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# Two singular integral equations involving confluent hypergeometric functions

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1. *Introduction.* Widder (1) obtained an inversion of the convolution transform

$$f(x) = \int_0^x L_n(x-t)\, g(t)\, dt \tag{1·1}$$

by the method of the Laplace transform, $L_n(x)$ being the Laguerre polynomial. Buschman (2) inverted a similar transform with a generalized Laguerre polynomial as kernel and also solved (3) the singular integral equation

$$F(t) = \int_0^t (t-u)^\alpha L_n^\alpha(t-u)\, G(u)\, du \tag{1·2}$$

using Mikusinski operators. Srivastava (4, 4a) solved singular integral equations with kernels involving $L_n^\alpha(x)$ and Whittaker functions $M_{k,\mu}(x)$.

More generally Wimp (5) proved, again by the use of the Laplace transform, that

$$f(x) = \int_0^t (t-u)^{c-1}\, \Phi(a;\, c;\, \lambda(t-u))\, g(u)\, du, \tag{1·3}$$

$$g(t) = \frac{1}{\Gamma(c)\,\Gamma(n-c)} \int_0^t (t-u)^{n-c-1}\, \Phi(-a;\, n-c;\, \lambda(t-u)) f^n(u)\, du \tag{1·4}$$

imply each other where $f(t) \in c^n$ and $f^\nu(0) = 0 \quad \text{for} \quad \nu = 0, 1, \ldots, n-1$.

In this paper we apply a technique based on the fractional integration operator $I^\mu$ (see section 2) to discuss the integral equations

$$\int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)}\, {}_1F_1(a;\, b;\, x-t) f(t)\, dt = g(x) \text{ p.p.}, \tag{1·5}$$

$$\int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)}\, {}_1F_1(a;\, b;\, t-x) f(t)\, dt = g(x) \text{ p.p.} \quad \text{for} \quad \operatorname{Re} b > 0, \quad x \in [0, \alpha], \quad \alpha < \infty. \tag{1·6}$$

The functions $f$ and $g$ belong to the class of measurable complex-valued functions of a real variable defined p.p. in $[0, \alpha]$. If two measurable functions are equal p.p., they are not regarded as distinct and are said to be equivalent.

Erdélyi (6) used the method of fractional integration to solve an integral equation involving a Legendre function, viz., the equation

$$\int_0^x (x^2 - t^2)^{\frac{1}{2}\lambda}\, P_\nu^{-\lambda}\left(\frac{x}{t}\right) g(t)\, dt = f(x), \tag{1·7}$$

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and Love (7) applied it to discuss the equation

$$\int_0^x \frac{(x-t)^{c-1}}{\Gamma(c)} F\left(a, b, c, 1-\frac{x}{t}\right) f(t)\, dt = g(x) \text{ p.p.} \tag{1·8}$$

After a preliminary discussion of the fractional integration operator $I^\mu$ in section 2, we introduce (linear) operators $K_{a,b}$ and $K^*_{a,b}$ in section 3. These operators map a linear space $L$ (see section 2) onto $L_b \subset L$. Theorems 3 and $3a$ contain main results about the operators and are used in the proofs of the theorems that follow. Theorem 1 ensures the existence of most of the integrals involved. The result of Theorem 2, though a simple analogue of Bateman integral formula ((8), 2·4 (2)) for Gauss's hypergeometric function, does not seem to have been noticed earlier.

In section 4, the existence of inverse operators $K^{-1}_{a,b}$ and $K^{*-1}_{a,b}$ is proved and this ensures uniqueness of the solutions of (1·5) and (1·6).

In section 5, it is shown that the existence of $I^{-b}g(x)$ is a necessary and sufficient condition for the existence of solutions of (1·5) and (1·6) for all complex $a$.

In section 6, explicit solutions of (1·5), (1·6) are obtained separately for the two ranges of values of $a$, namely $a > 0$ and $a \leqslant 0$. This discussion is spread over Theorems 9, $9a$, 10 and $10a$. Theorems 7 and 8 as existence theorems for the solutions of the integral equations

$$\int_0^x \frac{(x-t)^{\lambda-1}}{\Gamma(\lambda)} e^{-t} f(t)\, dt = e^{-x} \int_0^x \frac{(x-t)^{\lambda-1}}{\Gamma(\lambda)} G(t)\, dt \text{ p.p.}, \tag{1·9}$$

$$\int_0^x \frac{(x-t)^{\lambda-1}}{\Gamma(\lambda)} e^{t} f(t)\, dt = e^{x} \int_0^x \frac{(x-t)^{\lambda-1}}{\Gamma(\lambda)} G(t)\, dt \text{ p.p.} \tag{1·10}$$

play a useful role, apart from being of some interest in themselves, in obtaining solutions of (1·5) and (1·6). Incidentally the equations are explicitly solved for $f$ when $g$ is assumed known and vice versa.

In section 7, the last section of the paper, solutions of several equations are obtained directly from the results of section 6. Some of these equations have already been solved (see (1–3, 5)) though our solutions are expressed more explicitly (involving only the exponential function $e^z$ and the operator $I^\mu$) and under fewer restrictions; others are believed to be new.

2. *The operator $I^\mu$.*

*The space $L$.* Let $L$ denote the linear space of (equivalent classes of) complex-valued functions $f(x)$ which are Lebesgue-integrable on $[0, \alpha]$, $\alpha < \infty$. For $f(x) \in L$ and complex numbers $\mu$ with $\operatorname{Re}\mu > 0$, the Riemann–Liouville fractional integral of order $\mu$ is as usual defined by

$$I^\mu f(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t)\, dt \quad \text{for almost all} \quad x \in [0, \alpha]. \tag{2·1}$$

It is well known that $I^\mu f(x)$ exists p.p. and is in $L$, so that for $\operatorname{Re}\mu > 0$, (2·1) defines a linear operator

$$I^\mu : L \xrightarrow{\text{into}} L.$$

The inverse operator is usually denoted by $I^{-\mu}$ and exists since by (9)

$$I^\mu f = 0 \text{ p.p.} \Rightarrow f = 0 \text{ p.p.} \tag{2·2}$$

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Evidently for $\mathrm{Re}\,\mu < 0$, $I^\mu$ is to be understood as the inverse of $I^{-\mu}$ and $F = I^\mu f$ if $f = I^{-\mu}F$.

It easily follows that for complex numbers $\mu$ and $\nu$, with non-zero real components,

$$I^\mu I^\nu = I^\nu I^\mu = I^{\mu+\nu} \tag{2·3}$$

under suitable conditions.

If $\mathrm{Re}\,\mu = 0$, we define $I^\mu$ as

$$I^\mu = I^{-1}I^{1+\mu}, \tag{2·4}$$

so that with this definition, (2·3) holds for any two complex numbers $\mu$ and $\nu$ under, of course, appropriate conditions.

It is evident that for positive integral $\mu$, $I^\mu f(x)$ is the $\mu$ times repeated indefinite integral of $f(x)$, $I^{-\mu}f(x)$ is the $\mu$th order derivative and

$$I^0 f(x) = f(x).$$

3. *Some convolution integrals and operators.* We consider, in this section, some convolution integrals with kernels involving Kummer's confluent hypergeometric function ${}_1F_1$. As usual, ${}_1F_1(a;b;z)$ is defined by the series

$${}_1F_1(a;\,b;\,z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{z^n}{n!} \tag{3·1}$$

for all complex values of the parameters $a$ and $b$ except $b = 0, -1, -2, -3, \ldots$. The series is absolutely convergent for all finite $z$ so that ${}_1F_1(a;\,b;\,z)$ is an entire function of $z$.

