European Journal of Applied Mathematics (2026), **37**, pp. 1164–1199
doi:10.1017/S0956792525100260

CAMBRIDGE UNIVERSITY PRESS

PAPERS

Opinion dynamics with continuous age structure

Andrew Nugent[1,2], Susana N. Gomes[2] and Marie-Therese Wolfram[2]

[1]MathSys CDT, University of Warwick, Coventry, UK
[2]Mathematics Institute, University of Warwick, Coventry, UK
Corresponding author: Andrew Nugent; Email: a.nugent@warwick.ac.uk

Received: 06 March 2025; **Revised:** 03 November 2025; **Accepted:** 03 November 2025;
First published online: 4 December 2025

Keywords: opinion dynamics; age structured population; mean-field limit

2020 Mathematics Subject Classification: 35Q91 (Primary); 91D30 (Secondary)

Abstract

We extend a classical model of continuous opinion formation to explicitly include an age-structured population. We begin by considering a stochastic differential equation model which incorporates ageing dynamics and birth/death processes, in a bounded confidence type opinion formation model. We then derive and analyse the corresponding mean field partial differential equation and compare the complex dynamics on the microscopic and macroscopic levels using numerical simulations. We rigorously prove the existence of stationary states in the mean field model, but also demonstrate that these stationary states are not necessarily unique. Finally, we establish connections between this and other existing models in various scenarios.

Highlights

- Introduce an individual-based model for opinion dynamics with an explicit continuous age structure.
- Derive the mean-field partial differential equation and provide examples of novel complex dynamics.
- Rigorously prove the existence of steady states and discuss conditions for their uniqueness.
- Establish connections with several existing models in various scenarios.

1. Introduction

Opinion dynamics modelling aims to capture the mechanisms through which a population of individuals forms and updates their opinions, and answer questions about if, and on what scale, agreement emerges. The majority of models focus on consensus dynamics, in which interactions between individuals cause their opinions to move closer together. This raises the key question, initially posed by Axelrod [2], of why consensus is not more prevalent.

One commonly accepted answer is that individuals have bounded confidence, meaning that they are only willing to interact with those who already share a sufficiently similar opinion. The models of Hegselmann and Krause [26] and Deffuant and Weisbuch [12] both introduce this effect, and the resulting clustering of opinions has been well studied in agent-based models [3, 25, 37, 47], ordinary and stochastic differential equation models for individuals' opinions [4, 41, 43], and partial differential equation models describing population-level opinion distributions [21, 22, 52]. Furthermore, other contributing factors such as the presence of leaders or stubborn individuals [14, 54], the impact of network structure [1, 19, 32, 42] and the interactions between opinions on multiple topics [17, 31, 48] have been studied.

© The Author(s), 2025. Published by Cambridge University Press. This is an Open Access article, distributed under the terms of the Creative Commons Attribution licence (https://creativecommons.org/licenses/by/4.0/), which permits unrestricted re-use, distribution and reproduction, provided the original article is properly cited.

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

An important effect is the addition of noise, with the simplest approach being to add a small amount of noise to each interaction, causing opinion diffusion [21, 46]. Alternatively, noise may be added to the opinions communicated between individuals or to the size of opinion updates [43, 50], or as 'events' that may affect the entire population [9]. In 2009, Pineda et al. [45] introduced 'free will in the form of noisy perturbations', meaning that individuals' opinions were occasionally randomly updated to a new randomly selected opinion. With this type of noise, the opinion formation process continues to be dynamic, rather than remaining in the typically observed clusters. A similar approach has been taken in [7] and [24], in which this effect is described specifically as modelling the death/exit and birth/entry of individuals into the population.

In this paper, we model the effect of ageing more explicitly, defining an age for each individual and establishing at the microscopic (individual) level the mechanism through which individuals exit and re-enter the population with new opinions. In addition, we look at the macroscopic (population) level by describing the evolution of the joint density over age and opinion. This approach is closer to the way that births and deaths are typically included in population dynamics and certain epidemiological models (see, for example, [11, 29, 33, 44]), where individuals' ages are included explicitly in the model rather than being viewed solely as a source of noise.

It is worth noting that this approach is very different from including a memory-dependent transition rate between opinion states, as in e.g. [36, 49], in which context an individuals' 'age' reflects the length of time they have held their current opinion rather than the biological age considered here.

There are several motivations for considering age structure in opinion dynamics. Primarily, the entry and exit of individuals introduce new stochasticity and continue the dynamics after the traditional formation of clusters. We can ask questions regarding the long-term maintenance of a consensus state, the merging or emerging of clusters and the possibility of observing periodic behaviours. Two perspectives can be taken on timescales: studying both opinion change over the lifetime of an individual and changes over the long-term, multi-generational behaviour of a population. We can also incorporate the effects of different age groups interacting at different rates, having differing levels of confidence in each other and being more prevalent in the population. As we will see in various examples throughout this paper, the introduction of age structure adds a new richness to the opinion formation process through several effects that are not otherwise observed.

We begin by introducing the microscopic model in Section 2 and providing examples of interesting behaviours made possible by introducing continuous age structure. In Section 3, we derive the macroscopic PDE model and provide examples of the corresponding behaviours. We then focus on the steady states of the macroscopic model in Section 4, exploiting a connection with the classical mean-field model to show their existence, propose an efficient method to find them, and discuss when they are unique. In Section 5, we explore properties of the macroscopic system and examine the connection with other existing models, before concluding and discussing future directions in Section 6.

2. Microscopic model

Consider a finite population of N individuals. The state of individual i is described by the pair (a_i, x_i) where $a_i \in [0, A)$ is the age of individual i and $x_i \in U$ is their opinion. Throughout this paper, we typically normalise to $A = 1$ and consider $U = (-1, 1)$ to represent the level of (dis)agreement with some statement, but the model could be rescaled for any bounded interval $U \subset \mathbb{R}$ and $A > 0$. The initial age of each individual is chosen uniformly at random in $[0, A)$ and the initial opinions are chosen randomly according to some given distribution ρ_0. These states evolve according to

$$\mathrm{d}a_i = \tau \, \mathrm{d}t, \tag{1a}$$

$$\mathrm{d}x_i = \left(\frac{1}{N} \sum_{j=1}^{N} M(a_i, a_j)\, \phi(x_j - x_i)\, (x_j - x_i) \right) \mathrm{d}t + \sigma \, d\beta_i \qquad i = 1, \ldots, N, \tag{1b}$$

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

except that when an individual reaches age $a_i = A$, their age is reset to $a_i = 0$, and a new opinion is chosen according to a given age-zero opinion distribution μ. The interaction function $\phi : [-2, 2] \to [0, 1]$ is an odd function that describes the extent to which individuals interact based on the distance between their opinions. The age-interaction kernel $M : [0, A)^2 \to \mathbb{R}_{\geq 0}$ describes the strength/frequency of interactions between individuals of different ages, with the simplest case being $M \equiv 1$ in which individuals' ages do not affect their interactions. The constant τ determines the timescale on which individuals age (relative to the change in their opinions). The constant σ describes the amount of external noise affecting individuals' opinions, and β_i (for $i = 1 \ldots, N$) are N independent standard Brownian motions, modelling external influences on individuals' opinions. In general, we are concerned with relatively small σ, so that the opinion dynamics is mostly driven by opinion interactions. We impose reflecting boundary conditions at the boundary of U, here $x = \pm 1$. This model maintains a finite, fixed population size N.

Assumption 1. *We make the following assumptions throughout:*

A: The interaction function $\phi \in C^2(\mathbb{R})$.

B: The age-interaction kernel M is Lipschitz continuous.

Under these assumptions, the system (1) is well-posed [28]. The age jump times are deterministic and can be determined from the initial conditions. As initial ages are selected uniformly at random in $[0, A)$, the probability of two individuals having the same age, and therefore jumping (resetting their opinions) at the same time, is zero. Between each jump, the SDE system is well-posed as the vector field is Lipschitz continuous. Therefore, a solution can be constructed by splitting the time interval according to the known jump times and piecing together the solution in each time interval.

Note that due to the random replacement of individuals' opinions when they reach age A, this system cannot exhibit stationary states on the microscopic level. This provides one motivation for studying the mean-field model, which, as we will see in Section 4, exhibits possibly non-unique stationary states.

This model could be altered by introducing stochastic death timing, meaning that each agent dies according to a Poisson process with a prescribed age-dependent death rate $d(a)$. This would allow a more realistic age distribution in the population. However, as is discussed in greater detail in Section 5.6, a similar effect can be achieved by multiplying the age-interaction kernel $M(a, b)$ by the population-level age distribution resulting from this death rate. In fact, in the large-population limit, these approaches are equivalent, provided that the death rate $d(a)$ ensures bounded ages, which is a realistic assumption in the context of opinion dynamics.

In Figure 1, we provide a small number of examples that highlight several interesting features of the model (1). These examples all take $M \equiv 1$ and use a population of size $N = 500$. The interaction function ϕ as well as σ and τ are varied to show different behaviours, with the simulation length T changed accordingly. We use a mollified version of the classical bounded confidence interaction function [26] so that ϕ satisfies Assumption 1. Specifying $r_2 > r_1 > 0$ we take

$$\phi(r) = \begin{cases} 1 & \text{if } |r| < r_1, \\ \tilde{\phi}(r) & \text{if } r_1 \leq |r| \leq r_2, \\ 0 & \text{if } |r| > r_2, \end{cases} \tag{2}$$

where $\tilde{\phi}$ is a function that smoothly interpolates between 0 and 1.

Figure 1a shows an example with $\phi \equiv 1$, meaning all individuals interact with each other at all timepoints. The initial opinion distribution is given by

$$\rho_0(a, x) = \kappa \, e^{-10(1-x)^2}, \tag{3}$$

where κ is a normalisation constant. This initial distribution is heavily weighted towards positive opinions and we see in Figure 1a that initially a consensus forms at a positive opinion. However, as new individuals enter the population with uniformly distributed opinions, this consensus slowly shifts to lie near $x = 0$ with only minor deviations due to stochasticity.

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

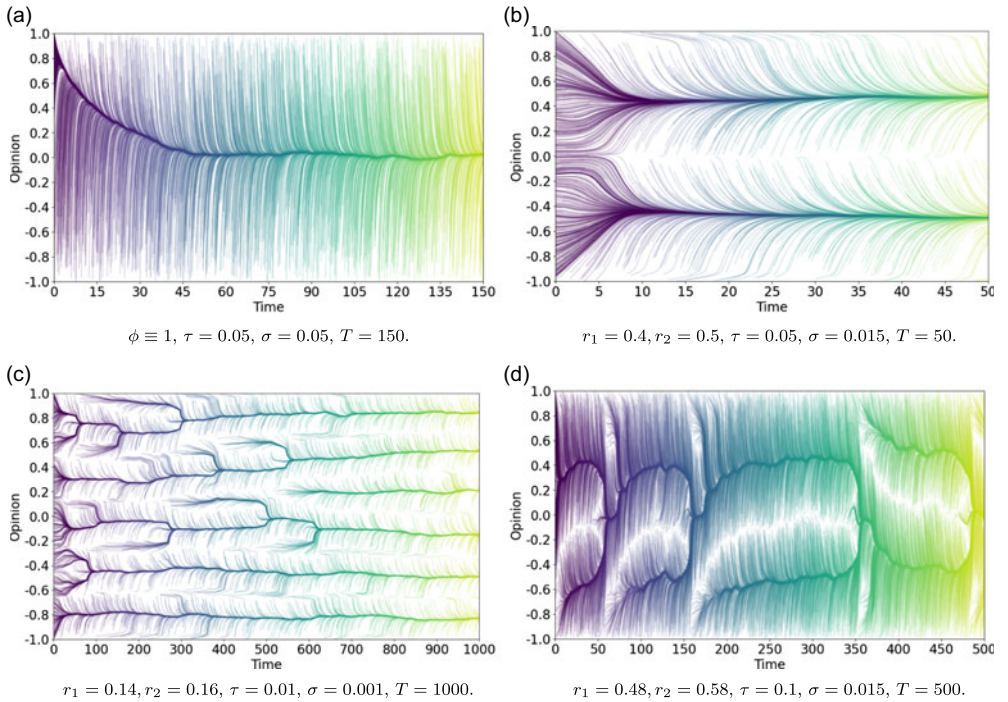


Figure 1. *Example numerical solutions to (1) using an Euler-Maruyama scheme with timestep* $\Delta t = 0.01$. *Individuals' opinion trajectories are plotted over time, with colour indicating the time at which individuals joined the population, with initially present individuals shown in purple. All examples use an interaction function* ϕ *given by (2),* $M \equiv 1$ *and a uniform* μ. *(b) and (c) also use a uniform* ρ_0, *while Figure 1a uses an exponentially distributed* ρ_0 *(3) and (d) uses a bimodal* ρ_0 *with peaks at* $x = 0$ *and* $x = -0.8$ *(4). The values of parameters* r_1, r_2, τ *and* σ *are varied to demonstrate different behaviours, with the final time T changed accordingly.*

In Figure 1b, we set $r_2 = 0.5$ and a uniform ρ_0. This causes the population to split into two clusters. The bounded confidence radius of these two clusters partitions the opinion space, meaning each individual joining the population simply joins the nearest cluster and these clusters are therefore stable over time. This indicates that even though individuals may die, there is the possibility of observing more structured macroscopically stable patterns.

In Figure 1c, we set $r_2 = 0.16$ and a uniform ρ_0, and observe many smaller clusters. There is significantly more merging of clusters due to stochasticity, with individuals joining the population acting as 'bridges' between nearby clusters, as seen in [6]. In addition, new individuals entering the population provide the opportunity for new clusters to emerge, often at more extreme opinions or in the vacuum created following cluster merges.

Figure 1d shows a population that does not settle towards a macroscopically stationary pattern. The initial opinion distribution is

$$\rho_0(a, x) = \kappa \left(e^{-10\,(0.8-x)^2} + e^{-10x^2} \right), \tag{4}$$

where κ is a normalisation constant. We set $r_2 = 0.58, r_1 = 0.48$, allowing slightly more interaction between clusters than Figure 1b. We now observe the repeated formation of two clusters that then combine in a brief but unstable consensus before reforming again. When considering the macroscopic PDE model, this may translate into a predictable, periodic pattern.

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

While the inclusion of births/deaths in the SDE model (1) creates various interesting behaviours, it also creates new challenges in its analysis as the process no longer reaches a stationary distribution on the individual level. However, as we have observed in Figure 1, stability can emerge on the macroscopic level; hence, we next consider a mean-field PDE model obtained in the large-population limit.

3. Macroscopic model

Motivated by the emergence of macroscopic behaviours observed in Section 2, we would now like to obtain a mean-field limit of the form $\rho(t, a, x)$, which describes the density of individuals with age a and opinion x at time t. In particular, we will be interested in the steady state behaviour of this model and the extent to which it mirrors the dynamics of the microscopic system (1).

We first set some notation to be used throughout this section. Define

$$\varphi(y - x) := \phi(y - x)\,(y - x)\,. \tag{5}$$

Note that since ϕ is odd, the same is true of φ. In addition, φ is also twice continuously differentiable.

Recall that $A > 0$ is the age at which individuals have their opinions reset, $U = (-1, 1)$ is the opinion space and denote by $U_A = (0, A) \times U$ the joint age-opinion space.

3.1. Formal derivation of mean-field limit

We begin by considering the model without noise (i.e. $\sigma = 0$) and deriving a PDE satisfied by the empirical density, defined by

$$\rho(t, a, x) = \frac{1}{N}\sum_{i=1}^{N} \delta_{a_i(t)}(a)\,\delta_{x_i(t)}(x), \tag{6}$$

where δ_z denotes a Dirac delta centred at z. Let $\zeta(a, x)$ be an infinitely differentiable, compactly supported test function. We formally compute

$$\begin{aligned}
&\int_U \int_0^A \partial_t \rho(t, a, x)\,\zeta(a, x)\,\mathrm{d}a\,\mathrm{d}x \\
&\quad= \frac{\mathrm{d}}{\mathrm{d}t}\int_U \int_0^A \frac{1}{N}\sum_{i=1}^{N} \delta_{a_i(t)}(a)\,\delta_{x_i(t)}(x)\,\zeta(a, x)\,\mathrm{d}a\,\mathrm{d}x \\
&\quad= \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{N}\sum_{i=1}^{N} \zeta\big(a_i(t), x_i(t)\big) \\
&\quad= \frac{1}{N}\sum_{i=1}^{N} \partial_a \zeta\big(a_i(t), x_i(t)\big)\,\tau + \frac{1}{N}\sum_{i=1}^{N} \partial_x \zeta\big(a_i(t), x_i(t)\big)\left(\frac{1}{N}\sum_{j=1}^{N} M(a_i, a_j)\,\varphi(x_j - x_i)\right) \\
&\quad= \int_U \int_0^A \frac{1}{N}\sum_{i=1}^{N} \delta_{a_i(t)}(a)\,\delta_{x_i(t)}(x)\left(\partial_a \zeta(a, x)\,\tau + \partial_x \zeta(a, x)\,\frac{1}{N}\sum_{j=1}^{N} M(a, a_j)\,\varphi(x_j - x)\right)\mathrm{d}a\,\mathrm{d}x \\
&\quad= \int_U \int_0^A \rho(t, a, x)\,\partial_a \zeta(a, x)\,\tau + \partial_x \zeta(a, x)\,\rho(t, a, x)\left(\int_U \int_0^A M(a, b)\,\varphi(y - x)\,\rho(t, b, y)\,\mathrm{d}b\,\mathrm{d}y\right)\mathrm{d}a\,\mathrm{d}x \\
&\quad= -\int_U \int_0^A \zeta(a, x)\,\partial_a \rho(t, a, x)\,\tau \\
&\qquad + \zeta(a, x)\,\partial_x\left(\rho(t, a, x)\left(\int_U \int_0^A M(a, b)\,\varphi(y - x)\,\rho(t, b, y)\,\mathrm{d}b\,\mathrm{d}y\right)\right)\mathrm{d}a\,\mathrm{d}x
\end{aligned}$$

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

Note that in the final line, the integration by parts produces no boundary terms since ζ is compactly supported. This gives that ρ is a weak solution to

$$\partial_t \rho(t,a,x) + \tau\,\partial_a \rho(t,a,x) + \partial_x \left(\rho(t,a,x) \left(\int_U \int_0^A M(a,b)\,\varphi(y-x)\,\rho(t,b,y)\,\mathrm{d}b\,\mathrm{d}y \right) \right) = 0\,. \tag{7}$$

When considering the model with noise, the derivation is more complex as the empirical density no longer satisfies the limiting PDE. Indeed, the stochastic effects only disappear in the large-population limit [30]. In this case, we note that the structure of (1) is similar to that of the second-order Cucker-Smale model for bird flocking [10] and the general second-order model discussed in [30], in which alignment and noise are only present in the evolution of velocities (here opinions). The derivation of the mean-field limit in this setting can be found in [23, 51]. While we do not show this rigorously here, the similarity in structure implies that the corresponding mean-field limit for (1) is given by

$$\partial_t \rho + \tau\,\partial_a \rho + \partial_x \left(\rho(t,a,x) \left(\int_U \int_0^A M(a,b)\,\varphi(y-x)\,\rho(t,b,y)\,\mathrm{d}b\,\mathrm{d}y \right) \right) - \frac{\sigma^2}{2}\partial_{xx}\rho = 0\,. \tag{8}$$

Defining the flux,

$$F[\rho](t,a,x) = \rho(t,a,x) \left(\int_U \int_0^A M(a,b)\,\varphi(y-x)\,\rho(t,b,y)\,\mathrm{d}b\,\mathrm{d}y - \frac{\sigma^2}{2}\partial_x \log\left(\rho(t,a,x)\right) \right), \tag{9}$$

then the PDE can be written as

$$\partial_t \rho + \tau\,\partial_a \rho + \partial_x F[\rho] = 0\,.$$

Note that the flux also incorporates the diffusion term.

