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Bifurcation to localized oscillations

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A singular perturbation method is applied to a system of two weakly coupled strongly nonlinear but non-identical oscillators. For certain parameter regimes, stable localized solutions exist for which the amplitude of one oscillator is an order of magnitude smaller than the other. The leading-order dynamics of the localized states is described by a new system of coupled equations for the phase difference and scaled amplitudes. The degree and stability of the localization has a non-trivial dependence on coupling strength, detuning, and the bifurcation parameter. Three distinct types of localized behaviour are obtained as solutions to these equations, corresponding to phase-locking, phase-drift, and phase-entrainment. Quantitative results for the phases and amplitudes of the oscillators and the stability of these phenomena are expressed in terms of the parameters of the model.

1 Introduction

Recent studies have been concerned with arrays of weakly coupled oscillators which can exhibit localization; that is, the amplitude of one or more of the oscillators is small compared to the amplitude(s) of the others. These studies have been mainly concerned with conservative systems. Existence of time-periodic, spatially localized solutions, which have been termed 'breathers' in the more general context of Ablowitz *et al.* [1], has been demonstrated by MacKay & Aubry [2] for general Hamiltonian or time-reversible systems. A general methodology for approximating localized 'nonsimilar' normal mode solutions for conservative n-degree-of-freedom systems with cyclic symmetries is outlined and applied by Vakakis & Cetinkaya [3]. These studies give informative results for localization in conservative arrays of identical oscillators with weak nearest-neighbor coupling. There are several important topics not addressed previously, such as phase dynamics of localized solutions, localization in non-conservative systems (coupled limit-cycle oscillators), and localization in a system where not all oscillators are identical, which we address in this paper. A system of two coupled limit-cycle oscillators is analyzed in detail. By developing an analytical theory for localized states, we expect to find necessary conditions for localization in larger arrays of coupled oscillators. Localization has been studied for systems modeling anharmonic crystals by MacKay & Aubry [2] and for large space engineering structures by Vakakis & Cetinkaya [3]. More recently Kuske & Erneux [4] have analysed localization in a system of two coupled solid state lasers, using the approach of this paper as a motivation and guide.

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There is a considerable literature on the phase dynamics of systems of coupled oscillators, and we review only work relevant to this paper. There are many applications for which the analytical study is performed on a reduced system which involves only phase equations, for example in the work of Ermentrout [7], Kopell [8] and Watanabe & Strogatz [9]. This is typically the case when the leading-order amplitude is assumed to be $O(1)$ and/or the oscillators are identical. Thus, these analyses do not correspond to localized solutions. In fact, we show in this paper that an analysis of both amplitude and phase equations is necessary for describing the dynamics of localized solutions.

We recall some results for two-degree-of-freedom systems and contrast them with the present study. Aronson *et al.* [10] consider a linearly coupled system of two identical oscillators. This system is of the same form as that considered in this paper, namely (2.1) below with $\lambda_1 = \lambda_2$ and $\omega_1 = \omega_2$. Aronson *et al.* [11] study a system of two oscillators, not necessarily identical, assuming $\lambda_1 = \lambda_2$. This assumption is the critical difference between the analyses of Aronsom *et al.* [10, 11] and that of this paper. As is shown in our analysis of localized solutions of this system, it is critical that $\lambda_1 \neq \lambda_2$ and $\lambda_2 \ll 1$. Although the oscillators are not necessarily identical in the study of Aronson [11], their analysis is primarily concerned with the phase dynamics, while in this paper we study the dynamics of amplitudes coupled with phase differences within the context of localization.

Neu [6] developed a technique for studying the phase dynamics of a weakly, linearly coupled system of two oscillators which are identical to leading-order. While this method is applied to a general class of oscillators, it does not allow a study of localized oscillations. That is, the leading-order amplitude of the oscillators is taken to be equal. In this context, the phase difference varies on a slow time scale and, to leading-order, does not depend on the correction to the amplitude. In contrast, localized behaviour is described by coupled amplitude and phase equations which vary on a fast time scale (see (2.6)).

In this paper a method based on singular perturbation techniques and ideas described by Neu [6] is developed for studies of two weakly coupled non-identical limit-cycle oscillators. We show that the degree of localization depends non-uniformly on the coupling strength, the detuning of the oscillators, and the bifurcation parameter λ_2. One complication in the analysis of localized solutions is the leading-order coupling of the phase difference and amplitude equations, which does not occur in the work of several authors [6–9]. These localized solutions can exhibit phase-locking, phase-entrainment (sometimes referred to as phase trapping) or phase-drift. In all of these cases quantitative results for the leading-order amplitudes and phases of the oscillators are obtained. The analysis yields the stable localized behaviours, which is also demonstrated with the numerical solution for the full system. The numerical results have been obtained using the *lsoda* package, namely the Livermore solver for ordinary differential equations, with automatic method switching for stiff and nonstiff problems.

One should note that the phenomenon of localization is different from that of oscillator death studied by Ermentrout [5], in which oscillations of one or more of the oscillators cease. In the case of oscillator death, sufficiently large coupling results in the oscillators pull each other from their limit cycles to a state of rest. In contrast, localization typically appears in systems of weakly coupled oscillators.

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The paper is organized as follows. In §2 we derive the phase and amplitude equations which correspond to the localized solutions. In §3 the different possible localized solutions are obtained together with the parameter values corresponding to each. The numerical simulations of the model agree with the description obtained in the analysis. In §4 we give a summary of the different localization states which are present in the system in terms of the parameters of the model.