THEOREM 1. *For* $\mathrm{Re}\,\mu > 0$ *and* $f(x)\in L$, *the improper integrals*

$$\int_0^x \frac{(x-t)^{\mu-1}}{\Gamma(\mu)}\,{}_1F_1(a;\,b;\,x-t)f(t)\,dt, \tag{3·2}$$

$$\int_0^x \frac{(x-t)^{\mu-1}}{\Gamma(\mu)}\,{}_1F_1(a;\,b;\,t-x)f(t)\,dt \tag{3·3}$$

*exist for almost all* $x$ *in* $[0,\alpha], \alpha < \infty$ *and define functions in* $L$.

For $\mathrm{Re}\,\mu > 0$, $t^{\mu-1}\in L$ and ${}_1F_1(a;\,b;\,t)$ being an entire function, $t^{\mu-1}\,{}_1F_1(a;\,b;\,t)\in L$. The integral (3·2) exists as the convolution of two functions in $L$ and hence

$$\int_0^x \left|\frac{(x-t)^{\mu-1}}{\Gamma(\mu)}\,{}_1F_1(a;\,b,\,x-t)f(t)\right| dt \tag{3·4}$$

exists for almost all $x$ in $[0,\alpha]$.

To show that (3·2) is in $L$, all that we need to do is to prove the finiteness of

$$\int_0^\alpha dx \int_0^x \left|\frac{(x-t)^{\mu-1}}{\Gamma(\mu)}\,{}_1F_1(a;\,b;\,x-t)f(t)\right| dt \tag{3·5}$$

and apply Fubini's theorem.

But (3·5) can be written as

$$\int_0^\alpha |f(t)|\,dt \int_t^\alpha \left|\frac{(x-t)^{\mu-1}}{\Gamma(\mu)}\,{}_1F_1(a;\,b;\,x-t)\right| dx$$

$$\leqslant \frac{1}{|\Gamma(\mu)|}\int_0^\alpha |f(t)|\,dt \int_0^\alpha |u^{\mu-1}\,{}_1F_1(a;\,b;\,u)|\,du, \tag{3·6}$$

putting $u = x-t$.

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The first integral on the right is obviously finite since $f(x) \in L$. The entire function ${}_1F_1(a; b; u)$ is bounded in $[0, \alpha]$; let

$$|{}_1F_1(a; b; u)| \leqslant M, \quad u \in [0, \alpha],$$

so that

$$\int_0^\alpha |u^{\mu-1}\, {}_1F_1(a; b; u)|\, du \leqslant M \int_0^\alpha u^{\operatorname{Re}\mu - 1}\, du$$

which is finite for $\operatorname{Re}\mu > 0$.

Thus the right-hand expression in (3·6) is proved finite when $\operatorname{Re}\mu > 0$.

The arguments used here can also be applied to establish the existence of (3·3) in $L$.

THEOREM 2. *If* $\operatorname{Re}\mu > 0$, $\operatorname{Re} b > 0$ *and* $z$ *is complex, then*

$$\int_0^1 \frac{(1-\sigma)^{\mu-1}}{\Gamma(\mu)} \frac{\sigma^{b-1}}{\Gamma(b)}\, {}_1F_1(a; b; z\sigma)\, d\sigma = \frac{1}{\Gamma(b+\mu)}\, {}_1F_1(a; b+\mu; z). \tag{3·7}$$

The integral $\displaystyle\int_0^1 \frac{(1-\sigma)^{\mu-1}}{\Gamma(\mu)} \frac{\sigma^{b-1}}{\Gamma(b)}\, {}_1F_1(a; b; z\sigma)\, d\sigma$

$$= \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \frac{1}{\Gamma(\mu)\,\Gamma(b)} \int_0^1 (1-\sigma)^{\mu-1}\, \sigma^{b+n-1}\, d\sigma,$$

using (3·1) and interchanging the order of integration and summation. The interchange is obviously justified by Lebesgue's theorem.

Evaluating the Eulerian integral

$$\int_0^1 (1-\sigma)^{\mu-1}\, \sigma^{b+n-1}\, d\sigma$$

in terms of gamma functions, it is easily seen that the integral in (3·7) is equal to

$$\frac{1}{\Gamma(b+\mu)} \sum_{n=0}^{\infty} \frac{(a)_n}{(b+\mu)_n} \frac{z^n}{n!} = \frac{1}{\Gamma(b+\mu)}\, {}_1F_1(a; b+\mu; z).$$

COROLLARY 2·1. *If* $\operatorname{Re}\mu > 0$, $\operatorname{Re} b > 0$ *and* $0 \leqslant t \leqslant x$, *then*

$$\int_t^x \frac{(x-v)^{\mu-1}}{\Gamma(\mu)} \frac{(v-t)^{b-1}}{\Gamma(b)}\, {}_1F_1(a; b; v-t)\, dv = \frac{(x-t)^{b+\mu-1}}{\Gamma(b+\mu)}\, {}_1F_1(a; b+\mu; x-t) \tag{3·8}$$

*and*

$$\int_t^x \frac{(x-v)^{\mu-1}}{\Gamma(\mu)} \frac{(v-t)^{b-1}}{\Gamma(b)}\, {}_1F_1(a; b; t-v)\, dv = \frac{(x-t)^{b+\mu-1}}{\Gamma(b+\mu)}\, {}_1F_1(a; b+\mu; t-x). \tag{3·9}$$

To prove (3·8), we put

$$\sigma = \frac{v-t}{x-y}, \quad z = x-t \text{ in (3·7)},$$

while the substitutions

$$\sigma = \frac{v-t}{x-t}, \quad z = t-x \text{ therein give (3·9)}.$$

*The operators* $K_{a,b}$ *and* $K^*_{a,b}$. For complex numbers $a$ and $b$ with $\operatorname{Re} b > 0$ and $f(x) \in L$, the integrals

$$\int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)}\, {}_1F_1(a; b; x-t) f(t)\, dt, \tag{3·10}$$

$$\int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)}\, {}_1F_1(a; b; t-x) f(t)\, dt \tag{3·11}$$

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exist p.p. in $L$, by Theorem 1. Setting

$$K_{a,b}f(x) = \int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)}\, {}_1F_1(a;\, b;\, x-t) f(t)\, dt,$$

$$K^*_{a,b}f(x) = \int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)}\, {}_1F_1(a;\, b;\, t-x) f(t)\, dt,$$

we define the (linear) operators

$$K_{a,b}\colon L \overset{\text{into}}{\to} L,$$

$$K^*_{a,b}\colon L \overset{\text{into}}{\to} L.$$

THEOREM 3. *If* $\operatorname{Re}\mu > 0$, $\operatorname{Re} b > 0$, *then for every* $f \in L$,

$$I^\mu K_{a,b} f = K_{a,b+\mu} f \text{ p.p.} \tag{3·12}$$

*and*

$$I^\mu K^*_{a,b} f = K^*_{a,b+\mu} f \text{ p.p.} \tag{3·13}$$

For $\operatorname{Re} b > 0$ and $f(x) \in L$, $K_{a,b} f \in L$. When $\operatorname{Re}\mu > 0$, $I^\mu K_{a,b} f$ exists in $L$ and p.p. equals

$$\begin{aligned}
&\int_0^x \frac{(x-v)^{\mu-1}}{\Gamma(\mu)}\, dv \int_0^v \frac{(v-t)^{b-1}}{\Gamma(b)}\, {}_1F_1(a;\, b;\, v-t) f(t)\, dt \\
&= \int_0^x f(t)\, dt \int_t^x \frac{(x-v)^{\mu-1}}{\Gamma(\mu)} \frac{(v-t)^{b-1}}{\Gamma(b)}\, {}_1F_1(a;\, b;\, v-t)\, dv \\
&= \int_0^x f(t) \left[ \frac{(x-t)^{b+\mu-1}}{\Gamma(b+\mu)}\, {}_1F_1(a;\, b+\mu;\, x-t) \right] dt, \quad \text{by (3·8)} \\
&= K_{a,b+\mu} f.
\end{aligned} \tag{3·14}$$

To justify the change in the order of integration in (3·14), we prove that

$$\int_0^x dv \int_0^v \left| \frac{(x-v)^{\mu-1}}{\Gamma(\mu)} \frac{(v-t)^{b-1}}{\Gamma(b)}\, {}_1F_1(a;\, b;\, v-t) f(t) \right| dt \tag{3·15}$$

exists finitely p.p. and apply Tonneli's theorem. From the finiteness of (3·4), it follows that

$$\Phi(v) = \int_0^v \left| \frac{(v-t)^{b-1}}{\Gamma(b)}\, {}_1F_1(a;\, b;\, v-t) f(t) \right| dt$$

exists in $L$. The double integral (3·15) can be written as

$$\frac{1}{|\Gamma(\mu)|} \int_0^x (x-v)^{\operatorname{Re}\mu - 1}\, \Phi(v)\, dv,$$

a convolution of two functions belonging to $L$, which exists finitely p.p. in $[0, \alpha]$.