We also need to specify the initial and boundary conditions. We assume that the initial condition is a given probability distribution $\rho_0(a,x)$. As we imposed reflecting boundary conditions in the SDE model, we impose no-flux boundary conditions on the boundary of U (here $x = \pm 1$) [15, 22]. It is also necessary to specify the opinion distribution at age-zero. As with the microscopic system, we assume that this follows a given distribution μ. In order to maintain a constant population size, we therefore set

$$\rho(t,0,x) = \mu(x) \int_U \rho(t,1,y)\,\mathrm{d}y,$$

as the integral

$$\int_U \rho(t,1,y)\,\mathrm{d}y,$$

describes the total volume of individuals whose opinions are reset at time t.

The complete initial-boundary value problem can be written

$$\partial_t \rho(t,a,x) + \tau\,\partial_a \rho(t,a,x) + \partial_x F[\rho](t,a,x) = 0 \tag{10a}$$

$$F[\rho](t,a,-1) = F[\rho](t,a,1) = 0 \tag{10b}$$

$$\rho(t,0,x) = \mu(x) \int_U \rho(t,1,y)\,\mathrm{d}y \tag{10c}$$

$$\rho(0,a,x) = \rho_0(a,x) \tag{10d}$$

Having derived this PDE problem, we next look at several example solutions before moving on to our analysis of its steady states.

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

Note that this age-structured PDE (10) can also be compared against the standard mean-field limit for the model without age structure:

$$\partial_t \rho(t,x) + \partial_x \left(\rho(t,x) \left(\int_U \varphi(y-x)\, \rho(t,y)\, \mathrm{d}y \right) \right) - \frac{\sigma^2}{2} \partial_{xx} \rho(t,x) = 0 \tag{11a}$$

$$\rho(t,1) \left(\int_U \varphi(y-1)\, \rho(t,y)\, \mathrm{d}y \right) - \frac{\sigma^2}{2} \partial_x \rho(t,1) = 0 \tag{11b}$$

$$\rho(t,-1) \left(\int_U \varphi(y+1)\, \rho(t,y)\, \mathrm{d}y \right) - \frac{\sigma^2}{2} \partial_x \rho(t,-1) = 0 \tag{11c}$$

$$\rho(0,x) = \rho_0(x)\,. \tag{11d}$$

3.2. Numerical simulations of the mean-field PDE

In this section, we show numerical solutions of (10), some of which mirror the examples in Section 2. A description of the numerical scheme used can be found in Appendix B.1.

In each of the examples shown in Figure 2, we take both the initial distribution $\rho_0(a,x)$ and the age-zero distribution μ to be uniform (with the exception of Figure 2f). We again use the mollified bounded confidence interaction function (2). By varying the confidence bounds, ageing rate and the strength of the noise/diffusion, we observe different clustering behaviours. In each figure, we show the evolution over time of the total opinion density,

$$P(t,x) = \int \rho(t,a,x)\, \mathrm{d}a, \tag{12}$$

for $x \in U,\ t \geq 0$. This allows a more direct comparison with the behaviours observed in the microscopic model.

Figures 2a and 2b show examples in which the population quickly moves towards one and two stable clusters, respectively, with Figure 2b mirroring the behaviour observed in the microscopic system in Figure 1b. Here, the ageing rate τ and level of noise σ have a relatively minor effect compared to the strong clustering. Reducing either τ or σ leads to more strongly peaked clusters, as individuals either have longer lifespans in which to concentrate their opinions or diffuse less away from the centre of the cluster. These effects can be seen by comparing Figures 2a and 2b, in the former τ and σ are relatively larger and the cluster(s) formed are therefore broader. In Figure 2c, we show an intermediate point between the formation of one and two clear clusters. A strong central cluster forms, initially bringing the whole population to consensus. However, as new individuals enter the population, those with more extreme opinions are not brought into this consensus. This leads to the formation of two smaller, less concentrated clusters at the extremes.

In Figure 2d, we consider a smaller confidence bound. Initially, the population remains near the uniform distribution due to the relatively strong diffusion, except at the boundary of the opinion space. As individuals at the boundary can only interact with those with less extreme opinions, small clusters form near the boundary, causing a subsequent decrease in density on the opposite side of the cluster. This, in turn, prompts the formation of another cluster closer to the centre, repeating until these clusters reach the centre of the opinion space. We also observe that by time $t = 400$ this central cluster cannot be maintained, and all individuals in this cluster either die or join the clusters on either side. As in Figure 2b, the population reaches a steady state with almost evenly spaced, although not evenly sized, clusters.

Figures 2e and 2f show instances of periodic behaviour, similar to that seen in the microscopic model in Figure 1d. In Figure 2e, as in 2c, a strong central cluster quickly forms, leaving a vacuum at the extreme opinions in which new, smaller clusters form. In this case, these eventually combine with the central cluster before the cycle repeats again. This effect is only possible due to the influx of new individuals to the population, as without it the initial consensus would simply persist. The parameter range for which

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

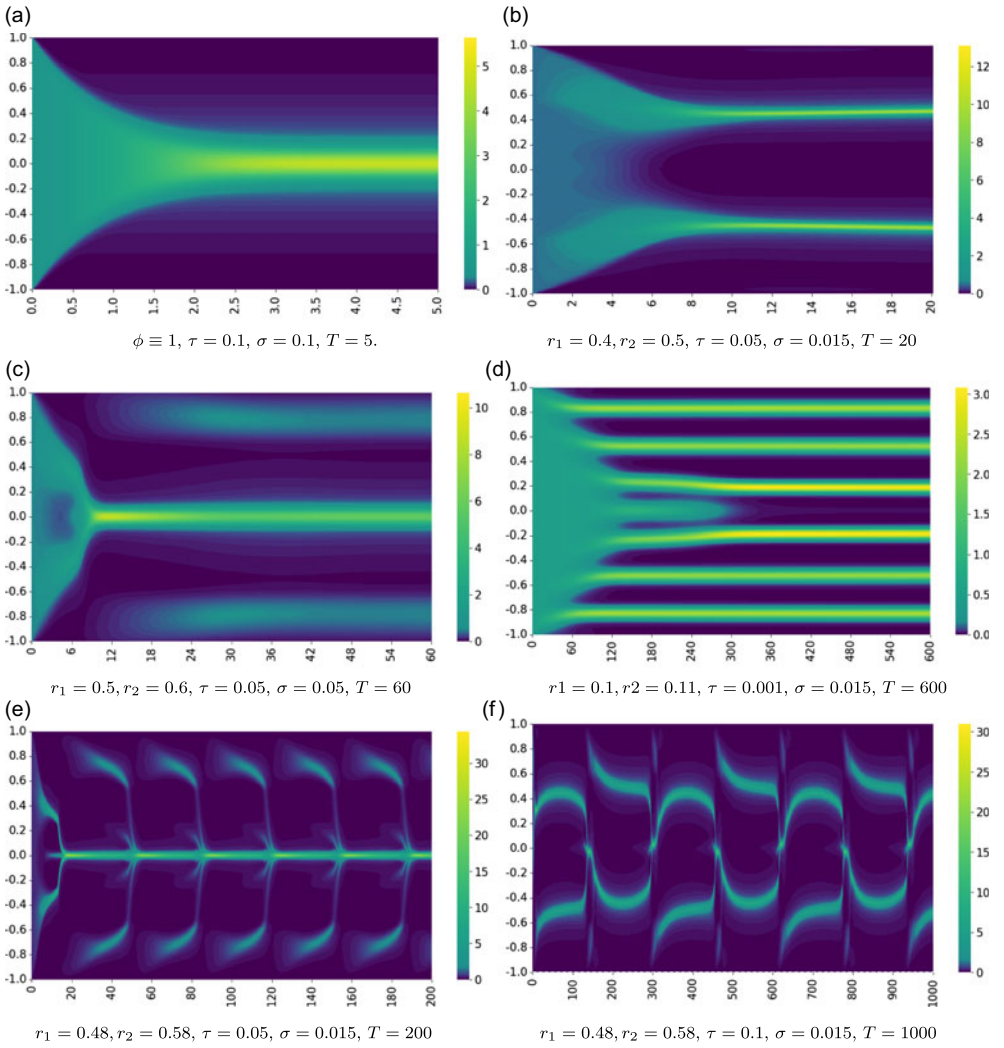


Figure 2. *Example numerical solutions to (10). Colour indicates the total opinion density $P(t, x)$ (12). The colour map is shifted to make areas of low density more clearly visible. All examples use an interaction function ϕ given by (2), $M \equiv 1$ and a uniform μ and ρ_0, with the exception of Figure 2f which uses (4) for ρ_0. The values of parameters r_1, r_2, τ and σ are varied to demonstrate different behaviours, with the final time T changed accordingly. All solutions use the numerical scheme described in Appendix B.1 with $J_x = J_a = 500$ and $\Delta t = 0.001$.*

this behaviour occurs is relatively small, as it lies on the boundary between the population forming into two clusters (as in Figure 2b) and the population forming one large central cluster and two smaller clusters that do not merge (as in Figure 2c).

In Figure 2f, we begin with a non-uniform initial condition given by (4), which has two peaks centred at 0 and -0.8. Due to the uniform influx of new individuals, these peaks move to become more symmetric. As in Figure 2e, these clusters periodically combine and reform. However, in this case, a lack of symmetry persists, meaning that after the two clusters combine, only one new extreme cluster forms (rather than one forming near each boundary). This leads to a periodic shifting between positive and negative clusters that form a brief off-centre consensus that cannot be maintained. This directly mirrors the behaviour observed on the individual level in Figure 1d.

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

These examples demonstrate the rich variety of complex behaviours that are made possible by the inclusion of a continuous age-structure. In the following Section 4, we move to studying the possible stationary states of this model.

4. Stationary states

To address the question of macroscopic stability in the SDE model (1), as observed for example in Figure 1b, and the long-term behaviour seen in Figures 2b and 2d, we hope to establish the existence of steady states of the PDE model (10). That is, we are looking for a probability density $\rho(a, x)$ such that

$$\tau\,\partial_a\rho(a,x) + \partial_x\left(\rho(a,x)\left(\int_U\int_0^A M(a,b)\,\varphi(y-x)\,\rho(b,y)\,\mathrm{d}b\,\mathrm{d}y\right)\right) - \frac{\sigma^2}{2}\partial_{xx}\rho(a,x) = 0, \tag{13a}$$

$$\rho(0,x) - \mu(x)\int_U \rho(1,y)\,\mathrm{d}y = 0, \tag{13b}$$

$$\rho(a,1)\left(\int_U\int_0^A M(a,b)\,\varphi(y-1)\,\rho(b,y)\,\mathrm{d}b\,\mathrm{d}y\right) - \frac{\sigma^2}{2}\partial_x\rho(a,1) = 0, \tag{13c}$$

$$\rho(a,-1)\left(\int_U\int_0^A M(a,b)\,\varphi(y+1)\,\rho(b,y)\,\mathrm{d}b\,\mathrm{d}y\right) - \frac{\sigma^2}{2}\partial_x\rho(a,-1) = 0\,. \tag{13d}$$

If we assume that initially there is a uniform density in age, then this will be preserved, meaning (13b) can be replaced with

$$\rho(0,x) = \mu(x)\,. \tag{14}$$

Note that (13a) is similar in structure to the classical mean-field limit, with the age variable a acting as 'time'. However, the integral over db inside the flux would then correspond to a time evolution in which individuals interacted with the future and past opinions of the population. Thus, the PDE (13) cannot be solved forward in age/time like the classical mean-field limit. To overcome this problem, we will instead construct a mapping whose fixed point corresponds to a solution of (13).

For certain parameter regimes, it can also take some time for the model to converge towards a steady state in numerical simulations, or we observe seemingly periodic behaviour. To this end, we aim to find a simpler method to identify steady states that avoids solving either (8) or (13) directly. In proving the validity of this method, we also prove the existence of steady states.

4.1. Existence of steady states

In this section, we make the following assumptions:

Assumption 2. *Assume the following:*

A: For all $a \in [0, A]$

$$\int_U \rho_0(a,x)\,\mathrm{d}x = 1\,.$$

B: The age-interaction kernel $M(a,b)$ *depends only upon* b. *That is, we can consider it as a function* $M : [0,A] \to \mathbb{R}_{\geq 0}$, $b \mapsto M(b)$.

C: The integral

$$\int_0^A M(b)\,\mathrm{d}b = 1\,. \tag{15}$$

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

Assumption 2A means that the initial ($t = 0$) age profile is uniform. As will be seen in Section 5.1, the age profile of the population satisfies a transport equation, so Assumption 2A ensures that the age profile remains uniform at all times. Assumption 2B ensures that all individuals view the population-level opinion density in the same way, regardless of their own age. This assumption will be crucial in performing the analysis below. As discussed in Section 5.6, even with these two assumptions, the model can still effectively capture realistic ageing effects and non-uniform population-level age distributions. Note that Assumption 2C can be achieved by appropriately rescaling A, τ and σ and is made to reduce the number of parameters.

As the age-interaction kernel M is Lipschitz continuous on a finite domain U, it is also bounded and we denote the maximum value of M by

$$M_{\max} = \max_{0 \le b \le A} M(b)\,.$$

We can now rewrite (13a) as

$$\tau\,\partial_a \rho(a,x) + \partial_x\left(\rho(a,x)\left(\int_U \varphi(y-x)\left(\int_0^A M(b)\,\rho(b,y)\,\mathrm{d}b\right)\mathrm{d}y\right)\right) - \frac{\sigma^2}{2}\partial_{xx}\rho(a,x) = 0\,.$$

To show the existence of a steady state, we will use a fixed-point argument. To this end, assume temporarily that the value of the following integral, which is constant in a, is known

$$\lambda(y) = \int_0^A M(b)\,\rho(b,y)\,\mathrm{d}b\,.$$

This integral describes the population-level opinion distribution with which individuals interact. From this, we define $\Lambda : U \to \mathbb{R}$ by

$$\Lambda(x) := \int_U \varphi(y-x)\,\lambda(y)\,\mathrm{d}y, \tag{16}$$

and let $\tilde{\rho}(a,x)$ be a solution to the initial-boundary value problem

$$\partial_a \rho(a,x) + \partial_x\Big(\rho(a,x)\Lambda(x)\Big) - \frac{\sigma^2}{2}\partial_{xx}\rho(a,x) = 0, \tag{17a}$$

$$\rho(0,x) = \mu(x), \tag{17b}$$

$$\rho(a,1)\,\Lambda(1) - \frac{\sigma^2}{2}\partial_x\rho(a,1) = 0, \tag{17c}$$

$$\rho(a,-1)\,\Lambda(-1) - \frac{\sigma^2}{2}\partial_x\rho(a,-1) = 0\,. \tag{17d}$$

From this solution, define

$$\tilde{\lambda}(y) = \int_0^A M(b)\,\tilde{\rho}(b,y)\,\mathrm{d}b, \tag{18}$$

and note that if $\tilde{\lambda} = \lambda$ we have in fact recovered (13a). That is, there is a one-to-one correspondence between stationary states of (10) and fixed points of the mapping $\lambda \mapsto \tilde{\lambda}$ described above. In Appendix A, we rigorously show the existence of such stationary states through the following steps:

Step 1: We determine conditions for the existence of solutions to (17) in an appropriate Sobolev space, applying results from [34] to show such existence in Proposition A.2.

