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MEAN TOPOLOGICAL DIMENSION OF A RANDOM DYNAMICAL SYSTEM FOR AMENABLE GROUPS

YU LIU AND XIAOJUN HUANG

College of Mathematics and Statistics, Chongqing University, Chongqing, China
Corresponding author: Xiaojun Huang, email: mathhxj@163.com

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Abstract We introduce the mean topological dimension of random bundle transformations associated with an infinite countable discrete amenable group action and show that continuous bundle random dynamical systems for amenable groups with finite fibre topological entropy have zero mean topological dimensions.

Keywords: bundle random dynamical system; mean topological dimension; metric mean dimension; amenable group

2020 Mathematics subject classification: 37A35, 37H05.

1. Introduction

Topological entropy is a fundamental concept in the theory of dynamical systems. Initially proposed by Adler, Konheim and McAndrew [2] as an invariant under topological conjugacy for the analysis of dynamical systems in compact topological spaces, it was later independently redefined in metric spaces by Bowen [9] and Dinaburg [17], providing a distinct yet equivalent characterization.

Shub and Weiss [51] introduced the concept of a small set, crucial in studying the small entropy factors of a topological dynamical system. Gromov [22] then introduced the mean topological dimension, an analog of the Lebesgue covering dimension, to investigate the dynamical properties of specific spaces, such as holomorphic maps and complex varieties. Lindenstrauss and Weiss [40] systematically explored the mean dimension of dynamical systems for $\mathbb{Z}$-actions, using it to refute an open question by Auslander [5] regarding the embeddability of every minimal system (X, T) into $[0,1]^{\mathbb{Z}}$. In [39], Lindenstrauss introduced a topological Rokhlin lemma and provided sufficient conditions for a positive resolution to the embedding problem for extensions of aperiodic minimal systems.

In the setting of random dynamical systems (RDSs), rather than considering iterations of just one map, we study the successive application of different transformations chosen at random. The basic framework was established by Ulam and von Neumann [53] and later by Kakutani [24] in proofs of the random ergodic theorem. Since the 1980s,

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mainly because of stochastic flows arising as solutions of stochastic differential equations, interest in the ergodic theory of random transformations has grown [3, 6–8, 14, 30–34, 38, 41–43, 57]. It was shown in [6] that the cornerstone for the entropy theory of random transformations is the Abramov–Rokhlin mixed entropy of the fibre of a skew-product transformation [1].

The theory of mean dimension has been developed in several related aspects. In the mean topological dimension theory of dynamical systems as studied in Ma [44], introducing the mean topological dimension for a continuous bundle RDS, which enables one to assign a quantity to systems with infinite-dimensional state spaces or infinite topological entropy.

The foundations of the theory of amenable group actions were set up in the pioneering paper [49] by Ornstein and Weiss and further developed by Rudolph and Weiss [50] and Danilenko [15]. For a delightful overview, refer to Benjy Weiss' survey article [56]. Global entropy theory for amenable group actions has also been discussed by Moulin Ollagnier [46]. Other related aspects were discussed in [16, 20, 21, 47, 48, 52, 55]. The connection between local entropy and combinatorial independence across orbits of sets in dynamical systems was studied systematically by Kerr and Li in [27, 28] for amenable group actions and in [29] for sofic group actions and has been discussed by Chung and Li in [12] for amenable group actions on compact groups by automorphisms.

Dooley and Zhang [18] extended the local theory of entropy to the setting of RDSs of countable amenable group actions. To date, most discussions of RDSs have concerned $\mathbb{R}$-actions, $\mathbb{Z}$-actions or $\mathbb{Z}^+$-actions. In slightly more precise terms, they made a systematic study of the local entropy theory of a continuous bundle RDS over an infinite countable discrete amenable group.

In view of the vigorous development of the RDSs for amenable groups, there is a natural question to ask:

Problem 1.1 *Whether the notion of mean topological dimension can be generalized to the setting of RDSs of countable amenable group actions?*

In the present paper, we follow [18, 44] and introduce the mean topological dimension of a continuous bundle RDS associated with an infinite countable discrete amenable group action, which enables one to assign a quantity to systems with infinite-dimensional state spaces or infinite topological entropy. We also define the lower metric mean dimension for a bundle RDS of countable amenable group actions, including the deterministic case when the probability measure is supported on a single point. Furthermore, we show that bundle RDSs of countable amenable group actions with finite fibre topological entropy have zero mean dimension.

The organization of this paper is as follows:

In § 2, we recall some background on continuous bundle RDSs associated with an infinite countable discrete amenable group action, covering dimension, and some basic results with respect to set-valued measurability, as well as introducing the notion of isomorphism for continuous bundle RDSs of countable amenable group actions. In § 3, we establish the measurability of $D(\alpha_F(\omega))$. In § 4, given a monotone sub-additive invariant family of random continuous functions, we introduce the concept of the mean topological dimension, discuss some basic properties and show that it is an isomorphism invariant.

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In § 5, we introduce the notion of the $\mathbb{S}$-lower metric mean dimension and point out that a finite fibre topological entropy implies that the mean topological dimension is zero.

2. Preliminaries

We review here the definitions of infinite countable discrete amenable group and in the process introduce some of the basic notation and terminology appearing throughout this paper.

Let G be an infinite countable discrete group and denote by e_G the identity of G. Denote by $\mathcal{F}_G$ the set of all non-empty finite subsets of G.

Definition 2.1. *G is called amenable if for each $K \in \mathcal{F}_G$ and any $\delta > 0$ there exists $F \in \mathcal{F}_G$ such that*

$$|F\Delta KF| < \delta|F|,$$

where $|\bullet|$ is the counting measure of the set $\bullet$, $KF = \{kf : k \in K, f \in F\}$ and $F\Delta KF = (F\backslash KF) \cup (KF\backslash F)$.

Definition 2.2. *A sequence $\{F_n : n \in \mathbb{N}\}$ in $\mathcal{F}_G$ is called a Følner sequence, if*

$$\lim_{n\to\infty} \frac{|gF_n\Delta F_n|}{|F_n|} = 0$$

for each $g \in G$.

It is not too hard to obtain the usual asymptotic invariance property from this, viz.: G is amenable if and only if G has a Følner sequence $\{F_n\}_{n\in\mathbb{N}}$. In the class of countable discrete groups, amenable groups include all solvable groups.

Throughout the current paper, we will assume that G is always an infinite countable discrete amenable group.

Let $(\Omega, \mathcal{F}, \mathbb{P}, G)$ be a complete probability space together with a group G of invertible $\mathbb{P}$-preserving transformations of $(\Omega, \mathcal{F}, \mathbb{P})$, where e_G acts as the identity transformation. Let X be a compact metric space with the distance function d and the Borel σ-algebra $\mathcal{B}(X)$. Let $\mathcal{E}$ be a measurable subset of $\Omega \times X$ with respect to the product σ-algebra $\mathcal{F} \otimes \mathcal{B}(X)$. The fibres $\mathcal{E}_\omega = \{x \in X : (\omega, x) \in \mathcal{E}\}, \omega \in \Omega$, are non-empty compact subsets of X.

A bundle RDS associated with $(\Omega, \mathcal{F}, \mathbb{P}, G)$ is a family

$$\mathbb{T} = \{T_{g,\omega} : \mathcal{E}_\omega \to \mathcal{E}_{g\omega} \mid g \in G, \omega \in \Omega\}$$

satisfying:

(1) for each $\omega \in \Omega$, the transformation $T_{e_G,\omega}$ is the identity over $\mathcal{E}_\omega$,

(2) for each $g \in G$, the map $(\mathcal{E}, (\mathcal{F} \times \mathcal{B}(X))_\mathcal{E}) \to (X, \mathcal{B}(X))$, given by $(\omega, x) \mapsto T_{g,\omega}(x)$, is measurable and

(3) for each $\omega \in \Omega$ and all $g_1, g_2 \in G, T_{g_2,g_1\omega} \circ T_{g_1,\omega} = T_{g_2g_1,\omega}$ (and so $T_{g^{-1},\omega} = \left(T_{g,g^{-1}\omega}\right)^{-1}$ for each $g \in G$).

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In this case, G has a natural measurable action on $\mathcal{E}$ with $\Theta_g(\omega, x) = (g\omega, T_{g,\omega}x)$ for each $g \in G$, called the corresponding skew product transformation.

If $T_{g,\omega}$ is a continuous map for each $g \in G$ (and so $T_{g,\omega} : \mathcal{E}_\omega \to \mathcal{E}_{g\omega}$ is a homeomorphism for $\mathbb{P}$-a.e. $\omega \in \Omega$ and each $g \in G$), then it is called a continuous bundle RDS.

By [11, Chapter III], the mapping $\omega \mapsto \mathcal{E}_\omega$ is measurable with respect to the Borel σ-algebra induced by the Hausdorff topology on the hyperspace 2^{X} of all non-empty compact subsets of X, and the distance function $d(x, \mathcal{E}_\omega)$ is measurable in $\omega \in \Omega$ for each $x \in X$.

These concepts generalize the classical concepts of dynamical systems as follows. We say that $(\Omega, \mathcal{F}, \mathbb{P}, G)$ is a trivial MDS, if Ω is a singleton.

For a compact metric space X, a continuous bundle RDS associated with a trivial measure dynamical system (MDS) means that there is a topological G-action (K, G) for some non-empty compact subset $K \subseteq X$, that is, the group G acts on K in the sense that there exists a family of homeomorphisms $\{F_g : g \in G\}$ of K such that $F_{g_2} \circ F_{g_1} = F_{g_2 g_1}$ for all $g_1, g_2 \in G$ and F_{e_G} acts as the identity transformation over K. The pair (K, G) is also called a topological dynamical G-system.

Definition 2.3. *A closed random set A is a measurable set-valued map A from $(\Omega, \mathcal{F})$ to $\left(2^X, \mathcal{B}\left(2^X\right)\right)$. An open random set U is a set-valued map U such that $\omega \mapsto X \backslash U(\omega)$ is a closed random set (see [3, Definition 1.6.1]).*

Let $\mathrm{C}^o_{\mathcal{E}}$ be the set of all finite covers of $\mathcal{E}$ consisting of subsets of open random sets [3, Definition 1.6.1]. Similarly, we denote by C^o_K the set of finite open covers of $K \subset X$. Denote by $\mathfrak{C}^o_{\mathcal{E}}$ the set of all countable open random covers α of $\mathcal{E}$ with $\alpha(\omega) = \{A(\omega) : A \in \alpha\} \in \mathrm{C}^o_{\mathcal{E}_\omega}$ for $\mathbb{P}$-a.a. ω, which have an increasing sequence $\{\Omega_1 \subset \Omega_2 \subset \cdots\} \subset \mathcal{F}$ such that $\mathbb{P}(\Omega_n) \to 1$ and $\alpha \cap (\Omega_n \times X)$ is a finite family for each $n \geq 1$ [19, Definition 11.1].

Suppose that $\alpha = \left\{A^{(i)} : i \in I\right\} \in \mathfrak{C}^o_{\mathcal{E}}$ and $F = \{g_1, ..., g_n\} \in \mathcal{F}_G$. Let

$$\alpha_F(\omega) = \bigvee_{i=1}^{n} \left(T_{g_i,\omega}\right)^{-1} \alpha\left(g_i\omega\right).$$

Then $\alpha_F(\omega)$ is a finite open cover of $\mathcal{E}_\omega$ for $\mathbb{P}$-a.a. ω consisting of sets

$$A^{(j_1,j_2,\ldots,j_n)}(\omega) = \bigcap_{i=1}^{n} \left(T_{g_i,\omega}\right)^{-1} A^{(j_i)}\left(g_i\omega\right), \quad j_i \in I,$$

(some of which may be empty).

By the measurability of Θ_{g_i}, for every $g_i \in F$, it is easy to prove that, for any $j = (j_1, j_2, \ldots, j_n) \in I^n$, the graph of $\omega \mapsto A^j(\omega)$, denoted by $A^j = \left\{(\omega, x) : x \in A^j(\omega)\right\}$, is measurable. In fact, $A^j = \bigcap_{i=1}^n \left(\Theta_{g_i}\right)^{-1} A^{(j_i)}$.

Let $K \subset X$ be compact and let α be a finite open cover of K. Write [40]

$$\mathrm{ord}(\alpha) = -1 + \sup_{x \in K} \sum_{U \in \alpha} 1_U(x) \quad \text{and} \quad D(\alpha) = \min_{\beta \succ \alpha} \mathrm{ord}(\beta),$$

where β runs over all finite open covers of K refining α.