Using (3·11) and (3·9), it is easy to see that the proof of the truth of (3·13) proceeds on similar lines.

We now show that the conclusion is still true if the condition $\operatorname{Re}\mu > 0$ is replaced by a more liberal condition $\operatorname{Re}(b+\mu) > 0$. Of course, $\operatorname{Re} b > 0$ is assumed throughout.

THEOREM 3*a*. *If* $\operatorname{Re} b > 0$, $\operatorname{Re}\mu > -\operatorname{Re} b$, $f \in L$, *then*

$$I^\mu K_{a,b} f = K_{a,b+\mu} f \text{ p.p.}, \tag{3·16}$$

$$I^\mu K^*_{a,b} f = K^*_{a,b+\mu} f \text{ p.p.} \tag{3·17}$$

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We shall prove (3·16) only; the proof of (3·17) follows the same manner.

It is enough to consider the cases when $\operatorname{Re}\mu < 0$ and $\operatorname{Re}\mu = 0$.

*Case* (i). When $\operatorname{Re}\mu < 0$, let $\mu = -\lambda$.

Since $\operatorname{Re}(b+\mu) > 0$, for $f(x) \in L$, $K_{a,b+\mu}f$ exists in $L$ and for almost all $x$ in $[0,\alpha]$,

$$\begin{aligned} I^{-\mu}K_{a,b+\mu}f &= I^{\lambda}K_{a,b-\lambda}f \\ &= K_{a,b}f, \end{aligned}$$

by Theorem 3.

But $K_{a,b}f$ exists in $L$ and therefore

$$I^{\mu}K_{a,b}f = K_{a,b+\mu}f \text{ p.p.}$$

*Case* (ii) .When $\operatorname{Re}\mu = 0$, by case (i), we have $I^{\mu-1}K_{a,b+1}f = K_{a,b+\mu}f$ p.p. But $I^{\mu-1}K_{a,b+1}f$ can be written as $I^{\mu}I^{-1}K_{a,b+1}f$ since $I^{-1}K_{a,b+1}f$ exists and by case (i), is equal to $K_{a,b}f$.

Hence $I^{\mu}K_{a,b}f = K_{a,b+\mu}f$ p.p.

4. *The inverse operators* $K_{a,b}^{-1}$ *and* $K_{a,b}^{*-1}$. We shall in this section establish the existence of the inverse operators $K_{a,b}^{-1}$ and $K_{a,b}^{*-1}$. This is the same thing as solving the uniqueness problem since it ensures that the integral equations (1·5) and (1·6) admit unique solutions.

By a well-known theorem, the inverse of a linear operator $K$ exists if and only if $Ky = 0 \Rightarrow y = 0$. We prove

THEOREM 4. *For* $x \in [0,\alpha]$,

$$K_{a,b}f(x) = 0 \text{ p.p.} \tag{4·1}$$

*implies that* $f(x) = 0$ p.p.

We consider the two cases when $\operatorname{Re}a > 0$ and when $\operatorname{Re}a \leqslant 0$.

If $\operatorname{Re}a > 0$, then for almost all $x \in [0,\alpha]$

$$I^{a}K_{a,b}f(x) = K_{a,b+a}f(x) = I^{b}K_{a,a}f(x), \quad \text{by Theorem 3.} \tag{4·2}$$

If $K_{a,b}f(x) = 0$ p.p., then $I^{b}K_{a,a}f(x) = 0$ p.p. which implies $K_{a,a}f(x) = 0$ p.p. by (2·2). Now

$$\begin{aligned} K_{a,a}f(x) &= \int_0^x \frac{(x-t)^{a-1}}{\Gamma(a)} e^{x-t}f(t)\,dt \\ &= e^{x}\int_0^x \frac{(x-t)^{a-1}}{\Gamma(a)} e^{-t}f(t)\,dt, \end{aligned}$$

so that $K_{a,a}f(x) = 0 \Rightarrow I^{a}e^{-x}f(x) = 0 \Rightarrow e^{-x}f(x) = 0$ p.p., using (2·2).

This obviously leads to $f(x) = 0$ p.p.

If $\operatorname{Re}a \leqslant 0$, then using Kummer's first theorem ((10), (1.4.1)), (4·1) can be written as

$$\int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)} {}_1F_1(b-a;\, b;\, t-x)\, e^{x-t}f(t)\,dt = 0 \text{ p.p.}, \quad x \in [0,\alpha], \tag{4·3}$$

i.e.

$$e^{x}K_{b-a,b}^{*}e^{-x}f(x) = 0 \text{ p.p.}$$

It is enough to prove that

$$K_{b-a,b}^{*}f(x) = 0 \Rightarrow f(x) = 0 \text{ p.p.}, \quad x \in [0,\alpha].$$

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Now $$I^{b-a}K^*_{b-a,b}f(x) = I^bK^*_{b-a,b-a}f(x),$$
using (2·3) and Theorem 3.

It is easy to see that
$$K^*_{b-a,b}f(x) = 0 \text{ p.p.}$$
implies that
$$K^*_{b-a,b-a}f(x) = 0 \text{ p.p.}$$
that is,
$$\int_0^x \frac{(x-t)^{c-1}}{\Gamma(c)}\,{}_1F_1(c;\,c;\,t-x)f(t)\,dt = 0 \text{ p.p.}$$
with $c = b-a$ so that $\operatorname{Re} c > 0$. Consequently
$$e^{-x}I^c[e^xf(x)] = 0 \text{ p.p.}$$
which leads to $f(x) = 0$ p.p.

THEOREM 4$a$. *For* $x \in [0, \alpha]$
$$K^*_{a,b}f(x) = 0 \Rightarrow f(x) = 0 \text{ p.p.}$$

*Case* (i). When $\operatorname{Re} a > 0$
$$I^aK^*_{a,b}f(x) = K^*_{a,b+a}f(x) = I^bK^*_{a,a}f(x),$$
where
$$K^*_{a,a}f(x) = e^{-x}\int_0^x \frac{(x-t)^{a-1}}{\Gamma(a)}\,e^tf(t)\,dt.$$

Using these results and (2·2), it follows that
$$K^*_{a,b}f(x) = 0 \Rightarrow e^{-x}I^ae^xf(x) = 0 \Rightarrow f(x) = 0 \text{ p.p.}$$

*Case* (ii). When $\operatorname{Re} a \leqslant 0$, using ((9), (1.4.1)),
$$K^*_{a,b}f(x) = \int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)}\,{}_1F_1(b-a;\,b;\,x-t)\,e^{t-x}f(t)\,dt$$
$$= e^{-x}K_{b-a,b}(e^xf(x)).$$
By an application of Theorem 4, the assertion is easily proved.

5. *Conditions for the existence of solutions of* (1·5) *and* (1·6). In this section we solve the existence problem. We prove that a necessary and sufficient condition for the integral equations (1·5) and (1·6) to admit unique solutions is the existence of $I^{-b}g(x)$ in $L$.