Step 2: Introduce the mapping $\lambda \mapsto \tilde{\lambda} =: \mathcal{F}(\lambda)$ and show that $\mathcal{F} : L^\infty(U) \to L^\infty(U)$ is well-defined. Specifically,

$$\mathcal{F}(\lambda)(x) = \int_0^A M(b)\rho(b,x)\,\mathrm{d}b\,.$$

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

Step 3: Find an appropriate set $K \subset L^\infty(U)$ so that we may apply Schauder's Fixed Point Theorem to $\mathcal{F} : K \to K$. Using Propositions A.3–A.6, we establish the existence of a constant C such that the set

$$K = \{\lambda \in L^\infty(U) : \lambda \geq 0 \text{ a.e.}, \|\lambda\|_{L^1(U)} = 1,\ \|\lambda\|_{L^\infty(U)} \leq C\}$$

satisfies the requirements.

This allows us to show the following theorem.

Theorem 4.1. *Let Assumption 1 and Assumption 2 hold and assume that $\mu \in L^2(U)$. Then there exists at least one solution $\rho \in W_2^{2,1}(U_A)$ to the steady state equation (13).*

Proof. Each solution to the steady state equation can be uniquely characterised by a fixed point of the mapping $\mathcal{F}$. We have shown in Proposition A.4 that the set $K \subset L^\infty(U)$ is invariant under $\mathcal{F}$. The set K is also convex and, by Proposition A.5, weakly compact and weakly closed. In addition, Proposition A.6 shows that $\mathcal{F}$ is weakly continuous on K. Hence applying Schauder's Fixed Point Theorem to $\mathcal{F} : K \to K$ with the weak topology on $L^\infty(U)$ gives the existence of at least one fixed point of $\mathcal{F}$ in the set K, and thus the existence of at least one solution to the steady state equation (13). □

A natural question to ask next is whether such a steady state is unique. More specifically, when τ is large, individuals are quickly replaced and the opinion distribution cannot deviate significantly from the age-zero distribution μ, so we may expect uniqueness of the steady state. Indeed, the following Proposition 4.1, proven in Appendix A, states that the mapping $\mathcal{F}$ is Lipschitz continuous on K.

Proposition 4.1. *Under the assumptions of Theorem 4.1, the mapping $\mathcal{F} : K \to K$ is Lipschitz continuous. Specifically there exists some constant c depending on U, ϕ, A and τ but independent of λ, such that for all $\lambda_1, \lambda_2 \in K$,*

$$\left\|\mathcal{F}(\lambda_1) - \mathcal{F}(\lambda_2)\right\|_{L^\infty(U)} \leq c\left(A^{1/2} + A^{-1/4}\right) M_{\max} \left\|\lambda_1 - \lambda_2\right\|_{L^\infty(U)}. \tag{19}$$

However, it is not possible to bound c_8 such that this Lipschitz constant is smaller than 1 as $\tau \to \infty$. Therefore, we cannot conclude that $\mathcal{F}$ is a contraction for sufficiently large τ (which would allow an application of the Banach Fixed-Point Theorem). Section 4.3 explores numerically an example in which steady states are and are not unique as τ varies.

4.2. Stationary age-zero density

We first describe a specific situation in which it is straightforward to identify a steady state. Assume that the age-zero distribution μ is a steady state of the classical mean-field PDE. That is,

$$\partial_x\left(\mu(x)\left(\int_U \varphi(y-x)\,\mu(x)\,\mathrm{d}y\right)\right) - \frac{\sigma^2}{2}\partial_{xx}\mu(x) = 0, \tag{20}$$

coupled with the appropriate no-flux boundary conditions. Define $\rho^*(a,x) = \mu(x)$ for all $a \in [0,A]$. Then we have,

$$\begin{aligned}
&\tau\,\partial_a\rho^*(a,x) + \partial_x\left(\rho^*(a,x)\left(\int_U\int_0^A M(a,b)\,\varphi(y-x)\,\rho^*(b,y)\,\mathrm{d}b\,\mathrm{d}y\right)\right) - \frac{\sigma^2}{2}\partial_{xx}\rho^*(a,x)\\
&= \tau\,\partial_a\mu(x) + \partial_x\left(\mu(x)\left(\int_U\int_0^A M(a,b)\,\varphi(y-x)\,\mu(y)\,\mathrm{d}b\,\mathrm{d}y\right)\right) - \frac{\sigma^2}{2}\partial_{xx}\mu(x)\\
&= \partial_x\left(\mu(x)\left(\int_U \varphi(y-x)\,\mu(y)\,\mathrm{d}y\right)\left(\int_0^A M(a,b)\,\mathrm{d}b\right)\right) - \frac{\sigma^2}{2}\partial_{xx}\mu(x)\\
&= \partial_x\left(\mu(x)\left(\int_U \varphi(y-x)\,\mu(y)\,\mathrm{d}y\right)\right) - \frac{\sigma^2}{2}\partial_{xx}\mu(x)\\
&= 0,
\end{aligned}$$

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

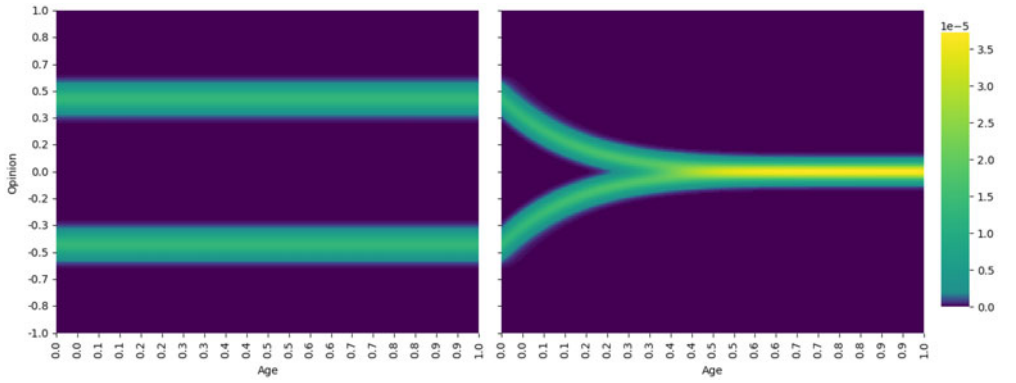


Figure 3. *Showing two alternative steady states of (10) over the joint age-opinion space. In both cases the interaction function ϕ is a smooth bounded confidence (2) with $r_1 = 0.5, r_2 = 0.6$, the diffusion coefficient $\sigma = 0.05$, the age-interaction kernel $M \equiv 1$ and the age-zero distribution $\mu = \mu^{(2)}$. The first corresponds to a fixed point of the mapping (A6) with $\lambda_0 = \mu^{(2)}$, while the second corresponds to a fixed point in which $\lambda_0 = \mu^{(1)}$ has a single cluster.*

hence (13a) is satisfied. In addition

$$\rho^*(0,x) - \mu(x) \int_U \rho^*(1,y)\,\mathrm{d}y = \mu(x) - \mu(x) \int_U \rho^*(1,y)\,\mathrm{d}y = \mu(x) - \mu(x) = 0$$

hence (13b) is satisfied. Finally, the no-flux boundary conditions for μ directly imply those for ρ^*, hence all conditions are satisfied for ρ^* to be a steady state. We now use this context to provide a specific example of non-uniqueness and demonstrate the impact of changing τ.

4.3. Non-uniqueness

As discussed at the end of Section 4.1, we cannot show that steady states are unique. That is, for given μ and ϕ, there may be multiple $\lambda \in K$ such that the corresponding solution $\rho \in W_2^{2,1}(U_A)$ of (17) also satisfies $\lambda = \mathcal{F}(\lambda)$. In Figure 3 below, we give a specific example in the setting discussed in the previous Section 4.2, in which μ is stationary in the classical mean-field limit.

Consider an age-zero distribution $\mu^{(2)}$ satisfying (20) in which the population is split into two equal clusters (as in e.g. [22]). Note that $\mu^{(2)}$ is found numerically using a scheme that applies the opinion operator described in Appendix B.1. As in the previous Section 4.2, define $\rho^*(a,x) = \mu^{(2)}(x)$ for all $a \in [0,A]$. As this ρ^* is known to be a steady state, we also have that $\mathcal{F}(\mu^{(2)}) = \mu^{(2)}$.

We can also consider iteratively applying $\mathcal{F}$ from a different starting point, using a numerical approximation of $\mathcal{F}$, described in Appendix B.2,. In particular, we begin from a solution $\mu^{(1)}$ of (20) with only a single cluster. We find that we do not converge towards $\mu^{(2)}$ but instead reach a different steady state in which the population is encouraged towards consensus at higher ages.

An example is shown in Figure 3, using a smoothed bounded confidence interaction function with $r_1 = 0.5, r_2 = 0.6$, $\tau = 0.15$, $\sigma = 0.05$ and $M \equiv 1$. Heatmaps show the stationary distribution over U_A, with age on the horizontal axis and opinion on the vertical axis. In the left panel, we begin at $\mu^{(2)}$ and the population remains in these two stable clusters. However, in the right panel, we begin the fixed point iteration from $\mu^{(1)}$ and observe a different stationary distribution. This shows that in certain cases, multiple stable behaviours could be observed on the macroscopic level.

We next investigate the effect of increasing τ. For each value of τ, we again begin the fixed point iteration at $\mu^{(1)}$. The final distribution reached is shown for various τ in Figure 4. For small values below $\tau = 0.23$ (including the example shown in the right of Figure 3), λ has a single peak, indicating

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

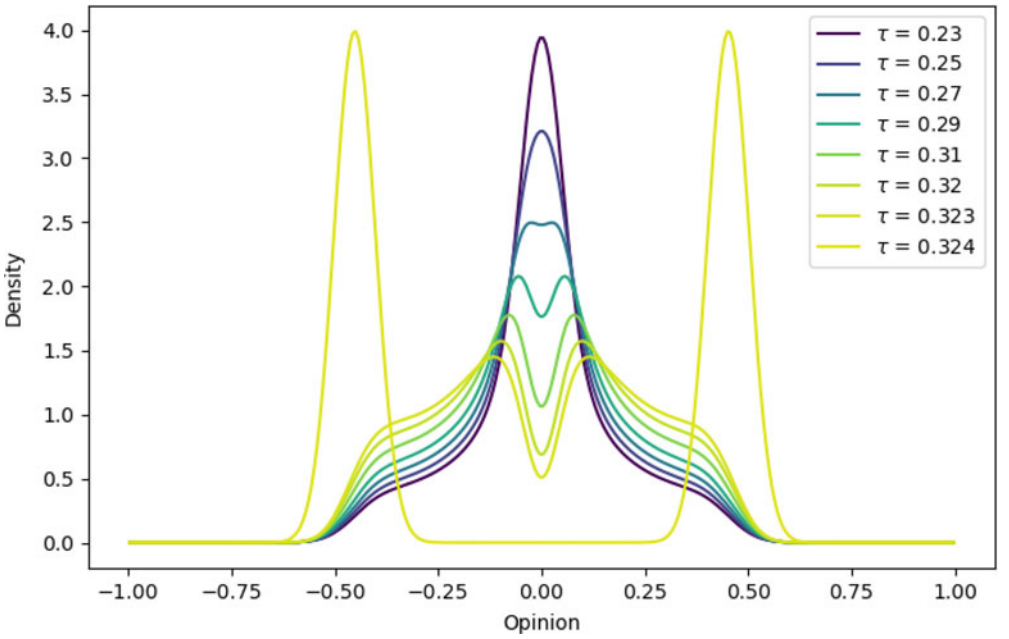


Figure 4. *The fixed point λ of the mapping $\mathcal{F}$ (A6), calculated numerically for various values of τ. The interaction function ϕ is a smooth bounded confidence (2) with $r_1 = 0.5$, $r_2 = 0.6$, the diffusion coefficient $\sigma = 0.05$, the age-interaction kernel $M \equiv 1$ and the age-zero distribution $\mu = \mu^{(2)}$. As τ is increased, there is a transition from a density with a single cluster to a density with two nearby peaks, then further to $\lambda = \mu^{(2)}$.*

that the population merges into a single cluster as individuals age. As τ is increased, the age at which individuals merge increases, causing λ to spread out. Eventually, this merging occurs extremely close to age $a = 1$ and λ has two peaks. Finally, for τ above approximately 0.323, the stationary state in which the clusters merge no longer exists, and $\lambda = \mu^{(2)}$ is the unique stationary state.

This example raises the interesting question of what clustering behaviours, merges and emergences are possible as the opinion profile changes (in a) across a steady state? This forms a part of a broader question about the type of patterns it is possible to observe in the joint age-opinion distribution at a steady state, and which of these are stable.

It is also natural to consider the stability of these stationary distributions. However, performing even a linear stability analysis of this system is not straightforward for several reasons: the stationary distributions are not known analytically, the linearised system remains non-local and perturbations must satisfy positivity and mass-preserving constraints, while also leaving the population-level age distribution unaffected. For these reasons, we reserve the stability analysis of these stationary distributions for future research.

4.4. Symmetry

In this section, we show that if $\mu(x)$ is symmetric around zero, then so is the steady state.

Let $\rho(a, x)$ be a solution to (13). Assume that for all $x \in [-1, 1]$ that $\rho(0, x) = \mu(x) = \mu(-x)$.

Define $\tilde{\rho}(a, \tilde{x}) = \rho(a, -x)$. Then

$$\partial_a \tilde{\rho}(a, x) = \partial_a \rho(a, -x)$$

$$= -\partial_x \left(\rho(a, -x) \left(\int_U \int_0^A M(a, b)\, \varphi(y + x)\, \rho(b, y)\, \mathrm{d}b\, \mathrm{d}y \right) \right) + \frac{\sigma^2}{2} \partial_{xx} \rho(a, -x)$$

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

$$= -\partial_x \left(\rho(a,-x) \left(\int_U \int_0^A M(a,b)\, \varphi(-y+x)\, \rho(b,-y)\, \mathrm{d}b\, \mathrm{d}y \right) \right) + \frac{\sigma^2}{2} \partial_{xx} \rho(a,-x)$$

$$= -\partial_x \left(-\rho(a,-x) \left(\int_U \int_0^A M(a,b)\, \varphi(y-x)\, \rho(b,-y)\, \mathrm{d}b\, \mathrm{d}y \right) \right) + \frac{\sigma^2}{2} \partial_{xx} \rho(a,-x)$$

$$= -\partial_x \left(-\tilde{\rho}(a,\tilde{x}) \left(\int_U \int_0^A M(a,b)\, \varphi(y-x)\, \tilde{\rho}(b,y)\, \mathrm{d}b\, \mathrm{d}y \right) \right) + \frac{\sigma^2}{2} \partial_{xx} \tilde{\rho}(a,\tilde{x})$$

$$= -\partial_{\tilde{x}} \left(\tilde{\rho}(a,\tilde{x}) \left(\int_U \int_0^A M(a,b)\, \varphi(y-x)\, \tilde{\rho}(b,y)\, \mathrm{d}b\, \mathrm{d}y \right) \right) + \frac{\sigma^2}{2} \partial_{xx} \tilde{\rho}(a,\tilde{x})\,.$$

The condition on μ also ensures that the boundary conditions are the same. Hence, the derivative (in a) of $\rho(a,x) - \rho(a,-x)$ is equal to zero everywhere in U and so $\rho(a,-x) = \tilde{\rho}(a,\tilde{x}) = \rho(a,x)$, meaning that (13) preserves the symmetry around $x = 0$ present in $\mu(x)$. Hence, if $\mu(x)$ is symmetric, then the steady state is also symmetric.

5. Macroscopic dynamics

To shed light on the dynamics of the full system (10), we show in this section key properties of the density ρ, then consider several scenarios in which solutions can be expressed in terms of simpler systems. For the duration of Section 5, we assume the solution $\rho(t,a,x)$ of (10) is sufficiently smooth to allow the exchange of integrals and derivatives as required.

5.1. Age transport and conservation of mass

Define the density

$$\pi(t,a) = \int_U \rho(t,a,x)\, \mathrm{d}x$$

which describes the total density of individuals with age a (across all opinions), which formally satisfies

$$\begin{aligned}
\partial_t \pi(t,a) &= \int_U \partial_t \rho(t,a,x)\, \mathrm{d}x \\
&= -\tau\, \partial_a \int_U \rho(t,a,x)\, \mathrm{d}x - \int_U \partial_x F[\rho](t,a,x)\, \mathrm{d}x \\
&= -\tau\, \partial_a \pi(t,a) - F[\rho](t,a,-1) + F[\rho](t,a,1) \\
&= -\tau\, \partial_a \pi(t,a)
\end{aligned}$$

with the final equality holding due to the no-flux boundary conditions. Hence, π satisfies the transport equation

$$\partial_t \pi(t,a) + \tau\, \partial_a \pi(t,a) = 0,$$

with periodic boundary condition $\pi(t,0) = \pi(t,1)$ arising from the age-zero boundary condition for ρ. Therefore,

$$\pi(t,a) = \pi(0, \{a - \tau t\})$$

where here $\{x\}$ denotes the fractional part of x. Thus, if $\tau(0,a) = 1$ for all $a \in [0,1]$, meaning there is initially a uniform age density, then this will also be preserved. In addition, we can also see that

$$\int_U \int_0^A \rho(t,a,x)\, \mathrm{d}a\, \mathrm{d}x$$

is constant in time and so the total mass is conserved.

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

We note also that a similar argument to that in Section 4.4 can be applied to the full system to show that if $\rho(0, a, x) = \rho(0, a, -x)$ and $\mu(x) = \mu(-x)$ for all $a \in [0, 1]$ and all $x \in [-1, 1]$, then $\rho(t, a, x) = \rho(t, a, -x)$ for all $t \geq 0$. That is, the PDE (10) preserves the symmetry around $x = 0$.