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For each $\omega \in \Omega$, write

$$D(\alpha_F(\omega)) = -1 + \min_{\beta} \sup_{x \in \mathcal{E}_\omega} \sum_{U \in \beta} 1_U(x),$$

where β runs over all finite open covers of $\mathcal{E}_\omega$ refining $\alpha_F(\omega)$. We will discuss the measurability and the subadditivity of the function $D(\alpha_F(\omega))$ in the next section. We need the following standard facts, which will be used often in this paper.

Lemma 2.4. *Let $\pi_\Omega : \Omega \times X \to \Omega$ be the natural projection and $Q \in \mathcal{F} \otimes \mathcal{B}(X)$. Then $\pi_\Omega Q \in \mathcal{F}$ and there exists a measurable function $q : \pi_\Omega Q \to X$ such that $(\omega, q(\omega)) \in Q$ for each $\omega \in \pi_\Omega Q$.*

Lemma 2.5. *Let Γ be a set-valued map from a measurable space $(\mathcal{T}, \mathscr{C})$ to compact subsets of X. Consider the following properties.*

(i) $\{t \in \mathcal{T} : \Gamma(t) = \emptyset\} \in \mathscr{C}$ *and* $\{t \in \mathcal{T} : \Gamma(t) \cap U \neq \emptyset\} \in \mathscr{C}$ *for any open* $U \subset X$.
(ii) Γ *is a measurable map from* $(\mathcal{T}, \mathscr{C})$ *to* $\left(2^X, \mathcal{B}\left(2^X\right)\right)$.
(iii) *The graph of* Γ *is* $\mathscr{C} \otimes \mathcal{B}(X)$ *measurable.*

Then (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii). Moreover, if $(\mathcal{T}, \mathscr{C}, m)$ is a complete probability space, then (iii) $\Rightarrow$ (i).

These results can be found in [11, Chapter III]. Lemma 2.4 is a combination of Theorems III.22 and III.23. Lemma 2.5 is from Definition 10, Proposition III.13 and a trivial application of Lemma 2.4; see also [14, Proposition 2.4]. We remark that the ω-section of G in Lemma 2.4 is not required to be a non-empty closed subset of X, which is a little different from the condition of the classical Kuratowski–Ryll–Nardzewski measurable selection theorem (see [11, Theorem III.8] or [26, Theorem 12.13]).

We now give the definition of the isomorphism in the category of continuous bundle RDSs. For simplicity, we only consider the case of two continuous bundle RDSs over the same measure-preserving group G. For the general case, see Arnold [3, 1.9].

Definition 2.6. *Let T, S be two continuous bundle RDSs with state space X, Y over $(\Omega, \mathcal{F}, \mathbb{P}, G)$ on $\mathcal{E}$ and $\mathcal{H}$, respectively, where X, Y are compact metric spaces. T is called isomorphic to S if there exists a family of homeomorphisms $\psi_\omega : \mathcal{E}_\omega \to \mathcal{H}_\omega$ such that $(\omega, x) \mapsto \psi_\omega(x)$ is measurable and ψ_ω preserves the random transformation, i.e. $S_{g,\omega} \circ \psi_\omega = \psi_{g\omega} \circ T_{g,\omega}$ (the following diagram commutates for each $g \in G$).*

$$\begin{array}{ccc} \mathcal{E}_\omega & \xrightarrow{T_{g,\omega}} & \mathcal{E}_{g\omega} \\ \downarrow{\scriptstyle \psi_\omega} & & \downarrow{\scriptstyle \psi_{g\omega}} \\ \mathcal{H}_\omega & \xrightarrow[S_{g,\omega}]{} & \mathcal{H}_{g\omega} \end{array}$$

Let $\Psi : \mathcal{E} \to \mathcal{H}$ denote $\Psi(\omega, x) = (\omega, \psi_\omega x)$. Denote by Θ_g, Λ_g the skew-product transformations of T, S for any $g \in G$, respectively. We also say that Θ_g is isomorphic to Λ_g via Ψ for any $g \in G$.

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In this definition, we do not require the measurability of $(\omega, y) \mapsto \psi_\omega^{-1}(y)$. In fact, Ψ is automatically bimeasurable by our assumption. Ψ is measurable, which can be easily obtained from the measurability of the map $(\omega, x) \mapsto \psi_\omega(x)$. In the other direction, set

$$H = \{(\omega, x, \psi_\omega(x)) \in \Omega \times X \times Y : (\omega, x) \in \mathcal{E}\}.$$

Then H is $\mathcal{F} \otimes \mathcal{B}(X) \otimes \mathcal{B}(Y)$ measurable by Lemma 2.5 as the graph of $(\omega, x) \mapsto \psi_\omega(x)$. Let π_Ω be the natural projection from $\Omega \times X \times Y$ onto Ω. For any $y \in Y, E \in \mathcal{B}(X)$,

$$\left\{\omega \in \Omega : \psi_\omega^{-1}(y) \in E\right\} = \pi_\Omega(H \cap (\Omega \times E \times \{y\}))$$

is $\mathcal{F}$ measurable by Lemma 2.4. So $\psi_\omega^{-1}(y)$ is measurable in ω. Note that ψ_ω is a homeomorphism, so $\psi_\omega^{-1}(y)$ is continuous in Y. It follows that $(\omega, y) \mapsto \psi_\omega^{-1}(y)$ is a Carathéodory map and hence is measurable [4, Definition 8.2.7, Lemma 8.2.6].

3. The function $D(\alpha_F(\omega))$

Recall that two families $(E_i)_{i\in I}$ and $(F_i)_{i\in I}$, with common index set I, are called combinatorially equivalent if

$$\bigcap_{i\in J} E_i \neq \emptyset \Longleftrightarrow \bigcap_{i\in J} F_i \neq \emptyset$$

for every subset $J \subset I$.

Remark 3.1. If α and β are families of finite open covers of a compact set K that are combinatorially equivalent, then $\mathrm{ord}(\alpha) = \mathrm{ord}(\beta)$.

Proof. Let $\mathrm{ord}(\alpha) = n - 1$, $\mathrm{ord}(\beta) = m - 1$. Assume $n < m$, then there exits $B_1, B_2, ..., B_m$, such that $\bigcap_{i=1}^m B_i \neq \emptyset$. Since α and β are combinatorially equivalent, then $\bigcap_{i=1}^m A_i \neq \emptyset$, that is $-1 + \sup_{x\in K} \sum_{A\in\alpha} 1_A(x) = n - 1 \neq m - 1$. It's a contradiction. □

The proof of the measurability of $D(\alpha_F(\omega))$ relies on the following lemmas [13, § 1.6].

Lemma 3.2. *Let K be a normal space. Let $(F_i)_{i\in I}$ be a finite family of closed subsets of K and $(U_i)_{i\in I}$ a family of open subsets of K such that $F_i \subset U_i$ for all $i \in I$. Then there exists a family $(V_i)_{i\in I}$ of open subsets of K satisfying the following conditions.*

(i) $F_i \subset V_i \subset \overline{V_i} \subset U_i$ *for all* $i \in I$.
(ii) The families $(F_i)_{i\in I}, (V_i)_{i\in I}$ *and* $\left(\overline{V_i}\right)_{i\in I}$ *are combinatorially equivalent.*

Lemma 3.3. *Let K be a normal space and let α be a finite open cover of K. Then*

$$D(\alpha) = \min_\gamma \mathrm{ord}(\gamma),$$

where γ runs over all finite closed covers of K that are finer than α.

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The proof of the following theorem comes from [44], but with some additions here.

Theorem 3.4 *For every $F \in \mathcal{F}_G$ and $\alpha \in \mathfrak{C}_{\mathcal{E}}^o$, the function $\omega \mapsto D(\alpha_F(\omega))$ is measurable.*

Proof. Let $\alpha = \{A^{(i)}\} \in \mathrm{C}_{\mathcal{E}}^o$, $F = \{g_1, g_2, ..., g_n\}$. J is an index set with $|J| = |\alpha|^n$. Define $f : 2^X \times \left(2^X\right)^{|J|} \to \{-1\} \cup \mathbb{N}$ by

$$f\left(K, \left(F^j\right)_{j\in J}\right) = -1 + \min_{\beta} \sup_{x\in K} \sum_{B\in\beta} 1_B(x),$$

where β runs over all families of finite open covers of K such that

$$\beta \succ \left(K\backslash F^j\right)_{j\in J} \quad \text{and} \quad K = \bigcup_{B\in\beta} B.$$

For $K = \emptyset$ or $\beta \in \emptyset$, we set $f = -1$. Obviously,

$$D(\alpha_F(\omega)) = f\left(\mathcal{E}_\omega, \left(\mathcal{E}_\omega\backslash A^j(\omega)\right)_{j\in J}\right) \geq 0.$$

Note that $\omega \mapsto \mathcal{E}_\omega$ is measurable and, for every $j \in J$, $\left\{(\omega, x) : x \in \mathcal{E}\backslash A^j(\omega)\right\} = \mathcal{E}\backslash A^j$, the graph of the multifunction $\omega \mapsto \mathcal{E}_\omega\backslash A^j(\omega)$ is measurable. In fact,

$$\mathcal{E}\backslash A^j = \mathcal{E}\backslash\left(\bigcap_{i=1}^n \left(\Theta g_i\right)^{-1} A^{(j_i)}\right) = \bigcup_{i=1}^n \left(\mathcal{E}\backslash\left(\left(\Theta g_i\right)^{-1} A^{(j_i)}\right)\right).$$

By the measurability of Θ_{g_i} and $A^{(j_i)} \in \alpha$, the graph of $\omega \to \mathcal{E}_\omega\backslash A^{j_i}(\omega)$, denoted by $\mathcal{E}\backslash A^{(j_i)}$, is measurable.

By Lemma 2.5, $\omega \mapsto \mathcal{E}_\omega\backslash A^j(\omega)$ is measurable for any $j \in J$. Recalling that the composition of two measurable functions is measurable, it suffices to prove that

$$\mathscr{Q}_q = \left\{\left(K, \left(F^j\right)_{j\in J}\right) : 0 \leq f\left(K, \left(F^j\right)_{j\in J}\right) \leq q\right\}$$

is measurable with respect to the product σ-algebra $\mathcal{B}\left(2^X\right) \otimes \mathcal{B}\left(2^X\right)^{\otimes|J|}$ for any $q \in \mathbb{N}$. In fact, $\mathscr{Q}_q$ is open in $2^X \times \left(2^X\right)^{|J|}$.

For any $\left(K_0, \left(F_0^j\right)_{j\in J}\right) \in \mathscr{Q}_q$, then $K_0 \neq \emptyset$, and there exists $\beta \in \mathrm{C}_{K_0}^o$, such that

$$\beta \succ \left(K_0\backslash F_0^j\right)_{j\in J}.$$

This implies that $\left(K_0\backslash F_0^j\right)_{j\in J}$ itself is a finite open covering on K_0 and has $D\left(\left(K_0\backslash F_0^j\right)_{j\in J}\right) \leq q$. By Lemma 3.3, there exists a family of finite closed subsets $\left(E^i\right)_{i\in I}$ of K_0 and a map $\varphi : I \to J$ such that

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$$\begin{cases} K_0 = \bigcup_{i\in I} E^i; \\ E^i \subset K_0 \backslash F_0^{\varphi(i)}, & \text{for all } i \in I; \\ \sum_{i\in I} 1_{E^i}(x) \le q+1, & \text{for all } x \in K_0. \end{cases}$$

Applying Lemma 3.2 to $E^i \subset K_0\backslash F_0^{\varphi(i)}, i \in I$, there exists a family of open subsets $\left(V^i\right)_{i\in I}$ of K_0 such that

$$E^i \subset V^i \subset \overline{V^i} \subset K_0\backslash F_0^{\varphi(i)} \quad \text{for all } i \in I \tag{3.1}$$

and $\left(E^i\right)_{i\in I}, \left(V^i\right)_{i\in I}$ and $\left(\overline{V^i}\right)_{i\in I}$ are combinatorially equivalent.

$$\sup_{x\in K_0} \sum_{i\in I} 1_{\overline{V}^i}(x) = \sup_{x\in K_0} \sum_{i\in I} 1_{V^i}(x) = \sup_{x\in K_0} \sum_{i\in I} 1_{E^i}(x) \le q+1. \tag{3.2}$$

Recall that, by Kuratowski [37, §17], [36, §42], the family of all sets $\{F : F \subset G\}$ and of all sets $\{F : F \cap G \neq \emptyset\}$, where G runs over all open subsets of X, is an open subbase for the Hausdorff metric topology, which coincides with the Vietoris topology on 2^X [35, Corollary 4.2.3]. Set

$$\mathscr{K}_1 = \left\{ K \in 2^X : K \subset \bigcup_{i\in I} V^i \right\},$$

$$\mathscr{K}_2 = \bigcap_{\substack{\Gamma\subset I \\ |\Gamma|>q+1}} \left\{ K \in 2^X : K \cap \bigcap_{i\in\Gamma} \overline{V^i} = \emptyset \right\}$$

and, for $j \in J$, set

$$\mathscr{F}_j = \left\{ F \in 2^X : F \subset \bigcap_{i\in\varphi^{-1}\{j\}} \left(X\backslash\overline{V^i}\right) \right\}.$$

Hence $\mathscr{K}_1, \mathscr{K}_2$ and $\mathscr{F}_j$ are all open in 2^{X}.

$$\mathscr{O} = (\mathscr{K}_1 \cap \mathscr{K}_2) \times \prod_{j\in J} \mathscr{F}_j$$

is open in $2^X \times \left(2^X\right)^{|J|}$, so that $\left(K_0, \left(F_0^j\right)_{j\in J}\right) \in \mathscr{O} \subset \mathscr{Q}_q$.