Denoting by $L_b$ the class of all the functions $g(x) \in L$ for which $I^{-b}g(x)$ exists in $L$, it will follow that $K_{a,b}$ defines a one-to-one (linear) mapping of $L$ onto $L_b$. The same remark is true for $K^*_{a,b}$.

THEOREM 5. *If* $f(x) \in L$, *then*
$$I^{-b}K_{a,b}f(x) = f(x) + a\int_0^x {}_1F_1(a+1;\,2;\,x-t)f(t)\,dt \quad \textit{for almost all } x \in [0, \alpha]. \quad (5{\cdot}1)$$

By Theorem 3$a$, for almost all $x \in [0, \alpha]$
$$I^{1-b}K_{a,b}f(x) = K_{a,1}f(x)$$
$$= \int_0^x {}_1F_1(a;\,1;\,x-t)f(t)\,dt.$$

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Since for $\operatorname{Re} b > 0$, $K_{a,b}f(x)$ exists in $L$ and also

$$\int_0^x {}_1F_1(a;\,1;\,x-t)f(t)\,dt$$

can be shown by a simple application of Theorem 1, to exist in $L$, we have

$$\begin{aligned} K_{a,b}f(x) &= I^{b-1}\int_0^x {}_1F_1(a;\,1;\,x-t)f(t)\,dt \\ &= I^b\left\{\frac{d}{dx}\int_0^x {}_1F_1(a;\,1;\,x-t)f(t)\,dt\right\} \\ &= I^b\left\{f(x)+a\int_0^x {}_1F_1(a+1;\,2;\,x-t)f(t)\,dt\right\}. \end{aligned} \tag{5·2}$$

Evidently both the functions within brackets in (5·2) exist in $L$. Hence $I^{-b}K_{a,b}f(x)$ exists in $L$ and p.p. equals

$$f(x)+a\int_0^x {}_1F_1(a+1;\,2;\,x-t)f(t)\,dt.$$

COROLLARY 5·1. *A necessary condition for the existence of a solution $f(x)\in L$ of the integral equation*

$$\int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)}\,{}_1F_1(a;\,b;\,x-t)f(t)\,dt = g(x) \quad \textit{for} \quad \operatorname{Re} b > 0, x\in[0,\alpha]\ \text{p.p.} \tag{5·3}$$

*is that $I^{-b}g(x)$ exists in $L$.*

COROLLARY 5·2. *If $\operatorname{Re} b > 0$ and $I^{-b}g(x)$ exists in $L$, then (5·3) is equivalent to a Volterra equation of the second kind, namely the equation*

$$f(x)+a\int_0^x {}_1F_1(a+1;\,2;\,x-t)f(t)\,dt = I^{-b}g(x)\ \text{p.p.} \tag{5·4}$$

THEOREM 5$a$. *For $f(x)\in L$,*

$$I^{-b}K^*_{a,b}f(x) = f(x)-a\int_0^x {}_1F_1(a+1;\,2;\,t-x)f(t)\,dt. \tag{5·5}$$

The proof of the theorem follows the manner of that for Theorem 5. For

$$\begin{aligned} I^{1-b}K^*_{a,b}f(x) &= K^*_{a,1}f(x) \\ &= \int_0^x {}_1F_1(a;\,1;\,t-x)f(t)\,dt, \end{aligned}$$

giving

$$\begin{aligned} I^{-b}K^*_{a,b}f(x) &= \frac{d}{dx}\int_0^x {}_1F_1(a;\,1;\,t-x)f(t)\,dt \\ &= f(x)-a\int_0^x {}_1F_1(a+1;\,2;\,t-x)f(t)\,dt. \end{aligned}$$

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COROLLARY 5·1*a*. *The existence of $I^{-b}g(x)$ in $L$ is a necessary condition for the integral equation*

$$\int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)}\,{}_1F_1(a;\,b;\,t-x)f(t)\,dt = g(x)\ \text{p.p.} \tag{5·6}$$

*to have a solution $f(x)\in L$.*

COROLLARY 5·2*a*. *If* $\operatorname{Re} b > 0$ *and $I^{-b}g(x)$ exists in $L$, then* (5·6) *is equivalent to the integral equation of Volterra-type*

$$f(x) - a\int_0^x {}_1F_1(a+1;\,2;\,t-x)f(t)\,dt = I^{-b}g(x)\ \text{p.p.} \tag{5·7}$$

THEOREM 6. *If* $\operatorname{Re} b > 0$, *then a sufficient condition for the integral equation*

$$\int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)}\,{}_1F_1(a;\,b;\,x-t)f(t)\,dt = g(x)\ \text{p.p.} \tag{5·8}$$

*to admit a unique solution $f(x)\in L$ is the existence of $I^{-b}g(x)$ in $L$.*

Consider the Volterra equation of the second kind

$$f(x) + a\int_0^x {}_1F_1(a+1;\,2;\,x-t)f(t)\,dt = I^{-b}g(x)\ \text{p.p.} \tag{5·9}$$

The function

$$|{}_1F_1(a+1;\,2;\,x-t)| \leqslant M \quad (0\leqslant x\leqslant\alpha),$$

$M$ being a constant. Also $I^{-b}g(x)$ is assumed to exist in $L$, so that by (11), the equation (5·9) has a unique solution $f(x)$ in $L$. By Corollary 5·2 this solution is a unique solution of (5·8) also.

THEOREM 6*a*. *The conditions of Theorem* 6 *are sufficient for*

$$\int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)}\,{}_1F_1(a;\,b;\,t-x)f(t)\,dt = g(x)\ \text{p.p.} \tag{5·10}$$

*to have a unique solution $f(x)$ in $L$.*

This theorem is proved by using the equation (5·7) and Corollary 5·2*a* almost the same way as Theorem 6 is proved using (5·4) and Corollary 5·2.

6. *Explicit solutions.* In this section, we obtain explicit solutions of (1·5) and (1·6) when $\operatorname{Re} b > 0$ and $I^{-b}g(x)$ exists in $L$. Equivalently, we determine explicitly the inverse operator $K_{a,b}^{-1}$ (as also $K_{a,b}^{*-1}$) with domain $L_b$ and range $L$:

$$K_{a,b}^{-1}\colon L_b \to L,$$

where

$$L_b = \{g\colon I^{-b}g(x)\in L\}.$$

We separately consider the case when $\operatorname{Re} a > 0$ in Theorems 9 and 9*a* and the case $\operatorname{Re} a \leqslant 0$ in Theorems 10 and 10*a*. In the next two theorems (i.e. 7 and 8) we solve two integral equations which, apart from being of independent interest, will be used in the subsequent discussion.