5.2. Evolution of the mean opinion

Based on the observation of a shifting consensus in Figure 1a, we next wish to consider how the mean opinion evolves in time. Define the mean opinion at age a and the overall mean opinion, respectively, by

$$m(t, a) = \int_U x\, \rho(t, a, x)\, \mathrm{d}x, \tag{21}$$

$$\bar{m}(t) = \int_0^A m(t, a)\, \mathrm{d}a = \int_0^A \int_U x\, \rho(t, a, x)\, \mathrm{d}x\, \mathrm{d}a\,. \tag{22}$$

Then we have

$$\begin{aligned}
\frac{\mathrm{d}\bar{m}}{\mathrm{d}t} &= \int_0^A \int_U x\, \partial_t \rho(t, a, x)\, \mathrm{d}x\, \mathrm{d}a \\
&= -\int_0^A \int_U x[\tau\, \partial_a \rho + \partial_x F[\rho](t, a, x)]\mathrm{d}x\, \mathrm{d}a \\
&= -\tau \int_U x \left(\int_0^A \partial_a \rho(t, a, x)\, \mathrm{d}a \right) \mathrm{d}x - \int_0^A \left(\int_U x\, \partial_x F[\rho](t, a, x)\, \mathrm{d}x \right)\, \mathrm{d}a \\
&= -\tau \int_U x(\rho(t, 0, x) - \rho(t, 1, x))\mathrm{d}x - \int_0^A (x\, F[\rho](t, a, x)\Big|_{x=-1}^{x=1} - \int_U F[\rho](t, a, x)\, \mathrm{d}x)\, \mathrm{d}a \\
&= \tau(m(t, 0) - m(t, 1)) + \int_0^A \int_U F[\rho](t, a, x)\, \mathrm{d}x\, \mathrm{d}a\,.
\end{aligned}$$

Note that the final equality holds as a result of the no-flux boundary conditions. Looking at the interaction term in the flux, we have

$$\begin{aligned}
&\int_0^A \int_U \rho(t, a, x) \left(\int_U \int_0^A M(a, b)\, \varphi(y - x)\, \rho(t, b, y)\, \mathrm{d}b\, \mathrm{d}y \right) \mathrm{d}x\, \mathrm{d}a \\
&\qquad = \int_0^A \int_0^A \int_U \int_U \varphi(y - x)\, M(a, b)\, \rho(t, b, y)\, \rho(t, a, x)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}a\, \mathrm{d}b
\end{aligned} \tag{23}$$

By comparing the effect of the two transformations $(x, a) \to (-x, a)$, $(y, b) \to (-y, b)$ and $(x, a) \to (-y, b)$, $(y, b) \to (-x, a)$, we see that the integral (23) is equal to zero when $M(a, b) = M(b, a)$ for all $a, b \in [0, A]$, meaning ages affect each other symmetrically. In this case, only the noise term remains in the flux and we obtain

$$\frac{\mathrm{d}\bar{m}}{\mathrm{d}t} = \tau \left(\int_U x\, \mu(x)\, \mathrm{d}x - m(t, A) \right) + \frac{\sigma^2}{2} \left(\int_0^A \rho(t, a, -1)\, \mathrm{d}a - \int_0^A \rho(t, a, 1)\, \mathrm{d}a \right)\,. \tag{24}$$

The first term describes evolution of the mean due to deaths/births, which brings the mean closer towards that of the age-zero distribution μ. The second term describes evolution of the mean due to interactions with the no-flux boundary conditions, which arise as individuals with opinions on the boundary can only move their opinions in one direction. These contributions help to explain the movement of clusters towards a more symmetric distribution when the mean of μ is zero, as is observed in Figure 2.

We next consider several scenarios in which a solution to the full problem (10) can be constructed from solutions of a different, typically simpler, system.

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

5.3. Convergence to consensus

We evaluate in a simple scenario how quickly the population converges to consensus. Assume $M \equiv 1$ and $\phi \equiv 1$, so that all individuals interact equally regardless of age or opinion. Assume also that the initial distribution ρ_0 and the age-zero distribution μ are both symmetric. In this case, the PDE (10) reduces to

$$\partial_t \rho(t,a,x) + \tau\, \partial_a \rho(t,a,x) + \partial_x \left(\rho(t,a,x) \left(\int_U \int_0^A (y-x)\, \rho(t,b,y)\, \mathrm{d}b\, \mathrm{d}y \right) \right) - \frac{\sigma^2}{2} \partial_{xx} \rho(t,a,x) = 0\,.$$

We first calculate

$$\begin{aligned} \int_U \int_0^A (y-x)\, \rho(t,b,y)\, \mathrm{d}b\, \mathrm{d}y &= \int_0^A \int_U y\, \rho(t,b,y)\, \mathrm{d}y\, \mathrm{d}b - x \int_U \int_0^A \rho(t,b,y)\, \mathrm{d}b\, \mathrm{d}y \\ &= \bar{m}(t) - x\,. \end{aligned}$$

As ρ_0 and μ are both symmetric, the solution will remain symmetric and so $\bar{m}(t) = 0$ for all $t \geq 0$. The PDE therefore reduces further to

$$\partial_t \rho(t,a,x) + \tau\, \partial_a \rho(t,a,x) + \partial_x \big(-x\, \rho(t,a,x) \big) - \frac{\sigma^2}{2} \partial_{xx} \rho(t,a,x) = 0\,. \tag{25}$$

Note that we have now effectively decoupled each age, as interactions between different ages arose only via the mean $\bar{m}$, which we know to be zero due to symmetry. Therefore, the only role played by age is to transport the density forward in a. If we momentarily ignore this age transport term in (25), we obtain the Fokker-Planck equation of an Ornstein–Uhlenbeck process [20] (with reflecting boundary conditions). We use this connection to construct a solution to (25).

For a given density $\varrho(x) \in L^2(U)$, let $\mathcal{S}_t[\varrho](x)$ denote the weak solution at time $t \geq 0$ of the following initial-boundary value problem

$$\partial_t p(t,x) + \partial_x \big(-x\, p(t,a,x) \big) - \frac{\sigma^2}{2} \partial_{xx} p(t,a,x) = 0 \qquad \text{in } U, \tag{26a}$$

$$x\, p(t,a,x) + \frac{\sigma^2}{2} \partial_x p(t,a,x) = 0 \qquad \text{on } \partial U = \{-1,1\}, \tag{26b}$$

$$p(0,x) = \varrho(x) \qquad \text{in } U\,. \tag{26c}$$

We consider $\varrho \in L^2(U)$ as we will choose below $\varrho = \mu$.

Then the solution to (25) is given by

$$\rho(t,a,x) = \begin{cases} \mathcal{S}_{a\tau^{-1}}[\mu](x) & \text{for } 0 \leq a \leq \tau t, \\ \mathcal{S}_t[\rho_0(a-\tau t, \cdot)](x) & \text{for } \tau t \leq a \leq A\,. \end{cases} \tag{27}$$

For 'large' values of a in the second case, $\rho(t,a,x)$ is given by the solution to (26) at the same time t but applied to $\rho_0(a-\tau t, \cdot)$ to account for the age transport. For 'smaller' values of a, $a - \tau t$ would be negative, meaning $\rho(t,a,x)$ is in fact the evolution of the age-zero distribution μ under (26) until time $a\tau^{-1}$ (since it has been this length of time since this distribution was at $a = 0$). Requiring $\rho_0(0,x) = \mu(x)$ ensures that there is no jump at $t = a\tau^{-1}$. Note that we can construct the solution in each section separately and concatenate them because the evolution at each age is independent.

For $t \geq \tau^{-1} A$ all values of $a \in [0,A]$ fall into the first case. As $\mathcal{S}_a[\mu](x)$ is independent of t, the solution is therefore stationary for $t \geq \tau^{-1} A$, with the stationary distribution being entirely independent of the initial distribution ρ_0.

Using this solution we can also examine the evolution of the opinion variance by studying

$$v(t) = \int_U x^2\, \mathcal{S}_t[\varrho](x)\, \mathrm{d}x\,.$$

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

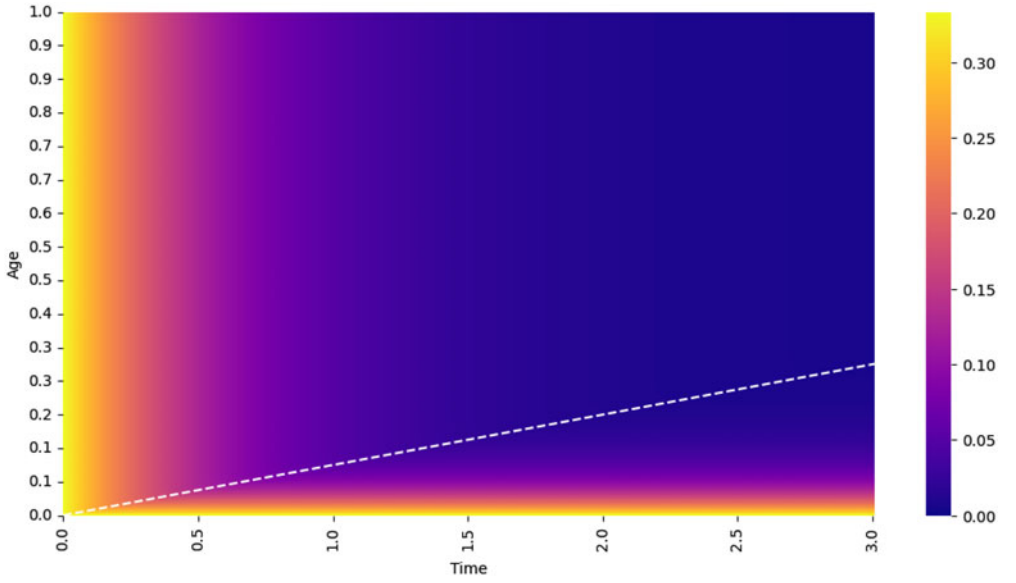


Figure 5. *Opinion variance at each age over time, calculated from the numerical solution of (10) with* $M \equiv 1$, $\phi \equiv 1$, $\tau = \sigma = 0.1$ *and both* ρ_0 *and* μ *uniformly distributed. A white dashed line shows when* $a = \tau t$, *which marks the transition between the two parts of the solution (27).*

Differentiating in time, then integrating by parts twice, we have

$$\begin{aligned}\frac{\mathrm{d}v}{\mathrm{d}t} &= \int_U x^2\, \partial_x \left(x\, \mathcal{S}_t[\varrho](x) + \frac{\sigma^2}{2} \partial_x \mathcal{S}_t[\varrho](x) \right) \mathrm{d}x, \\ &= -2 \int_U x \left(x\, \mathcal{S}_t[\varrho](x) + \frac{\sigma^2}{2} \partial_x \mathcal{S}_t[\varrho](x) \right) \mathrm{d}x, \\ &= -2v(t) - \sigma^2 \left(\mathcal{S}_t[\varrho](1) - \mathcal{S}_t[\varrho](-1) - \int_U \mathcal{S}_t[\varrho](x)\, \mathrm{d}x \right).\end{aligned}$$

As (26) also preserves symmetry, if ϱ is symmetric then $\mathcal{S}_t[\varrho](1) = \mathcal{S}_t[\varrho](-1)$. Moreover, the mass is preserved, so if ϱ has mass 1, then so does $\mathcal{S}_t[\varrho]$. Note that both these properties are true of the choices for ϱ in (27). In such a case, we conclude that $v(t)$ satisfies

$$\frac{\mathrm{d}v}{\mathrm{d}t} = \sigma^2 - 2v \quad \Rightarrow \quad v(t) = \frac{\sigma^2}{2} + \left(v(0) - \frac{\sigma^2}{2} \right) e^{-2t}.$$

Hence we conclude that the opinion density $\rho(t, a, x)$ has mean zero and variance

$$\int_U x^2\, \rho(t, a, x)\, \mathrm{d}x = \begin{cases} \dfrac{\sigma^2}{2} + \left(\displaystyle\int_U x^2\, \mu(x)\, \mathrm{d}x - \frac{\sigma^2}{2} \right) e^{-2a\tau^{-1}} & \text{for } 0 \le a \le \tau t, \\ \dfrac{\sigma^2}{2} + \left(\displaystyle\int_U x^2\, \rho_0(a - \tau t, x)\, \mathrm{d}x - \frac{\sigma^2}{2} \right) e^{-2t} & \text{for } \tau t \le a \le A\,. \end{cases} \tag{28}$$

Figure 5 shows the opinion variance at each age over time, calculated from the numerical solution of 10 with $M \equiv 1$, $\phi \equiv 1$, $\tau = \sigma = 0.1$ and both ρ_0 and μ uniformly distributed. This matches exactly with the theoretical variance (28). Initially, there is a high variance at all ages due to the uniform initial condition, but as the population moves towards consensus, the only source of variance is from μ, which decays as individuals age.

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

5.4. No inter-age interactions

We now consider the other extreme and assume that the age-interaction kernel $M(a,b)=\delta_a(b)$, that is, individuals only interact with others of exactly the same age. Note that this violates Assumption 1 and is not a reasonable choice for the microscopic model as all individuals almost surely have distinct ages; therefore, there would simply be no opinion evolution. However, in the macroscopic setting, we will show that a solution for (10) can still be constructed.

For $M(a,b)=\delta_a(b)$ the flux becomes,

$$F[\rho](t,a,x)=\rho(t,a,x)\left(\int_U \varphi(y-x)\,\rho(t,a,y)\,\mathrm{d}y\right)-\frac{\sigma^2}{2}\partial_x\rho(t,a,x),$$

as each age a only interacts with the density $\rho(t,a,x)$ at that same age. We therefore expect that the solution will match that of the standard mean-field model 11, transported forwards in age. We proceed in much the same way as in the previous Section 5.3.

For a given density $\varrho(x)\in L^2(U)$, let $\mathcal{S}_t^{(2)}[\varrho](x)$ denote the weak solution at time $t\geq 0$ of the classical mean-field limit, given by the following initial-boundary value problem

$$\partial_t p(t,x)+\partial_x G[p](t,x)=0 \tag{29a}$$

$$G[p](t,-1)=G[p](t,1)=0 \tag{29b}$$

$$p(0,x)=\varrho(x) \tag{29c}$$

with flux term

$$G[p](t,x)=p(t,x)\left(\int_U \varphi(y-x)\,p(t,y)\,\mathrm{d}y\right)-\frac{\sigma^2}{2}\partial_x p(t,x)\,. \tag{30}$$

Then, similarly to the construction in Section 5.3, the solution to (10) for $M(a,b)=\delta_a(b)$ is given by

$$\rho(t,a,x)=\begin{cases}\mathcal{S}^{(2)}_{a\tau^{-1}}[\mu](x) & \text{for } 0\leq a\leq \tau t,\\ \mathcal{S}^{(2)}_{t}[\rho_0(a-\tau t,\cdot)](x) & \text{for } \tau t\leq a\leq A\,.\end{cases} \tag{31}$$

Again, the distribution is stationary for $t\geq \tau^{-1}A$ and independent of the initial distribution ρ_0.

In both this example, where individuals interact only with those of exactly the same age, and the previous example of Section 5.3, where all individuals interact at all times, we exploit the decoupling of ages to construct the solution of (10) from the solution to simpler problems.

5.5. No ageing

In our next scenario, we consider how the full problem (10) can again be reduced in the case that $\tau=0$. While this may appear to remove the ageing effect entirely, it still allows for differing opinion distributions at different ages and certain non-trivial age interactions (specifically, we may still reduce the problem in the case that age-interactions have the form $M(a,b)=M(b)$).

Setting $\tau=0$ in (10), we instead have

$$\partial_t\rho(t,a,x)+\partial_x F[\rho](t,a,x)=0 \tag{32a}$$

with the same boundary conditions as in (10). Assume that $M(a,b)=M(b)$, in which case the flux (9) reduces to

$$F[\rho](t,a,x)=\rho(t,a,x)\left(\int_U \varphi(y-x)\,\left(\int_0^A M(b)\,\rho(t,b,y)\,\mathrm{d}b\right)\,\mathrm{d}y\right)-\frac{\sigma^2}{2}\partial_x\rho(t,a,x)$$

We first solve for

$$u(t,x)=\int_0^A M(b)\,\rho(t,b,x)\,\mathrm{d}b,$$

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

and use this to decouple the ages in (32). Formally, we have

$$\begin{aligned}\partial_t u(t,x) &= \int_0^A M(b)\,\partial_t \rho(t,b,x)\,\mathrm{d}b \\ &= -\int_0^A M(b)\,\partial_x\left(\rho(t,b,x)\left(\int_U \varphi(y-x)\,u(t,y)\,\mathrm{d}y\right)\right) - M(b)\frac{\sigma^2}{2}\partial_{xx}\rho(t,b,x)\,\mathrm{d}b, \\ &= -\partial_x\left(\int_0^A M(b)\,\rho(t,b,x)\left(\int_U \varphi(y-x)\,u(t,y)\,\mathrm{d}y\right)\,\mathrm{d}b\right) + \frac{\sigma^2}{2}\partial_{xx}\left(\int_0^A M(b)\rho(t,b,x)\,\mathrm{d}b\right).\end{aligned}$$

It is at this point crucial that $u(t,y)$ is independent of age, as we may now factorise the following integral,

$$\int_0^A M(b)\,\rho(t,b,x)\left(\int_U \varphi(y-x)\,u(t,y)\,\mathrm{d}y\right)\mathrm{d}b = \left(\int_0^A M(b)\,\rho(t,b,x)\,\mathrm{d}b\right)\left(\int_U \varphi(y-x)\,u(t,y)\,\mathrm{d}y\right).$$

If $M(a,b)$ depended on both a and b, rather than only on b, we would require u to be a function of (t,a,x) and the above factorisation would not be possible. Using this factorisation, we conclude,

$$\partial_t u(t,x) + \partial_x\left(u(t,x)\left(\int_U \varphi(y-x)\,u(t,y)\,\mathrm{d}y\right)\right) - \frac{\sigma^2}{2}\partial_{xx}u(t,x) = 0, \tag{33}$$

with appropriate no-flux boundary conditions. This is again the standard mean-field limit of the model without age structure 11. Using the notation of Section 5.4, we have

$$u(t,x) = \mathcal{S}_t^{(2)}[u(0,\cdot)](x) = \mathcal{S}_t^{(2)}\left[\int_0^A M(b)\,\rho_0(a,\cdot)\right](x).$$

Since $u(t,x)$ is now known, we may replace the flux term in (32) with

$$F_u[\rho](t,a,x) = \rho(t,a,x)\left(\int_U \varphi(y-x)\,u(t,y)\,\mathrm{d}y\right) - \frac{\sigma^2}{2}\partial_x\rho(t,a,x).$$

Using this flux term, the evolution of $\rho(t,a,x)$ is now independent of the value of ρ at any other age. Hence we can instead specify a family of problems: for each $a \in (0,1]$ find a solution $\rho^{(a)}(t,x)$ of

$$\partial_t \rho^{(a)}(t,x) + \partial_x F_u\left[\rho^{(a)}\right](t,x) = 0 \tag{34a}$$

$$F_u\left[\rho^{(a)}\right](t,-1) = F_u\left[\rho^{(a)}\right](t,1) = 0 \tag{34b}$$

$$\rho^{(a)}(0,x) = \rho_0(a,x) \tag{34c}$$

while for $a = 0$ we have the constant

$$\rho^{(0)}(t,x) = \mu(x)\int_U \rho(0,1,y)\,\mathrm{d}y.$$

Similarly to the fixed point argument used to show the existence of steady states in Section 4, we have again reduced the problem to a (family of) linear second-order parabolic PDE(s). Since the problems (34) differ only in their initial conditions $\rho_0(a,x)$ they will be continuous in a if ρ_0 is, except at $a = 0$ where the constant solution may create a discontinuity.

