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A Lower Bound on the Size of Lipschitz Subsets in Dimension 3

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A set $S \subset \mathbb{R}^d$ is *C-Lipschitz in the x_i-coordinate*, where $C > 0$ is a real number, if, for every two points $a, b \in S$, we have $|a_i - b_i| \leqslant C \max\{|a_j - b_j| : j = 1, 2, \ldots, d,\ j \neq i\}$. Motivated by a problem of Laczkovich, the author asked whether every n-point set in $\mathbb{R}^d$ contains a subset of size at least $cn^{1-1/d}$ that is C-Lipschitz in one of the coordinates, for suitable constants C and $c > 0$ (depending on d). This was answered negatively by Alberti, Csörnyei and Preiss. Here it is observed that a combinatorial result of Ruzsa and Szemerédi implies the existence of a 2-Lipschitz subset of size $n^{1/2}\varphi(n)$ in every n-point set in $\mathbb{R}^3$, where $\varphi(n) \to \infty$ as $n \to \infty$.

1. Introduction

Let us call a set $S \subset \mathbb{R}^d$ *C-Lipschitz in the x_i-coordinate*, where $C > 0$ is a real number, if, for every two points $a, b \in S$, we have $|a_i - b_i| \leqslant C \max\{|a_j - b_j| : j \in [d],\ j \neq i\}$ (where $[d] = \{1, 2, \ldots, d\}$). Equivalently, S is C-Lipschitz in the x_d-coordinate, say, if there is a C-Lipschitz function $\mathbb{R}^{d-1} \to \mathbb{R}$ whose graph contains S. (Here the distance in $\mathbb{R}^{d-1}$ is measured in the ℓ_∞ norm, but very similar considerations apply with respect to the Euclidean norm, up to the values of the constants.)

Let $\mathrm{LS}_{d,C}(n)$ denote the maximum integer k such that, for every n-point set $P \subset \mathbb{R}^d$, there exists $i \in [d]$ and a k-point subset $S \subseteq P$ that is C-Lipschitz in the x_i-coordinate. It is not difficult to show that $\mathrm{LS}_{2,1}(n) \geqslant n^{1/2}$ (a proof is sketched below), and this implies $\mathrm{LS}_{d,1}(n) \geqslant n^{1/2}$ (by projection onto the first two coordinates).

The following question was raised in [5]: Given $d \geqslant 3$, are there constants $C > 0$ and $c > 0$ such that $\mathrm{LS}_{d,C}(n) \geqslant cn^{1-1/d}$ for all n? The grid $[m]^d$ shows that $\mathrm{LS}_{d,C}(m^d) \leqslant m^{d-1}$

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(for all C), and it seems natural to suspect the grid to be an extremal example. Moreover, for any n-point set in $\mathbb{R}^d$, at least one of the 'shadows', *i.e.*, projections to coordinate hyperplanes, has at least $n^{1-1/d}$ points [8], and so there is a C-Lipschitz subset in some coordinate system with that many points for *some* C (depending on the set).

The problem of C-Lipschitz subsets was motivated by a beautiful question of Laczkovich [4]: Is it true that, for any set $E \subseteq \mathbb{R}^d$ of positive Lebesgue measure, there exists a Lipschitz mapping $f : \mathbb{R}^d \to Q = [0,1]^d$ with $f(E) = Q$? A positive answer for $d = 2$ was obtained by Preiss [6]. An alternative proof, based on the fact that $\mathrm{LS}_{2,1}(n) \geqslant n^{1/2}$, was given in [5]. It was also noted that $\mathrm{LS}_{d,1}(n) \geqslant \mathrm{const} \cdot n^{1-1/d}$ would imply a positive answer to Laczkovich's question in the same way. However, the suggested bound $\mathrm{LS}_{d,1}(n) \geqslant \mathrm{const} \cdot n^{1-1/d}$ turned out to be false in a very strong sense.

First of all, Tardos observed that $\mathrm{LS}_{3,1}(n) = O(n^{1/2})$, as is witnessed by the set $\{(i, j, i+j);\ i, j \in [m]\}$. This example still leaves the possibility that $\mathrm{LS}_{3,C}(n)$ could be of the order $n^{2/3}$ for $C = 2$, say. However, this was recently disproved by Alberti, Csörnyei and Preiss [1]. Their example shows that, for any C, we have $\mathrm{LS}_{3,C}(n) \leqslant n^{2/3-\delta}$ for some $\delta = \delta(C) > 0$. (The problem was also considered by Szabó and Tardos [9], who suggested some combinatorial generalizations and provided negative examples for them, which do not seem to transfer to the original problem, however.)

In this note, we show that $\mathrm{LS}_{3,2}(n)$ exceeds the trivial lower bound of $n^{1/2}$ by a factor tending to infinity.

Theorem 1.1. *We have* $\mathrm{LS}_{3,2}(n) \geqslant n^{1/2}\varphi(n)$, *where* $\varphi(n)$ *is a function tending to* ∞ *as* $n \to \infty$.

