Now, if $n = p_1^{a_1}p_2^{a_2}p_3^{a_3}\ldots p_k^{a_k}$, then

$$d(n) = (1 + a_1)(1 + a_2)(1 + a_3)\ldots(1 + a_k).$$

Using this relation in (3), we obtain

$$d(L(n)) = (1 + 1)^{\pi(n) - \pi(\sqrt{n})}(1 + 2)^{\pi(\sqrt{n}) - \pi(\sqrt[3]{n})}(1 + 3)^{\pi(\sqrt[3]{n}) - \pi(\sqrt[4]{n})}\ldots.$$

Simplifying further we obtain

$$d(L(n)) = \left(2^{\pi(n) - \pi(\sqrt{n})}\right)\left(3^{\pi(\sqrt{n}) - \pi(\sqrt[3]{n})}\right)\left(4^{\pi(\sqrt[3]{n}) - \pi(\sqrt[4]{n})}\right)\ldots.$$

Finally this becomes

$$d(L(n)) = \prod_{k=1}^{\infty}(k + 1)^{\pi(\sqrt[k]{n}) - \pi(\sqrt[k+1]{n})}$$

which, after some algebra, gives us

$$d(L(n)) = \prod_{k=1}^{\infty}\left(1 + \frac{1}{k}\right)^{\pi(\sqrt[k]{n})}$$

as required.

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## 106.02 The rational distance problem revisited

The rational distance problem states that there are no points in the plane of the unit square having rational distances to the four vertices of the square. This is an open problem; see Guy [1, Problem D19, pp. 283-287]. Barbara [2] gave an elementary proof for the case where the point lies on the boundary of the square. In this Note, we present a self-contained proof for the case where the point lies on the axes of symmetry of the square. We begin with a lemma.

*Lemma* 1: If $x, y, z$ are rational numbers such that

$$x^4 - y^4 = z^2, \tag{1}$$

then $xyz = 0$.

*Proof*: Assume that $(x, y, z)$ is a rational solution to (1) with $xyz \neq 0$. Write $x = \frac{x'}{m}$, $y = \frac{y'}{m}$, $z = \frac{z'}{m}$, where $x', y', z', m \in \mathbb{Z}$. Then $(x', y', mz')$ is an integer solution to (1). Of course $x'y'mz' \neq 0$ since $xyz \neq 0$. Therefore $(|x'|, |y'|, |mz'|)$ is a positive integer solution to (1). Among the positive integer solutions to (1), let $(X, Y, Z)$ be a solution with the smallest $X$. Since $X^4 = Y^4 + Z^2$, and $X$ is the smallest, we have $\gcd(X, Y, Z) = 1$; otherwise $X, Y, Z$ share a prime factor $p$ for which $p^2$ must divide $Z$ so that

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$\left(\frac{X}{p}, \frac{Y}{p}, \frac{Z}{p^2}\right)$ is another positive integer solution to (1) with a smaller $x$. We also have $2 \nmid X$; otherwise, since $\gcd(X, Y, Z) = 1$, (1) implies $Y$, $Z$ must be odd, so that equation $X^4 = Y^4 + Z^2$ is impossible modulo 4. Since $(Z, Y^2, X^2)$ is a primitive Pythagorean triple (see [2] for Pythagorean triples) there exist coprime positive integers $a$, $b$ such that

$$X^2 = a^2 + b^2, \qquad Y^2 = a^2 - b^2, \qquad Z = 2ab \tag{2}$$

or

$$X^2 = a^2 + b^2, \qquad Y^2 = 2ab, \qquad Z = a^2 - b^2. \tag{3}$$

In (2), then

$$(XY)^2 = a^4 - b^4.$$

On the other hand, $a^2 < X^2$ so that $a < X$, contradicting the minimum assumption on $X$.

In (3), since $Y^2 = 2ab$, we can find $p, q \in \mathbb{Z}^+$ such that

$$Y = 2pq, \qquad a = 2p^2, \qquad b = q^2,$$

or

$$Y = 2pq, \qquad a = q^2, \qquad b = 2p^2.$$

In both cases

$$X^2 = 4p^4 + q^4.$$

Hence

$$(X^2 - q^2)(X^2 + q^2) = 4p^4. \tag{4}$$

Since $\gcd(a, b) = 1$, we have $\gcd(2p, q) = 1$. Thus $X$, $q$ are odd. Therefore (4) implies

$$X^2 - q^2 = 2u^4, \qquad X^2 + q^2 = 2v^4$$

where $u, v \in \mathbb{Z}^+$. Hence

$$X^2 = u^4 + v^4, \qquad q^2 = v^4 - u^4.$$

So

$$(Xq)^2 = \left(v^2\right)^4 - \left(u^2\right)^4.$$

Notice that $2v^2 \leqslant 2v^4 = X^2 + q^2 < 2X^2$. Thus $v < X$, contradicting the minimum assumption on $X$. The proof is complete.

Now, we solve the rational distance problem in the case when the point lies on a symmetrical axis of the unit square. There are three types of symmetrical axes: the diagonal axes, the horizontal axis, and the vertical axis. Since a diagonal of a unit square has length $\sqrt{2}$, an irrational number, we only need to consider the case of the vertical axis or the horizontal axis. Without loss of generality, we consider the vertical axis case. Consider

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Figure 1. (By scaling, we assume that each side of the square has length 2 instead of 1.)