Note that a similar analysis is not possible when $\tau > 0$ since the evolution of u would then be influenced by the boundary at $a = A$. Specifically, it would require knowledge of the opinion density at age A as this would be replaced continuously by μ as individuals die, in the same way that the mean opinion at age A is continuously replaced by the mean of μ in (24).

5.6. McKendrick ageing

To conclude our analysis of the dynamics of (10), we now compare against the ageing structure utilised in the McKendrick (MK) equation, a commonly used model for age-structured population dynamics in

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

the life sciences [29, 33, 38]. The MK equation allows for more complex population dynamics than the simple age-transport in (10); therefore, we are interested in cases where (10) can still incorporate some of these effects.

For a population of animals/individuals, the MK equation describes the density $\pi(t, a)$ of an age $a \in [0, \infty)$ at time $t \geq 0$. The initial-boundary value problem reads

$$\partial_t \pi(t, a) + \tau \partial_a \pi(t, a) = -\tau \, d(a) \, \pi(t, a), \tag{35a}$$

$$\pi(t, 0) = \int_0^\infty d(a) \, \pi(t, a) \, \mathrm{d}a, \tag{35b}$$

$$\pi(0, a) = \pi_0(a), \tag{35c}$$

where $d : [0, \infty) \to \mathbb{R}$ is the age-dependent death rate and $\pi_0 \in L^\infty(\mathbb{R}^+)$ is the initial age profile. Note that in the standard MK model (11), the constant τ could be removed using a time rescaling; it is retained here to simplify the comparison with the opinion formation model (10). The boundary condition (35b) is chosen to ensure a constant total population size. In addition, we assume that

$$\int_0^\infty \pi_0(a) \, \mathrm{d}a = 1,$$

so that

$$\int_0^\infty \pi(t, a) \, \mathrm{d}a = 1,$$

for all $t \geq 0$.

In the same way that (10) can be viewed as an extension of the age transport equation to include opinion dynamics, this setup can now be similarly extended to give

$$\partial_t \rho(t, a, x) + \tau \partial_a \rho(t, a, x) + \partial_x \tilde{F}[\rho](t, a, x) = -\tau \, d(a) \, \rho(t, a, x), \tag{36a}$$

$$\tilde{F}[\rho](t, a, -1) = \tilde{F}[\rho](t, a, 1) = 0, \tag{36b}$$

$$\rho(t, 0, x) = \mu(x) \int_U \int_0^\infty d(a) \, \rho(t, a, x) \, \mathrm{d}a \, \mathrm{d}x, \tag{36c}$$

$$\rho(0, a, x) = \rho_0(a, x), \tag{36d}$$

where the definition of $\tilde{F}[\rho]$ is adapted slightly from (9) as

$$\tilde{F}[\rho](t, a, x) = \rho(t, a, x) \left(\int_U \int_0^\infty M(a, b) \, \varphi(y - x) \, \rho(t, b, y) \, \mathrm{d}b \, \mathrm{d}y - \frac{\sigma^2}{2} \partial_x \log\left(\rho(t, a, x)\right) \right). \tag{37}$$

In addition, for a solution ρ of (36), the age density

$$\pi(t, a) = \int_U \rho(t, a, x) \, \mathrm{d}x$$

satisfies the MK equation (35) with initial condition

$$\pi_0(a) = \int_U \rho_0(a, x) \, \mathrm{d}x,$$

and π can thus be solved independently of ρ (e.g. using the method of characteristics). Using this solution, we can replace or update the boundary condition (36c) to obtain

$$\partial_t \rho(t, a, x) + \tau \partial_a \rho(t, a, x) + \partial_x \tilde{F}[\rho](t, a, x) = -\tau \, d(a) \, \rho(t, a, x), \tag{38a}$$

$$\tilde{F}[\rho](t, a, -1) = \tilde{F}[\rho](t, a, 1) = 0, \tag{38b}$$

$$\rho(t, 0, x) = \mu(x) \left(\int_0^\infty d(a) \, \pi(t, a) \, \mathrm{d}a \right), \tag{38c}$$

$$\rho(0, a, x) = \rho_0(a, x) \, . \tag{38d}$$

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

This approach allows for greater flexibility in specifying the population's age profile, with the drawback that ages can become arbitrarily large. Indeed, steady states of the MK equation (35) have the form

$$\pi(a) = \pi(0)\ \exp\left(-\int_0^a d(b)\,\mathrm{d}b\right).$$

So $\pi(a)$ will be compactly supported (on $[0, A]$) if and only if there exists some $A > 0$ such that

$$\lim_{a \to A^-} d(a) = \infty,$$

that is, the death rate diverges. The following Proposition 5.1, proven in Appendix A, shows that in such cases a solution to (38) can be constructed from a solution to (10). That is, the original model (10) with a uniform age profile can also capture the behaviour of the model (38), which uses the more realistic MK ageing.

Proposition 5.1. *Assume that μ, d, ρ_0 and M are given and let $\pi(a)$ be the corresponding stationary age profile of (35), which is assumed to have compact support in* $[0, 1]$. *Let q be a solution of (10) with age interactions given by $M'(a, b) = M(a, b)\,\pi(b)$. Then ρ defined by*

$$\rho(t, a, x) = q(t, a, x)\,\pi(a) \tag{39}$$

is a solution of (38).

Therefore, by incorporating the stationary age profile π into the age interaction term M', the original model (10) can capture some effects from the model using McKendrick ageing (38), in particular, a non-uniform age profile. It should be noted that this only applies in the case that π is compactly supported, although in the context of opinion dynamics, this is a realistic assumption.

Note that if $M(a, b)$ is independent of a then so is $M'(a, b)$, hence Assumption 2 still holds and the analysis of steady states in Section 4 is still applicable.

This raises the question of how a non-stationary age density $\pi(t, a)$ might affect the opinion formation process. For example, we observe in many numerical solutions to (10) that the population becomes increasingly clustered with higher ages. Therefore, a population with an increasing proportion of older individuals might become more influenced by these clusters, while a population with an increasing proportion of younger individuals might be increasingly influenced by the age-zero distribution μ.

This section has analysed several features of the macroscopic dynamics, establishing properties such as transport of the population-level age distribution, conservation of mass and mean dynamics. These provide some insight into the macroscopic behaviours observed in Section 3.2. We have also provided connections to several existing models, allowing the solution to (10) to be expressed in terms of solutions to other models in certain cases. These connections provide a possible avenue for further analysis. Finally, we have compared against the structure of the McKendrick equation for population dynamics, demonstrating when there is an equivalence between these approaches and when they differ, opening additional directions for future work.

6. Conclusion

Throughout this paper, we have observed in both the microscopic SDE model as well as the macroscopic PDE model for opinion dynamics that introducing an explicit continuous age structure allows for a richer variety of opinion dynamics. The non-linearity introduced by the birth and death process opens the possibility of a shifting consensus, the emergence of new clusters and periodic behaviour. Such new possibilities naturally generate several open questions and challenges.

To begin, while the similarity in structure with the Cucker-Smale model and derivation performed in Section 3.1 indicate the convergence of the SDE model (1) to the PDE model (10), it remains to show this rigorously for this specific model. This would also establish the existence (and uniqueness) of the

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

solution to (10) in an appropriate space, which would allow the formal computations made in Section 5 to be made rigorous.

In demonstrating the existence of steady states and discussing their uniqueness, we have begun to classify the model's behaviours; however, there is still significant work to be done in this regard. For instance, to determine precise conditions for the uniqueness of steady states of (10). Moreover, it would be useful to define an equivalence between steady states that describes the various patterns in Figure 2. Finally, it remains to prove that the repeated patterns observed in FigureS 2f and 2e are indeed periodic solutions, with the broader goal of determining when and what periodic solutions are possible.

In considering the applicability of this model and its ability to predict real population-level age-structured opinion formation, there are various sources of data that may be useful. For example, the POLYMOD data set has been widely used as a measure of contact frequency between individuals of different ages [39, 40]. As described in Section 5.6, this age-interaction kernel could also be used to incorporate a fitted age distribution. In addition, even in the case that age-structured data is not available, model outputs could be aggregated over ages as in Figure 2 to provide a population-level opinion distribution that could be compared against survey data. As the model can display a wide variety of dynamics, it would be extremely useful to identify appropriate parameters and test its predictive capacity.

Lastly, one could consider the introduction of noise on the PDE level, giving a stochastic partial differential equation (SPDE). Such an approach was considered by Wehlitz et al. in [53] to mimic the movement of whole clusters, as observed in Figure 1c, that is normally lost on the PDE level (as the stochasticity in the SDE becomes deterministic diffusion in the PDE). By using an SPDE, one could recreate the movement of clusters and study their merging and re-emergence.

While there are several open directions for future research, it is already clear that the introduction of a continuous age structure leads to novel opinion dynamics that are interesting from both an applied and mathematical perspective. We hope that this framework will contribute to the development of more realistic opinion formation models that capture, and begin to explain, the persistent heterogeneity of opinions seen in the real world.

Funding statement. AN was supported by the Engineering and Physical Sciences Research Council through the Mathematics of Systems II Centre for Doctoral Training at the University of Warwick (reference EP/S022244/1). MTW acknowledges partial support from the EPSRC Small Grant EPSRC EP/X010503/1.

Competing interests. The authors declare no competing interests.

References

[1] Amblard, F. & Deffuant, G. (2004) The role of network topology on extremism propagation with the relative agreement opinion dynamics. *Phys. A: Stat. Mech. Appl.* **343**, 725–738.
[2] Axelrod, R. (1997) The dissemination of culture: A model with local convergence and global polarization. *J. Confl. Resolut.* **41**(2), 203–226.
[3] Blondel, V. D., Hendrickx, J. M. & Tsitsiklis, J. N. (2007) On the 2r conjecture for multi-agent systems. In: *European Control Conference (ECC)*, IEEE, pp. 874–881.
[4] Blondel, V. D., Hendrickx, J. M. & Tsitsiklis, J. N. (2010) Continuous-time average-preserving opinion dynamics with opinion-dependent communications. *SIAM J. Control Optim.* **48**(8), 5214–5240.
[5] Brezis, H. & Brézis, H. (2011) *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Springer.
[6] Cahill, P. H. & Gottwald, G. A. (2024) A modified Hegselmann-Krause model for interacting voters and political parties, arXiv preprint arXiv: 2410.13378.
[7] Carletti, T., Fanelli, D., Guarino, A., Bagnoli, F. & Guazzini, A. (2008) Birth and death in a continuous opinion dynamics model: The consensus case. *Eur. Phys. J. B* **64**, 285–292.
[8] Clarke, F. (2013) *Functional Analysis, Calculus of Variations and Optimal Control*, Vol. 264, Springer.
[9] Condie, S. A. & Condie, C. M. (2021) Stochastic events can explain sustained clustering and polarisation of opinions in social networks. *Sci. Rep.-UK* **11**(1), 1355.
[10] Cucker, F. & Smale, S. (2007) Emergent behavior in flocks. *IEEE Trans. Automat. Control* **52**(5), 852–862.
[11] De Reggi, S., Scarabel, F. & Vermiglio, R. (2023) Approximating reproduction numbers: A general numerical method for age-structured models, arXiv preprint arXiv: 2312.13477.
[12] Deffuant, G., Neau, D., Amblard, F. & Weisbuch, G. (2000) Mixing beliefs among interacting agents. *Adv. Complex Syst.* **3**(01n04), 87–98.

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

[13] Diestel, J. & Uhl, J. (1977) *Vector measures*, mathematical surveys and monographs, American Mathematical Society., **URL:**. *Available at:* https://books.google.co.uk/books?id=fEKZAwAAQBAJ.
[14] Düring, B. & Wolfram, M.-T. (2015) Opinion dynamics: Inhomogeneous Boltzmann-type equations modelling opinion leadership and political segregation. *Proc. R. Soc. A: Math., Phys. Eng. Sci.* **471**(2182), 20150345.
[15] Erban, R. & Chapman, S. J. (2007) Reactive boundary conditions for stochastic simulations of reaction–diffusion processes. *Phys. Biol.* **4**(1), 16.
[16] Evans, L. C. (2022) *Partial Differential Equations*, Vol. 19, American Mathematical Society.
[17] Fortunato, S., Latora, V., Pluchino, A. & Rapisarda, A. (2005) Vector opinion dynamics in a bounded confidence consensus model. *Int. J. Mod. Phys. C* **16**(10), 1535–1551.
[18] Friedman, A. (2008) *Partial Differential Equations of Parabolic Type*, Courier Dover Publications.
[19] Gabbay, M. (2007) The effects of nonlinear interactions and network structure in small group opinion dynamics. *Phys. A: Stat. Mech. Appl.* **378**(1), 118–126.
[20] Gardiner, C. W. (1985) Handbook of stochastic methods for physics, chemistry and the natural sciences.Springer series in synergetics.
[21] Garnier, J., Papanicolaou, G. & Yang, T.-W. (2017) Consensus convergence with stochastic effects. *Vietnam J. Math.* **45**, 51–75.
[22] Goddard, B. D., Gooding, B., Short, H. & Pavliotis, G. (2022) Noisy bounded confidence models for opinion dynamics: The effect of boundary conditions on phase transitions. *IMA J. Appl. Math.* **87**(1), 80–110.
[23] Graham, C., Kurtz, T. G., Méléard, S., et al. (1996) Asymptotic behaviour of some interacting particle systems; Mckean-Vlasov and Boltzmann models', Probabilistic Models for Nonlinear Partial Differential Equations: Lectures given at the 1st Session of the Centro Internazionale Matematico Estivo (CIME) held in Montecatini Terme, Italy, May 22-30, 1995, pp. 42–95.
[24] Grauwin, S. & Jensen, P. (2012) Opinion group formation and dynamics: Structures that last from nonlasting entities. *Phys. Rev. E – Stat., Nonlinear, Soft Matter Phys.* **85**(6), 066113.
[25] Hegselmann, R. & Krause, U. (2015) Opinion dynamics under the influence of radical groups, charismatic leaders, and other constant signals: A simple unifying model. *Netw. Heterogeneous Media* **10**(3), 477.
[26] Hegselmann, R. & Krause, U. (2002) Opinion dynamics and bounded confidence models, analysis, and simulation. *J. Artif. Soc. Soc. Simul.* **5**(3).
[27] Holden, H. (2010) *Splitting Methods for Partial Differential Equations with rough Solutions: Analysis and MATLAB Programs*, Vol. 11. European Mathematical Society.
[28] Ikeda, N. & Watanabe, S. (2014) *Stochastic Differential Equations and Diffusion Processes*, Vol. 24, Elsevier,
[29] Inaba, H. (2017) *Age-Structured Population Dynamics in Demography and Epidemiology*, Springer.
[30] Jabin, P.-E. & Wang, Z. (2017) Mean field limit for stochastic particle systems. *Adv. Theory, Models, Appl., Act. Part.* **1**, 379–402.
[31] Jacobmeier, D. (2005) Multidimensional consensus model on a barabási–albert network. *Int. J. Mod. Phys. C* **16**(04), 633–646.
[32] Kan, U., Feng, M. & Porter, M. A. (2023) An adaptive bounded-confidence model of opinion dynamics on networks. *J. Complex Netw.* **11**(1), 415–444.
[33] Keyfitz, B. L. & Keyfitz, N. (1997) The mckendrick partial differential equation and its uses in epidemiology and population study. *Math. Comput. Model* **26**(6), 1–9.
[34] Ladyzhenskaia, O. A., Solonnikov, V. A. & Ural'tseva, N. N. (1968) *Linear and Quasi-Linear Equations of Parabolic Type*, Vol. 23. American Mathematical Society.
[35] LeVeque, R. J. (2002) *Finite Volume Methods for Hyperbolic Problems*, Vol. 31. Cambridge University Press.
[36] Llabrés, J., Oliver-Bonafoux, S., Anteneodo, C. & Toral, R. (2024) Aging in some opinion formation models: A comparative study. *Physics* **6**(2), 515–528.
[37] Lorenz, J. (2007) Continuous opinion dynamics under bounded confidence: A survey. *Int. J. Mod. Phys. C* **18**(12), 1819–1838.
[38] M'kendrick, A. (1925) Applications of mathematics to medical problems. *Proc. Edinb. Math. Soc.* **44**, 98–130.
[39] Mossong, J., Hens, N., Jit, M., Beutels, P., Auranen, K., Mikolajczyk, R., Massari, M., Salmaso, S., Tomba, G. S., Wallinga, J., Heijne, J., Sadkowska-Todys, M., Rosinska, M. & Edmunds, W. J. (2017) POLYMOD social contact data (1.1), [Data set].
[40] Mossong, J., Hens, N., Jit, M., et al. (2008) Social contacts and mixing patterns relevant to the spread of infectious diseases. *PLos Med.* **5**(3), e74.
[41] Motsch, S. & Tadmor, E. (2014) Heterophilious dynamics enhances consensus. *SIAM Rev.* **56**(4), 577–621.
[42] Nugent, A., Gomes, S. N. & Wolfram, M.-T. (2023) On evolving network models and their influence on opinion formation. *Phys. D: Nonlinear Phenomena* **456**, 133914.
[43] Nugent, A., Gomes, S. N. & Wolfram, M.-T. (2024) Bridging the gap between agent based models and continuous opinion dynamics. *Phys. A: Stat. Mech. Appl.* **651**, 129886.
[44] Perthame, B. (2006) *Transport Equations in Biology*, Springer Science & Business Media.
[45] Pineda, M., Toral, R. & Hernandez-Garcia, E. (2009) Noisy continuous-opinion dynamics. *J. Stat. Mech.: Theory Exp.* **2009**(08), P08001.
[46] Pineda, M., Toral, R. & Hernández-García, E. (2011) Diffusing opinions in bounded confidence processes. *Eur. Phys. J. D* **62**, 109–117.
[47] Proskurnikov, A. V. & Tempo, R. (2018) A tutorial on modeling and analysis of dynamic social networks. part ii. *Annu. Rev. Control* **45**, 166–190.