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Indeed, $\left(K_0, \left(F_0^j\right)_{j\in J}\right) \in \mathscr{O}$. For any $\Gamma \subset I$ with $|\Gamma| > q+1$, if $K_0 \cap \bigcap_{i\in\Gamma} \overline{V^i} \neq \emptyset$, then there exists $y \in K_0$ and $y \in \bigcap_{i\in\Gamma} \overline{V^i}$. By $y \in K_0 = \bigcup_{i\in I} E^i$ and (3.2) then $\sum_{i\in I} 1_{\overline{V}^i}(y) \leq q+1$. By $y \in \bigcap_{i\in\Gamma} \overline{V^i}$, then

$$\sum_{i\in I} 1_{\overline{V^i}}(y) \geq \sum_{i\in\Gamma} 1_{\overline{V^i}}(y) \geq |\Gamma| > q+1,$$

a contradiction.

Combining $K_0 = \bigcup_{i\in I} E^i \subset \bigcup_{i\in I} V^i$, we know that $K_0 \in \mathscr{K}_1 \cap \mathscr{K}_2$.

From (3.1), for any $i \in I$, $F_0^{\varphi(i)} \subset X\backslash\overline{V}^i$. Then for $j = \varphi(i)$,

$$F_0^j \subset \bigcap_{i\in\varphi^{-1}\{j\}} (X\backslash\overline{V}^i), \quad \text{i.e.} \quad \left(F_0^j\right)_{j\in J} \in \prod_{j\in J} \mathscr{F}_j.$$

Hence, $\left(K_0, \left(F_0^j\right)_{j\in J}\right) \in \mathscr{O}$.

Next, for any $\left(K, \left(F^j\right)_{j\in J}\right) \in \mathscr{O}$, we want to get $\left(K, \left(F^j\right)_{j\in J}\right) \in \mathscr{Q}_q$. By $K \in \mathscr{K}_1 \bigcap \mathscr{K}_2$, then

$$K \subset \bigcup_{i\in I} V^i \subset \bigcup_{i\in I} \overline{V}^i \quad \text{and} \quad \sup_{x\in K} \sum_{i\in I} \overline{V}^i \leq q+1.$$

It follows from $(F_j)_{j\in J} \in \prod_{j\in J} \mathscr{F}_j$ that for any $j \in J$,

$$F^j \subset \bigcap_{i\in\varphi^{-1}\{j\}} \left(X\backslash\overline{V}^i\right),$$

so we get for any $j \in J$, there exists $i \in \varphi^{-1}\{j\}$ such that $K \bigcap \overline{V}^i \subset K\backslash F^j$, that is,

$$\left(K \cap \overline{V}^i\right)_{i\in I} \succ (K\backslash F^j)_{j\in J}.$$

By Lemma 3.3 again, this implies that

$$f\left(K, \left(F^j\right)_{j\in J}\right) = -1 + \min_{\beta} \sup_{x\in K} \sum_{B\in\beta} 1_B(x) \leq -1 + \sup_{x\in K} \sum_{i\in I} 1_{\overline{V}^i}(x) \leq q.$$

Hence, $\left(K, \left(F^j\right)_{j\in J}\right) \in \mathscr{Q}_q$.

Since $\left(K_0, \left(F_0^j\right)_{j\in J}\right)$ is arbitrary, $\mathscr{Q}_q$ is open in $2^X \times \left(2^X\right)^{|J|}$.

To prove that the open set $\mathscr{Q}_q$ is measurable, we simply observe that (X, d) is a compact metric space, so [36, § 42, V], [45] 2^X is also a compact metric space (and hence separable) with the Hausdorff metric ($\emptyset$ is an isolated point in 2^X).

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Hence [25, Lemma 1.2] the product σ-algebra coincides with the σ-algebra generated by the product topology

$$\mathcal{B}\left(2^X\right) \otimes \mathcal{B}\left(2^X\right)^{\otimes |J|} = \mathcal{B}\left(2^X \times \left(2^X\right)^{|J|}\right).$$

It follows that the open set $\mathscr{Q}_q$ is $\mathcal{B}\left(2^X\right) \otimes \mathcal{B}\left(2^X\right)^{\otimes |J|}$ measurable.

Let $\alpha \in \mathfrak{C}_{\mathcal{E}}^o$ and $\{\Omega_1 \subset \Omega_2 \subset \cdots\} \subset \mathcal{F}$ be an increasing sequence in the definition of $\mathfrak{C}_{\mathcal{E}}^o$. Set

$$\alpha^r = [\alpha \cap (\Omega_r \times X)] \bigcup [(\Omega_r^c \times X) \cap \mathcal{E}].$$

It is obvious that $D\left(\alpha_F(\omega)\right) = D\left(\alpha_F^r(\omega)\right)$ for any $\omega \in \bigcap_{i=1}^n \left(g_i\right)^{-1} \Omega_r$. Note that

$$\bigcap_{i=1}^n \left(g_i\right)^{-1} \Omega_r = \Omega \setminus \bigcup_{i=1}^n \{\omega : g_i\omega \notin \Omega_r\},$$

while $\mathbb{P}\{\omega : g_i\omega \notin \Omega_r\} = 1 - \mathbb{P}(\Omega_r) \to 0$. Hence for large enough r, $\mathbb{P}$-a.e. ω,

$$D(\alpha_F(\omega)) = D(\alpha_F^r(\omega)).$$

The proof of this theorem is complete. □

Corollary 3.5. *For any finite random cover $\alpha \in \mathrm{C}_{\mathcal{E}}^o, D(\alpha(\omega))$ is integrable.*

Proof. Assume that $|\alpha| = l$.

$$D(\alpha(\omega)) \leq \mathrm{ord}(\alpha(\omega)) \leq l - 1 \Longrightarrow \int D(\alpha(\omega)) d\mathbb{P}(\omega) \leq l - 1.$$

□

4. Mean topological dimension

Before introducing the notion of mean topological dimension for a continuous bundle RDS, we first review some properties of $D(\alpha)$. These lemmas can be found in [40] and [13]. In this chapter, we will assume that G is always an infinite countable discrete amenable group admitting a tiling Fø lner sequence.

Lemma 4.1. *Let X be a topological space. Let α and β be finite open covers of X such that $\alpha \succ \beta$. Then $D(\alpha) \geq D(\beta)$.*

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Lemma 4.2. *Let X and Y be topological spaces and let $f : X \to Y$ be a continuous map. Let β be a finite open cover of Y. Then*

$$D\left(f^{-1}(\beta)\right) \leq D(\beta).$$

Lemma 4.3. *Let X be a normal space. Let α and β be finite open covers of X. Then*

$$D(\alpha \vee \beta) \leq D(\alpha) + D(\beta).$$

Let X and Y be topological spaces. Let α be a finite open cover of X. A continuous map $f : X \to Y$ is said to be α-compatible if there exists a finite open cover β of Y such that $f^{-1}(\beta) \succ \alpha$. We will use the notation $f \succ \alpha$ to denote that f is α-compatible.

Lemma 4.4. *Let X be a compact space and Y be a topological space. Let $f : X \to Y$ be a continuous function such that for every $y \in Y, f^{-1}\{y\}$ is a subset of some $U \in \alpha$. Then* f *is α-compatible.*

Lemma 4.5. *Let X be a topological space and let α be a finite open cover of X. Suppose that there is a topological space Y and an α-compatible continuous map $f : X \to Y$. Then $D(\alpha) \leq \dim Y$.*

The following lemma [3, Theorem 3.3.2] can be seen as a random version of Fekete's subadditive lemma.

Lemma 4.6. (Kingman's subadditive ergodic theorem) *Let ϑ be a measure-preserving transformation on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $(f_n)_{n\geq 1}$ be a sequence of non-negative random variables such that*

$$f_{n+m}(\omega) \leq f_n(\omega) + f_m\left(\vartheta^n \omega\right).$$

If $f_n \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ for all n*, then there exists an integrable function $f : \Omega \to \mathbb{R}$ such that, for $\mathbb{P}$-a.a. $\omega, n^{-1} f_n(\omega) \to f(\omega)$. Moreover, the convergence also holds in $\mathbb{L}^1$ and satisfies*

$$\int_\Omega f(\omega) d\mathbb{P}(\omega) = \lim_{n\to\infty} \frac{1}{n} \int_\Omega f_n(\omega) d\mathbb{P}(\omega) = \inf_{n\geq 1} \frac{1}{n} \int_\Omega f_n(\omega) d\mathbb{P}(\omega) < \infty.$$

If, in addition, ϑ is ergodic, then $f(\omega) = \int f d\mathbb{P}$ for $\mathbb{P}$-a.a. ω.

In contrast to the amenable group $\mathbb{Z}$, a general infinite countable discrete amenable group may have a complicated combinatorial structure.

Let $f : \mathcal{F}_G \to \mathbb{R}$ be a function. Following [23], we say that f is:

(1) monotone, if $f(E) \leq f(F)$ for any $E, F \in \mathcal{F}_G$ satisfying $E \subseteq F$;
(2) non-negative, if $f(F) \geq 0$ for any $F \in \mathcal{F}_G$;
(3) G-invariant, if $f(Fg) = f(F)$ for any $F \in \mathcal{F}_G$ and $g \in G$;
(4) sub-additive, if $f(E \cup F) \leq f(E) + f(F)$ for any $E, F \in \mathcal{F}_G$ satisfying $E \cap F = \emptyset$.

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The following convergence property is well known (see, for example, [23, Lemma 2.4] or [40, Theorem 6.1]).

Proposition 4.7. *Let $f : \mathcal{F}_G \to \mathbb{R}$ be a monotone non-negative* G*-invariant sub-additive function. Then for any Fø lner sequence $\{F_n : n \in \mathbb{N}\}$ of* G*, the sequence $\left\{\frac{f(F_n)}{|F_n|} : n \in \mathbb{N}\right\}$ converges and the value of the limit is independent of the choice of the Fø lner sequence $\{F_n : n \in \mathbb{N}\}$.*

For a function f as in Proposition 4.7, in general, we cannot conclude that the limit of the sequence $\left\{\frac{f(F_n)}{|F_n|} : n \in \mathbb{N}\right\}$ is its infimum. This is shown by [19, Example 2.7].

Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\}$ be a family of functions in $\mathbb{L}^1(Y, \mathcal{D}, \nu)$. We make some natural assumptions on the family $\mathbf{D}$ which will enable us to state analogues of Theorem 4.6.