2 Formulation and amplitude equations

We consider the model

$$\begin{aligned}
\dot{x}_1 &= \lambda_1 x_1 + \omega_1 y_1 - x_1(x_1^2 + y_1^2) + \epsilon(x_2 - x_1) \\
\dot{y}_1 &= \lambda_1 y_1 - \omega_1 x_1 - y_1(x_1^2 + y_1^2) + \epsilon(y_2 - y_1) \\
\dot{x}_2 &= \lambda_2 x_2 + \omega_2 y_2 - x_2(x_2^2 + y_2^2) + \epsilon(x_1 - x_2) \\
\dot{y}_2 &= \lambda_2 y_2 - \omega_2 x_2 - y_2(x_2^2 + y_2^2) + \epsilon(y_1 - y_2)
\end{aligned} \tag{2.1}$$

where the coupling is weak, $\epsilon \ll 1$. This system has been studied by Aronson *et al.* [10, 11] as a canonical model for coupled oscillations in biological, physiological and chemical phenomena, and we refer the reader there for further discussions of the applications. As noted in the Introduction, a significant feature in these previous analyses is the fact that $\lambda_1 = \lambda_2$.

In the uncoupled case $\epsilon = 0$, a limit cycle solution exists for each oscillator and is given by

$$x_j = r_j \cos\theta_j, \quad y_j = r_j \sin\theta_j, \quad \theta_j = -\omega_j t. \tag{2.2}$$

For $\lambda_j < 0$ the only solution of this form is $r_j = 0$, that is, the oscillators are at rest. However, for $\lambda_j > 0$ the solution $r_j = 0$ is unstable, while the solution $r_j = \sqrt{\lambda_j}$ is stable.

A preliminary analysis of (2.1) shows that for $0 < \lambda_1 = O(1)$ and $0 < \lambda_2 = O(1)$ localized solutions are not stable if $\epsilon \to 0$. This can be demonstrated using a regular perturbation expansion in ϵ.

In looking for localized solutions for the coupled case ($\epsilon \neq 0$), it is therefore appropriate first to consider the case $\lambda_1 > 0$, $\lambda_1 = O(1)$ and $\lambda_2 \ll 1$. If $\epsilon = 0$, the first oscillator has $O(1)$ amplitude while the amplitude of the second is small ($O(\sqrt{\lambda_2})$). This suggests seeking a solution of (2.1) of the form

$$\begin{aligned}
x_1 &= \sqrt{\lambda_1}(1 + \epsilon A_1(t)) \cos\theta_1(t) \\
y_1 &= \sqrt{\lambda_1}(1 + \epsilon A_1(t)) \sin\theta_1(t) \\
x_2 &= \delta A_2(t) \cos\theta_2(t) \\
y_2 &= \delta A_2(t) \sin\theta_2(t), \quad \delta \ll 1.
\end{aligned} \tag{2.3}$$

The strength of the localization is described by the parameter δ which depends non-uniformly on the relative magnitude of the three independent parameters ϵ, $\omega_2 - \omega_1$ and λ_2. It is shown below that it is consistent to assume that the magnitude of δ is at least as large as that of ϵ. Substituting (2.3) into (2.1), we find the following equations for the

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amplitudes and phases:

$$\begin{aligned}
\dot{\theta}_1 &= -\omega_1 + \epsilon\delta \frac{A_2 \sin(\theta_2 - \theta_1)}{\sqrt{\lambda_1}(1 + \epsilon A_1)}, \\
\dot{A}_1 &= A_1(-2\lambda_1 - \epsilon) - 3\epsilon\lambda_1 A_1^2 - \epsilon^2 \lambda_1 A_1^3 - 1 + \delta \frac{A_2}{\sqrt{\lambda_1}} \cos(\theta_2 - \theta_1), \\
\dot{\theta}_2 &= -\omega_2 - \frac{\epsilon}{\delta}\sqrt{\lambda_1}\frac{\sin(\theta_2 - \theta_1)}{A_2} - \frac{\epsilon^2}{\delta}\frac{A_1}{A_2}\sqrt{\lambda_1}\sin(\theta_2 - \theta_1), \\
\dot{A}_2 &= (\lambda_2 - \delta^2 A_2^2)A_2 + \frac{\epsilon}{\delta}\sqrt{\lambda_1}(1 + \epsilon A_1)\cos(\theta_2 - \theta_1) - \epsilon A_2.
\end{aligned} \tag{2.4}$$

It is well known [11] that this system can be reduced to a three-dimensional system. We demonstrate this in a manner which highlights the difference between the analysis of localized and non-localized solutions. This demonstration also leads to a convenient formulation for the asymptotic analysis.