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

[48] Rodriguez, N., Bollen, J. & Ahn, Y.-Y. (2016) Collective dynamics of belief evolution under cognitive coherence and social conformity. *PLoS One* **11**(11), e0165910.
[49] Stark, H.-U., Tessone, C. J. & Schweitzer, F. (2008) Decelerating microdynamics can accelerate macrodynamics in the voter model. *Phys. Rev. Lett.* **101**(1), 018701.
[50] Steiglechner, P., Keijzer, M. A., Smaldino, E., Moser, P., D, & Merico, A. (2024) Noise and opinion dynamics: How ambiguity promotes pro-majority consensus in the presence of confirmation bias. *R. Soc. Open Sci.* **11**(4), 231071.
[51] Sznitman, A.-S. (1991) Topics in propagation of chaos, *Ecole d'été de probabilités de Saint-Flour XIX—1989* **1464**:165–251.
[52] Wang, C., Li, Q., Chazelle, B., et al. (2017) Noisy Hegselmann–Krause systems: Phase transition and the 2r-conjecture. *J. Stat. Phys.* **166**(5), 1209–1225.
[53] Wehlitz, N., Sadeghi, M., Montefusco, A., Schütte, C., Pavliotis, G. A. & Winkelmann, S. (2024) Approximating particle-based clustering dynamics by stochastic PDEs, arXiv preprint arXiv: 2407.18952.
[54] Zhao, Y., Zhang, L., Tang, M. & Kou, G. (2016) Bounded confidence opinion dynamics with opinion leaders and environmental noises. *Comput. Oper. Res.* **74**, 205–213.

Appendix

A. Proofs

A.1. Steps required for the proof of Theorem 4.1

Step 1: Determine conditions for the existence of solution to (17) in an appropriate space.

We will see that it is sufficient to consider weak solutions of (17). We therefore begin by defining and setting the notation for weak derivatives and several useful function spaces, following the definitions in [16] and [34].

For $1 \le p \le \infty$, let $L^p(U_A)$ be the Banach space of all measurable functions on U_A with finite norm

$$\|u\|_{L^p(U_A)} = \begin{cases} \left(\int_0^A \int_U |u(a,x)|^p \, \mathrm{d}x \, \mathrm{d}a \right)^{1/p} & p < \infty, \\ \operatorname{ess\,sup}_{(a,x)\in U_A} |u(a,x)| & p = \infty \, . \end{cases}$$

We use the shorthand notation $\|u\|_p$ for this norm.

Suppose $u \in L^1(U_A)$. We say that $v \in L^1(U_A)$ is the weak partial x-derivative of u, written $D_x u = v$ provided

$$\int_0^A \int_U u(a,x)\, D_x \zeta(a,x) \, \mathrm{d}x \, \mathrm{d}a = - \int_0^A \int_U v(a,x)\, \zeta(a,x) \, \mathrm{d}x \, \mathrm{d}a$$

for all test functions $\zeta : U_A \to \mathbb{R}$ that are infinitely differentiable with compact support in U_A. The second weak x-derivative and the weak a-derivative, denoted D_x^2 and D_a, respectively, are defined similarly.

Denote by $W_2^{2,1}(U_A)$ the Sobolev space consisting of the elements of $L^2(U_A)$ having weak derivatives of the form $D_a^r D_x^s$ for any r and s with $2r + s \le 2$. The norm in this space is

$$\|u\|_{W_2^{2,1}(U_A)} = \left(\|u\|_2^2 + \|D_x u\|_2^2 + \|D_x^2 u\|_2^2 + \|D_a u\|_2^2 \right)^{1/2} \, .$$

While $u \in W_2^{2,1}(U_A)$ are functions of both age and opinion, it will at times be convenient to think of them instead as mapping ages $(0, A)$ into a space of functions over opinions. Denote by $H^1(U)$ the Sobolev space consisting of the elements of $L^2(U)$ with weak x-derivative and norm

$$\|v\|_{H^1(U)} = \left(\|v\|_{L^2(U)}^2 + \|D_x v\|_{L^2(U)}^2 \right)^{1/2} \, .$$

For any $u \in W_2^{2,1}(U_A)$, $u(a, \cdot)$ must lie in $H^1(U)$ for almost every $a \in (0, A)$. Thus, we can associate u with a function $\underline{u} : (0, A) \to H^1(U)$ given by $\underline{u}(a) = u(a, \cdot)$ (after possibly redefining u on a set of measure zero). Such functions are elements of a Bochner space: for $1 \le p < \infty$ let $L^p(0, A\,; H^1(U))$ denote the Bochner space of functions $\underline{u} : (0, A) \to H^1(U)$ with finite norm

$$\|\underline{u}\|_{L^p(0,A;H^1(U))} := \left(\int_0^A \|\underline{u}(a)\|_{H^1(U)}^p \, \mathrm{d}a \right)^{1/p} \, .$$

The following lemma gives a comparison of these norms.

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

Lemma A.1. *For any $u \in W_2^{2,1}(U_A)$, the function $\underline{u} : [0, A] \to H^1(U)$ defined by $\underline{u}(a) = u(a, \cdot)$ is an element of $L^2(0, A\,; H^1(U))$. In addition,*

$$\|\underline{u}\|_{L^2(0,A;H^1(U))} \leq \|u\|_{W_2^{2,1}(U_A)}\,. \tag{A1}$$

Proof. As $u \in W_2^{2,1}(U_A)$, the required weak derivative $D_x\underline{u}(a)$ exists for almost all a. We then compute directly

$$\begin{aligned}
\|\underline{u}\|_{L^2(0,A;H^1(U))} &= \left(\int_0^A \|\underline{u}(a)\|^2_{H^1(U)}\, \mathrm{d}a \right)^{1/2} \\
&= \left(\int_0^A \|\underline{u}(a)\|^2_{L^2(U)} + \|D_x\underline{u}(a)\|^2_{L^2(U)}\, \mathrm{d}a \right)^{1/2} \\
&= \left(\int_0^A \int_U u(a,x)^2 + \big(D_x u(a,x)\big)^2\, \mathrm{d}x\, \mathrm{d}a \right)^{1/2} \\
&\leq \left(\int_0^A \int_U u(a,x)^2\, \mathrm{d}x\, \mathrm{d}a \right)^{1/2} + \left(\int_0^A \int_U \big(D_x u(a,x)\big)^2\, \mathrm{d}x\, \mathrm{d}a \right)^{1/2} \\
&\leq \|u\|_{W_2^{2,1}(U_A)}
\end{aligned}$$

Hence $\underline{u} \in L^2(0, A\,; H^1(U))$ and we have shown (A1). □

We aim to apply Theorem 9.1 from Chapter IV Section 9 of [34] to guarantee the existence and uniqueness of weak solutions to (17). Using the notation established above, we now clarify the notion of a weak solution to (17) as an element $\rho \in W_2^{2,1}(U_A)$.

Assume first that (17) has a classical solution $\rho(a, x)$. Take a test function $\zeta(a, x)$ that is twice continuously differentiable, multiply $\rho(a, x)$ by $\zeta(a, x)$ and integrate over U_A to give

$$\begin{aligned}
0 &= \int_0^A \int_U \left(\tau\, \partial_a \rho + \partial_x(\rho\Lambda) - \frac{\sigma^2}{2} \partial_{xx}\rho \right) \zeta\, \mathrm{d}x\, \mathrm{d}a \\
&= \tau \int_0^A \int_U \partial_a \rho\, \zeta\, \mathrm{d}x\, \mathrm{d}a + \int_0^A \int_U \partial_x \left(\rho\Lambda - \frac{\sigma^2}{2} \partial_x \rho \right) \zeta\, \mathrm{d}x\, \mathrm{d}a \\
&= \tau \int_0^A \int_U \partial_a \rho\, \zeta\, \mathrm{d}x\, \mathrm{d}a + \int_0^A \left[\left(\rho\Lambda - \frac{\sigma^2}{2} \partial_x \rho \right) \zeta \right]_{x=-1}^{x=1} \mathrm{d}a - \int_0^A \int_U \left(\rho\Lambda - \frac{\sigma^2}{2} \partial_x \rho \right) \partial_x \zeta\, \mathrm{d}x\, \mathrm{d}a
\end{aligned} \tag{A2}$$

For an element $\rho \in W_2^{2,1}(U_A)$ the weak partial derivatives $D_a\rho$ and $D_x\rho$ exist and, since ρ is twice weakly differentiable in x we may apply the trace theorem to ρ and $D_x\rho$ to ensure their values at the boundary $x \in \{-1, 1\}$ are well-defined. Thus, we can prescribe the boundary condition

$$\rho\Lambda - \frac{\sigma^2}{2} D_x \rho = 0,$$

on $S_A = (0, A) \times \{-1, 1\}$, hence

$$\int_0^A \left[\left(\rho\Lambda - \frac{\sigma^2}{2} D_x \rho \right) \zeta \right]_{x=-1}^{x=1} \mathrm{d}a = 0\,.$$

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

We therefore call $\rho \in W_2^{2,1}(U_A)$ a weak solution of (17) if the following holds for any test function ζ:

$$\int_0^A \int_U D_a\rho\, \zeta \, \mathrm{d}x\, \mathrm{d}a - \frac{1}{\tau}\int_0^A \int_U \left(\rho\Lambda - \frac{\sigma^2}{2}D_x\rho\right) D_x\zeta \, \mathrm{d}x\, \mathrm{d}a = 0, \tag{A3a}$$

$$\rho(a,1)\Lambda(a,1) - \frac{\sigma^2}{2}D_x\rho(a,1) = 0, \tag{A3b}$$

$$\rho(a,-1)\Lambda(a,-1) - \frac{\sigma^2}{2}D_x\rho(a,-1) = 0, \tag{A3c}$$

$$\rho(0,x) = \mu(x)\,. \tag{A3d}$$

In order to apply the desired existence and uniqueness theorem, we will require the following conditions:

(L1) $\Lambda \in L^3(U_A)$.

(L2) $\Lambda' \in L^2(U_A)$, where $'$ denotes a derivative.

(L3) $\|\Lambda\|_{L_3(U_{t,t+s})} \to 0$ as $s \to 0$.

(L4) $\|\Lambda'\|_{L^2(U_{t,t+s})} \to 0$ as $s \to 0$.

(L5) $\mu \in L^2(U)$.

If we assume that $\lambda \in L^\infty(U)$ then we have immediately that $\Lambda \in L^\infty(U) \subset L^3(U)$, with the inclusion holding since U is bounded. In addition as $\phi \in C^2(U)$ we may use that $\lambda \in L^\infty(U)$ to exchange the integration and differentiation below to obtain

$$\begin{aligned}\Lambda'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\int_U \varphi(y-x)\,\lambda(y)\,\mathrm{d}y,\\ &= \int_U \frac{\mathrm{d}}{\mathrm{d}x}\varphi(y-x)\,\lambda(y)\,\mathrm{d}y,\\ &= -\int_U \Big(\phi'(y-x)\,(y-x) + \phi(y-x)\Big)\,\lambda(y)\,\mathrm{d}y,\end{aligned} \tag{A4}$$

hence $\Lambda' \in L^\infty(U) \subset L^2(U)$.

In addition to the assumption that $\lambda \in L^\infty(U)$, we note that we intend to define $\mathcal{F}(\lambda)$ as the integral of ρ, which is itself a probability density. Hence, we also assume that λ is positive almost everywhere with

$$\|\lambda\|_{L^1(U)} = \int_U \lambda(y)\,\mathrm{d}y = 1\,.$$

To summarise, we assume that

$$\lambda \in K_1 := \{\lambda \in L^\infty(U) : \lambda \geq 0 \text{ a.e.}, \|\lambda\|_{L^1(U)} = 1\}\,.$$

From this, we can now write the following Proposition A.1.

Proposition A.1. *The functions Λ and Λ', defined by (16) and (A4) respectively, are bounded uniformly with respect to $\lambda \in K_1$. That is, there exists a constant $C_{\phi'}$, depending on ϕ' and U only, such that for all $\lambda \in K_1$ and all $x \in U$,*

$$|\Lambda(x)| \leq 2 \quad \textit{and} \quad |\Lambda'(x)| \leq C_{\phi'}\,.$$

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

Proof. For any fixed $x \in U$, we have

$$\begin{aligned} |\Lambda(x)| &= \left| \int_U \varphi(y-x)\,\lambda(y)\,\mathrm{d}y \right| \\ &\leq \int_U \left|\varphi(y-x)\right| \left|\lambda(y)\right| \mathrm{d}y \\ &\leq 2 \int_U \left|\lambda(y)\right| \mathrm{d}y \\ &= 2\,. \end{aligned}$$

Since $|\phi| \leq 1$ and ϕ' is continuous, and therefore bounded on U, there is also some constant $C_{\phi'}$, which depends on ϕ and U only, such that

$$|\phi'(y-x)\,(y-x) + \phi(y-x)| \leq C_{\phi'}$$

for all x, y. Hence an almost identical argument to that above shows the second inequality. □

Since both Λ and Λ' are constant in time, we have therefore satisfied conditions (L1)–(L4) above. It will be necessary to later verify that the set K_1, or some subset of it, is invariant under $\mathcal{F}$ to ensure that these conditions are maintained.

Proposition A.2. *For any $\lambda \in K_1$ and $\mu \in L^2(U)$, there exists a unique solution $\rho \in W_2^{2,1}(U_A)$ to the initial-boundary value problem (17). Moreover there exists a constant c_1, depending on μ but not on λ, such that*

$$\|\rho\|_{W_2^{2,1}(U_A)} \leq c_1\, \|\mu\|_{L^2(U)}. \tag{A5}$$

Proof. Using the results above, we can directly apply Theorem 9.1 from Chapter IV Section 9 of [34] (with $f \equiv 0, \Phi \equiv 0$). Note that this also tells us that the constant c_1 depends upon the final age A but is bounded for any $A < \infty$. The constant c_1 also depends upon the coefficients of (17) but since, by Proposition A.1, these coefficients are bounded uniformly for $\lambda \in K_1$, we may also take a constant c_1 such that (A5) holds for any $\lambda \in K_1$. □

Step 2: Introduce the mapping $\lambda \mapsto \tilde{\lambda} =: \mathcal{F}(\lambda)$ and show that $\mathcal{F} : L^\infty(U) \to L^\infty(U)$ is well-defined.

Letting $\rho \in W_2^{2,1}(U_A)$ be the unique solution to (17) with a given $\lambda \in L^\infty(U)$, we define

$$\mathcal{F}(\lambda) := \int_0^A M(b)\, \underline{\rho}(a)\, \mathrm{d}a, \tag{A6}$$

where the integral is a Bochner integral (see, for example, §2.2 of [13] for further details). As $\rho \in W_2^{2,1}(U_A)$, (A1) gives that $\underline{\rho}$ is a well-defined element of $L^2(0, A\,; H^1(U))$. As $[0, A]$ is bounded, $L^2(0, A\,; H^1(U)) \subset L^1(0, A\,; H^1(U))$ and so Hölder's inequality gives

$$\|\underline{\rho}\|_{L^1(0,A;H^1(U))} \leq \sqrt{A}\|\underline{\rho}\|_{L^2(0,A;H^1(U))}\,. \tag{A7}$$

For each age $a \in (0, A)$, $M(a)$ is simply a constant, thus

$$\|M\underline{\rho}\|_{L^1(0,A;H^1(U))} = \int_0^A \|M(a)\,\underline{\rho}(a)\|_{H^1(U)}^2\,\mathrm{d}a = \int_0^A M(a)\,\|\underline{\rho}(a)\|_{H^1(U)}^2\,\mathrm{d}a \leq M_{\max} \int_0^A \|\underline{\rho}(a)\|_{H^1(U)}^2\,\mathrm{d}a\,.$$

Hence $M\underline{\rho} \in L^1(0, A\,; H^1(U))$, so Theorem 2 in §2.2 of [13] gives that the Bochner integral (A6) is well-defined.

Step 3: Find an appropriate set $K \subset L^\infty(U)$ so that we may apply Schauder's Fixed Point Theorem to $\mathcal{F} : K \to K$.