We say that $\mathbf{D}$ is:

(1) non-negative if: each element from $\mathbf{D}$ is non-negative;
(2) G-invariant if: $d_{Fg}(y) = d_F(gy)$ for ν-a.e. $y \in Y$ and all $F \in \mathcal{F}_G, g \in G$;
(3) G-bi-invariant if: it is G-invariant, and in addition $d_{gF}(y) = d_{Fg}(y)$ for ν-a.e. $y \in Y$ and all $F \in \mathcal{F}_G, g \in G$;
(4) monotone if: $d_E(y) \leq d_F(y)$ for ν-a.e. $y \in Y$ and all $\emptyset \neq E \subseteq F \in \mathcal{F}_G$;
(5) sub-additive (sup-additive, respectively) if: $d_{E\cup F}(y) \leq d_E(y)+d_F(y)$ for ν-a.e. $y \in Y$ and all disjoint $E, F \in \mathcal{F}_G$ (-D is sub-additive, respectively);
(6) strongly sub-additive (strongly sup-additive, respectively) if $d_{E\cap F}(y) + d_{E\cup F}(y) \leq d_E(y) + d_F(y)$ for ν-a.e. $y \in Y$ and all $E, F \in \mathcal{F}_G$, here by convention we set $d_\emptyset(y) = 0$ for each $y \in Y$ (-D is strongly sub-additive, respectively).

Proposition 4.8. *Let $\mathbf{D} = \{d_F : F \in \mathcal{F}_G\} \subseteq \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ be a sub-additive* G*-invariant family and $\mathbb{P}$ be a* G*-invariant probability measure on Ω. Then, for the function*

$$f : \mathcal{F}_G \to \mathbb{R}, F \mapsto \int_\Omega d_F(\omega) d\mathbb{P}(\omega),$$

$f(Eg) = f(E)$ and $f(E \cup F) \leq f(E) + f(F)$ whenever $g \in G$ and $E, F \in \mathcal{F}_G$ satisfy $E \cap F = \emptyset$. Moreover, if $\boldsymbol{D}$ is monotone then $\boldsymbol{D}$ is non-negative, and so f *is a monotone non-negative sub-additive* G*-invariant function.*

Let $\alpha \in \mathfrak{C}^o_{\mathcal{E}}$ and $\mathbf{D} = \{D(\alpha_F(\omega)) : F \in \mathcal{F}_G\}$ be a family functions on $(\Omega, \mathcal{F}, \mathbb{P})$. By Lemmas 4.2 and 4.3, for $\mathbb{P}$-*a.e.* $\omega \in \Omega$ and $E, F \in \mathcal{F}_G$ with $E \cap F = \emptyset$,

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$$\begin{aligned} D\left(\alpha_{E\cup F}(\omega)\right) &= D\left(\bigvee_{g\in E\cup F}\left(T_{g,\omega}\right)^{-1}\alpha\left(g\omega\right)\right) \\ &= D\left(\bigvee_{g\in E}\left(T_{g,\omega}\right)^{-1}\alpha\left(g\omega\right)\vee\bigvee_{g\in F}\left(T_{g,\omega}\right)^{-1}\alpha\left(g\omega\right)\right) \\ &\leq D\left(\bigvee_{g\in E}\left(T_{g,\omega}\right)^{-1}\alpha\left(g\omega\right)\right)+D\left(\bigvee_{g\in F}\left(T_{g,\omega}\right)^{-1}\alpha\left(g\omega\right)\right) \\ &= D\left(\alpha_E(\omega)\right)+D\left(\alpha_F(\omega)\right). \end{aligned}$$

For $\mathbb{P}$-*a.e.* $\omega\in\Omega$ and all $F\in\mathcal{F}_G$, $g\in G$,

$$\begin{aligned} D\left(\alpha_{Eg}(\omega)\right) &= D\left(\bigvee_{h\in Eg}\left(T_{h,\omega}\right)^{-1}\alpha\left(h\omega\right)\right)=D\left(\bigvee_{h\in E}\left(T_{hg,\omega}\right)^{-1}\alpha\left(hg\omega\right)\right) \\ &= D\left(\bigvee_{h\in E}\left(T_{g,\omega}\right)^{-1}\circ\left(T_{h,g\omega}\right)^{-1}\alpha\left(h(g\omega)\right)\right) \\ &\leq D\left(\bigvee_{h\in E}\left(T_{h,g\omega}\right)^{-1}\alpha\left(h(g\omega)\right)\right) \\ &= D\left(\alpha_E(g\omega)\right). \end{aligned}$$

Similarly,

$$\begin{aligned} D\left(\alpha_E(g\omega)\right) &= D\left(\bigvee_{h\in E}\left(T_{h,g\omega}\right)^{-1}\alpha\left(h(g\omega)\right)\right)=D\left(\bigvee_{hg^{-1}\in E}\left(T_{hg^{-1},g\omega}\right)^{-1}\alpha\left(h\omega\right)\right) \\ &= D\left(\bigvee_{hg^{-1}\in E}\left(T_{h,\omega}\circ T_{g^{-1},g\omega}\right)^{-1}\alpha\left(h\omega\right)\right) \\ &\leq D\left(\alpha_{Eg}(\omega)\right). \end{aligned}$$

Thus,

$$D(\alpha_{Eg}(\omega))=D(\alpha_E(g\omega)).$$

For all $\emptyset\neq E\subseteq F\in\mathcal{F}_G$, then $D\left(\alpha_E(\omega)\right)\leq D\left(\alpha_F(\omega)\right)$ by $\alpha_F(\omega)\succ\alpha_E(\omega)$ for $\mathbb{P}$-a.e. $\omega\in\Omega$.

Therefore, $\mathbf{D}=\{D(\alpha_F(\omega)):F\in\mathcal{F}_G\}$ is monotone non-negative subadditive and G-invariant function.

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Set

$$\mathcal{D}=\left\{\alpha\in\mathfrak{C}_{\mathcal{E}}^{o}:\int D(\alpha(\omega))d\mathbb{P}(\omega)<\infty\right\}.$$

By Corollary 3.5, $\mathrm{C}_{\mathcal{E}}^{o}\subset\mathcal{D}$. Note also that by Lemma 4.3, if $\alpha,\beta\in\mathcal{D}$, then $\alpha\vee\beta\in\mathcal{D}$. Thus $\mathcal{D}$ is a directed set.

Suppose $\alpha\in\mathcal{D}$, for any $F\in\mathcal{F}_G$, let $F=\{g_1,g_2,...,g_n\}$.

$$\begin{aligned}D\left(\alpha_F(\omega)\right)&=D\left(\bigvee_{g\in F}\left(T_{g,\omega}\right)^{-1}\alpha\left(g\omega\right)\right)\\&=D\left(\left(T_{g_1,\omega}\right)^{-1}\alpha\left(g_1\omega\right)\vee\left(T_{g_2,\omega}\right)^{-1}\alpha\left(g_2\omega\right)\vee\cdots\vee\left(T_{g_n,\omega}\right)^{-1}\alpha\left(g_n\omega\right)\right)\\&\leq D\left(\alpha(g_1\omega)\right)+D\left(\alpha(g_2\omega)\right)+\cdots+D\left(\alpha(g_n\omega)\right).\end{aligned}$$

Since for $i=1,2,...,n$,

$$\int D\left(\alpha(g_i\omega)\right)d\mathbb{P}(\omega)=\int D(\alpha(\omega))d\mathbb{P}(g_i^{-1}\omega)=\int D\left(\alpha(\omega)\right)d\mathbb{P}(\omega),$$

that is,

$$\int D\left(\alpha_F(\omega)\right)d\mathbb{P}(\omega)\leq n\int D(\alpha(\omega))d\mathbb{P}(\omega).$$

Then $D\left(\alpha_F(\omega)\right)$ is integrable for every $F\in\mathcal{F}_G$.

Now write $D(\alpha,T,G)$ for the $\mathbb{P}$-almost surely (a.s.) limit of $\frac{1}{|F_n|}\int_\Omega D\left(\alpha_{F_n}(\omega)\right)d\mathbb{P}(\omega)$. Then, by Propositions 4.7 and 4.8,

$$D(\alpha,T,G)=\lim_{n\to\infty}\frac{1}{|F_n|}\int_\Omega D\left(\alpha_{F_n}(\omega)\right)d\mathbb{P}(\omega),$$

for any Fø lner sequence $\{F_n:n\in\mathbb{N}\}$ of G and the value of the limit is independent of the choice of the Fø lner sequence $\{F_n:n\in\mathbb{N}\}$.

Remark 4.9. $0\leq D(\alpha,T,G)\leq\int D(\alpha(\omega))d\mathbb{P}$.

Remark 4.10. $\alpha(\omega)\succ\beta(\omega),\mathbb{P}$-a.a. $\omega\Longrightarrow D(\alpha,T,G)\geq D(\beta,T,G)$.

Proposition 4.11. *Let $\beta\in\mathcal{D}$ with an increasing sequence $(\Omega_r)_{r\geq1}\subset\mathcal{F}$ satisfying $\mathbb{P}\left(\Omega_r\right)\to1$, where each $\beta\cap\left(\Omega_r\times X\right),r\geq1$, is a finite family.*

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Let

$$\beta^r = [\beta \cap (\Omega_r \times X)] \bigcup [(\Omega_r^c \times X) \cap \mathcal{E}] \in \mathrm{C}^o_{\mathcal{E}},$$

then

$$D(\beta^r, T, G) \to D(\beta, T, G).$$

Proof. It is not hard to check that $D(\beta^r, T, G)$ is increasing in r. In fact, if $r_1 < r_2$, by the definition of β^r,

$$\beta^{r_1}(\omega) = \begin{cases} \beta(\omega), & \text{if } \omega \in \Omega_{r_1}; \\ \mathcal{E}_\omega, & \text{if } \omega \in \Omega_{r_2} \backslash \Omega_{r_1}; \\ \mathcal{E}_\omega, & \text{if } \omega \notin \Omega_{r_2} \end{cases} \quad \text{and} \quad \beta^{r_2}(\omega) = \begin{cases} \beta(\omega), & \text{if } \omega \in \Omega_{r_1}; \\ \beta(\omega), & \text{if } \omega \in \Omega_{r_2} \backslash \Omega_{r_1}; \\ \mathcal{E}_\omega, & \text{if } \omega \notin \Omega_{r_2}, \end{cases}$$

we have: for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\beta^{r_2}(\omega) \succ \beta^{r_1}(\omega)$.

Recall Remark 4.10, so

$$D(\beta^{r_1}, T, G) \leq D(\beta^{r_2}, T, G).$$

And that each member is pointwise less than $D(\beta, T, \omega)$ in ω by Lemma 4.1. It follows that

$$\lim_{r \to \infty} D(\beta^r, T, G) \leq D(\beta, T, G). \tag{4.1}$$

In the other direction, for any Fø lner sequences $F_n = \{g_1, g_2, ..., g_{k_n}\}$, set

$$I_{r,F_n}(\omega) = \{1 \leq i \leq k_n : g_i\omega \in \Omega_r\},$$
$$I^{r,F_n}(\omega) = \{1 \leq i \leq k_n : g_i\omega \notin \Omega_r\}$$

and note that, by Lemmas 4.1 and 4.3,

$$\begin{aligned} D(\beta_{F_n}(\omega)) &= D\left(\bigvee_{i=1}^{k_n} (T_{g_i,\omega})^{-1} \beta(g_i\omega)\right) \\ &\leq D\left(\bigvee_{i \in I_{r,F_n}(\omega)} (T_{g_i,\omega})^{-1} \beta(g_i\omega)\right) + D\left(\bigvee_{i \in I^{r,F_n}(\omega)} (T_{g_i,\omega})^{-1} \beta(g_i\omega)\right) \\ &= D\left(\beta^r_{F_n}(\omega)\right) + D\left(\bigvee_{i \in I^{r,F_n}(\omega)} (T_{g_i,\omega})^{-1} \beta(g_i\omega)\right) \\ &\leq D\left(\beta^r_{F_n}(\omega)\right) + \sum_{i \in I^{r,F_n}(\omega)} D(\beta(g_i\omega)). \end{aligned}$$

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It follows that

$$\begin{aligned} D(\beta,T,G) &= \lim_{n\to\infty}\frac{1}{|F_n|}\int_\Omega D\left(\beta_{F_n}(\omega)\right)d\mathbb{P}(\omega) \\ &\le \lim_{n\to\infty}\frac{1}{|F_n|}\int_\Omega D\left(\beta^r_{F_n}(\omega)\right)d\mathbb{P}(\omega) \\ &\quad + \limsup_{n\to\infty}\frac{1}{|F_n|}\int\sum_{i=1}^{k_n}1_{\Omega\setminus\Omega_r}(g_i\omega)\,D\left(\beta\left(g_i\omega\right)\right)d\mathbb{P}(\omega) \\ &\le D(\beta^r,T,G)+\int 1_{\Omega\setminus\Omega_r}(\omega)D(\beta(\omega))d\mathbb{P}(\omega). \end{aligned}$$

By the assumption that $\mathbb{P}(\Omega_r)\to 1$ and $\beta\in\mathcal{D}$, according to Lebesgue's dominated convergence theorem, we have

$$\lim_{r\to\infty}\int 1_{\Omega\setminus\Omega_r}(\omega)D(\beta(\omega))d\mathbb{P}(\omega)=0$$

and hence

$$D(\beta,T,G)\le\lim_{r\to\infty}D(\beta^r,T,G). \tag{4.2}$$

Combining (4.1) with (4.2) we complete the proof. □

Proposition 4.12. *For every $\beta\in\mathcal{D}$ and $\epsilon>0$, there exists a finite open cover $\mathcal{V}$ of X such that*

$$D(\mathcal{V},T,G)\ge D(\beta,T,G)-\epsilon,$$

where

$$D(\mathcal{V},T,G)=\lim_{n\to\infty}\frac{1}{|F_n|}\int_\Omega D\left(\mathcal{V}_{F_n}(\omega)\right)d\mathbb{P}(\omega)$$

with $\mathcal{V}(\omega)=\mathcal{V}\cap\mathcal{E}_\omega$ for any Fø lner sequence $\{F_n:n\in\mathbb{N}\}$ of G.