To leading-order the equations for A_2 and θ_2 decouple from the equations for A_1 and θ_1. For clarity we demonstrate this for $\delta = \epsilon$ and $\lambda_2 \ll 1$, noting that a similar result can be obtained for $\epsilon \ll \delta \ll 1$ and $\lambda_2 \ll 1$. Then the leading order equations are obtained for $\delta = \epsilon \to 0$ and $\lambda_2 = 0$:

$$\begin{aligned}
\dot{\theta}_1 &= -\omega_1 + O(\epsilon) \\
\dot{A}_1 &= A_1(-2\lambda_1) - 1 + O(\epsilon) \\
\dot{\theta}_2 &= -\omega_2 - \sqrt{\lambda_1}\frac{\sin(\theta_2 - \theta_1)}{A_2} + O(\epsilon) \\
\dot{A}_2 &= \sqrt{\lambda_1}\cos(\theta_2 - \theta_1) + O(\epsilon).
\end{aligned} \tag{2.5}$$

It is possible to solve for the leading-order behaviour of θ_1 and A_1 and substitute this in the equations for θ_2 and A_2. Typically one is interested in conditions for phase-locking and phase-drift, so that it is convenient to look at the equation for the phase difference $\psi = \theta_2 - \theta_1$ rather than θ_2. From (2.5) we find that the leading-order equations for A_2 and ψ are coupled:

$$\begin{aligned}
\dot{\psi} &= \omega_1 - \omega_2 - \sqrt{\lambda_1}\frac{\sin\psi}{A_2} + O(\epsilon) \\
\dot{A}_2 &= \sqrt{\lambda_1}\cos\psi + O(\epsilon).
\end{aligned} \tag{2.6}$$

Both A_2 and ψ depend on the original time scale t. This is significantly different from previous analyses of non-localized behaviour, for example in References [6]–[9], which lead to a decoupled equation for the phase difference which varies on a slow time scale. Thus, the leading-order equations (2.6) demonstrate that a description of localized behaviour requires a study of coupled phase and amplitude equations.

Note that the equations (2.6) describe simple harmonic motion. Neglecting $O(\epsilon)$ terms and setting $u = A_2\cos\psi$, $v = A_2\sin\psi$ and $\omega = \omega_2 - \omega_1$, these equations become

$$\begin{aligned}
\dot{u} &= \omega v + \sqrt{\lambda_1} \\
\dot{v} &= -\omega u.
\end{aligned} \tag{2.7}$$

For $\omega \neq 0$ the point $(u, v) = (0, -\sqrt{\lambda_1}/\omega)$ corresponds to a phase-locked state, where $\psi =$ constant. The possible states for which ψ is not a constant are solutions of (2.7) which lie

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on circles of radius r centered at $(0, -\sqrt{\lambda_1}/\omega)$,

$$u = r\cos\omega t, \quad v = -\sqrt{\lambda}/\omega - r\sin\omega t. \tag{2.8}$$

In the following section we show that phase-entrainment, for which $0 < |\psi| < \pi$ and $r < \sqrt{\lambda}/\omega$, is described by a limit cycle in the $A_2 - \psi$ plane. Phase drift corresponds to $r > \sqrt{\lambda}/\omega$ and $|\psi|$ increasing with time. Higher-order terms are included in the following analysis in order to determine the stability of these localized solutions.

As motivated by (2.7), it is mathematically useful to introduce the change of variables $U(t) = A_2(t)\cos\psi(t)$ and $V(t) = A_2(t)\sin\psi(t)$. Now (2.4) is transformed into

$$\begin{aligned}
\dot{\theta}_1 &= -\omega_1 + \epsilon\delta\frac{V}{\sqrt{\lambda_1}(1+\epsilon A_1)} \\
\dot{A}_1 &= (-2\lambda_1 - \epsilon)A_1 - 1 + \delta\frac{U}{\sqrt{\lambda_1}} - 3\epsilon\lambda_1 A_1^2 - \epsilon^2\lambda_1 A_1^3
\end{aligned} \tag{2.9}$$

$$\dot{U} = \omega V + \frac{\epsilon}{\delta}\sqrt{\lambda_1}(1+\epsilon A_1) + \epsilon\delta\frac{V^2}{\sqrt{\lambda_1}(1+\epsilon A_1)} + (c - \delta^2(U^2+V^2))U$$

$$\dot{V} = -\omega U - \epsilon\delta\frac{UV}{\sqrt{\lambda_1}(1+\epsilon A_1)} + (c - \delta^2(U^2+V^2))V, \tag{2.10}$$

where

$$\omega = \omega_2 - \omega_1, \quad c = \lambda_2 - \epsilon. \tag{2.11}$$

The notation U and V is used for the solution of the full equations (2.10) to distinguish it from the solution of the reduced equations (2.7). Equations (2.10) can be combined into one equation by introducing $B = U + iV$, thus obtaining the three-dimensional system mentioned above. The equation for B is given in (3.4). Higher order terms have been kept in these equations since, as demonstrated in the next section, they play an important role in determining the stability of phase-locked, phase-entrained, phase-drift conditions. The results demonstrate that phase-locked solutions are stable for ω below a critical value. Then ψ takes a constant value between 0 to $\pi/2$, which varies continuously with ω, λ_2, and ϵ. The phase-locked solution goes unstable when the difference $|\omega_2 - \omega_1|$ is sufficiently large and the resonant effects of the coupling and nonlinear interaction result in phase-entrainment or drift.

3 Localized solutions

In this section the solutions for (2.9)-(2.10) are determined for varying values of the parameters $\omega \equiv \omega_2 - \omega_1$, the difference in the linear frequencies of the oscillators, and $\lambda_2 \ll 1$.