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

The above also shows that $\mathcal{F}(\lambda) \in H^1(U)$, although this alone does not guarantee that $\mathcal{F}(\lambda)$ remains in $L^\infty(U)$ as required. For this, we apply the following Lemma A.2.

Lemma A.2. *There exists a constant c_2 such that for any $u \in W_2^{2,1}(U_A)$ the following estimate holds*

$$\left\| \int_0^A \underline{u}(a)\, da \right\|_{L^\infty(U)} \leq c_2 \sqrt{A}\, \|u\|_{W_2^{2,1}(U_A)}\,. \tag{A8}$$

Proof. We apply a version of the Gagliardo–Nirenberg interpolation inequality to compare the L^∞ norm to the H^1 norm on a bounded interval U. The specific form used can be found in the Comments on Chapter 8 of [5], setting $p = \infty, q = r = 2, a = 1/2$. There exists a constant c_2 such that for any $v \in H^1(U)$

$$\|v\|_{L^\infty(U)} \leq c_2 \|v\|_{L^2(U)}^{1/2} \|u\|_{H^1(U)}^{1/2}\,.$$

Note that $\|v\|_{L^2(U)} \leq \|v\|_{H^1(U)}$ so we can replace the above with

$$\|v\|_{L^\infty(U)} \leq c_2 \|v\|_{H^1(U)}\,. \tag{A9}$$

Applying this to $\underline{u}(a)$ for almost every $0 \leq a \leq A$ gives

$$\int_0^A \|\underline{u}(a)\|_{L^\infty(U)}\, \mathrm{d}a \leq c_2 \int_0^A \|\underline{u}(a)\|_{H^1(U)}\, \mathrm{d}a < \infty, \tag{A10}$$

since $\underline{u} \in L^1(0, A\,; H^1(U))$. Hence $\underline{u}$ is integrable in $L^\infty(U)$ and so, combining the following inequalities gives

$$\begin{aligned}
\left\| \int_0^A \underline{u}(a)\, \mathrm{d}a \right\|_{L^\infty(U)} &\leq \int_0^A \|\underline{u}(a)\|_{L^\infty(U)}\, \mathrm{d}a && \text{Section 2, Theorem 4 of Diestel \& Uhl [13]} \\
&\leq c_2 \int_0^A \|\underline{u}(a)\|_{H^1(U)}\, \mathrm{d}a && \text{by (A10)} \\
&\leq c_2 \sqrt{A} \|\underline{u}\|_{L^2(0,A;H^1(U))} && \text{by (A7)} \\
&\leq c_2 \sqrt{A} \|u\|_{W_2^{2,1}(U_A)} && \text{by (A1)}
\end{aligned}$$

thus establishing the estimate. □

We can now combine Lemma A.2 with an estimate on the solution ρ of (17) to bound $\mathcal{F}(\lambda)$ uniformly in $L^\infty(U)$. Specifically, we show:

Proposition A.3. *Equation (A6) defines a mapping $\mathcal{F} : L^\infty(U) \to L^\infty(U)$. Moreover, this mapping is uniformly bounded in $L^\infty(U)$ as there exists some constant C such that for any $\lambda \in L^\infty(U)$,*

$$\|\mathcal{F}(\lambda)\|_{L^\infty(U)} \leq C\,. \tag{A11}$$

Proof. Let $\rho \in W_2^{2,1}(U_A)$ be the unique solution of (17) for a given $\lambda \in L^\infty(U)$ and $\underline{\rho}$ the corresponding function in $L^1(0, A\,; H^1(U))$. Then

$$\begin{aligned}
\|\mathcal{F}(\lambda)\|_{L^\infty(U)} &= \left\| \int_0^A M(a)\, \underline{\rho}(a)\, \mathrm{d}a \right\|_{L^\infty(U)} \\
&\leq M_{\max} \left\| \int_0^A \underline{\rho}(a)\, \mathrm{d}a \right\|_{L^\infty(U)} \\
&\leq c_2 \sqrt{A} M_{\max}\, \|\rho\|_{W_2^{2,1}(U_A)}, && \text{by Lemma A.2} \\
&\leq c_1 c_2 \sqrt{A} M_{\max}\, \|\mu\|_{L^2(U)}, && \text{by (A5).}
\end{aligned}$$

Letting $C = c_1 c_2 \sqrt{A} M_{\max}\, \|\mu\|_{L^2(U)}$ completes the proof. □

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

Next, we use properties of the PDE (17) to look for a subset $K \subset L^\infty(U)$ that is invariant under $\mathcal{F}$, with the goal of applying Schauder's Fixed Point Theorem to conclude the existence of a fixed point of $\mathcal{F}$ in K.

Proposition A.4. *The following set*

$$K = \{\lambda \in L^\infty(U) : \lambda \geq 0 \text{ a.e.}, \|\lambda\|_{L^1(U)} = 1,\ \|\lambda\|_{L^\infty(U)} \leq C\} \tag{A12}$$

is invariant under $\mathcal{F}$.

Proof. Applying a maximum principle, for example Lemma 5 in Chapter 2 of [18], gives that $\rho \geq 0$ and so $\lambda \geq 0$ (almost everywhere). As (17) preserves mass, we also have that $\|\lambda\|_1 = \|\mu\|_1 = 1$. The final property is given by Proposition A.3. □

A major obstacle is that, in the norm topology on $L^\infty(U)$, it is not clear if the set K is compact. Alternatively, we may be able to apply Schauder's Fixed Point Theorem in the weak$*$ topology on $L^\infty(U)$. The following Proposition A.5 and Proposition A.6 guarantee the necessary properties.

Proposition A.5. *The set K is weakly compact and weakly closed in* $L^\infty(U)$.

Proof. The set K is a subset of

$$K' = \{\lambda \in L^\infty(U) : \|\lambda\|_{L^\infty(U)} \leq C\},$$

which itself is weakly compact. This is a consequence of the Banach-Alaoglu Theorem, see for example Theorem 3.14 and Corollary 3.15 in [8].

Denote by $\langle \xi, u \rangle$ the pairing of $\xi \in L^1(U)$ with $u \in L^\infty(U)$. Let λ^n be a sequence of elements of K converging weakly in $L^\infty(U)$ to some $\lambda \in L^\infty(U)$. We wish to show that $\lambda \in K$.

Assume that λ is not positive almost everywhere, then there exists some set $V \subset U$ with strictly positive Lebesgue measure on which λ is strictly negative. Let χ_V denote the indicator function on this set, then $\chi_V \in L^1(U)$ and so $\langle \chi_V, \lambda^n \rangle \to \langle \chi_V, \lambda \rangle$. However for every $n \in \mathbb{N}$ the integral $\langle \chi_V, \lambda^n \rangle \geq 0$, since χ_V is positive and λ^n is positive almost everywhere, while the limit $\langle \chi_V, \lambda \rangle < 0$ by construction. This is a contradiction, hence $\lambda \geq 0$ a.e.

As $\lambda \geq 0$ a.e. we have that

$$\|\lambda\|_{L^1(U)} = \int_U |\lambda(x)| \, \mathrm{d}x = \int_U \lambda(x) \, \mathrm{d}x = \langle \mathbf{1}, \lambda \rangle$$

where $\mathbf{1} \in L^1(U)$ denotes the constant function with value 1. Since λ^n converges weakly to λ,

$$\langle \mathbf{1}, \lambda \rangle = \lim_{n \to \infty} \langle \mathbf{1}, \lambda^n \rangle = \lim_{n \to \infty} \|\lambda^n\|_{L^1(U)} = 1 \, .$$

The final property required of λ is $\|\lambda\|_{L^\infty(U)} \leq C$. This follows from the fact that the unit ball in $L^\infty(U)$, and furthermore any dilation of the unit ball, is weakly closed. Thus λ satisfies all requirements to be an element of K and so K is weakly closed. □

We next show that the mapping $\mathcal{F}$ is continuous in the weak$*$ topology. To do so, we first require the following Lemmas.

Lemma A.3. *There exist constants* $\ell_1, \ell_2 \geq 0$ *such that for any* $\lambda \in K$ *the corresponding* Λ *and* Λ' *are Lipschitz continuous with Lipschitz constants* ℓ_1, ℓ_2 *respectively.*

Proof. Recall that $\varphi(y - x) := \phi(y - x)\,(y - x)$. Since $\phi \in C^2(U)$ both φ and φ' are Lipschitz continuous with constants denoted ℓ_1 and ℓ_2 respectively.

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

Now for any $\lambda \in K$ and any $x, z \in \overline{U}$ consider,

$$\begin{aligned}
|\Lambda(x) - \Lambda(z)| &= \left| \int_U \varphi(y-x)\,\lambda(y)\,\mathrm{d}y - \int_U \varphi(y-z)\,\lambda(y)\,\mathrm{d}y \right|, \\
&\leq \int_U |\varphi(y-x) - \varphi(y-z)|\,\lambda(y)\,\mathrm{d}y, \\
&\leq \int_U \ell_1 |x-z|\,\lambda(y)\,\mathrm{d}y, \\
&= \ell_1 |x-z| \int_U \lambda(y)\,\mathrm{d}y, \\
&= \ell_1 |x-z|\,.
\end{aligned}$$

An identical argument holds for Λ' with Lipschitz constant ℓ_2. □

Lemma A.4. *Let $U \subset \mathbb{R}$ be a bounded interval and f^n a sequence of functions $f^n : \overline{U} \to \mathbb{R}$ that are uniformly Lipschitz continuous with Lipschitz constant L. Assume that f^n converges pointwise everywhere in $\overline{U}$ to a function $f : \overline{U} \to \mathbb{R}$ that is also Lipschitz continuous with Lipschitz constant L. Then f^n converges to f in $L^\infty(U)$.*

Proof. We show that f^n converges to f uniformly, from which convergence in $L^\infty(U)$ follows. Fix any $\varepsilon > 0$. For each $y \in \overline{U}$ let $B(y, \frac{\varepsilon}{4L})$ be the open ball of radius $\frac{\varepsilon}{4L}$ centred at y. As $\overline{U}$ is compact there exists a finite collection of points $y_1, \ldots, y_m$ such that the union $\cup_{i=1}^m B(y_i, \frac{\varepsilon}{4L})$ covers $\overline{U}$.

For each $i = 1, \ldots, m$ take $x \in B(y_i, \frac{\varepsilon}{4L})$ and consider

$$\begin{aligned}
|f^n(x) - f(x)| &\leq |f^n(x) - f^n(y_i)| + |f^n(y_i) - f(y_i)| + |f(y_i) - f(x)| \\
&\leq L|x - y_i| + |f^n(y_i) - f(y_i)| + L|y_i - x| \\
&\leq \frac{\varepsilon}{2} + |f^n(y_i) - f(y_i)|
\end{aligned}$$

Since $f^n(y_i)$ converges to $f(y_i)$, there exists $N_i \in \mathbb{N}$ such that for all $n \geq N_i$, $|f^n(y_i) - f(y_i)| \leq \frac{\varepsilon}{2}$. So for $n \geq N_i$ we have $|f^n(x) - f(x)| \leq \varepsilon$ for all $x \in B(y_i, \frac{\varepsilon}{4L})$.

Define $N = \max_{i=1,\ldots,N}$. Then for $n \geq N$ we have $|f^n(x) - f(x)| \leq \varepsilon$ for all $x \in \cup_{i=1}^m B(y_i, \frac{\varepsilon}{4L}) \supset \overline{U}$. Hence f^n converges to f uniformly in $\overline{U}$ and therefore also in $L^\infty(U)$. □

Proposition A.6. *The mapping $\mathcal{F} : K \to K$ is continuous in the weak∗ topology on $L^\infty(U)$.*

Proof. Fix some $\lambda \in K$ and consider a sequence $\lambda^n \in K$ converging weakly to λ in $L^\infty(U)$, meaning that for any $\xi \in L^1(U)$,

$$\int_U \lambda^n(y)\,\xi(y)\,\mathrm{d}y \to \int_U \lambda(y)\,\xi(y)\,\mathrm{d}y\,.$$

Therefore, by fixing some $x \in \overline{U}$ and taking $\xi(y) = \phi(y-x)\,(y-x)$ and $\xi(y) = -\phi'(y-x)\,(y-x) - \phi(y-x)$, we have that $\Lambda^n(x)$ and $(\Lambda^n)'(x)$ converge pointwise to $\Lambda(x)$ and $\Lambda'(x)$, respectively, everywhere in $\overline{U}$. Applying Lemmas A.3 and A.4 gives that $\Lambda^n(x)$ and $(\Lambda^n)'(x)$ converge to $\Lambda(x)$ and $\Lambda'(x)$, respectively, in $L^\infty(U)$. We now show that the corresponding solutions to (17) converge.

For each $n \in \mathbb{N}$ let $\rho^n \in W_2^{2,1}(U_A)$ denote the solution to (17) using interaction density λ^n and similarly let $\rho \in W_2^{2,1}(U_A)$ be the solution using λ. We will next show that ρ^n converges to ρ in $W_2^{2,1}(U_A)$. Define $q^n = \rho^n - \rho \in W_2^{2,1}(U_A)$. Combining the corresponding versions of (17) gives that q^n is a weak solution

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

to the following initial-boundary value problem:

$$\tau\,\partial_a q^n(a,x) + \partial_x\left(q^n(a,x)\Lambda^n(x)\right) - \frac{\sigma^2}{2}\partial_{xx}q^n(a,x) = -\partial_x\left(\rho(a,x)\left(\Lambda^n(x)-\Lambda(x)\right)\right), \tag{A13a}$$

$$q^n(0,x) = 0, \tag{A13b}$$

$$q^n(a,1)\,\Lambda^n(1) - \frac{\sigma^2}{2}\partial_x q^n(a,1) = -\rho(a,1)\left(\Lambda^n(1)-\Lambda(1)\right), \tag{A13c}$$

$$q^n(a,-1)\,\Lambda^n(-1) - \frac{\sigma^2}{2}\partial_x q^n(a,-1) = -\rho(a,-1)\left(\Lambda^n(-1)-\Lambda(-1)\right). \tag{A13d}$$

We wish to apply Theorem 9.1 from Chapter IV Section 9 of [34] to obtain a bound on $\|q^n\|_{W_2^{2,1}(U_A)}$ of the form

$$\|q^n\|_{W_2^{2,1}(U_A)} \le c_3\left(\|f^n\|_2 + \|\Phi^n\|_{W_2^{3/2,3/4}(S_A)} + A^{-3/4}\|\Phi^n\|_{L^2(S_A)}\right),$$

where

$$f^n(a,x) = -\partial_x\left(\frac{1}{\tau}\rho(a,x)\left(\Lambda^n(x)-\Lambda(x)\right)\right),$$
$$\Phi^n(a,x) = -\rho(a,x)\left(\Lambda^n(x)-\Lambda(x)\right).$$

First consider f^n. Using Hölder's inequality, we have the following bound,

$$\begin{aligned}\|f^n\|_2 &= \frac{1}{\tau}\,\|\,\partial_x\rho(a,x)(\Lambda^n(x)-\Lambda(x)) + \rho(a,x)((\Lambda^n)'(x)-\Lambda'(x))\|_2\\ &\le \frac{1}{\tau}\,\|\,\partial_x\rho\|_2\,\left\|\Lambda^n-\Lambda\right\|_\infty + \frac{1}{\tau}\,\|\rho\|_2\,\|(\Lambda^n)'-\Lambda'\|_\infty \qquad \text{(A14)}\\ &\le \frac{1}{\tau}\,\|\rho\|_{W_2^{2,1}(U_A)}\left(\|\Lambda^n-\Lambda\|_\infty + \|(\Lambda^n)'-\Lambda'\|_\infty\right). \qquad \text{(A15)}\end{aligned}$$

We have already established that Λ^n and $(\Lambda^n)'$ converge to Λ and Λ', respectively, in $L^\infty(U)$ and therefore in $L^\infty(U_A)$ (as they are constant in a). Moreover, $\|\rho\|_{W_2^{2,1}(U_A)}$ is a finite constant since the solution $\rho \in W_2^{2,1}(U_A)$. Hence $\|f^n\|_2$ is finite and converges to zero as $n \to \infty$.

Next we require that $\Phi^n \in W_2^{3/2,3/4}(S_A)$. Applying Lemma 3.4 from Chapter 2 Section 3 of [34] gives the existence of a constant c_4 such that for any $u \in W_2^{2,1}(U_A)$

$$\|u\|_{W_2^{3/2,3/4}(S_A)} \le c_4\|u\|_{W_2^{2,1}(U_A)}.$$

Since Λ^n and Λ are continuous and $\rho \in W_2^{2,1}(U_A)$, $\Phi^n \in W_2^{2,1}(U_A)$ for all $n \in \mathbb{N}$. Hence $\Phi^n \in W_2^{3/2,3/4}(S_A)$ and,

$$\begin{aligned}\|\Phi^n\|_{W_2^{3/2,3/4}(S_A)} &\le c_4\|\Phi^n\|_{W_2^{2,1}(U_A)},\\ &\le c_4(\|\Phi^n\|_2 + \|D_a\Phi^n\|_2 + \|D_x\Phi^n\|_2 + \|D_{x^2}\Phi^n\|_2),\\ &\le c_5\|\rho\|_{W_2^{2,1}(U_A)}(\|\Lambda^n-\Lambda\|_\infty + \|(\Lambda^n)'-\Lambda'\|_\infty + \|(\Lambda^n)''-\Lambda''\|_\infty),\end{aligned}$$

with the final inequality obtained by applying Hölder's inequality to each component of $\|\Phi^n\|_{W_2^{2,1}(U_A)}$ and using $\|\rho\|_{W_2^{2,1}(U_A)}$ as an upper bound for the L^2 norm of ρ and its weak derivatives. Using an identical argument as that for Λ^n and $(\Lambda^n)'$, $(\Lambda^n)''$ converges to Λ'' in $L^\infty(U)$. Hence $\|\Phi^n\|_{W_2^{3/2,3/4}(S_A)} \to 0$. This argument also shows that $\|\Phi^n\|_{L^2(S_A)} \to 0$.