THEOREM 7. *If* $\operatorname{Re}\lambda > 0$, *then the integral equation*

$$\int_0^x \frac{(x-t)^{\lambda-1}}{\Gamma(\lambda)}\,e^{-t}f(t)\,dt = e^{-x}\int_0^x \frac{(x-t)^{\lambda-1}}{\Gamma(\lambda)}\,G(t)\,dt\ \text{p.p.}$$

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*that is, the equation*

$$I^{\lambda}e^{-x}f(x) = e^{-x}I^{\lambda}G(x) \text{ p.p.} \tag{6·1}$$

*has for each $G(x)$ in $L$ a solution $f(x)$ in $L$, given by*

$$f(x) = G(x) - \lambda\int_0^x {}_1F_1(\lambda+1;\,2;\,t-x)\,e^{-(t-x)}\,dt. \tag{6·2}$$

*Conversely for each $f(x)\in L$, the equation has a solution in $L$, given by*

$$G(x) = f(x) + \lambda\int_0^x {}_1F_1(\lambda+1;\,2;\,x-t)f(t)\,dt. \tag{6·3}$$

If $G(x)\in L$, then $e^{-x}G(x)\in L$. From Theorem 3$a$, for almost all $x\in[0,\alpha]$,

$$I^{1-\lambda}K^*_{\lambda,\lambda}(G(x)\,e^{-x}) = K^*_{\lambda,1}(G(x)\,e^{-x}),$$

i.e.
$$I^{1-\lambda}\int_0^x \frac{(x-t)^{\lambda-1}}{\Gamma(\lambda)}\,e^{t-x}G(t)\,e^{-t}\,dt = \int_0^x {}_1F_1(\lambda;\,1;\,t-x)\,G(t)\,e^{-t}\,dt,$$

i.e.
$$e^{-x}\int_0^x \frac{(x-t)^{\lambda-1}}{\Gamma(\lambda)}\,G(t)\,dt = I^{\lambda}\frac{d}{dx}\int_0^x {}_1F_1(\lambda;\,1;\,t-x)\,G(t)\,e^{-t}\,dt$$

$$= I^{\lambda}\left\{G(x)\,e^{-x} - \lambda\int_0^x {}_1F_1(\lambda+1;\,2;\,t-x)\,G(t)\,e^{-t}\,dt\right\},$$

$$e^{-x}I^{\lambda}G(x) = I^{\lambda}\left[e^{-x}\left\{G(x) - \lambda\int_0^x {}_1F_1(\lambda+1;\,2;\,t-x)\,e^{-(t-x)}G(t)\,dt\right\}\right]. \tag{6·4}$$

The integral on the right of (6·4) being a convolution of ${}_1F_1(\lambda+1;\,2;\,-t)\,e^{-t}$ and $G(t)$, both belonging to $L$ exists and is easily shown to be in $L$. Thus

$$G(x) - \lambda\int_0^x {}_1F_1(\lambda+1;\,2;\,t-x)\,e^{-(t-x)}G(t)\,dt$$

exists in $L$ and

$$f(x) = G(x) - \lambda\int_0^x {}_1F_1(\lambda+1;\,2;\,t-x)\,e^{-(t-x)}G(t)\,dt$$

satisfies (6·1).

Again, if $f(x)\in L$, by Theorem 3$a$, p.p. in $[0,\alpha]$

$$I^{1-\lambda}K_{\lambda,\lambda}f(x) = K_{\lambda,1}f(x)$$

so that
$$\int_0^x \frac{(x-t)^{\lambda-1}}{\Gamma(\lambda)}\,e^{x-t}f(t)\,dt = I^{\lambda-1}\int_0^x {}_1F_1(\lambda;\,1;\,x-t)f(t)\,dt$$

$$= I^{\lambda}\frac{d}{dx}\int_0^x {}_1F_1(\lambda;\,1;\,x-t)f(t)\,dt.$$

Thus
$$e^{x}I^{\lambda}(e^{-x}f(x)) = I^{\lambda}\left\{f(x) + \lambda\int_0^x {}_1F_1(\lambda+1;\,2;\,x-t)f(t)\,dt\right\}.$$

Reasoning as above, it follows that

$$G(x) = f(x) + \lambda\int_0^x {}_1F_1(\lambda+1;\,2;\,x-t)f(t)\,dt$$

is a solution in $L$, of the equation (6·1).

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THEOREM 8. *If* $\operatorname{Re}\lambda > 0$, *then the integral equation*

$$I^{\lambda}e^{x}f(x) = e^{x}I^{\lambda}G(x) \text{ p.p.} \tag{6·5}$$

*has for each* $G(x) \in L$, *a solution in* $L$ *given by*

$$f(x) = G(x) + \lambda \int_0^x {}_1F_1(\lambda+1;\, 2;\, x-t)\, e^{-(x-t)}\, G(t)\, dt \tag{6·6}$$

*and also for each* $f(x) \in L$, *a solution*

$$G(x) = f(x) - \lambda \int_0^x {}_1F_1(\lambda+1;\, 2;\, t-x) f(t)\, dt. \tag{6·7}$$

The proof of the theorem follows the manner of the previous one. For

$$I^{1-\lambda}K_{\lambda,\lambda}(e^{x}G(x)) = K_{\lambda,1}(e^{x}G(x)) \text{ p.p.}$$

so that

$$\int_0^x \frac{(x-t)^{\lambda-1}}{\Gamma(\lambda)}\, e^{x-t}e^{t}G(t)\, dt = I^{\lambda}\frac{d}{dx}\int_0^x {}_1F_1(\lambda;\, 1;\, x-t)\, e^{t}G(t)\, dt \text{ p.p.}$$

giving

$$e^{x}I^{\lambda}G(x) = I^{\lambda}\left[e^{x}\left\{G(x) + \lambda\int_0^x {}_1F_1(\lambda+1;\, 2;\, x-t)\, e^{-(x-t)}G(t)\, dt\right\}\right] \text{ p.p.}$$

Also

$$I^{1-\lambda}K^{*}_{\lambda,\lambda}f(x) = K^{*}_{\lambda,1}f(x) \text{ p.p.}$$

leads to

$$e^{-x}I^{\lambda}e^{x}f(x) = I^{\lambda}\left[f(x) - \lambda\int_0^x {}_1F_1(\lambda+1;\, 2;\, t-x) f(t)\, dt\right] \text{ p.p.}$$

THEOREM 9. *If* $\operatorname{Re} b > 0$, $\operatorname{Re} a > 0$, $g(x) \in L_b$, *then the integral equation*

$$\int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)}\, {}_1F_1(a;\, b;\, x-t) f(t)\, dt = g(x) \text{ p.p.} \tag{6·8}$$

*that is*

$$K_{a,b}f = g \text{ p.p.} \tag{6·9}$$

*has a unique solution (in fact a class of equivalent solutions)* $f(x) \in L$, *explicitly expressible by*

$$f(x) = e^{x}I^{-a}e^{-x}I^{a-b}g(x). \tag{6·10}$$

LEMMA 9·1. *If* $\operatorname{Re} b > 0$, $\operatorname{Re} a > 0$ *and* $f \in L$, *then* $K_{a,b}f(x) = I^{b-a}e^{x}I^{a}e^{-x}f(x)$ *for almost all* $x \in [0, \alpha]$.

By Theorem 3,

$$I^{a}K_{a,b}f(x) = K_{a,a+b}f(x) = I^{b}K_{a,a}f(x). \tag{6·11}$$

But

$$K_{a,a}f(x) = \int_0^x \frac{(x-t)^{a-1}}{\Gamma(a)}\, e^{x-t}f(t)\, dt$$

$$= e^{x}I^{a}(e^{-x}f(x))$$

so that (6·11) gives

$$I^{a}K_{a,b}f(x) = I^{b}e^{x}I^{a}e^{-x}f(x)$$

for almost all $x$ in $[0, \alpha]$. Thus

$$K_{a,b}f(x) = I^{-a}[I^{b}e^{x}I^{a}e^{-x}f(x)]$$

$$= I^{b-a}e^{x}I^{a}e^{-x}f(x) \quad \text{p.p. in } (0, \alpha].$$

*Proof of the theorem.* From Theorem 7 with $\lambda = a$ and $G(x) = I^{-b}g(x)$, the equation

$$I^{a}e^{-x}f(x) = e^{-x}I^{a}[I^{-b}g(x)] \tag{6·12}$$



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has for each $I^{-b}g(x)$ in $L$, a solution $f(x) \in L$. Since $\operatorname{Re} b > 0$ and $I^{-b}g(x)$ exists in $L$ the phrase 'for each $I^{-b}g(x)$ in $L$' can be replaced by 'for each $g(x) \in L_b$'.

Clearly (6·12) can be rewritten as

$$I^a e^{-x} f(x) = e^{-x} I^{a-b} g(x) \tag{6·13}$$

and admits a solution $f(x)$ which is expressible by

$$f(x) = e^x I^{-a} e^{-x} I^{a-b} g(x). \tag{6·14}$$

The existence of $I^{-a}e^{-x}I^{a-b}g(x)$ is ensured by Theorem 7.