First we note that the leading-order solution of (2.9) for θ_1 and A_1 is given by

$$\begin{aligned}
&\theta_1 \sim -\omega_1 t, \\
&A_1 \sim a_0 e^{(-2\lambda_1 - \epsilon)t} - \frac{1}{(2\lambda_1 + \epsilon)} + \epsilon A_{1,1} + \delta A_{1,2} \equiv A_{1,0} + \epsilon A_{1,1} + \delta A_{1,2}.
\end{aligned} \tag{3.1}$$

The reader may note that ϵ has been included in the definition of $A_{1,0}$. This is only a matter of convenience; we could include the corresponding terms in the higher-order

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contributions to A_1. These terms do not play a role in the stability analysis of localized solutions, which is the main concern of the following analysis. Therefore (3.1) is a convenient way to write A_1 which makes the following analysis cleaner, but does not change the results.

The term $A_{1,2}$ depends on U and is determined below for cases in which it plays a significant role in the higher-order contributions to U and V. Next we concentrate on the solution for U and V, or equivalently A_2 and ψ.

3.1 The case $\omega = \omega_2 - \omega_1 = 0$

For $\omega = 0$ it is more straightforward to determine the solution of (2.4) for A_2 and ψ than to study (2.10). In this case we will find that the correct ordering of small parameters is either $\epsilon = O(\delta^3)$ or $\epsilon \ll \delta^3$, depending on the circumstances. The equilibrium solution and the correct scaling for δ are obtained by solving the leading-order problem for A_2 and ψ,

$$\dot{\psi} \sim -\frac{\epsilon}{\delta}\sqrt{\lambda_1}\frac{\sin\psi}{A_2}$$
$$\dot{A}_2 \sim (\lambda_2 - \delta^2 A_2^2)A_2 + \frac{\epsilon}{\delta}\sqrt{\lambda_1}\cos\psi. \tag{3.2}$$

There is a stable equilibrium point for the system at

$$\psi = 0$$
$$A_2 = R_2 \quad \text{where } (\lambda_2 R_2 - \delta^2 R_2^3) + \frac{\epsilon}{\delta}\sqrt{\lambda_1} = 0. \tag{3.3}$$

Since $\psi = 0$, the oscillations are phase-locked with $\theta_1 = \theta_2$. The magnitude δ of the amplitude of the second oscillator, indicating the degree of localization, depends on the relative magnitudes of λ_2 and ϵ. For $\lambda_2 \ll \epsilon^{2/3}$ the stable solution corresponds to $\delta = \epsilon^{1/3}$ and $R_2 = (\sqrt{\lambda_1})^{1/3} + O(\lambda_2/\epsilon^{2/3})$. For $\lambda_2 = O(\epsilon^{2/3})$ or larger, $\delta = \sqrt{\lambda_2}$, with $R_2 \sim 1$ for $\lambda_2 \gg \epsilon^{2/3}$. These phase-locked solutions are observed in a numerical solution of the full system (2.1), as shown in Fig. 1. Note that the amplitudes of the oscillators are as predicted by the analysis; $\lambda_2 \ll \epsilon^{2/3}$ and the amplitude of the second oscillator is $O(\epsilon^{1/3})$.

3.2 The case $\omega = \omega_2 - \omega_1 = O(1)$, $|\lambda_2 - \epsilon| = O(\epsilon^2)$

For $\omega \neq 0$ we consider (2.10) and formulate an equation for $B = U + iV$,

$$\dot{B}(t) = -i\omega B(t) + \frac{\epsilon}{\delta}\sqrt{\lambda_1}(1 + \epsilon A_1) - \frac{\epsilon\delta}{2}\frac{B(B - B^*)}{\sqrt{\lambda_1}(1 + \epsilon A_1)} + (c - \delta^2|B|^2)B. \tag{3.4}$$

The parameter $c = \lambda_2 - \epsilon$ is small and the case $c = O(\epsilon^2)$ is considered in detail, for which it is appropriate to take $\delta = \epsilon$. The result for $c \gg \epsilon^2$ is obtained in a similar manner and is discussed briefly in §3c.

The leading-order solution, B_0, follows from substituting $A_1 = A_{1,0}$ in (3.4) and neglect-

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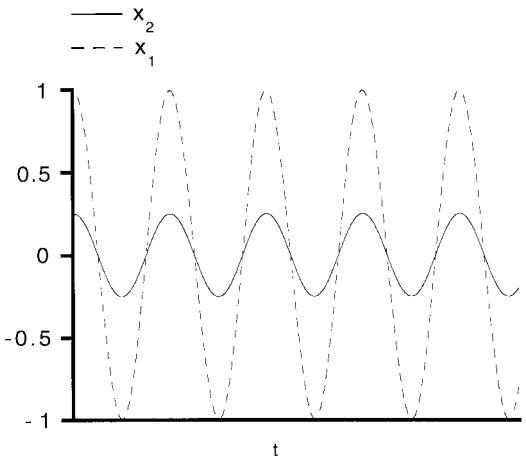


FIGURE 1. Stable phase-locked solution obtained numerically from (2.1) for $\omega = 0$, $\lambda_1 = 1$, $\lambda_2 \ll \epsilon^{2/3}$ ($\epsilon = 1/64$). Note that $\max(x_2) = O(\epsilon^{1/3})$, as predicted by the analysis for this range of parameters (see §3a).

ing $O(\epsilon^2)$ terms,

$$B_0 = \frac{\sqrt{\lambda_1}}{i\omega} + \frac{\epsilon\sqrt{\lambda_1}}{i\omega(2\lambda_1)} + a(\tau)e^{-i\omega t} + c_1 e^{(-2\lambda_1-\epsilon)t}. \tag{3.5}$$