Hence we can apply Theorem 9.1 from Chapter IV Section 9 of [34] to give the existence of a unique solution $q^n \in W_2^{2,1}$ to (A13). Moreover, there exists a constant c_3 such that,

$$\|q^n\|_{W_2^{2,1}(U_A)} \le c_3\left(\|f^n\|_2 + \|\Phi^n\|_{W_2^{3/2,3/4}(S_A)} + A^{-3/4}\|\Phi^n\|_{L^2(S_A)}\right), \tag{A16}$$

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

so $\|q^n\|_{W_2^{2,1}(U_A)} \to 0$ as $n \to \infty$, meaning that ρ^n converges to ρ in $W_2^{2,1}$. Applying Lemma A.2 to $\rho^n - \rho$ we then have that

$$\left\|\mathcal{F}(\lambda^n) - \mathcal{F}(\lambda)\right\|_{L^\infty(U)} \leq c_2\sqrt{A}M_{\max}\,\|\rho^n - \rho\|_{W_2^{2,1}(U_A)} \to 0\,.$$

Hence $\mathcal{F}(\lambda^n)$ converges (in norm) to $\mathcal{F}(\lambda)$ in $L^\infty(U)$, and so also converges weakly in $L^\infty(U)$. Thus, the mapping $\mathcal{F}: K \to K$ is continuous in the weak topology on $L^\infty(U)$.

This concludes the necessary steps to prove Theorem 4.1.

A.2. Further proofs

Proof of Proposition 4.1

Proof. Consider $\lambda_1, \lambda_2 \in K$ and let ρ_1 and ρ_2 be the corresponding solutions to (17). Applying Lemma A.2 to $\rho_1 - \rho_2$ we have that

$$\left\|\mathcal{F}(\lambda_1) - \mathcal{F}(\lambda_2)\right\|_{L^\infty(U)} \leq c_2\sqrt{A}M_{\max}\,\|\rho_1 - \rho_2\|_{W_2^{2,1}(U_A)}\,.$$

We wish to bound the terms on the right-hand side so that we may compare this directly to $\left\|\lambda_1 - \lambda_2\right\|_{L^\infty(U)}$.

Following the same approach as in the proof of Proposition A.6, the equivalent of (A16) gives

$$\|\rho_1 - \rho_2\|_{W_2^{2,1}(U_A)} \leq c_3\left(\|f\|_2 + \|\Phi\|_{W_2^{3/2,3/4}(S_A)} + A^{-3/4}\|\Phi\|_{L^2(S_A)}\right) \tag{A17}$$

where

$$\begin{aligned} f(a,x) &= -\partial_x\left(\rho_1(a,x)\left(\Lambda_1(x) - \Lambda_2(x)\right)\right), \\ \Phi(a,x) &= -\rho_1(a,x)\left(\Lambda_1(x) - \Lambda_2(x)\right)\,. \end{aligned}$$

Recall from the proof of Proposition A.6 that each of the norms appearing in the right-hand side of (A17) can be bounded by a constant multiplied by

$$\max\{1,\tau^{-1}\}\,\|\rho_1\|_{W_2^{2,1}(U_A)}\left(\|\Lambda_1 - \Lambda_2\|_\infty + \|(\Lambda_1)' - (\Lambda_2)'\|_\infty + \|(\Lambda_1)'' - (\Lambda_2)''\|_\infty\right)\,.$$

As $|\varphi|$ is bounded above by 2 we have

$$\begin{aligned} \left\|\Lambda_1 - \Lambda_2\right\|_\infty &= \left|\int_U \varphi(y-x)\,\lambda_1(y)\,\mathrm{d}y - \int_U \varphi(y-x)\,\lambda_2(y)\,\mathrm{d}y\right|, \\ &\leq \int_U |\varphi(y-x)|\,|\lambda_1(y) - \lambda_2(y)|\,\mathrm{d}y, \\ &\leq 2\int_U |\lambda_1(y) - \lambda_2(y)|\,\mathrm{d}y, \\ &\leq 4\left\|\lambda_1 - \lambda_2\right\|_{L^\infty(U)}\,. \end{aligned}$$

Similarly, $\left\|\Lambda_1' - \Lambda_2'\right\|_\infty \leq c_6\left\|\lambda_1 - \lambda_2\right\|_{L^\infty(U)}$ and $\left\|\Lambda_1'' - \Lambda_2''\right\|_\infty \leq c_7\left\|\lambda_1 - \lambda_2\right\|_{L^\infty(U)}$ for some constants c_6, c_7 depending only upon U and ϕ (and its derivatives).

As the bound (A5) given by

$$\|\rho\|_{W_2^{2,1}(U_A)} \leq c_1\,\|\mu\|_{L^2(U)},$$

is independent of λ, we can therefore bound each of the norms appearing in the right-hand side of (A17) by a constant multiplied by

$$\|\mu\|_{L^2(U)}\left\|\lambda_1 - \lambda_2\right\|_{L^\infty(U)}\,.$$

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

Therefore, there exists a constant c_8, depending upon U, ϕ, A and τ, such that

$$\left\|\mathcal{F}(\lambda_1) - \mathcal{F}(\lambda_2)\right\|_{L^\infty(U)} \leq c_8\sqrt{A}M_{\max}\left(1 + A^{-3/4}\right)\left\|\lambda_1 - \lambda_2\right\|_{L^\infty(U)}.$$

Hence, we see that $\mathcal{F}$ is Lipschitz continuous and satisfies (19). □

Proof of Proposition 5.1

Proof. From (39), we have the following

$$\begin{aligned}
\partial_t\rho(t,a,x) &= \partial_t q(t,a,x)\,\pi(a)\\
&= -\left(\tau\,\partial_a q(t,a,x) + \partial_x F[q](t,a,x)\right)\,\pi(a)\\
\partial_a\rho(t,a,x) &= \partial_a q(t,a,x)\,\pi(a) + q(t,a,x)\,\partial_a\pi(a)\\
&= \partial_a q(t,a,x)\,\pi(a) - q(t,a,x)\,d(a)\,\pi(a)\\
&= \left(\partial_a q(t,a,x) - q(t,a,x)\,d(a)\right)\,\pi(a)\\
\partial_x\rho(t,a,x) &= \partial_x q(t,a,x)\,\pi(a)
\end{aligned}$$

Next, we compare the flux term in (38) with that in (10),

$$\begin{aligned}
&\tilde{F}[\rho](t,a,x)\\
&= \rho(t,a,x)\int_U \varphi(y-x)\left(\int_0^\infty M(a,b)\,\rho(t,b,y)\,\mathrm{d}b\right)\,\mathrm{d}y - \frac{\sigma^2}{2}\partial_x\rho(t,a,x),\\
&= q(t,a,x)\,\pi(a)\int_U \varphi(y-x)\left(\int_0^\infty M(a,b)\,q(t,b,y)\,\pi(b)\,\mathrm{d}b\right)\,\mathrm{d}y - \frac{\sigma^2}{2}\partial_x q(t,a,x)\,\pi(a)\\
&= \pi(a)\left(q(t,a,x)\int_U \varphi(y-x)\left(\int_0^1 M(a,b)\,q(t,b,y)\,\pi(b)\,\mathrm{d}b\right)\,\mathrm{d}y - \frac{\sigma^2}{2}\partial_x q(t,a,x)\right)\\
&= \pi(a)\left(q(t,a,x)\int_U \varphi(y-x)\left(\int_0^1 M(a,b)\,\pi(b)\,q(t,b,y)\,\mathrm{d}b\right)\,\mathrm{d}y - \frac{\sigma^2}{2}\partial_x q(t,a,x)\right)\\
&= \pi(a)\left(q(t,a,x)\int_U \varphi(y-x)\left(\int_0^1 M'(a,b)\,q(t,b,y)\,\mathrm{d}b\right)\,\mathrm{d}y - \frac{\sigma^2}{2}\partial_x q(t,a,x)\right)\\
&= \pi(a)\,F[q](t,a,x).
\end{aligned}$$

Combining this with the derivatives above gives

$$\begin{aligned}
\partial_t\rho(t,a,x) + \tau\,\partial_a\rho(t,a,x) + \partial_x\tilde{F}[\rho](t,a,x) &= \pi(a)(-\tau\,\partial_a q(t,a,x) - \partial_x F[q](t,a,x)\\
&\quad + \tau\,\partial_a q(t,a,x) - \tau\,q(t,a,x)\,d(a) + \partial_x F[q](t,a,x))\\
&= -\tau\,p(t,a,x)\,d(a)
\end{aligned}$$

hence we have

$$\partial_t\rho(t,a,x) + \partial_a\rho(t,a,x) + \partial_x\tilde{F}[\rho](t,a,x) = -\tau\,\rho(t,a,x)\,d(a)$$

as required. In addition the correspondence $\tilde{F}[\rho](t,a,x) = \pi(a)\,F[q](t,a/A,x)$ immediately ensures that the no-flux boundary conditions at $x = \pm 1$ are met. Furthermore, both equations share the same initial ($t = 0$) condition. The last condition to check is the age-zero distribution. From (39) we have

$$\begin{aligned}
\rho(t,0,x) &= q(t,0,x)\,\pi(0)\\
&= \mu(x)\,\pi(0)\int_U q(t,1,y)\,\mathrm{d}y\\
&= \mu(x)\,\pi(0),
\end{aligned}$$

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

since the age profile for q is uniform. Since π is stationary, it maintains a constant population size and so satisfies

$$\pi(0) = \int_0^\infty d(a)\,\pi(a)\,\mathrm{d}a,$$

which gives

$$\rho(t,0,x) = \mu(x)\left(\int_0^\infty d(a)\,\pi(a)\,\mathrm{d}a\right),$$

as required. □

B. Numerical Schemes

B.1. Full System

The overall approach is to apply Strang Splitting [27] to allow separate numerical schemes for ageing and opinion formation. At each timestep, we apply a half-step in age transport, a full step in opinion formation, then a further half-step in age transport. For ageing, we solve

$$\partial_t \rho + \tau\,\partial_a \rho = 0,$$

while for opinion formation, we solve

$$\partial_t \rho + \partial_x\left(\rho(t,a,x)\left(\int_U \int_0^A M(a,b)\,\varphi(y-x)\,\rho(t,b,y)\,\mathrm{d}b\,\mathrm{d}y\right)\right) - \frac{\sigma^2}{2}\partial_{xx}\rho = 0\,.$$

In the following, we assume $M \equiv 1$ for simplicity, but note where an adaptation would be made to account for more complex age interaction kernels.

Fix $J_t, J_x, J_a \geq 1$ the number of discretisation points for time, opinion and age, respectively.

Let $\Delta t = J_t^{-1},\ \Delta x = 2J_x^{-1}$ and $\Delta a = J_a^{-1}$. Denote $x_i = i\,\Delta_x$ for $i = 0, \ldots, J_x$.

Half age step:

Denote by μ^{J_x} the discretised version of the age zero distribution μ into J_x opinion sections, given by

$$\mu_i^{J_x} = \int_{x_{i-1}}^{x_i} \mu(x)\,\mathrm{d}x,$$

for $i = 1, \ldots, J_x$.

Let $\kappa = \frac{\tau}{2}\frac{\Delta t}{\Delta a}$ and define the matrices $B^1 \in \mathbb{R}^{J_a \times J_a}$ and $B^2 \in \mathbb{R}^{J_x \times J_a}$ by

$$B^1 = \begin{pmatrix} 0 & \kappa & 0 & 0 & 0 & 0 \\ 0 & 1-\kappa & \kappa & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1-\kappa & \kappa \\ 0 & 0 & 0 & \ldots & 0 & 1-\kappa \end{pmatrix} \quad \text{and} \quad B^2 = \begin{pmatrix} \mu_1^{J_x} & 0 & 0 & 0 & 0 & 0 \\ \mu_2^{J_x} & 0 & 0 & \ldots & 0 & 0 \\ \mu_3^{J_x} & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mu_{J_x-1}^{J_x} & 0 & 0 & \ldots & 0 & 0 \\ \mu_{J_x}^{J_x} & 0 & 0 & \ldots & 0 & 0 \end{pmatrix},$$

then we have that a half update step corresponds to

$$S^a(\rho) = \rho B^1 + \left(\mathbf{1}_{1\times J_x}\,\rho\,e_{J_a}\right)B^2$$

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

where $\mathbf{1}_{n\times m}$ is an $n \times m$ matrix of ones. The matrix B^1 handles the transport in age for ages strictly above zero, while B^2 sets the age-zero distribution. The factor $\left(\mathbf{1}_{1\times J_x}\ \rho\ e_{J_a}\right)$ simply sums the age-one distribution to ensure the total mass of ρ is preserved.

Opinion step:

The opinion step applied a finite volumes scheme [35]. Define the constant matrix $\Phi \in \mathbb{R}^{(J_x-1)\times J_x}$, which describes opinion interactions, by

$$\Phi_{ij} = \int_{x_{j-1}}^{x_j} \phi(y - x_i)\,(y - x_i)\,\mathrm{d}y\,.$$

for $i = 1, \ldots, J_x - 1$ and $j = 1, \ldots, J_x$.

To help approximate the fluxes define $F^1, F^2 \in \mathbb{R}^{(J_x-1)\times J_x}$ as follows

$$F^1 = -\frac{1}{2}\begin{pmatrix} 1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 1 & 1 \end{pmatrix}, \quad F^2 = \frac{1}{\Delta x}\begin{pmatrix} -1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & -1 & 1 \end{pmatrix}.$$

The matrix F^1 approximates the value of ρ in the midpoint of each section (over which finite volumes are taken), while F^2 approximates the spatial derivative appearing in the flux.

We now define the matrix $C \in \mathbb{R}^{J_x\times(J_x-1)}$ that combines the fluxes (accounting for the no-flux boundary conditions)

$$C = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & -1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & -1 \end{pmatrix}.$$

An opinion update step then corresponds to

$$S^o(\rho) = \rho + \frac{\Delta t}{\Delta x} C\left(\left(F^1\rho\right)\left(\Phi\rho\mathbf{1}_{M\times M}\right) + \frac{\sigma^2}{2}F^2\rho\right). \tag{B1}$$

If M is not always 1 then $\mathbf{1}_{J_a\times J_a}$ would be replaced with an approximation of M.

Complete Step

A full step is given by

$$\rho(t + \Delta t) = (S^a \circ S^o \circ S^a)(\rho(t)),$$

where $\circ$ denotes function composition.

B.2. For steady states

We now adapt the scheme for a known interaction density λ. Recall that in (17), age plays the role of 'time'.

As for μ, denote by λ^{J_x} the discretised version of λ into J_x opinion sections, given by

$$\lambda_i^{J_x} = \int_{x_{i-1}}^{x_i} \lambda(x)\,\mathrm{d}x,$$

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press

for $i = 1, \ldots, J_x$. We then replace (A17) with

$$S^o(\rho) = \rho + \frac{\Delta a}{\Delta x} C \left((F^1 \rho) \star (\Phi \lambda^{J_x}) + \frac{\sigma^2}{2} F^2 \rho \right). \tag{B2}$$

where $\star$ denotes elementwise multiplication.

This can be simplified by defining

$$\Lambda^{J_x} = \mathrm{diag}(\Phi \lambda^{J_x}),$$

which gives

$$\begin{aligned} S^o(\rho) &= \rho + \frac{\Delta a}{\tau \Delta x} C \left((\Lambda^{J_x} F^1 \rho) + \frac{\sigma^2}{2} F^2 \rho \right), \\ &= \rho + \frac{\Delta a}{\tau \Delta x} C \left(\Lambda^{J_x} F^1 \rho + \frac{\sigma^2}{2} F^2 \rho \right), \\ &= \rho + \frac{\Delta a}{\tau \Delta x} C \left(\Lambda^{J_x} F^1 + \frac{\sigma^2}{2} F^2 \right) \rho, \\ &= \left(\mathrm{Id} + \frac{\Delta a}{\tau \Delta x} C \left(\Lambda^{J_x} F^1 + \frac{\sigma^2}{2} F^2 \right) \right) \rho . \end{aligned}$$

where Id is the $J_x \times J_x$ identity matrix. Hence the approximation $\rho_k = \rho(k\, \Delta a)$ satisfies

$$\rho_k = \left(\mathrm{Id} + \frac{\Delta a}{\tau \Delta x} C \left(\Lambda^{J_x} F^1 + \frac{\sigma^2}{2} F^2 \right) \right)^k \mu^{J_x} .$$

For ease of notation, define

$$\Omega = \frac{1}{\tau \Delta x} C \left(\Lambda^{J_x} F^1 + \frac{\sigma^2}{2} F^2 \right)$$

then we have

$$\rho_k = \left(\mathrm{Id} + \frac{\Omega}{J_a} \right)^k \mu \approx \left(\mathrm{Id} + \frac{\Omega}{J_a} \right)^{a J_a} \mu \approx e^{a\Omega} \mu$$

for $a \in [0, 1]$.

This expression can be used to rapidly solve (17) and therefore iterate a numerical approximation of $\mathcal{F}$ to find steady states. Due to the simpler nature of (17) and the corresponding numerical scheme, this is significantly faster and more stable than solving the PDE for a large T. In addition, since the PDE may exhibit periodic solutions, solving for large T may never yield a steady state.

Cite this article: Nugent A., Gomes S. N. and Wolfram M.-T. (2026). Opinion dynamics with continuous age structure. *European Journal of Applied Mathematics*, **37**, 1164–1199. https://doi.org/10.1017/S0956792525100260

https://doi.org/10.1017/S0956792525100260 Published online by Cambridge University Press