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A note on residually finite groups†

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We shall prove the following

THEOREM. *There are exactly* $2^{2^{\aleph_0}}$ *isomorphism classes of residually finite groups with every proper quotient finite.*

As Wilson (3) has remarked, a residually finite group G with every proper quotient finite has cardinal at most $2^{\aleph_0}$. This is because G has a family $(H_i)_{i\in\omega}$ of non-trivial normal subgroups with $H_i > H_{i+1}$ for all i, and G can be embedded in the Cartesian product $\prod_{i\in\omega} G/H_i$ of the finite groups G/H_i. However, there are no more than $2^{2^{\aleph_0}}$ isomorphism classes of groups of cardinal at most $2^{\aleph_0}$; thus it only remains for us to display $2^{2^{\aleph_0}}$ non-isomorphic residually finite groups with every proper quotient finite. For this we need the

LEMMA. *Let* $n\in\omega$ *and* $n \geqslant 3$. *Let* A *be a local ring containing no nth roots of* 1 *except* 1. *Then for any non-trivial normal subgroup* H *of* $SL(n, A)$, *there is a non-zero ideal* $\mathfrak{a}$ *of* A *such that* H *contains the kernel of the map*: $SL(n, A)\to SL(n, A/\mathfrak{a})$ *induced by the natural map*: $A\to A/\mathfrak{a}$.

Proof. We note first that $\zeta(SL(n, A)) = 1$. From Theorem 1 and Proposition 3 of Klingenberg (2) it follows that, for some ideal $\mathfrak{a}$ of A, the kernels of the maps

$$GL(n, A)\to \frac{GL(n, A/\mathfrak{a})}{\zeta(GL(n, A/\mathfrak{a}))}$$

and

$$SL(n, A)\to SL(n, A/\mathfrak{a})$$

bound H above and below. But if $\mathfrak{a} = 0$, we have

$$H \leqslant \zeta(GL(n, A)) \cap SL(n, A) \leqslant \zeta(SL(n, A)) = 1$$

and the proof is finished.

† The author died in April 1970, and the present note has been prepared for publication from one of his manuscripts.

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Now let $n \in \omega$ and $n \geqslant 3$; let p be a prime with n prime to $p-1$. Write $\mathbf{Q}_p$ for the p-adic numbers and $\mathfrak{o}_p$ for the p-adic integers, and for each subfield $\mathfrak{k}$ of $\mathbf{Q}_p$ let

$$A_{\mathfrak{k}} = \mathfrak{k} \cap \mathfrak{o}_p.$$

Then each $A_{\mathfrak{k}}$ is a discrete valuation ring with maximal ideal generated by p, and, for each ideal $\mathfrak{a} \neq 0$ of $A_{\mathfrak{k}}$, both $A_{\mathfrak{k}}/\mathfrak{a}$ and $SL(n, A_{\mathfrak{k}}/\mathfrak{a})$ are finite. Because the non-zero ideals of $A_{\mathfrak{k}}$ have trivial intersection, it follows that $SL(n, A_{\mathfrak{k}})$ is residually finite, and the Lemma shows that every proper quotient of $SL(n, A_{\mathfrak{k}})$ is finite.

We consider $SL(n, \mathfrak{o}_p)$ as a topological group with all non-trivial normal subgroups of $SL(n, \mathfrak{o}_p)$ as a base of neighbourhoods of 1. For each ideal $\mathfrak{a}$ of $A_{\mathfrak{k}}$, we have $\mathfrak{a} = A_{\mathfrak{k}} \cap \mathfrak{a}\mathfrak{o}_p$, so that the kernel of the map: $SL(n, A_{\mathfrak{k}}) \to SL(n, A_{\mathfrak{k}}/\mathfrak{a})$ is the intersection of $SL(n, A_{\mathfrak{k}})$ with the kernel of the map: $SL(n, \mathfrak{o}_p) \to SL(n, \mathfrak{o}_p/\mathfrak{a}\mathfrak{o}_p)$. This remark and the Lemma together show that all the non-trivial normal subgroups of $SL(n, A_{\mathfrak{k}})$ form a base of neighbourhoods of 1 for the topology on $SL(n, A_{\mathfrak{k}})$ induced by the topology on $SL(n, \mathfrak{o}_p)$. Thus a group isomorphism: $SL(n, A_{\mathfrak{k}}) \to SL(n, A_{\mathfrak{k}'})$ for subfields $\mathfrak{k}$, $\mathfrak{k}'$ of $\mathbf{Q}_p$ is already a topological group isomorphism. However $SL(n, A_{\mathfrak{k}})$ is dense in $SL(n, \mathfrak{o}_p)$ (this is not quite immediate, but an *ad hoc* proof is not hard to construct; otherwise it follows directly from the characterization of $SL(n, A_{\mathfrak{k}})$ by transvections given in Klingenberg (2), Theorem 1); thus by Proposition 5 of Bourbaki (1) there is one and only one (topological) group morphism: $SL(n, \mathfrak{o}_p) \to SL(n, \mathfrak{o}_p)$ such that the diagram

$$\begin{array}{ccc} SL(n, A_{\mathfrak{k}}) & \longrightarrow & SL(n, A_{\mathfrak{k}'}) \\ \subseteq \downarrow & & \subseteq \downarrow \\ SL(n, \mathfrak{o}_p) & \longrightarrow & SL(n, \mathfrak{o}_p) \end{array}$$

commutes. Now $SL(n, A_{\mathbf{Q}})$ is countable and dense in $SL(n, \mathfrak{o}_p)$; hence there are at most $2^{\aleph_0}$ continuous maps: $SL(n, \mathfrak{o}_p) \to SL(n, \mathfrak{o}_p)$. But $\mathbf{Q}_p$ has cardinal $2^{\aleph_0}$ and so it has absolute transcendence degree $2^{\aleph_0}$, and it has $2^{2^{\aleph_0}}$ subfields $\mathfrak{k}$. It follows that the groups $SL(n, A_{\mathfrak{k}})$ fall into exactly $2^{2^{\aleph_0}}$ (group) isomorphism classes.

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