Proof. Recall that the Lebesgue number $\lambda(\beta(\omega))$ of an open cover $\beta(\omega)$ on the fibre $\mathcal{E}_\omega$ is the largest λ such that, for every $x\in\mathcal{E}_\omega$, there is a $B(\omega)\in\beta(\omega)$ such that $d\left(x,\mathcal{E}_\omega\setminus B(\omega)\right)\ge\lambda$. Suppose that $\beta=\left\{B^{(1)},B^{(2)},\ldots,B^{(l_q)}\right\}$ on Ω_q, where $(\Omega_q)_{q\ge1}\subset\mathcal{F}$ is the family of increasing sets in the definition of $\mathfrak{C}^o_{\mathcal{E}}$. When $\omega\in\Omega_q$, $\beta(\omega)\in \mathrm{C}^o_{\mathcal{E}_\omega}$ and its Lebesgue number is

$$\lambda(\beta(\omega))=\frac{1}{l_q}\min_{x\in\mathcal{E}_\omega}\sum_{i=1}^{l_q}d\left(x,\mathcal{E}_\omega\setminus B^{(i)}(\omega)\right)\quad\text{on }\Omega_q.$$

Set $d(x,\emptyset)=\infty$, for convenience.

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Note that $(\omega, x) \mapsto d\left(x, \mathcal{E}_\omega \backslash B^{(i)}(\omega)\right)$ is a Carathéodory function and hence $\mathcal{F} \otimes \mathcal{B}(X)$ measurable [11, Lemma III.14]. By Lemma III.39 from [11], $\lambda(\beta(\omega))$ is measurable.

By the compactness of $\mathcal{E}_\omega$, we see that $\lambda(\beta(\omega)) > 0$ for all ω. Set

$$\Omega_{p,q} = \left\{\omega \in \Omega_q : \lambda(\beta(\omega)) \geq \frac{1}{p}\right\}.$$

Then $\bigcup_{p=1}^{\infty} \Omega_{p,q} = \Omega_q$. Let $\Omega_{0,q} = \emptyset, \widetilde{\Omega}_{p,q} = \Omega_{p,q} \backslash \Omega_{p-1,q}$ and $\widetilde{\Omega}_p = \bigcup_{q \geq 1} \widetilde{\Omega}_{p,q}$.

Now we can choose a maximal set $S_p \subset X$ of n_p points in X such that $d(x, y) \geq 1/(4p)$ for all $x, y \in S_p, x \neq y$, and let $\mathcal{U}_p$ be the cover of X by open balls of radius $1/(2p)$ centred at points of S_p. In fact, by the maximum of S_p, we have: for any $x \in X$, there exists a $s \in S_p$ such that $d(x, s) < 1/(4p) < 1/(2p)$. Define an open random cover α by

$$\alpha = \left\{\left(\widetilde{\Omega}_p \times U\right) \cap \mathcal{E} : U \in \mathcal{U}_p\right\} \quad \text{on } \widetilde{\Omega}_p \subset \Omega.$$

Then, for $\omega \in \widetilde{\Omega}_p$,

$$\operatorname{diam} \alpha(\omega) \leq \operatorname{diam} \mathcal{U}_p < \frac{1}{p} \leq \lambda(\beta(\omega)),$$

which implies that $\alpha(\omega) \succ \beta(\omega)$. Now $\lambda(\alpha(\omega)) \geq 1/(2p)$ by the construction of S_p. To see this, we only need check that for any $x \in \mathcal{E}_\omega$, there is a $A(\omega) \in \alpha(\omega)$ such that $d\left(x, \mathcal{E}_\omega \backslash A(\omega)\right) \geq 1/(2p)$. By the maximum of S_p, for any $x \in \mathcal{E}_\omega$, there exists a $s_x \in S_p$, such that $d(x, s_x) < 1/(4p)$, that is $x \in B\left(s_x, 1/(4p)\right) \subset B\left(s_x, 1/(2p)\right) \in \alpha(\omega)$. Then $\lambda\left(\alpha(\omega)\right) \geq 1/(2p)$.

Set $\Omega_r = \bigcup_{p=1}^{r} \widetilde{\Omega}_p$ and let $\alpha^r = [\alpha \cap (\Omega_r \times X)] \bigcup [(\Omega_r^c \times X) \cap \mathcal{E}]$. It is not hard to check that the sequence of non-negative random variables $D\left(\alpha^r_{F_n}(\omega)\right)$ increases to $D\left(\alpha_{F_n}(\omega)\right)$ for every Fø lner sequence $\{F_n : n \in \mathbb{N}\}$, $\mathbb{P}$-a.a. ω, as $r \to \infty$. Again, choose a sequence $(\mathcal{V}^r)_{r \geq 1}$ of finite open covers of X such that diam $\mathcal{V}^r < 1/(4r)$. Recall that $\mathcal{V}^r(\omega) = \mathcal{V}^r \cap \mathcal{E}_\omega$, so $\mathcal{V}^r(\omega) \succ \alpha^r(\omega)$ for $\mathbb{P}$-a.a. ω. In fact, if $\omega \in \Omega_r = \bigcup_{p=1}^{r} \widetilde{\Omega}_p$, then $\lambda\left(\alpha(\omega)\right) \geq 1/(2p) \geq 1/(2r)$. By the definition of $\alpha^r(\omega)$, for $\omega \in \Omega_r$, we have $\alpha^r(\omega) = \alpha(\omega)$, then $\mathcal{V}^r(\omega) \succ \alpha^r(\omega)$ for $\omega \in \Omega_r$. If $\omega \in \Omega_r^c$, then $\alpha^r(\omega) = \{\mathcal{E}_\omega\}$. so, $\mathcal{V}^r(\omega) \succ \alpha^r(\omega)$ for $\mathbb{P}$-a.a. ω. Hence,

$$D\left(\mathcal{V}^r_{F_n}(\omega)\right) \geq D\left(\alpha^r_{F_n}(\omega)\right) \to D\left(\alpha_{F_n}(\omega)\right) \geq D\left(\beta_{F_n}(\omega)\right).$$

It follows from Proposition 4.11 that

$$D\left(\mathcal{V}^r, T, G\right) \geq D\left(\alpha^r, T, G\right) \to D(\alpha, T, G) \geq D(\beta, T, G).$$

Hence, for a sufficiently large value of $r, \mathcal{V}^r$ will satisfy our requirements. □

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Definition 4.13. *The mean topological dimension of a bundle RDS* T *on* $\mathcal{E}$ *for an amenable group* G*, denoted by* $\mathbb{E}\mathrm{midm}(T,G)$*, is defined by*

$$\mathbb{E}\mathrm{mdim}(T,G)=\mathbb{E}\,\mathrm{mdim}(\mathcal{E},T,G)=\sup_{\alpha}D(\alpha,T,G)$$

where α *runs over all countable random covers in* $\mathcal{D}$*.*

Remark 4.14. $0\le\mathbb{E}\,\mathrm{mdim}(T,G)\le\infty$.

Remark 4.15. By Proposition 4.12, one can obtain the same quantity $\mathbb{E}\,\mathrm{mdim}(T,G)$ in Definition 4.13 when α runs over all finite open random covers of $\mathcal{E}$, or all finite open covers $\mathcal{V}$ of X.

Proposition 4.16. *Let* T,S *be two continuous bundle RDSs on* $\mathcal{E}$ *and* $\mathcal{H}$*, respectively. If* T *and* S *are isomorphic, then* $\mathbb{E}\,\mathrm{mdim}(T,G)=\mathbb{E}\,\mathrm{mdim}(S,G)$*.*

Proof. Suppose T and S are isomorphic via ψ_ω and let $\Psi:\mathcal{E}\to\mathcal{G}$ be given by $\Psi(\omega,x)=(\omega,\psi_\omega x)$. Let $\alpha\in\mathrm{C}^o_{\mathcal{E}}$. Then $\beta=\Psi(\alpha)\in\mathrm{C}^o_{\mathcal{H}}$. As $S_{g,\omega}\circ\psi_\omega=\psi_{g\omega}\circ T_{g,\omega}$, the homeomorphism ψ_ω sends $D\left(\alpha_{F_n}(\omega)\right)$ to $D\left(\beta_{F_n}(\omega)\right)$ for any Fø lner sequence $\{F_n:n\in\mathbb{N}\}$ of G. In fact,

$$\begin{aligned}D\left(\alpha_{F_n}(\omega)\right)&=D\left(\bigvee_{g\in F_n}\left(T_{g,\omega}\right)^{-1}\alpha\left(g\omega\right)\right)=D\left(\bigvee_{g\in F_n}\left(T_{g,\omega}\right)^{-1}\circ\left(\psi_{g\omega}\right)^{-1}\beta\left(g\omega\right)\right)\\&=D\left(\bigvee_{g\in F_n}\left(\psi_\omega\right)^{-1}\circ\left(S_{g,\omega}\right)^{-1}\beta\left(g\omega\right)\right)\\&\le D\left(\bigvee_{g\in F_n}\left(S_{g,\omega}\right)^{-1}\beta\left(g\omega\right)\right)=D\left(\beta_{F_n}(\omega)\right).\end{aligned}$$

Similarly,

$$\begin{aligned}D\left(\beta_{F_n}(\omega)\right)&=D\left(\bigvee_{g\in F_n}\left(S_{g,\omega}\right)^{-1}\beta\left(g\omega\right)\right)=D\left(\bigvee_{g\in F_n}S_{g^{-1},g\omega}\circ\psi_{g\omega}\alpha\left(g\omega\right)\right)\\&=D\left(\bigvee_{g\in F_n}\psi_\omega\circ T_{g^{-1},g\omega}\alpha\left(g\omega\right)\right)\\&\le D\left(\bigvee_{g\in F_n}\left(T_{g,\omega}\right)^{-1}\alpha\left(g\omega\right)\right)=D\left(\alpha_{F_n}(\omega)\right).\end{aligned}$$

It follows that $D\left(\alpha_{F_n}(\omega)\right)=D\left(\beta_{F_n}(\omega)\right)$ and hence $D(\alpha,T,G)=D(\beta,S,G)$. Since $\alpha\mapsto\Psi(\alpha)$ provides a bijective correspondence between the finite open random covers of $\mathcal{E}$ and those of $\mathcal{H}$, by Remark 4.15 we deduce that $\mathbb{E}\,\mathrm{mdim}(T,G)=\mathbb{E}\,\mathrm{mdim}(S,G)$.

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Proposition 4.17. *If* X *has finite topological dimension, then* $\mathrm{Emdim}(T,G)=0$.

Proof. Let $\alpha\in\mathfrak{C}^o_{\mathcal{E}}$. Then for any Fø lner sequence of infinite amenable group F_n,

$$D(\alpha_{F_n}(\omega))\le \dim\mathcal{E}_\omega\le \dim X<\infty$$

and

$$D(\alpha,T,G)=\lim_{n\to\infty}\frac{1}{|F_n|}\int_\Omega D\left(\alpha_{F_n}(\omega)\right)d\mathbb{P}(\omega).$$

Since $|F_n|\to\infty$ as $n\to\infty$ for any Fø lner sequence of infinite amenable group, then $\mathbb{E}\mathrm{mdim}(T,G)=0$. □

Proposition 4.18. *If a closed random set* $\mathcal{C}\subset\mathcal{E}$ *is forward invariant, i.e. for any* $g\in G$

$$\mathcal{C}_\omega\subset\left(T_{g,\omega}\right)^{-1}\mathcal{C}_{g\omega}\quad \mathbb{P}\text{-}a.a.\ \omega,$$

then $\mathrm{Emdim}\left(\mathcal{C},T|_{\mathcal{C}},G\right)\le\mathbb{E}\,\mathrm{mdim}(\mathcal{E},T,G)$.