Again by Theorem 7, for each $f(x) \in L$, the equation (6·12) has a solution $I^{-b}g(x) \in L$ and of course,

$$g(x) = I^{b-a} e^x I^{-a} e^{-x} f(x). \tag{6·15}$$

Hence by Lemma 9·1, (6·14) is a solution of (6·8).

THEOREM 9$a$. *Under the conditions of Theorem* 9, *the equation*

$$\int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)} {}_1F_1(a; b; t-x) f(t)\, dt = g(x) \text{ p.p.} \tag{6·16}$$

*admits a unique solution* $f(x) \in L$, *expressible by*

$$f(x) = e^{-x} I^{-a} e^x I^{a-b} g(x).$$

LEMMA 9·1$a$. *If* $\operatorname{Re} a > 0$, $\operatorname{Re} b > 0$ *and* $f \in L$, *then*

$$K^*_{a,b} f(x) = I^{b-a} e^{-x} I^a e^x f(x). \tag{6·17}$$

Using Theorem 3,

$$\begin{aligned} I^a K^*_{a,b} f(x) &= K^*_{a,a+b} f(x) \\ &= I^b K^*_{a,a} f(x) \\ &= I^b [e^{-x} I^a (e^x f(x))]. \end{aligned}$$

Hence

$$K^*_{a,b} f(x) = I^{b-a} e^{-x} I^a e^x f(x).$$

The existence of the various functions involved is easily proved.

*Proof of the theorem.* Setting $G(x) = I^{-b}g(x)$, the integral equation

$$I^a e^x f(x) = e^x I^a G(x) \tag{6·18}$$

has by Theorem 8, for each $g(x)$ in $L$, a solution $f(x)$ in $L$, which shows that $I^{-a}e^x I^{a-b}g(x)$ exists in $L$ and

$$f(x) = e^{-x} I^{-a} e^x I^{a-b} g(x). \tag{6·19}$$

By another application of Theorem 7, it follows that

$$g(x) = I^{b-a} e^{-x} I^a e^x f(x), \tag{6·20}$$

if $f(x)$ is in $L$. Thus (6·19) is a solution of (6·18) whenever (6·20) is true. By Lemma 9·1$a$, $f(x)$ in (6·19) is a solution of (6·16).

THEOREM 10. *If* $\operatorname{Re} b > 0$, $\operatorname{Re} a \leqslant 0$ *and* $I^{-b}g(x)$ *exists in* $L$, *then the integral equation*

$$\int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)} {}_1F_1(a; b; x-t) f(t)\, dt = g(x) \text{ p.p.} \tag{6·21}$$

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*admits a unique solution $f(x)$ in $L$, expressible as*

$$f(x) = I^{a-b}e^{x}I^{-a}e^{-x}g(x). \tag{6·22}$$

By Kummer's first theorem ((10), (1.4.1)), (6·21) can be written as

$$\int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)}\,{}_1F_1(b-a;\,b;\,t-x)\,e^{x-t}f(t)\,dt = g(x)\text{ p.p.}, \tag{6·23}$$

i.e.
$$\int_0^x \frac{(x-t)^{b_1-1}}{\Gamma(b_1)}\,{}_1F_1(a_1;\,b_1;\,t-x)f_1(t)\,dt = g_1(x)\text{ p.p.}, \tag{6·24}$$

where
$$a_1 = b-a, \quad b_1 = b, \quad f_1(t) = e^{-t}f(t),$$
$$g_1(x) = e^{-x}g(x).$$

Since $\operatorname{Re} a_1 > 0$, Theorem 9$a$ is applicable to (6·24) if we show that the existence of $I^{-b}g(x)$ in $L$ implies the existence of $I^{-b_1}g_1(x)$ in $L$. Now from Theorem 7 with $\lambda = b$, $G(x) = I^{-b}g(x)$, the equation
$$I^{b}e^{-x}\phi(x) = e^{-x}I^{b}I^{-b}g(x)$$
$$= e^{-x}g(x)$$
has for each $I^{-b}g(x)$ in $L$, a solution $\phi(x)$ in $L$ and
$$e^{-x}\phi(x) = I^{-b}e^{-x}g(x)$$
$$= I^{-b}g_1(x).$$
Since $\phi(x)$ exists in $L$ when $I^{-b}g(x)$ exists in $L$, so does $I^{-b}g_1(x)$.

The solution of (6·24) is therefore given by
$$e^{-x}I^{-a_1}e^{x}I^{a_1-b_1}g_1(x) \tag{6·25}$$
so that
$$f(x) = I^{a-b}e^{x}I^{-a}e^{-x}g(x) \tag{6·26}$$
is a solution of (6·21).

THEOREM 10$a$. *Under the conditions of Theorem* 10, *the equation*

$$\int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)}\,{}_1F_1(a;\,b;\,t-x)f(t)\,dt = g(x)\text{ p.p.} \tag{6·27}$$

*admits a unique solution*
$$f(x) = I^{a-b}e^{-x}I^{-a}e^{x}g(x)$$
*belonging to $L$.*

Using Kummer's first theorem, (6·27) is

$$\int_0^x \frac{(x-t)^{b_1-1}}{\Gamma(b_1)}\,{}_1F_1(a_1;\,b_1;\,x-t)f_1(t)\,dt = g_1(x)\text{ p.p.}, \tag{6·28}$$

where now $a_1 = b-a$, $b_1 = b$, $f_1(t) = e^{t}f(t)$, $g_1(t) = e^{t}g(t)$.

We first show that under the given conditions $I^{-b_1}g_1(x)$ exists in $L$. Applying Theorem 8, for given $I^{-b}g(x)$ in $L$, there exists a solution $\phi(x)$ in $L$, of the equation
$$I^{b}e^{x}\phi(x) = e^{x}I^{b}[I^{-b}g(x)],$$
i.e.
$$I^{b}e^{x}\phi(x) = e^{x}g(x),$$
which proves the existence of $I^{-b}e^{x}g(x)$, i.e. of $I^{-b_1}g_1(x)$ in $L$.



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An application of Theorem 9 to the equation (6·28) leads to

$$f_1(x) = e^x I^{-a_1} e^{-x} I^{-a_1-b_1} g_1(x).$$

Consequently

$$f(x) = I^{a-b} e^{-x} I^{-a} e^x g(x).$$

We can sum up the discussion in terms of the inverse operators $K_{a,b}^{-1}$ and $K_{a,b}^{*-1}$ on $L_b$ onto $L$ as follows:

The inverses $K_{a,b}^{-1}$, $K_{a,b}^{*-1}$ exist for all complex values of $a$ and are defined by

$$K_{a,b}^{-1} g(x) = e^x I^{-a} e^{-x} I^{a-b} g(x) \quad \text{for} \quad \operatorname{Re} a > 0,$$

$$K_{a,b}^{-1} g(x) = I^{a-b} e^x I^{-a} e^{-x} g(x) \quad \text{for} \quad \operatorname{Re} a \leqslant 0,$$

$$K_{a,b}^{*-1} g(x) = e^{-x} I^{-a} e^x I^{a-b} e^x g(x) \quad \text{for} \quad \operatorname{Re} a > 0,$$

$$K_{a,b}^{*-1} g(x) = I^{a-b} e^{-x} I^{-a} e^x g(x) \quad \text{for} \quad \operatorname{Re} a \leqslant 0.$$

7. *Applications.* As remarked in section 1, from the results obtained in the preceding section we get the solutions of several integral equations involving certain classical polynomials and other special functions in the kernels.