The long-time behaviour of the solution B is considered below. Therefore it is unnecessary to determine the coefficient c_1 of the decaying terms. A long time scale $\tau = \epsilon^2 t$ is introduced in the amplitude $a(\tau)$ of the oscillatory terms in order to employ the method of multiple scales to study the stability of the periodic solutions (3.5). Then B is written as a perturbation expansion $B = B_0 + \epsilon^2 B_2 + \ldots$ and an equation for $a(\tau)$ is determined from the solvability condition that eliminates secular terms (growing in time) from the solution B_2:

$$a'(\tau) = (\kappa - |a|^2 - \frac{2\sqrt{\lambda_1}}{\omega^2})a(\tau), \tag{3.6}$$

where

$$\kappa = \frac{c}{\epsilon^2} + \frac{1}{2(2\lambda_1 - i\omega)} + \frac{3}{2}\frac{i}{\omega} \equiv \kappa_r + i\kappa_i. \tag{3.7}$$

The details of the derivation of (3.6) are given in the Appendix. From (3.6) one can determine the long time behaviour of B_0. Of course, one could compute higher-order terms in the perturbation analysis, obtaining higher-order corrections to the stability results and the asymptotic expansion of B. Since these terms are of higher-order for all time, they do not affect the leading-order stability results described below.

The behaviour of the solution depends on the sign of

$$q \equiv \kappa_r - \frac{2\lambda_1}{\omega^2}. \tag{3.8}$$

For $\omega^2 < 2\lambda_1/\kappa_r$, $a(\tau) \to 0$ as $\tau \to \infty$, corresponding to

$$\begin{aligned} A_2 &\sim \frac{\sqrt{\lambda_1}}{\omega}, \\ \psi &\sim -\pi/2. \end{aligned} \tag{3.9}$$

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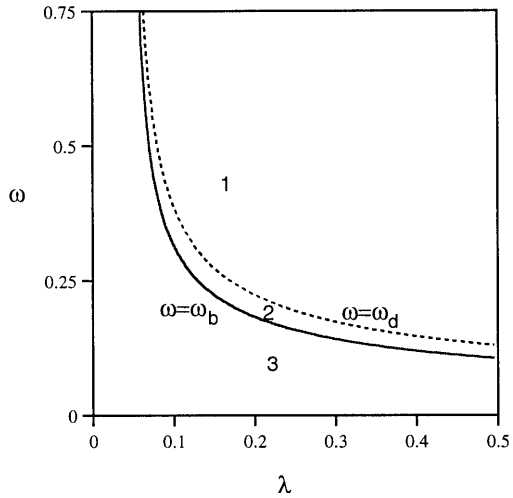


FIGURE 2. The graph of the critical values ω_b (3.10) and ω_d (3.13) as functions λ_2. For values of λ_2 and ω in region 3 the phase-locked solutions are stable. In region 2 phase-entrainment solutions are stable and in region 1 phase-drift solutions are stable. In regions 1 and 2 the phase-locked solutions are unstable.

This phase-locked solution is stable for $|\omega| < \omega_b$, where ω_b^2 is obtained by setting $q = 0$ in (3.8),

$$\omega_b^2 = \frac{2\sqrt{\lambda_1}}{\kappa_r} = \frac{2\lambda_1\epsilon^2(2((2\lambda_1)^2 + \omega_b^2))}{(\lambda_2 - \epsilon)(2((2\lambda_1)^2 + \omega_b^2)) + 2\lambda_1\epsilon^2}. \tag{3.10}$$

Figure 2 indicates the behaviour of ω_b as a function of λ_2. The solution (3.9) can be obtained numerically from (2.1).

For $|\omega| > \omega_b$ ($q > 0$), the phase-locked solution loses stability to the phase-entrainment and phase-drift solutions, as shown in Fig. 3. Then the solution of (3.6) is

$$a(\tau) \sim \sqrt{q}e^{i\kappa_i\tau}. \tag{3.11}$$

The phase $\kappa_i\tau$ introduces a correction to the frequency for the oscillatory term in (3.5). The solution (3.11) in terms of $B_0(t) \sim A_2e^{i\psi(t)}$ is

$$B_0 = \frac{\sqrt{\lambda}}{i\omega} + \sqrt{q}e^{-i(\omega - \epsilon^2\kappa_i)t}. \tag{3.12}$$

For $q > 0$, it is clear that A_2 and ψ vary with time. This solution is stable for $|\omega| > \omega_b$.

For $\omega_b < |\omega| < \omega_d$ the phase difference ψ is bounded ($0 < |\psi| < \pi$) for all time, which corresponds to phase-entrainment, an intermediate behaviour between phase-locking and phase-drift. For these values of ω the solution is a limit cycle in the $A_2 - \psi$ plane. The cutoff value ω_d for stable phase-entrainment is defined by

$$\omega_d^2 = \frac{3\lambda_1}{\kappa_r} + O(\epsilon) \tag{3.13}$$

and is shown in Figs. 2 and 3.

