PROCEEDINGS OF THE
Cambridge Philosophical Society

VOL. 58 JANUARY, 1962 PART 1

TWO PROBLEMS ON CONVEX BODIES

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Received 10 April 1961

1. We solve two problems on convex bodies stated on p. 38 of S. M. Ulam's book, *A collection of mathematical problems* (New York, 1960).

PROBLEM 1. This problem is due to Mazur. In three-dimensional Euclidean space there is given a convex surface W and a point O in its interior. Consider the set V of all points P defined by the requirement that the length of the interval OP is equal to the area of the plane section of W through O and perpendicular to OP. Is the (centrally symmetric) set V a convex surface?

The answer is *no*, and we shall prove some strong negative theorems. Call interior points (of W) for which the above property is true (i.e., with that specific point being the point O of the definition above) C-points, interior points for which it is false N-points. We consider the analogous problem in n-dimensional Euclidean space, the words 'area of a plane section' in the statement of the problem being replaced by '$(n-1)$-dimensional measure of the section on the $(n-1)$-dimensional hyperplane'. We shall prove:

THEOREM 1. *Every n-dimensional convex body contains some N-points.*

THEOREM 2. *There exist n-dimensional convex bodies for which every interior point is an N-point.*

There is a known positive result, namely, that if W is centrally symmetric with O as centre, then O is a C-point. (H. Busemann, *Proc. Nat. Acad. Sci. U.S.A.* 35 (1949), 27–31.) However, we have

THEOREM 3. *There exist centrally symmetric convex bodies for which the centre is the only C-point.*

2. *Proof of Theorem* 1. (See Fig. 1.) There exists an $(n-1)$-dimensional hyperplane H_1, with only one point in common with the body.† Then some hyperplane H_2, parallel to H_1 and sufficiently close to it, cuts the body in a set of arbitrarily small $(n-1)$-dimensional measure. For, take Cartesian coordinates, with origin O at the point common to H_1 and the body W, and take $H_1 \equiv x_1 = 0$, and W lying in $x_1 \geqslant 0$, then the cylinder

$$x_2^2 + x_3^2 + \ldots + x_n^2 = \epsilon^{2/n}$$

† See, for example, H. G. Eggleston, *Convexity* (Cambridge, 1958), p. 32, Ex. 1·11. Such a point is called a regular point.

https://doi.org/10.1017/S030500410003615X Published online by Cambridge University Press

and W have in common only points for which $x_1 > 0$. Let the least x_1 for such points be δ ($\delta > 0$, the body being closed). Then the hyperplane $H_2 \equiv x_1 = \delta$ cuts W in a set whose $(n-1)$-dimensional measure is less than $K\epsilon$, where K is a constant depending only on n.

Take ϵ small and fixed. Through some fixed point Q of H_2, there exist two hyperplanes H_3, H_4, such that H_2, H_3, H_4 have an $(n-2)$-dimensional hyperspace in common, and such that the angle α between them is not small, and such that the sections of W cut out upon H_3, H_4 are not of small $(n-1)$-dimensional measure. All this is clearly possible if the volume of W is non-zero.

