the first to give a uniform shear on the free ends, and the second to compensate the non-uniform displacement due to the first. In the second case the function to be represented is given numerically, requiring tedious calculation of the Fourier coefficients. Each series affects the other and an iterative process is necessary, the first approximation involving ten and eleven terms of the series respectively, and the next approximation one and eleven more terms.

The results of recent experiments with bubble rafts by Mr Lomer in the Cavendish Laboratory agree qualitatively with the theoretical solution. The rafts were strained until failure occurred by the formation of dislocations, either singly from the boundary or in pairs within the raft. It was found that the first failure occurred consistently near the centre for h in the range $\frac{1}{2}$ to $\frac{2}{3}$ only, and that for smaller values of h dislocations started at or near the edges. Some of these were probably due to the humps in the graph of the shear stress, and others to stress concentrations at irregularities in the boundary.

Summary. A solution is given of a problem already solved by Inglis, a rectangular plate stressed by the uniform longitudinal relative displacement of two opposite edges, the other pair of sides remaining free. The method used here yields a more accurate solution with greater rapidity than did the previous paper. The general form of the solution is given for a plate of any dimensions, and a plate of sides in the ratio 2 : 1 is considered in more detail. An approximate solution is obtained by taking four terms of an infinite series, and the consequent errors discussed and compared with those of Inglis.

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