I. *Kernels involving Laguerre polynomials. If* $\operatorname{Re}\alpha > -1$ *and* $I^{-\alpha-1} g(x)$ *exists in* $L$, *then the integral equations*

$$\int_0^x (x-t)^\alpha L_n^\alpha(x-t) f(t)\, dt = g(x) \text{ p.p.} \tag{7·1}$$

*and*

$$\int_0^x (x-t)^\alpha L_n^\alpha(t-x) f(t)\, dt = g(x) \text{ p.p.} \tag{7·2}$$

*have unique solutions in* $L$, *given respectively by*

$$f(x) = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} I^{-n-\alpha-1} e^x I^n e^{-x} g(x) \tag{7·3}$$

*and*

$$f(x) = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} I^{-n-\alpha-1} e^{-x} I^n e^x g(x), \tag{7·4}$$

$L_n^\alpha(x)$ *being, in the standard notation, the generalized Laguerre polynomial.*

In particular, when $\alpha = 0$, the solution of

$$\int_0^x L_n(x-t) f(t)\, dt = g(x) \text{ p.p.} \tag{7·5}$$

is expressible by

$$f(x) = I^{-n-1} e^x I^n e^{-x} g(x)$$

$$= (d/dx)^{n+1} e^x I^n [e^{-x} g(x)]. \tag{7·6}$$

It is well known that

$$L_n^\alpha(x) = \frac{(1+\alpha)_n}{n!} {}_1F_1(-n;\, 1+\alpha;\, x). \tag{7·7}$$

If $\operatorname{Re}\alpha > -1$ and $I^{-\alpha-1} g(x)$ is in $L$ we apply Theorem 10 with $a = -n$, $b = \alpha + 1$ and use (7·7). Thus the integral equation

$$\frac{n!}{(1+\alpha)_n} \int_0^x \frac{(x-t)^\alpha}{\Gamma(1+\alpha)} L_n^\alpha(x-t) f(t)\, dt = g_1(x), \quad x \in [0, \alpha] \text{ p.p.} \tag{7·8}$$

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has the solution in $L$, given by

$$f(x) = I^{-n-\alpha-1}e^x I^n e^{-x} g_1(x).$$

Putting
$$g(x) = \frac{\Gamma(1+\alpha+n)}{n!} g_1(x),$$

the solution of
$$\int_0^x (x-t)^\alpha L_n^\alpha(x-t) f(t)\,dt = g(x) \text{ p.p.}$$

is given by
$$f(x) = \frac{n!}{\Gamma(n+\alpha+1)} I^{-n-\alpha-1}e^x I^n e^{-x} g(x).$$

To solve (7·2), we use Theorem 10*a*.

II. *Kernels involving Whittaker functions.*

(i) *If* $\operatorname{Re} m > 0$, $\operatorname{Re} k \leqslant 0$, $I^{-2\operatorname{Re} m}g(x)$ *exists in* $L$, *the integral equation*

$$\int_0^x (x-t)^{m-1} M_{k,m-\frac{1}{2}}(x-t) f(t)\,dt = g(x) \text{ p.p.} \tag{7·9}$$

*has a solution*
$$f(x) = \frac{1}{\Gamma(2m)} e^{-\frac{1}{2}x} I^{-m+k} e^{-x} I^{-m-k} e^{\frac{1}{2}x} g(x). \tag{7·10}$$

It is a standard result ((10), (1.9.7)) that

$${}_1F_1(m-k;\, 2m;\, x) = e^{\frac{1}{2}x} x^{-m} M_{k,m-\frac{1}{2}}(x). \tag{7·11}$$

When $\operatorname{Re} m > 0$, $\operatorname{Re} k \leqslant 0$, we apply Theorem 9 to obtain the solution of

$$\int_0^x \frac{(x-t)^{2m-1}}{\Gamma(2m)} {}_1F_1(m-k;\, 2m;\, x-t) f_1(t) = g_1(x) \text{ p.p.} \tag{7·12}$$

in the form
$$f_1(x) = e^x I^{-(m-k)} e^{-x} I^{-m-k} g_1(x). \tag{7·13}$$

Using (7·11), the equation (7·12) is equivalent to

$$\int_0^x (x-t)^{m-1} M_{k,m-\frac{1}{2}}(x-t) f(t) = g(x) \text{ p.p.}, \tag{7·14}$$

with $f(x) = e^{-\frac{1}{2}x} f_1(x)$ and $g(x) = \Gamma(2m)\, e^{-\frac{1}{2}x} g_1(x)$.

From (7·13) it at once follows that

$$f(x) = \frac{1}{\Gamma(2m)} e^{-\frac{1}{2}x} I^{-m+k} e^{-x} I^{-m-k} e^{\frac{1}{2}x} g(x).$$

It is easily verified that the existence in $L$ of $I^{-2\operatorname{Re} m}g(x)$ implies the existence of $I^{-2\operatorname{Re} m}g_1(x)$, thus justifying an application of Theorem 9 to (7·12).

(ii) *If* $\operatorname{Re} k \geqslant \operatorname{Re} m > 0$ *and* $I^{-2\operatorname{Re} m}g(x)$ *exists in* $L$, *the integral equation*

$$\int_0^x (x-t)^{m-1} M_{k,m-\frac{1}{2}}(x-t) f(t)\,dt = g(x) \text{ p.p.} \tag{7·15}$$

*has in* $L$, *a solution*
$$f(x) = \frac{e^{-\frac{1}{2}x}}{\Gamma(2m)} I^{-m-k} e^x I^{-m+k} e^{\frac{1}{2}x} g(x). \tag{7·16}$$

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By Theorem 10,

$$\int_0^x \frac{(x-t)^{2m-1}}{\Gamma(2m)}\,{}_1F_1(m-k;\,2m;\,x-t)f_1(t)\,dt = g_1(x)\ \text{p.p.} \tag{7·17}$$

has the solution

$$f_1(x) = I^{-m-k}e^x I^{-m+k}e^{-x}g_1(x). \tag{7·18}$$

Using (7·11), it follows that

$$\int_0^x \frac{(x-t)^{2m-1}}{\Gamma(2m)}\,e^{\frac{1}{2}(x-t)}(x-t)^{-m}\,M_{k,\,m-\frac{1}{2}}(x-t)f_1(t) = g_1(x)\ \text{p.p.},$$

that is, the equation

$$\int_0^x (x-t)^{m-1}\,M_{k,\,m-\frac{1}{2}}(x-t)f(t)\,dt = g(x)\ \text{p.p.}, \tag{7·19}$$

where $f(x) = f_1(x)\,e^{-\frac{1}{2}x}$, $g(x) = \Gamma(2m)\,e^{-\frac{1}{2}x}g_1(x)$ has the solution $f(x)$ given by

$$e^{\frac{1}{2}x}f(x) = \frac{1}{\Gamma(2m)}\,I^{-m-k}e^x I^{-m+k}e^{\frac{1}{2}x}g(x).$$

III. *Kernels involving modified Bessel functions.*

*If* $\operatorname{Re} a > 0$ *and* $I^{-2a}g(x)$ *exists, then the integral equation*

$$\int_0^x \frac{(x-t)^{\alpha-\frac{1}{2}}}{\Gamma(a)}\,I_{a-\frac{1}{2}\,\frac{1}{2}}(x-t)f(t)\,dt = g(x)\ \text{p.p.} \tag{7·20}$$

*admits a solution expressible by*

$$f(x) = \frac{\sqrt{\pi}}{2^{2a-1}}\,e^{\frac{1}{2}x}I^{-a}e^{-x}I^{-a}e^{\frac{1}{2}x}g(x),$$