For values of $|\omega| > \omega_d$, the solution (3.12) is a phase-drift solution. For the phase-drift solution the phase difference $|\psi|$ passes through $n\pi$ for n odd as $|\psi|$ grows with time. This

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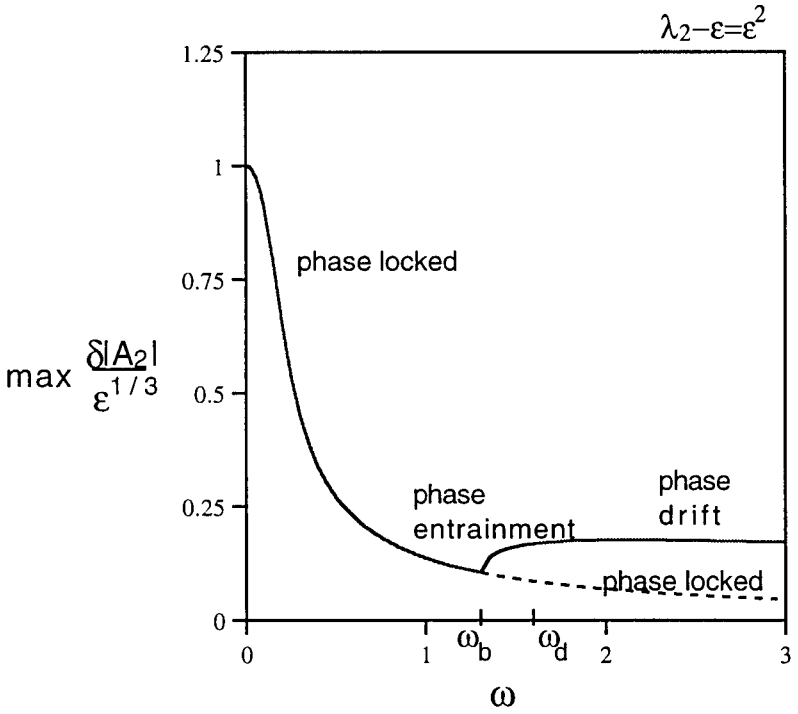


FIGURE 3. Maximum magnitude of $\delta A_2/\epsilon^{1/3}$ vs. the difference $\omega = \omega_2 - \omega_1$ for $\lambda_2 - \epsilon = \epsilon^2$ ($\epsilon = .05$) (see Section 3b). The solid (dotted) line corresponds to stable (unstable) solutions. For $|\omega| < \omega_b$ the phase-locked solutions are stable. For $\omega_d > |\omega| > \omega_b$ the phase-locked solutions are unstable, and the phase-entrainment solutions are stable. For $|\omega| > \omega_d$ the phase-drift solutions are stable.

can be seen from the expressions for A_2 and ψ obtained from (3.12),

$$A_2(t) = \sqrt{q}\cos(\Omega t + \psi(t)) - \frac{\sqrt{\lambda_1}}{\omega}\sin\psi(t) + O(\epsilon)$$
$$\sqrt{q}\cos(\Omega t)\sin\psi(t) = -\left(\sqrt{q}\sin(\Omega t) + \frac{\sqrt{\lambda_1}}{\omega}\right)\cos\psi(t) + O(\epsilon), \tag{3.14}$$

where $\Omega = \omega - \epsilon^2\kappa_i$. For $|\omega| > \omega_d$ the right hand side of (3.14) vanishes for some t (neglecting $O(\epsilon)$ terms). Then $\psi(t)$ passes through $n\pi$ and $|\psi|$ increases with time. The solution described by (3.14) is confirmed by Fig. 4, which shows a numerical solution of the full system (2.1).

To summarize, we relate the results of this section to (2.8). We have demonstrated that the phase-locked solution, described by (3.9) and (2.8) with $r = 0$, is stable for $\omega < \omega_b$. The phase-entrained solution, (3.12) and (2.8) with $r < \sqrt{\lambda_1}/|\omega|$, is stable for $\omega_b < |\omega| < \omega_d$. The phase-drift solution, (3.12) and (2.8) with $r > \sqrt{\lambda_1}/|\omega|$, is stable for $\omega_d < |\omega|$. In contrast to the transition at ω_b, the transition at ω_d from phase-entrainment to phase-drift is not a bifurcation.

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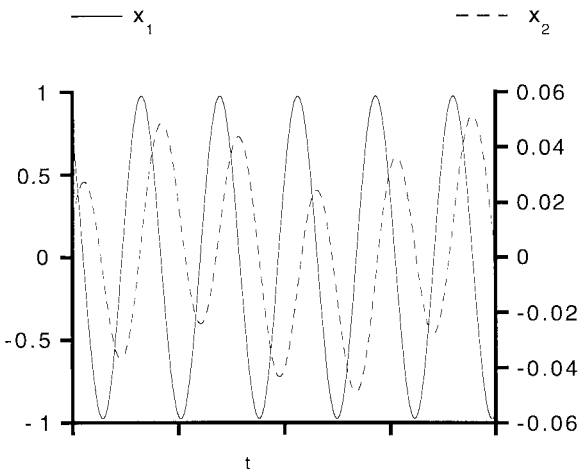


FIGURE 4. Numerical solution of (2.1) as described to leading-order in (3.14) for $\omega^2 > \omega_b^2$. For $\omega^2 > \omega_b^2$ the phase-locked solution is no longer stable.