The lower bound for $\varphi(n)$ obtained from our proof has order of magnitude approximately $\log^* n$ (where the iterated logarithm function $\log^* x$ is defined by $\log^* x = 0$ for $x \leqslant 1$ and $\log^* x = 1 + \log^*(\log_2 x)$ for $x > 1$). This particular approach cannot yield anything better than about $e^{\sqrt{\log n}}$ for $\varphi(n)$. This follows from a lower bound for a lemma of Ruzsa and Szemerédi used in the proof below (the lower bound is based on a well-known construction, due to Behrend [2], of a dense set of integers without 3-term arithmetic progressions).

2. Proof

Let $C \geqslant 1$ be fixed. For points $a, b \in \mathbb{R}^d$, we define $a \prec_i b$ if $b_i - a_i > C \cdot \max\{|a_j - b_j| : j \in [d],\ j \neq i\}$. That is, if we think of the x_i-axis as being vertical, b lies 'steep above' a. Further, we write $a \preceq_i b$ if $a = b$ or $a \prec_i b$. Clearly, each $\preceq_i$ is a partial ordering on $\mathbb{R}^d$. We note that $S \subseteq \mathbb{R}^d$ is C-Lipschitz in the x_i-coordinate if and only if it is an antichain in the ordering $\preceq_i$ (no two elements of S are comparable under $\preceq_i$).

Since $C \geqslant 1$, we have $\prec_i \cap \prec_j = \emptyset$ whenever $i \neq j$. In particular, for $d = 2$, every chain in $\preceq_1$ is an antichain in $\preceq_2$. By a simple consequence of a well-known theorem of Dilworth, every n-point partially ordered set contains an antichain of at least $n^{1/2}$ elements or a chain of at least $n^{1/2}$ elements. Applying this to $\preceq_1$, we have the proof of $\mathrm{LS}_{2,1}(n) \geqslant n^{1/2}$ promised above.

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Now we prove Theorem 1.1. Let $d = 3$ and consider the partial orderings $\preceq_1, \preceq_2, \preceq_3$ as above with $C = 2$. All the required combinatorial information about them, besides the fact $\prec_i \cap \prec_j = \emptyset$ mentioned above, is contained in the following.

Lemma 2.1. *Let $S_i = \{\{a, b\} : a \preceq_i b\}$ be the symmetrization of the relation $\preceq_i$. Then there are no three points $a, b, c \in \mathbb{R}^3$ with $\{a, b\} \in S_1$, $\{b, c\} \in S_2$, and $\{a, c\} \in S_3$.*

Proof. Informally, if we go from a almost parallel to the x_1-axis and then continue almost parallel to the x_2-axis, we cannot get back to a in a direction almost parallel to the x_3-axis. Less informally, suppose that a, b, c contradict the claim in the lemma. By symmetry, we may assume $|a_1 - b_1| \geqslant |b_2 - c_2|$. Then $|a_3 - b_3| < \frac{1}{2}|a_1 - b_1|$ and $|b_3 - c_3| < \frac{1}{2}|b_2 - c_2| \leqslant \frac{1}{2}|a_1 - b_1|$, which gives $|a_3 - c_3| < |a_1 - b_1|$. But $|a_1 - c_1| \geqslant |a_1 - b_1| - |b_1 - c_1| > |a_1 - b_1| - \frac{1}{2}|b_2 - c_2| \geqslant |a_1 - b_1| - \frac{1}{2}|a_1 - b_1| = \frac{1}{2}|a_1 - b_1|$. □

To prove Theorem 1.1, let $P \subset \mathbb{R}^3$ be an n-point set. Suppose that none of the partially ordered sets $(P, \preceq_i)$ has an antichain of size larger than s. Then, using Dilworth's theorem [3] in its full strength, we have decompositions $P = P_{i1} \dot\cup P_{i2} \dot\cup \cdots \dot\cup P_{is}$, $i = 1, 2, 3$, where each P_{ik} is a chain in $(P, \preceq_i)$.

We define a 3-uniform hypergraph (system of triples) H. The vertex set $V(H) = [3] \times [s]$, and the edge set $E(H) = \{\{(1, k_1), (2, k_2), (3, k_3)\} : P_{1k_1} \cap P_{2k_2} \cap P_{3k_3} \neq \emptyset\}$. Every point $p \in P$ determines exactly one triple $e_p \in E(H)$ (since p lies in exactly one P_{1k_1}, exactly one P_{2k_2}, and exactly one P_{3k_3}), so there are n edges and $3s$ vertices. Moreover, we note that H has the following two properties.

(C1) Every two edges intersect in at most one vertex (in other words, an edge is uniquely determined by any two of its three vertices). This is because $|P_{ik} \cap P_{j\ell}| \leqslant 1$ whenever $i \neq j$.

(C2) There is no triangle, *i.e.*, no three distinct pairwise intersecting edges. A little thought reveals that this is a re-formulation of Lemma 2.1.

Ruzsa and Szemerédi [7] proved, using Szemerédi's well-known Regularity Lemma, that any system of triples on m vertices satisfying (C1) and (C2) has at most $o(m^2)$ triples. Therefore, in our setting, $n = o(s^2)$ and Theorem 1.1 follows. □

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