*where* $I_a(x)$ *is the modified Bessel function of the first kind.*

By Theorem 9, if $\operatorname{Re} a > 0$,

$$\int_0^x \frac{(x-t)^{2a-1}}{\Gamma(2a)}\,{}_1F_1(a;\,2a;\,x-t)f_1(t)\,dt = g_1(x)\ \text{p.p.} \tag{7·21}$$

has for its solution

$$f_1(x) = e^x I^{-a}e^{-x}I^{-a}g_1(x). \tag{7·22}$$

By Kummer's second theorem ((10), (1.8.3)),

$${}_1F_1(a;\,2a;\,x) = e^{\frac{1}{2}x}(\tfrac{1}{4}x)^{\frac{1}{2}-a}\,\Gamma(a+\tfrac{1}{2})\,I_{a-\frac{1}{2}}(\tfrac{1}{2}x)$$

so that (7·21) is equivalent to

$$\Gamma(a+\tfrac{1}{2})\int_0^x \frac{(x-t)^{2a-1}}{\Gamma(2a)}\,e^{\frac{1}{2}(x-t)}\left(\frac{x-t}{4}\right)^{\frac{1}{2}-a} I_{a-\frac{1}{2}\,\frac{1}{2}}(x-t)f_1(t) = g_1(x)\ \text{p.p.},$$

i.e.

$$\int_0^x \frac{(x-t)^{a-\frac{1}{2}}}{\Gamma(a)}\,I_{a-\frac{1}{2}\,\frac{1}{2}}(x-t)f(t)\,dt = g(x)\ \text{p.p.}, \tag{7·23}$$

where

$$f(t) = e^{-\frac{1}{2}t}f_1(t), \quad g(x) = (2^{2a-1}/\sqrt{\pi})\,e^{-\frac{1}{2}x}g_1(x),$$

on using Legendre's duplication formula

$$\Gamma(2a) = (2^{2a-1}/\sqrt{\pi})\,\Gamma(a)\,\Gamma(a+\tfrac{1}{2}).$$

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Hence $$e^{\frac{1}{2}x}f(x) = (\sqrt{\pi}/2^{2a-1})\, e^x I^{-a} e^{-x} I^{-a} e^{\frac{1}{2}x} g(x).$$

Evidently existence in $L$ of $I^{-2a}g(x)$ ensures that of $I^{-2a}g_1(x)$.

IV. *Kernels involving incomplete Gamma function.* We define the incomplete Gamma function $\gamma(a,x)$ as

$$\gamma(a,x) = (1/a)\, x^a\, {}_1F_1(a;\, a+1;\, -x). \tag{7·24}$$

If $\operatorname{Re} a > 0$, it is equivalent to the usual definition as the incomplete Eulerian integral of the second kind; that is

$$\gamma(a,x) = \int_0^x e^{-t} t^{a-1}\, dt \quad (\operatorname{Re} a > 0). \tag{7·25}$$

For $\operatorname{Re} a \leqslant 0$, the integral is not convergent and only (7·24) is to be understood as the definition.

*If* $\operatorname{Re} a > 0$ *and* $I^{-a-1}g(x)$ *exists, then*

$$\frac{1}{\Gamma(a)} \int_0^x \gamma(a, x-t) f(t)\, dt = g(x) \text{ p.p.} \tag{7·26}$$

*has the unique solution in* $L$, *given by*

$$f(x) = e^{-x} I^{-a} e^x I^{-1} g(x). \tag{7·27}$$

*If* $-1 < \operatorname{Re} a \leqslant 0$, *then*

$$\frac{1}{\Gamma(a)} \int_0^x \gamma(a, x-t) f(t)\, dt = g(x) \text{ p.p.} \tag{7·28}$$

*has the solution*

$$f(x) = I^{-1} e^{-x} I^{-a} e^x g(x). \tag{7·29}$$

If $\operatorname{Re} a > 0$, by Theorem 9$a$, the equation

$$\int_0^x \frac{(x-t)^a}{\Gamma(a+1)}\, {}_1F_1(a;\, a+1;\, t-x) f(t)\, dt = g(x) \text{ p.p.} \tag{7·30}$$

has the solution

$$f(x) = e^{-x} I^{-a} e^x I^{-1} g(x).$$

Using (7·24), the equation can be rewritten as

$$\frac{1}{\Gamma(a)} \int_0^x \gamma(a, x-t) f(t) = g(x) \text{ p.p.}$$

If $-1 < \operatorname{Re} a \leqslant 0$, then applying Theorem 10$a$, we get the result.

V. *Kernels involving Hermite polynomials. If* $I^{-\frac{1}{2}}g(x)$ *exists in* $L$, *the integral equation*

$$\int_0^x \frac{(x-t)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} H_{2m}\sqrt{(x-t)}\, f(t)\, dt = g(x) \text{ p.p.} \tag{7·31}$$

*has a solution in* $L$, *expressible by*

$$f(x) = \frac{(-1)^m m!}{(2m)!}\, I^{-m-\frac{1}{2}} e^x I^m g(x)\, e^{-x}. \tag{7·32}$$

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*If $I^{-\frac{3}{2}}g(x)$ exists in $L$, the integral equation*

$$\int_0^x H_{2m+1}\{\sqrt{(x-t)}\}f(t)\,dt = g(x) \text{ p.p.} \tag{7·33}$$

*has the solution*

$$f(x) = \frac{(-1)^m m!}{(2m+1)!\sqrt{\pi}} I^{-m-\frac{3}{2}}e^x I^m e^{-x} g(x), \tag{7·34}$$

*where $H_{2m}(x)$ and $H_{2m+1}(x)$ are Hermite polynomials.*

To obtain the solution of (6·31), we apply Theorem 10 to the integral equation

$$\int_0^x \frac{(x-t)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} {}_1F_1(-m;\tfrac{1}{2};x-t)f(t)\,dt = g_1(x) \text{ p.p.} \tag{7·35}$$

whose solution is expressible as

$$f(x) = I^{-m-\frac{1}{2}}e^x I^m e^{-x} g_1(x). \tag{7·36}$$

Using the result ((12), 10.13 (17)),

$$m!\,H_{2m}(x) = (-1)^m (2m)!\,{}_1F_1(-m;\tfrac{1}{2};x^2),$$

(7·35) can be written as

$$\int_0^x \frac{(x-t)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} \frac{(-1)^m m!}{(2m)!} H_{2m}\{\sqrt{(x-t)}\}f(t)\,dt = g_1(x) \text{ p.p.},$$

i.e.

$$\int_0^x \frac{(x-t)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})} H_{2m}\{\sqrt{(x-t)}\}f(t)\,dt = g(x) \text{ p.p.}, \tag{7·37}$$

where

$$g(x) = \frac{(-1)^m (2m)!}{m!} g_1(x).$$

The solution of (7·37) is given by

$$f(x) = \frac{(-1)^m m!}{(2m)!} I^{-m-\frac{1}{2}}e^x I^m e^{-x} g(x).$$

To solve (7·33), consider the equation

$$\int_0^x \frac{(x-t)^{\frac{3}{2}-1}}{\Gamma(\frac{3}{2})} {}_1F_1(-m;\tfrac{3}{2};x-t)f(t)\,dt = g_1(x) \text{ p.p.} \tag{7·38}$$

whose solution, by Theorem 10, is

$$f(x) = I^{-m-\frac{3}{2}}e^x I^m e^{-x} g_1(x). \tag{7·39}$$

But we have the relation ((12), 10.13 (18))

$$m!\,H_{2m+1}(x) = (-1)^m (2m+1)!\,2x\,{}_1F_1(-m;\tfrac{3}{2};x^2)$$

so that (7·38) is equivalent to

$$\int_0^x H_{2m+1}\{\sqrt{(x-t)}\}f(t)\,dt = g(x) \text{ p.p.} \tag{7·40}$$

with

$$g(x) = \frac{(-1)^m (2m+1)!\sqrt{\pi}}{m!} g_1(x)$$

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and the solution of (7·40) is given by

$$f(x) = \frac{(-1)^m m!}{\sqrt{\pi}(2m+1)!} I^{-m-\frac{3}{2}} e^x I^m e^{-x} g(x).$$

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