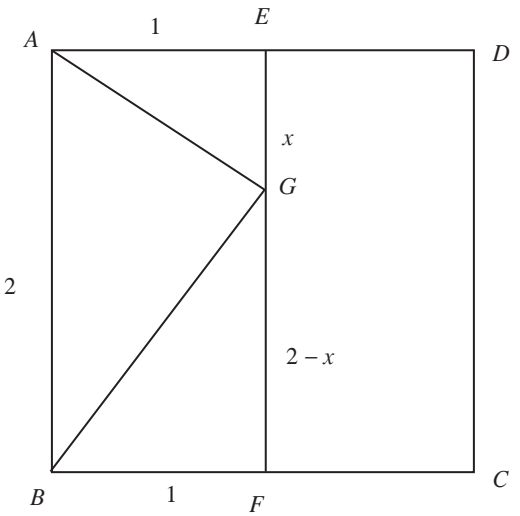



FIGURE 1

Each side of the square $ABCD$ has length 2. The point $G$ lies on the line $EF$ with $EG = x$ and $AE = BF = 1$. Assume that $GA\,(= GD)$ and $GB\,(= GC)$ are rational numbers. Then

$$x^2 + 1 = u^2, \qquad (2 - x)^2 + 1 = v^2, \tag{5}$$

where $u, v \in \mathbb{Q}$. So $4 - 4x = v^2 - u^2$. Hence $x \in \mathbb{Q}$. From (5), we have

$$\frac{x^2 - 4x + 5}{x^2 + 1} = t^2, \tag{6}$$

where $t = \dfrac{v}{u} \in \mathbb{Q}$. Thus

$$\left(t^2 - 1\right)x^2 + 4x + t^2 - 5 = 0.$$

Because $x$ is a rational number, it follows that $4 - \left(t^2 - 1\right)\left(t^2 - 5\right)$ is a rational square. Let $4 - \left(t^2 - 1\right)\left(t^2 - 5\right) = s^2$, where $s \in \mathbb{Q}$. Then

$$-\left(t^4 - 6t^2 + 1\right) = s^2. \tag{7}$$

Notice the following identity:

$$\left(t^4 - 6t^2 + 1\right)^2 + \left(4t\left(t^2 - 1\right)\right)^2 = \left(t^2 + 1\right)^4.$$

Therefore

$$\left(t^2 + 1\right)^4 - s^4 = \left(4t\left(t^2 - 1\right)\right)^2.$$

So $\left(t^2 + 1, s, 4t\left(t^2 - 1\right)\right)$ is a rational solution to (1). Hence Lemma 1 implies $s = 0$ or $t = 0$ or $t^2 - 1 = 0$. If $s = 0$, then (7) forces

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$t^4 - 6t^2 + 1 = 0$, so that $t^2 = 3 \pm 2\sqrt{2}$, impossible since $t \in \mathbb{Q}$. If $t = 0$, then (7) forces $s^2 = -1$, impossible. If $t^2 = 1$, then (6) forces $x = 1$. Then (5) forces $u^2 = 2$, impossible since $u \in \mathbb{Q}$. So there are no points on the line $EF$ with rational distances to $A$, $B$, $C$, $D$. The proof is complete.

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## 106.03 Real-rooted polynomials and a generalised Hermite-Sylvester theorem

A polynomial is *real-rooted* if all of its roots are real. A real-valued function $\Phi(x_1, \ldots, x_n)$ is *positive semidefinite* if $\Phi(x_1, \ldots, x_n) \geqslant 0$ for all vectors $(x_1, \ldots, x_n) \in \mathbb{R}^n$. An $n$-ary $m$-adic form is a polynomial in $n$ variables that is homogeneous of degree $m$.

Let $f(t)$ be a polynomial of degree $n \geqslant 1$ with real coefficients, and let $\lambda_1, \ldots, \lambda_n$ be the (not necessarily distinct) roots of $f(t)$. Hermite and Sylvester [1, 2] proved that the polynomial $f(t)$ is real-rooted if, and only if, the quadratic form in $n$ variables

$$\Phi_2(x_1, \ldots, x_n) = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \sum_{\ell=1}^{n} \lambda_\ell^{j_1+j_2-2} x_{j_1} x_{j_2}$$

is positive semidefinite. This Note generalises the Hermite-Sylvester theorem. We define, for every positive integer $m$, the following $n$-ary $m$-adic form:

$$\Phi_m(x_1, \ldots, x_n) = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_m=1}^{n} \sum_{\ell=1}^{n} \lambda_\ell^{\sum_{k=1}^{m}(j_k-1)} x_{j_1} x_{j_2} \cdots x_{j_m}. \qquad (1)$$

We shall prove that the polynomial $f(t)$ is real-rooted if, and only if, $\Phi_m(x_1, \ldots, x_n)$ is positive semidefinite for all positive even integers $m$.

*Lemma* 1: For all positive integers $m$,

$$\Phi_m(x_1, \ldots, x_n) = \sum_{\ell=1}^{n} p(\lambda_\ell)^m \qquad (2)$$

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