Proof. Since $\mathcal{C}$ is forward invariant, $(\mathcal{C},T|_{\mathcal{C}})$ is a continuous bundle RDS. For every $\alpha=\left\{A^{(i)}:i\in I\right\}\in\mathfrak{C}^o_{\mathcal{C}}$, let $B^{(i)}=\mathcal{E}\backslash\left(\mathcal{C}\backslash A^{(i)}\right)$. Then $\beta=\left\{B^{(i)}:i\in I\right\}\in\mathfrak{C}^o_{\mathcal{E}}$. We claim that $D\left(\alpha_{F_n}(\omega)\right)\le D\left(\beta_{F_n}(\omega)\right)$. It suffices to prove that, for every Fø lner sequence and $\mathbb{P}$-a.a. ω, if $\gamma\succ\beta_{F_n}(\omega)$ is an open cover of $\mathcal{E}_\omega$, then there exists an open cover $\gamma'\succ\alpha_{F_n}(\omega)$ of $\mathcal{C}_\omega$ such that $\mathrm{ord}\left(\gamma'\right)\le\mathrm{ord}(\gamma)$. In other words, it suffices to prove that

$$\sum_{C\in\gamma'}1_C(x)\le\sum_{U\in\gamma}1_U(x),\quad x\in\mathcal{C}_\omega\subset\mathcal{E}_\omega.$$

Indeed, we can take $\gamma'=\gamma\cap\mathcal{C}_\omega$. □

5. Metric mean dimension

For each $F\in\mathcal{F}_G$ and a positive $\epsilon>0$, define a family of metrics d^ω_F on $\mathcal{E}_\omega$ by the formula

$$d^\omega_F(x,y)=\max_{g\in F}d\left(T_{g,\omega}x,T_{g,\omega}y\right),\quad x,y\in\mathcal{E}_\omega.$$

Set

$$B_{d^\omega_F}(x,\epsilon)=\left\{y\in\mathcal{E}_\omega:d^\omega_F(x,y)<\epsilon\right\}.$$

Fix ω and let $\alpha=\{A_i:i=1,2,\ldots,l\}$ be a finite open cover of $\mathcal{E}_\omega$. Define the mesh of α with respect to the metric d^ω_F by

$$\mathrm{mesh}\left(\alpha,d^\omega_F\right)=\max_{1\le i\le l}\mathrm{diam}\left(A_i,d^\omega_F\right),$$

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where

$$\operatorname{diam}(A_i, d_F^\omega) = \sup\{d_F^\omega(x,y) : x, y \in A_i\}.$$

Let $\operatorname{cov}(\omega, \epsilon, F)$ be the minimal cardinality of finite open covers of $\mathcal{E}_\omega$ by sets of d_F^ω diameter less than ϵ, that is,

$$\operatorname{cov}(\omega, \epsilon, F) = \inf\left\{|\alpha| : \alpha \in \mathrm{C}^o_{\mathcal{E}_\omega}, \operatorname{mesh}(\alpha, d_F^\omega) < \epsilon\right\}.$$

Definition 5.1. *A set $H \subset \mathcal{E}_\omega$ is said to be (ω, ϵ, F)-separated if $x, y \in H, x \neq y$ implies that $d_F^\omega(x,y) > \epsilon$.*

We define

$$\operatorname{sep}(\omega, \epsilon, F) = \sup\{|H| : H \subset \mathcal{E}_\omega \text{is an } (\omega, \epsilon, F)\text{-separated set}\}$$

as the maximum cardinality of (ω, ϵ, F)-separated sets of $\mathcal{E}_\omega$.

Definition 5.2. *A set $H \subset \mathcal{E}_\omega$ is said to be (ω, ϵ, F)-spanning if for any $x \in \mathcal{E}_\omega$, there exists a $y \in H$ such that $d_F^\omega(x,y) \leq \epsilon$.*

We define

$$\operatorname{span}(\omega, \epsilon, F) = \inf\{|H| : H \subset \mathcal{E}_\omega \text{ is an } (\omega, \epsilon, F)\text{-spanning set}\}$$

as the minimum cardinality of (ω, ϵ, F)-spanning sets of $\mathcal{E}_\omega$.

As in the deterministic case [10, Lemma 2.5.1], we have the following facts.

Remark 5.3.

$$\operatorname{cov}(\omega, 2\epsilon, F) \leq \operatorname{span}(\omega, \epsilon, F) \leq \operatorname{sep}(\omega, \epsilon, F) \leq \operatorname{cov}(\omega, \epsilon, F). \tag{5.1}$$

Proof. Let H be a spanning set of minimal cardinality, then

$$\mathcal{E}_\omega \subset \bigcup_{h \in H} \bar{B}_{d_F^\omega}(h, \epsilon).$$

By compactness, then there exists $\epsilon_1 < \epsilon$, such that, the finite balls of radius ϵ_1 centred at the points of H over $\mathcal{E}_\omega$. Hence

$$\operatorname{cov}(\omega, 2\epsilon, F) \leq \operatorname{span}(\omega, \epsilon, F).$$

Let H be a separated set of maximal cardinality, then for any $x \in \mathcal{E}_\omega$, $H \bigcup \{x\}$ is not a (ω, ϵ, F)- separated set. That is, there exists $z \in H$ such that $d_F^\omega(x,z) \leq \epsilon$, then H be a (ω, ϵ, F)-spanning set. Hence,

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$$\operatorname{span}(\omega, \epsilon, F) \leq \operatorname{sep}(\omega, \epsilon, F).$$

Suppose $\alpha \in \mathrm{C}^o_{\mathcal{E}_\omega}$ of minimal cardinality with $\operatorname{mesh}(\alpha, d^\omega_F) < \epsilon$. Suppose $H \subset \mathcal{E}_\omega$ to be an (ω, ϵ, F)-separated set of maximal cardinality. If $|H| > |\alpha|$, then by the drawer principle, H must have at least two elements in α. It's a contradiction.

Hence,

$$\operatorname{sep}(\omega, \epsilon, F) \leq \operatorname{cov}(\omega, \epsilon, F).$$

□

Moreover, let $\alpha, \gamma \in \mathrm{C}^o_{\mathcal{E}}$, for any $\omega \in \Omega$, denote by $N(\alpha(\omega))$ the minimal cardinality of a subfamily of $\alpha(\omega)$ covering $\mathcal{E}_\omega$, equivalently,

$$N(\alpha(\omega)) = \min_{\beta \subset \alpha(\omega)} \left\{ |\beta| : \beta \in \mathrm{C}^o_{\mathcal{E}_\omega} \right\}.$$

Let $\lambda(\omega)$ be the Lebesgue number of $\alpha(\omega)$, and assume that $\operatorname{mesh}(\gamma(\omega), d^\omega_F) = \delta(\omega)$. In case similar to deterministic systems [54, Theorem 7.7], for $F \in \mathcal{F}_G$, we have the following inequality.

Remark 5.4.

$$N\left(\alpha_F(\omega)\right) \leq \operatorname{span}(\omega, \lambda/2, F) \leq \operatorname{sep}(\omega, \lambda/2, F).$$
$$\operatorname{span}(\omega, \delta, F) \leq \operatorname{sep}(\omega, \delta, F) \leq N\left(\gamma_F(\omega)\right).$$

Proof. In fact, if A is an $(\omega, \lambda/2, F)$-spanning set with minimal cardinality, then $\mathcal{E}_\omega \subset \bigcup_{a \in A} \bar{B}_{d^\omega_F}(a, \lambda/2)$. And let $F = \{g_1, g_2, ..., g_n\}$, then

$$\bar{B}_{d^\omega_F}(a, \lambda/2) = \bigcap_{k=1}^{n} \left\{ x : d\left(T_{g_k,\omega} x, T_{g_k,\omega} a\right) \leq \frac{1}{2}\lambda \right\},$$

and

$$\operatorname{diam} \bar{B}_{d^\omega_F}(a, \lambda/2) \leq \lambda.$$

Therefore, $\bar{B}_{d^\omega_F}(a, \lambda/2)$ must lie in some element of $\alpha_F(\omega)$. Hence $N\left(\alpha_F(\omega)\right) \leq \operatorname{span}(\omega, \lambda/2, F)$. The other side has been previously proven.

For the second inequality, it is only necessary to prove the inequality on the right-hand side. Let H be an (ω, δ, F)-separated set with maximal cardinality.

If $|H| > N\left(\gamma_F(\omega)\right)$, then by the drawer principle, at least two points in H fall into the same element of $\gamma_F(\omega)$. This contradicts our assumption about H. □

Referring to [19, Proposition 5.4], it is not difficult to prove that $\omega \mapsto N\left(\alpha_F(\omega)\right)$ is measurable. Furthermore, $\{\log N\left(\alpha_F(\omega)\right) : F \in \mathcal{F}_\mathcal{G}\}$ be a monotone sub-additive G-invariant family in $L^1(\Omega, \mathcal{F}, \mathbb{P})$. Thus by Propositions 4.7 and 4.8, we have the following definitions.

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Definition 5.5. *Let $\omega \in \Omega$, and $F \in \mathcal{F}_G$, one has the fibre topological entropy of T with respect to α and the fibre topological entropy of T, respectively, by*

$$h_{top}(T,\alpha,G) = \lim_{n\to\infty} \frac{1}{|F_n|} \int_{\Omega} logN(\alpha_{F_n}(\omega))d\mathbb{P}(\omega)$$

and

$$h_{top}(T,G) = \sup_{\gamma\in C_{\mathcal{E}}^{o}} h_{top}(T,\gamma,G).$$

Remark that the values of these invariants are all independent of the selection of the Fø lner sequence $\{F_n : n \in \mathbb{N}\}$.

Notice that for any $F \in \mathcal{F}_G$ and a positive random variable $\epsilon = \epsilon(\omega)$ the function $\mathrm{sep}(\omega,\epsilon,F)$ is measurable in ω (even for random $\epsilon(\omega)$) [32, Lemma 1.2].

Proposition 5.6. *For any $F \in \mathcal{F}_G$ and a positive random variable $\epsilon = \epsilon(\omega)$ the function $\mathrm{sep}(\omega,\epsilon,F)$ is measurable in ω.*

Proof. For $q, n \in \mathbb{N}_+$, set

$$\begin{aligned}
D_q &= \{(\omega, x_1, \ldots, x_q) : \omega \in \Omega, x_i \in \mathcal{E}_\omega, \forall i\},\\
E_q^F &= \{(\omega, x_1, \ldots, x_q) \in D_q : d_F^\omega(x_i, x_j) > \varepsilon, \forall i \neq j\},\\
E_q^{F,k} &= \left\{(\omega, x_1, \ldots, x_q) \in D_q : d_F^\omega(x_i, x_j) \geqslant \varepsilon(1+k^{-1}), \forall i \neq j\right\},\\
E_q^F(\omega) &= \left\{(x_1, \ldots, x_q) : (\omega, x_1, \ldots, x_q) \in E_q^F\right\},\\
E_q^{F,k}(\omega) &= \left\{(x_1, \ldots, x_q) : (\omega, x_1, \ldots, x_q) \in E_q^{F,k}\right\}.
\end{aligned}$$

Observe that $D_q \in \mathcal{F} \times (\mathcal{B}(X))^q$, where $(\mathcal{B}(X))^q$ is the product σ-algebra on the product X^q of q copies of X. This follows from Theorem III.30 in [11] since

$$d_q((x_1, \ldots, x_q), (y_1, \ldots, y_q)) = \sum_{i=1}^{q} d(x_i, y_i)$$

is the distance function on X^q; and if $\mathcal{E}_\omega^q$ denotes the product of q copies of $\mathcal{E}_\omega$, then

$$d_q((x_1, \ldots, x_q), \mathcal{E}_\omega^q) = \sum_{i=1}^{q} d(x_i, \mathcal{E}_\omega)$$

is measurable in ω for each $(x_1, \ldots, x_q) \in X^q$. Next, define $q(q-1)/2$ measurable functions $\psi_{ij}, 1 \leqslant i < j \leqslant q$, on D_q by $\psi_{ij}(\omega, x_1, \ldots, x_q) = d_F^\omega(x_i, x_j)$. Then

$$E_q^{F,k} = \bigcap_{1\leqslant i<j\leqslant q} \psi_{ij}^{-1}\left[\varepsilon(1+k^{-1}), \infty\right) \in \mathcal{F} \times (\mathcal{B}(X))^q.$$

By continuity of the RDS T, each $E_q^{F,k}(\omega)$ is a closed subset of $\mathcal{E}_\omega^q$, and so it is compact. Clearly, $E_q^{F,k} \uparrow E_q^F$ and $E_q^{F,k}(\omega) \uparrow E_q^F(\omega)$ as $k \uparrow \infty$, in particular, $E_q^F \in \mathcal{F} \times (\mathcal{B}(X))^q$.