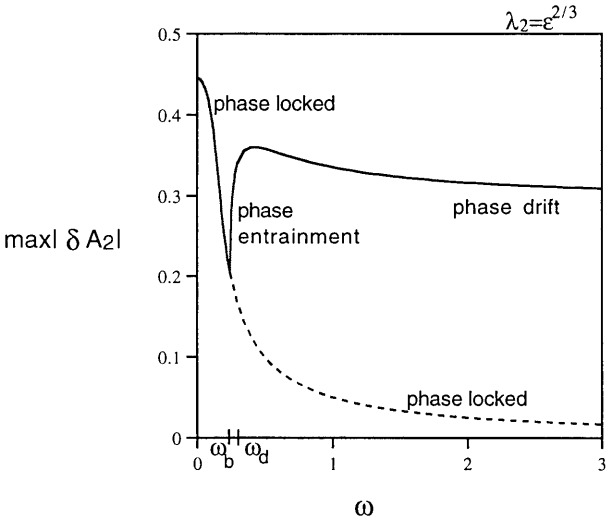


FIGURE 5. Maximum value of δA_2 as a function of ω with $\lambda_2 = O(\epsilon^{2/3})$ ($\epsilon = .05$) (see §3c), with stability indicated as in Fig. 3.

3.3 The case $\omega = \omega_2 - \omega_1 = O(1)$, $|\lambda_2 - \epsilon| \gg \epsilon^2$

For $\lambda_2 - \epsilon = c \gg \epsilon^2$, it is appropriate to take $\delta^2 = O(c)$ in (3.4) and expand B in a perturbation expansion, as in §3b. Then we obtain a similar result for the stability of phase-locked and phase-drift solutions, with q replaced by q_2:

$$q_2 = \frac{c}{\delta^2} - \frac{2\lambda_1 \epsilon^2}{\omega^2 \delta^2}. \tag{3.15}$$

Figure 5 shows the bifurcation from stable phase-locked solutions to stable phase-entrainment and phase-drift solutions for these parameter values.

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For

$$\omega^2 < \omega_b^2 = \frac{2\lambda_1\epsilon^2}{c}, \tag{3.16}$$

the phase-locked solution is stable. For $\omega_b < |\omega| < \omega_d$, where

$$\omega_d^2 = \frac{3\lambda_1\epsilon^2}{c}, \tag{3.17}$$

the phase-entrainment solution is stable. Since $\omega_b \ll 1$ and $\omega_d \ll 1$ in this case, we postpone the discussion of these solutions to §3d, where the case $|\omega| \ll 1$ is considered.

For $|\omega| > \omega_d$, phase-drift solutions are stable:

$$A_2 = \sqrt{q_2} + O(\epsilon\lambda_2^{-1/2}) \tag{3.18}$$

$$\psi = -\omega t + O(\epsilon\lambda_2^{-1/2}). \tag{3.19}$$

Neglecting the $O(\epsilon\lambda_2^{-1/2})$ terms for large enough λ_2, this result corresponds to two uncoupled limit cycle oscillators. To obtain the higher-order corrections for $\lambda_2 \to 1$ (nonlocalized oscillations), one can apply an averaging approach, (or refer to the analysis of Aronson *et al.* [10, 11]).

3.4 The case $\omega \ll 1$

From Fig. 2 it is clear that for increasing λ_2 the value of ω_b decreases, so that $\omega_b \ll 1$ for sufficiently large λ_2. Therefore, we must consider both phase-locked and phase-drift solutions for $\omega \ll 1$.

For $\omega_b < |\omega| \ll 1$ the solution describes either phase-entrainment or phase-drift. Then the magnitude of B is given by

$$|B|^2 \sim \frac{\epsilon^2}{\delta^2}\frac{\lambda_1}{\omega^2} + 2\frac{\epsilon}{\delta}\frac{\sqrt{\lambda_1}}{\omega}a\sin\omega t + a^2, \tag{3.20}$$

where $a = \sqrt{q}$ and $\delta = \epsilon$ for $c = O(\epsilon^2)$, and $a = \sqrt{q_2}$ and $\delta = \sqrt{\lambda_2}$ for $\lambda_2 \ll \epsilon$ (see §3c and 3d).

For $|\omega| < \omega_b$ and $\omega \ll 1$, we consider only steady state solutions of (3.4). Then $|B|$ satisfies

$$\delta^4|B|^6 - 2\delta^2 c|B|^4 + (\omega^2 + c^2)|B|^2 - \frac{\epsilon^2}{\delta^2}\lambda_1 = 0. \tag{3.21}$$

We determine the appropriate scalings of δ by balancing terms in (3.21), considering the magnitude of λ_2 and ω relative to ϵ, and matching with the results for $\omega = 0$ and $\omega = O(1)$ above. For example, for $|\lambda_2 - \epsilon| \gg \epsilon^2$, the phase-locked solution is described by $|B|^2 = R_0^2$ and ψ =constant, where

$$R_0^2(\delta^2 R_0^2 - c)^2 \sim \frac{\epsilon^2}{\delta^2}\lambda_1 - \omega^2 R_0^2$$

$$\tan\psi \sim \frac{\omega}{c - \delta^2 R_0^2}. \tag{3.22}$$

In the limit $\omega \to 0$ the result (3.22) is (3.3), with $\delta = O(\sqrt{\lambda_2})$. Similarly, for $|\lambda_2 - \epsilon| = O(\epsilon^2)$ we match B obtained from (3.21) with (3.3) and (3.9) to obtain the appropriate scaling

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Table 1. *The parameter δ as a function of ω and λ. The magnitude of δ describes the degree of localization.*

	$\lambda_2 \ll \epsilon^{2/3}$	$\lambda_2 = O(\epsilon^{2/3})$ or larger
$\omega \ll \epsilon^{2/3}$	$\delta = \epsilon^{1/3}$	$\delta = \sqrt{\lambda_2}$
$\omega = O(\epsilon^{2/3})$ or larger	$\delta = \frac{\epsilon}{\omega}$	$\delta = \sqrt{\lambda_2}$

of δ. Thus we obtain the degree of localization in terms of all relevant values of ω, λ, and ϵ. These results are summarized in Table 1. The maximum magnitudes of the stable solutions are shown in the bifurcation diagrams, Figs. 3 and 5, for representative values of λ_2 and ω.