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Let, again, $\operatorname{sep}(\omega, \epsilon, F)$ denote the largest cardinality of an (ω, ε, F)-separated set in $\mathcal{E}_\omega$ and set $t_{F,k}(\omega, \varepsilon) = \max\{q : E_q^{F,k}(\omega) \neq \emptyset\}$. By Theorem III.23 in [11], it follows that

$$\{\omega : t_{F,k}(\omega, \varepsilon) \geqslant q\} = \{\omega : E_q^{F,k}(\omega) \neq \emptyset\} = \Pr_\Omega E_q^{F,k} \in \mathcal{F}.$$

□

Set

$$S(\omega, \epsilon, G) = \liminf_{n\to\infty} \frac{1}{|F_n|} \log \operatorname{sep}(\omega, \epsilon, F_n),$$

for any Fø lner sequence $\{F_n : n \in \mathbb{N}\}$ of G.

Obviously, S is monotone non-decreasing as non-random $\epsilon \to 0$, and, for the purpose of measuring how fast it increases, we define the lower metric mean dimension of T on the fibre $\mathcal{E}_\omega$ as

$$\underline{\mathbb{E}\operatorname{mdim}_{\mathrm{M}}}(T, G, \omega) = \liminf_{\epsilon\to 0} \frac{1}{|\log \epsilon|} S(\omega, \epsilon, G).$$

It follows from Remark 5.3 and the fact that $\lim_{\epsilon\to 0}(\log \epsilon / \log 2\epsilon) = 1$ that

$$\underline{\mathbb{E}\operatorname{mdim}_{\mathrm{M}}}(T, G, \omega) = \liminf_{\epsilon\to 0} \frac{1}{|\log \epsilon|} \liminf_{n\to\infty} \frac{1}{|F_n|} \log \operatorname{cov}(\omega, \epsilon, F_n),$$

for any Fø lner sequence $\{F_n : n \in \mathbb{N}\}$ of G.

To satisfy the measurability of the function, we define a slightly smaller-scale lower metric mean dimension. Set

$$\mathbb{S} = \Big\{\{\epsilon_m\}_{m=1}^{+\infty} : \{\epsilon_m\} \text{ is a sequence of positive numbers that is monotonically} \times \text{ decreasing with } \lim_{m\to\infty} \epsilon_m = 0\Big\}.$$

We define the $\mathbb{S}$-lower metric mean dimension of T on the fibre $\mathcal{E}_\omega$ as

$$\underline{\mathbb{E}\operatorname{mdim}}_{\mathrm{M}}^{\mathbb{S}}(T, G, \omega) = \liminf_{m\to\infty} \frac{1}{|\log \epsilon_m|} S(\omega, \epsilon_m, G).$$

Since $\operatorname{sep}(\omega, \epsilon_m, F_n)$ is measurable, it follows that $\underline{\mathbb{E}\operatorname{mdim}}_{\mathrm{M}}^{\mathbb{S}}(T, G, \omega)$ is measurable as well. We have the following definition.

Definition 5.7. *The* $\mathbb{S}$*-lower metric mean topological dimension of bundle RDS* T *for a countable amenable group* G*, denoted by* $\underline{\mathbb{E}\operatorname{mdim}}_{\mathrm{M}}^{\mathbb{S}}(T, d, G)$*, is defined by*

$$\underline{\mathbb{E}\operatorname{mdim}}_{\mathrm{M}}^{\mathbb{S}}(T, d, G) = \int \underline{\mathbb{E}\operatorname{mdim}}_{\mathrm{M}}^{\mathbb{S}}(T, G, \omega) d\mathbb{P}(\omega).$$

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Let $\alpha, \gamma \in \mathrm{C}^o_{\mathcal{E}}$. Let $2\epsilon(\omega)$ be the Lebesgue number of $\alpha(\omega)$, and assume that $\mathrm{mesh}(\gamma(\omega), d_F^\omega) = \epsilon(\omega)$, for any $\omega \in \Omega$. Then by Remark 5.4,

$$N\left(\alpha_F^\epsilon(\omega)\right) \leq \mathrm{span}(\omega, \epsilon, F) \leq \mathrm{sep}(\omega, \epsilon, F) \leq N\left(\gamma_F^\epsilon(\omega)\right).$$

Hence,

$$h_{\mathrm{top}}(T, G) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{|F_n|} \int \log \mathrm{sep}(\omega, \epsilon, F_n) d\mathbb{P}(\omega).$$

Compare

$$\underline{\mathbb{E}\,\mathrm{mdim}}_{\mathrm{M}}^{\mathbb{S}}(T, d, G) = \int \liminf_{m \to \infty} \frac{1}{|\log \epsilon_m|} \liminf_{n \to \infty} \frac{1}{|F_n|} \log \mathrm{sep}\left(\omega, \epsilon_m, F_n\right) d\mathbb{P}(\omega)$$

with

$$h_{\mathrm{top}}(T, G) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{|F_n|} \int \log \mathrm{sep}(\omega, \epsilon, F_n) d\mathbb{P}(\omega).$$

We can immediately state the relationship between the metric mean dimension and the fibre topological entropy of a system.

Proposition 5.8. *If* $\underline{\mathbb{E}\,\mathrm{mdim}}_{\mathrm{M}}^{\mathbb{S}}(T, d, G) \neq 0$, *then* $h_{top}\ (T, G) = \infty$.

Proof. Set

$$S(\epsilon, G) = \liminf_{\epsilon \to 0} \frac{1}{|F_n|} \int \log \mathrm{sep}(\omega, \epsilon, F_n) \mathbb{P}(\omega).$$

Let $\epsilon = \frac{1}{m}$, then

$$\begin{aligned}
\underline{\mathbb{E}\,\mathrm{mdim}}_{\mathrm{M}}^{\mathbb{S}}(T, G, \omega) &\leq \liminf_{m \to \infty} \frac{1}{|\log m|} \liminf_{n \to \infty} \frac{1}{|F_n|} \log \mathrm{sep}\left(\omega, \frac{1}{m}, F_n\right) \\
&= \liminf_{m \to \infty} \frac{1}{|\log m|} \liminf_{n \to \infty} \frac{1}{|F_n|} \log \mathrm{cov}\left(\omega, \frac{1}{m}, F_n\right).
\end{aligned}$$

From Fatou's lemma, we have

$$\begin{aligned}
\underline{\mathbb{E}\,\mathrm{mdim}}_{\mathrm{M}}^{\mathbb{S}}(T, d, G) &\leq \int \liminf_{m \to \infty} \frac{1}{|\log m|} \liminf_{n \to \infty} \frac{1}{|F_n|} \log \mathrm{sep}\left(\omega, \frac{1}{m}, F_n\right) d\mathbb{P}(\omega) \\
&\leq \liminf_{m \to \infty} \frac{1}{|\log m|} \int \liminf_{n \to \infty} \frac{1}{|F_n|} \log \mathrm{sep}\left(\omega, \frac{1}{m}, F_n\right) d\mathbb{P}(\omega) \\
&\leq \liminf_{m \to \infty} \frac{1}{|\log m|} \liminf_{n \to \infty} \frac{1}{|F_n|} \int \log \mathrm{sep}\left(\omega, \frac{1}{m}, F_n\right) d\mathbb{P}(\omega) \\
&= \liminf_{m \to \infty} \frac{1}{|\log m|} S\left(\frac{1}{m}, G\right).
\end{aligned}$$

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If $\underline{\mathbb{E}\,\mathrm{mdim}}_{\mathrm{M}}^{\mathbb{S}}(T,d,G)>0$, then

$$\liminf_{m\to\infty}\frac{1}{|\log m|}S\left(\frac{1}{m},G\right)>0,$$

that is,

$$\lim_{m\to\infty}S\left(\frac{1}{m},G\right)=\infty.$$

Since $S(\epsilon,G)$ is monotone non-decreasing as non-random $\epsilon\to 0$, then we have

$$h_{\mathrm{top}}(T,G)=\lim_{\epsilon\to 0}S\left(\epsilon,G\right)=\lim_{m\to\infty}S\left(\frac{1}{m},G\right).$$

Hence

$$h_{\mathrm{top}}(T,G)=\lim_{\epsilon\to 0}S\left(\epsilon,G\right)=\infty.$$

□

Corollary 5.9. *If* T *has finite fibre topological entropy, then* $\underline{\mathbb{E}\,\mathrm{mdim}}_{\mathrm{M}}^{\mathbb{S}}(T,d,G)=0$.

We conclude this section by pointing out that the mean topological dimension is not larger than the $\mathbb{S}$-lower metric mean dimension for a continuous bundle RDS for amenable group as well as in the deterministic case. The main idea from Lindenstrauss and Weiss still works for the random transformations between different fibres [40, Theorem 4.2].

By Remark 4.15, it suffices to prove that, for any $\mathcal{A}=\{A_i\}_{i=1}^{l}\in \mathrm{C}_X^o$,

$$D(\mathcal{A},T,G)=\lim_{n\to\infty}\frac{1}{|F_n|}\int D\left(\mathcal{A}_{F_n}(\omega)\right)d\mathbb{P}(\omega)\leq \underline{\mathbb{E}\,\mathrm{mdim}}_{\mathrm{M}}^{\mathbb{S}}(T,d,G).$$

By Lemma 4.1, without loss of generality, we can refine $\mathcal{A}$ to be of the form

$$\mathcal{A}=\bigvee_{i=1}^{r}\{U_i,V_i\}$$

where, for every $1\leq i\leq r$, $\{U_i,V_i\}$ is a two element open cover of X. Let $U_i(\omega)=U_i\cap\mathcal{E}_\omega$ and $V_i(\omega)=V_i\cap\mathcal{E}_\omega$.

Define $\varphi_i^\omega:\mathcal{E}_\omega\to[0,1]$ by

$$\varphi_i^\omega(x)=\frac{d\left(x,X\backslash V_i(\omega)\right)}{d\left(x,X\backslash V_i(\omega)\right)+d\left(x,X\backslash U_i(\omega)\right)}$$

and let $\Phi^\omega(x)=(\varphi_1^\omega(x),\varphi_2^\omega(x),\ldots,\varphi_r^\omega(x))$.

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And

$$U_i(\omega) = (\varphi_i^\omega)^{-1}[0,1),$$
$$V_i(\omega) = (\varphi_i^\omega)^{-1}(0,1].$$

Notice that φ_i^ω is Lipschitz. In fact, let $A(i,\omega) = X\backslash V_i(\omega)$ and $B(i,\omega) = X\backslash U_i(\omega)$, then

$$\varphi_i^\omega(x) = \frac{d(x,A)}{d(x,A)+d(x,B)}.$$

One has

$$|\varphi_i^\omega(x) - \varphi_i^\omega(y)| = \frac{|d(x,A)\,d(y,B) - d(x,B)\,d(y,A)|}{(d(x,A)+d(x,B))\,(d(y,A)+d(y,B))}.$$

If

$$d(x,A)d(y,B) - d(x,B)d(y,A) \geq 0.$$

As A and B are closed, then $d(x,A) \leq d(x,y) + d(y,A)$ and by the symmetry it is clear to see that $d(y,B) \leq d(x,y) + d(x,B)$. Then

$$\begin{aligned}|\varphi_i^\omega(x) - \varphi_i^\omega(y)| &\leq \frac{(d(x,y)+d(y,A))\,d(y,B) + (d(x,y)-d(y,B))\,d(y,A)}{(d(x,A)+d(x,B))\,(d(y,A)+(y,B))}\\ &= \frac{d(x,y)\,(d(y,A)+d(y,B))}{(d(x,A)+d(x,B))\,(d(y,A)+d(y,B))}\\ &= \frac{d(x,y)}{d(x,A)+d(x,B)}.\end{aligned}$$

Similarly, if

$$d(x,A)d(y,B) - d(x,B)d(y,A) < 0,$$

one has

$$|\varphi_i^\omega(x) - \varphi_i^\omega(y)| \leq \frac{d(x,y)}{d(y,A)+d(y,B)}.$$

That is,

$$|\varphi_i^\omega(x) - \varphi_i^\omega(y)| \leq \max\left\{\frac{d(x,y)}{d(x,A)+d(x,B)}, \frac{d(x,y)}{d(y,A)+d(y,B)}\right\}.$$

Since

$$d(A,B) \leq \min\{d(x,A)+d(x,B), d(y,A)+d(y,B)\},$$

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then

$$|\varphi_i^\omega(x) - \varphi_i^\omega(y)| \le \frac{1}{d(A,B)} \cdot d(x,y).$$

Therefore, φ_i^ω is Lipschitz and the Lipschitz constant is $1/\left(d\left(X\backslash V_i(\omega), X\backslash U_i(\omega)\right)\right)$.

Let L_ω be a bound on the Lipschitz constants of all the $(\varphi_i^\omega)_{i=1}^r$. For any Fø lner sequence F_n of G, $F_n = \{g_1, g_2, ..., g_b\}$, where $b = b(n)$.

Define $R_{F_n}^\omega \colon \mathcal{E}_\omega \to [0,1]^{rb}$ by

$$\begin{aligned} R_{F_n}^\omega(x) &= \left(\Phi^{g_1\omega}\left(T_{g_1,\omega}x\right), \ldots, \Phi^{g_b\omega}\left(T_{g_b,\omega}x\right)\right) \\ &= (\varphi_1^{g_1\omega}(T_{g_1,\omega}x), \varphi_2^{g_1\omega}(T_{g_1,\omega}x), \ldots, \varphi_r^{g_1\omega}(T_{g_1,\omega}x), \\ &\qquad \vdots \\ &\qquad \varphi_1^{g_b\omega}\left(T_{g_b,\omega}x\right), \varphi_2^{g_b\omega}\left(T_{g_b,\omega}x\right), \ldots, \varphi_r^{g_b\omega}\left(T_{g_b,\omega}x\right)). \end{aligned}$$

From Lemma 4.4, we see that $R_{F_n}^\omega \succ \mathcal{A}_{F_n}(\omega)$, and

$$\left\|R_{F_n}^\omega(x) - R_{F_n}^\omega(y)\right\|_\infty \le L_\omega d_{F_n}^\omega(x,y).$$

As usual, if $S \subset \{1, 2, \ldots, rb\}$, denote by $R_{F_n}^\omega(x)_S \in [0,1]^{|S|}$ the projection of $R_{F_n}^\omega(x)$ to the coordinates in the index set S. Let $D(\omega) = \underline{\mathbb{E}\,\mathrm{mdim}}_{\mathrm{M}}^{\mathbb{S}}(T, G, \omega)$. We need the following two fibrewise lemmas, which correspond to [40, Lemmas 4.3 and 4.4].

Lemma 5.10. *Let $\delta > 0$, $D(\omega) = \underline{\mathbb{E}\,\mathrm{mdim}}_{\mathrm{M}}^{\mathbb{S}}(T, G, \omega)$. For any Fø lner sequence $\mathrm{F_n}$ of G, $F_n = \{g_1, g_2, ..., g_b\}$, where $b = b(n)$. If n is larger than some $N(\omega, \delta)$, there is a $\xi(\omega) \in (0,1)^{rb}$ such that, for any $|S| \ge (D(\omega) + \delta)b$,*

$$\xi(\omega)_S \notin R_{F_n}^\omega\left(\mathcal{E}_\omega\right)_S.$$

Proof. Notice that

$$D(\omega) = \underline{\mathbb{E}\,\mathrm{mdim}}_{\mathrm{M}}^{\mathbb{S}}(T, G, \omega) = \liminf_{m\to\infty} \frac{1}{|\log \epsilon_m|} \liminf_{n\to\infty} \frac{1}{|F_n|} \log \mathrm{cov}\left(\omega, \epsilon_m, F_n\right).$$

Fix ω and let m is large enough such that,

$$0 < \epsilon_m < \left(2^r \left(2L_\omega\right)^{D(\omega)}\right)^{-2/\delta},$$

and

$$\frac{1}{|\log \epsilon_m|} \liminf_{n\to\infty} \frac{1}{|F_n|} \log \mathrm{cov}\left(\omega, \epsilon_m, F_n\right) \le D(\omega) + \delta/4.$$

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If n is large enough (depending on ϵ_m and δ), $\mathcal{E}_\omega$ can be covered by $\epsilon_m^{-(D(\omega)+\delta/2)b}$ balls of the form

$$B_{d^\omega_{F_n}}(x,\epsilon_m)=\left\{x'\in\mathcal{E}_\omega : d^\omega_{F_n}(x,x')<\epsilon_m\right\}.$$

Since L_ω is a Lipschitz constant for all $(\varphi_i^\omega)_{i=1}^r$,

$$R^\omega_{F_n}\left(B_{d^\omega_{F_n}}(x,\epsilon_m)\right)\subset\left\{a\in[0,1]^{rb} : \|R^\omega_{F_n}(x)-a\|_\infty<L_\omega\epsilon_m\right\},$$

the set $R^\omega_{F_n}(\mathcal{E}_\omega)$ can be covered by $\epsilon_m^{-(D(\omega)+\delta/2)b}$ balls in the $\|\cdot\|_\infty$ norm of radius $L_\omega\epsilon_m$. Enumerate these balls as $B(\omega,k), k=1,2\ldots,K=\epsilon_m^{-(D(\omega)+\delta/2)b}$.

Choose ξ with uniform probability in $[0,1]^{rb}$. For any $S\subset\{1,2,...,rb\}$,

$$P\left(\xi_S\in R^\omega_{F_n}(\mathcal{E}_\omega)_S\right)\le\sum_{k=1}^{K}P\left(\xi_S\in B(\omega,k)|_S\right)\le\epsilon_m^{-(D(\omega)+\delta/2)b}\left(2L_\omega\epsilon_m\right)^{|S|}.$$

Hence

$$\begin{aligned}
&P\left(\exists S:|S|\ge(D(\omega)+\delta)b\text{and }\xi_S\in R^\omega_{F_n}(\mathcal{E}_\omega)_S\right)\\
&\quad\le\sum_{|S|\ge(D(\omega)+\delta)b}P\left(\xi_S\in R^\omega_{F_n}(\mathcal{E}_\omega)_S\right)\\
&\quad\le\#\left\{S:|S|\ge(D(\omega)+\delta)b\right\}\times\epsilon_m^{-(D(\omega)+\delta/2)b}\left(2L_\omega\epsilon_m\right)^{(D(\omega)+\delta)b}\\
&\quad\le 2^{rb}\left(\left(2L_\omega\right)^{D(\omega)}\epsilon_m^{\delta/2}\right)^b\ll 1,
\end{aligned}$$

and so, with high probability, a random ξ will satisfy the requirements. □

Lemma 5.11. *For any Fø lner sequence* $\mathrm{F_n}$ *of G,* $F_n=\{g_1,g_2,...,g_b\}$, *where* $b=b(n)$. *If* $\pi:R^\omega_{F_n}(\mathcal{E}_\omega)\to[0,1]^{rb}$ *satisfies for both* $\mathrm{a}=0$ *and 1, and, for all* $\xi\in[0,1]^{rb}$,

$$\{1\le k\le rb:\xi_k=a\}\subset\{1\le k\le rb:\pi(\xi)_k=a\},$$

then $\pi\circ R^\omega_{F_n}$ *is compatible with* $\mathcal{A}_{F_n}(\omega)$.

Proof. Indeed, if $\xi\in[0,1]^{rb}$, define for $0\le j<N$ and $1\le i<r$,

$$W_{i,j}=\begin{cases}\left(T_{g_j,\omega}\right)^{-1}U_i, & \text{if } \xi_{jr+i}=0;\\ \left(T_{g_j,\omega}\right)^{-1}V_i, & \text{otherwise.}\end{cases}$$

It is easy to see that

$$\left(\pi\circ R^\omega_{F_n}\right)^{-1}(\xi)\subset\bigcap_{\substack{1\le i<r\\1\le j\le b}}W_{i,j}\in\mathcal{A}_{F_n}(\omega).$$

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□

Theorem 5.12 *For any continuous bundle RDS* T *for a countable amenable group* G, $\mathbb{E}\mathrm{mdim}(T,G) \leq \underline{\mathbb{E}\,\mathrm{mdim}}_{\mathrm{M}}^{\mathbb{S}}(T,d,G)$.

Proof. Fix ω and let $\delta>0$. For any Fø lner sequence F_n of G, $F_n = \{g_1, g_2, ..., g_b\}$, where $b = b(n)$. Find $\bar{\xi}, b$ as in Lemma 5.10. Let

$$\Phi = \left\{\xi \in [0,1]^{rb} : \#\left\{k : \xi_k = \bar{\xi}_k\right\} > (D(\omega)+\delta)b\right\},$$

thus $R^{\omega}_{F_n}(\mathcal{E}_\omega) \subset [0,1]^{rb}\backslash\Phi$. For brevity, we shall denote the latter set as Φ^c,

$$\Phi^c = \left\{\xi \in [0,1]^{rb} : \#\left\{k : \xi_k \neq \bar{\xi}_k\right\} \geq rb - \lfloor (D(\omega)+\delta)b\rfloor\right\}.$$

We shall construct a continuous retraction π of Φ^c onto the $\lfloor (D(\omega)+\delta)b\rfloor + 1$ skeleton of the cube $I = [0,1]^{rb}$. For $k = 1, 2, \ldots$, we denote by J_k the set

$$J_k = \{\xi \in I : \#\{1 \leq i \leq rb : \xi_i \in \{0,1\}\} \geq k\},$$

and $\dim J_k = rb - k$.

Since $\bar{\xi}$ is in the interior of I, one can define $\pi_1 : I\backslash\{\bar{\xi}\} \to J_1$ by mapping each ξ to the intersection of the ray starting at $\bar{\xi}$ and passing through ξ and J_1. For each of the $(rb-1)$-dimensional cubes I' that comprise J_1, we can define a retraction on I' in a similar fashion using as a centre the projection of $\bar{\xi}$ onto I'. This will define a continuous retraction π_2 of Φ^c into J_2. As long as there is some intersection of Φ with the cubes in J_k, this process can be continued, and thus we get finally a continuous projection π of Φ^c onto J_{k_0}, with

$$k_0 = rb - \lfloor (D(\omega)+\delta)b\rfloor - 1.$$

Clearly, π satisfies the conditions of Lemma 5.11. Thus $\pi \circ R^{\omega}_{F_n} \succ \mathcal{A}_{F_n}(\omega)$.

Also, as $R^{\omega}_{F_n}(\mathcal{E}_\omega) \subset \Phi^c$,

$$\pi \circ R^{\omega}_{F_n} \subset J_{k_0},$$

the latter having topological dimension $\lfloor (D(\omega)+\delta)b\rfloor + 1$. To summarize: we have constructed an $\mathcal{A}_{F_n}(\omega)$ compatible function from $\mathcal{E}_\omega$ to a space of topological dimension $\leq (D(\omega)+\delta)b + 1$, and so by Lemma 4.5

$$\mathcal{D}\left(\mathcal{A}_{F_n}(\omega)\right) \leq \dim J_{k_0} \leq (D(\omega)+\delta)b + 1,$$

then

$$\lim_{n\to\infty} \frac{1}{|F_n|} \int \mathcal{D}\left(\mathcal{A}_{F_n}(\omega)\right) d\mathbb{P}(\omega) \leq \underline{\mathbb{E}\mathrm{mdim}}_{\mathrm{M}}^{\mathbb{S}}(T,d,G) + \delta.$$

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As $\delta > 0$ can be made as small as we like, we see that indeed

$$\mathbb{E}\mathrm{mdim}(T, G) \leq \underline{\mathbb{E}\mathrm{mdim}}_{\mathrm{M}}^{\mathbb{S}}(T, d, G).$$

□

Corollary 5.13. *If* T *has finite fibre topological entropy, then* $\mathbb{E}\mathrm{mdim}(T, G) = 0$.

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