4 Summary

We have developed an analytical theory for understanding localized states in coupled-limit-cycle oscillators by studying the canonical model (2.1). Coupled phase and amplitude equations are obtained for describing the leading-order behaviour. Our analysis demonstrates that a combination of control or bifurcation parameters (such as λ_2 in (2.1)), detuning of the coupled oscillators ($\omega = \omega_2 - \omega_1$), and strength of the coupling (ϵ) determines the stability and degree of localization and the type of localized state. The localized solutions can exhibit phase-locking, phase-entrainment (sometimes referred to as phase trapping), or phase-drift. We give the parameter values for which the localized phase-locked solutions bifurcate to those of phase-entrainment and phase-drift. In all of these cases quantitative results for the leading-order amplitudes and phases of the oscillators are given. Numerical simulations of the full system validate the predictions of the analysis.

The localization depends on the parameters in a non-uniform manner, which we summarize in the figures and Table 1. For $\lambda_2 - \epsilon = O(\epsilon^2)$ the system is strongly localized ($\delta A_2 = O(\epsilon)$) for values of $\omega = \omega_2 - \omega_1$ large relative to ϵ. For small values of ω the phase-locked solution is stable, and the amplitude of the second oscillator is larger ($O(\epsilon^{1/3})$). In Figure 3 we show the transition to strongly localized solutions and the bifurcation from phase-locked solutions to phase-drift solutions for increasing values of ω and $\lambda_2 - \epsilon = O(\epsilon^2)$.

For ω of $o(1)$ and sufficiently small λ_2, the phase-locked solution is stable and localized, with $\delta A_2 = O(\epsilon/\omega)$. For larger values of λ_2 ($\lambda_2 - \epsilon \gg \epsilon^2$), the magnitude of δA_2 is $O(\sqrt{\lambda_2})$ for both the phase-locked and phase-drift solutions (small and large values of ω). In Fig. 5 we show the amplitude for these solutions and the bifurcation from phase-locked to the phase-drift solutions. For $\lambda_2 \to 1$ the analysis of §3 is no longer valid, since it is based on the assumption that the solution is localized. As we have shown in the previous section, the analysis of this paper yields the leading-order result for $1 \gg \lambda_2 \gg \epsilon$, which corresponds to the uncoupled case $\epsilon = 0$ to leading-order. To consider the case of $\lambda_2 \to 1$ one could apply the method of averaging or proceed as in Aronson *et al.* [11] with $r_1 \neq r_2$.

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The approach developed in this paper can be generalized to study localization in other arrays of coupled oscillators. Similar analyses yield results for localized states in coupled solid state lasers, studied by Kuske & Erneux [4] and in conservative systems of coupled oscillators, as demonstrated by other methods of Vakakis & Cetinkaya [3] and Pierre & Shaw [12]. Applying the methods described in this paper to these applications has advantages over the methods of Vakakis & Cetinkaya [3] and Pierre & Shaw [12]. For example, a singular perturbation expansion yields the degree of localization in terms of both the coupling parameter and the initial conditions, on which the localized states depend. Amplitude and phase equations can be derived, whose leading-order solution is valid on time scales of $O(\epsilon^{-1})$. One can consider higher-order terms in these equations to determine whether the solution does indeed remain localized.

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Appendix A

The expansion $B = B_0 + \epsilon^2 B_2 \ldots$ is substituted in (3.4) and the coefficient of ϵ^2 is set equal to zero. Thus,

$$B_{2t} + i\omega B_2 = -B_{0\tau} + \sqrt{\lambda_1}\left(A_{1,1} + A_{1,2}\right) - \frac{B_0}{2\sqrt{\lambda_1}}(B_0 - B_0^*) + \left(\frac{c}{\epsilon^2} - |B_0|^2\right) B_0 \quad \text{(A 1)}$$

An equation for $a(\tau)$ is determined from the solvability condition which eliminates secular terms (growing in time) from the solution B_2. This is equivalent to setting the coefficient of $e^{-i\omega t}$ on the right-hand side of (A 1) equal to zero. Since this involves terms from A_1, it is necessary to determine which terms from A_1 contribute to the solvability condition. Substituting (3.1) in (2.9), we find that $A_{1,1}$ does not include such oscillatory terms, while $A_{1,2}$ does:

$$A_{1,2} = \frac{a(\tau)}{2\sqrt{\lambda_1}(2\lambda_1 + \epsilon - i\omega)} e^{-i\omega t} + \frac{\bar{a}(\tau)}{2\sqrt{\lambda_1}(2\lambda_1 + \epsilon + i\omega)} e^{i\omega t} + N(t). \quad \text{(A 2)}$$

Here $N(t)$ are non-oscillatory terms.

Substituting for B_0 and $A_{1,2}$ in (A 1) and applying the solvability conditions to eliminate solutions for B which grow with time yields (3.6) for $a(\tau)$.

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