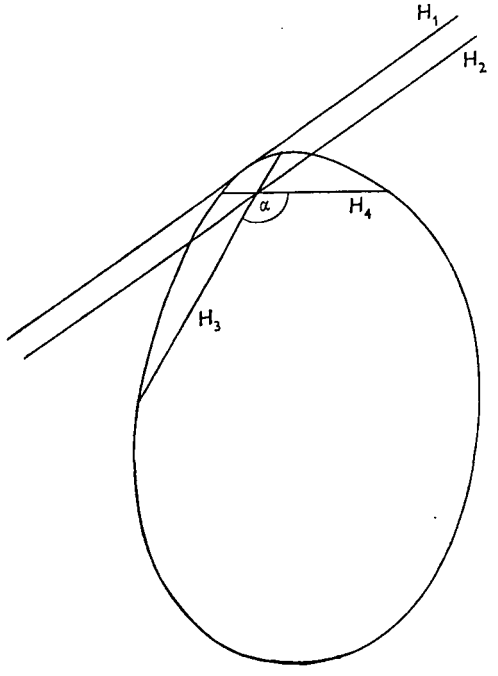


Fig. 1

Then clearly those points on the body V (constructed round Q as centre), that correspond to H_2, H_3, H_4, lie on an arc concave towards Q, i.e. Q is an N-point. This proves Theorem 1, and 'debunks' the original conjecture.

3. We now need two lemmas.

LEMMA 1. *Let a, b be two non-parallel coplanar lines, P a point in their plane, lying on neither; a line L is drawn through P, at angle θ to some fixed direction, θ being such that the intercept cut off on l by a, b includes the point P. Then, if r be the length of this intercept, the equation connecting r and θ, if interpreted in polar coordinates, is an arc of a hyperbola, concave towards the origin.*

Proof. (See Fig. 2.) We have, for $-\gamma < \theta < \beta$,

$$r = r_1 + r_2 = c \sin\beta \operatorname{cosec}(\beta - \theta) + d \sin\gamma \operatorname{cosec}(\gamma + \theta).$$

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This is the equation of a hyperbola, and $\theta = \beta, \theta = -\gamma$ are the asymptotes. So, since $r > 0$ in the range of θ considered, the arc is concave towards the origin.

LEMMA 2. *If O is an interior point of a two-dimensional convex body, of which the boundary contains two non-parallel lines, m, n, such that all lines in some (non-zero) angle at O cut both m and n (not produced), then O is an N-point of the two-dimensional body.*

Proof. The points (of V) constructed corresponding to those lines through O that cut m and n, give rise to a section of a hyperbola, concave towards O; hence O is an N-point.

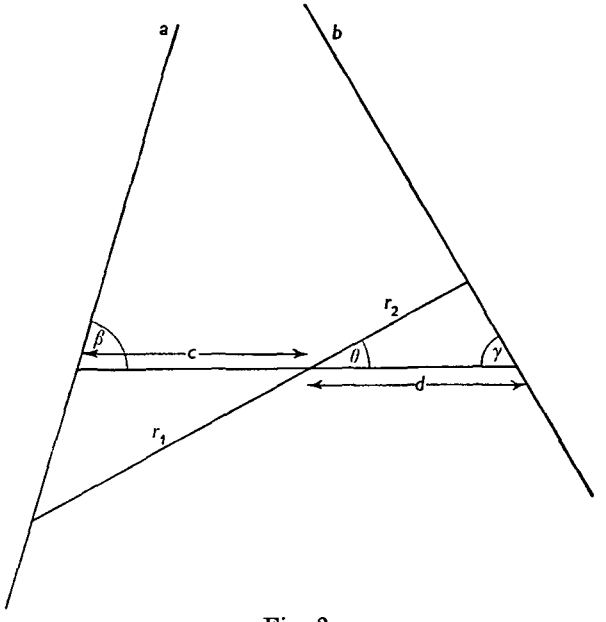


Fig. 2

Proofs of Theorems 2 and 3. The bodies we construct are polygonal prisms. If the Cartesian system of coordinates is $(x_1, x_2, \ldots, x_n)$, consider a body given by:

(i) x_1, x_2 satisfy certain inequalities such as to restrict (x_1, x_2) to the interior of a polygon Π (with a finite number of sides) in the (x_1, x_2)-plane; and

(ii) $|x_3| \leqslant A_3$, $|x_4| \leqslant A_4$, ..., $|x_n| \leqslant A_n$, the A's being positive constants.

Now, consider hyperplanes perpendicular to lines in the (x_1, x_2)-plane. It is clear that a point is an N-point if it is an N-point for the two-dimensional (x_1, x_2) section of the body.

To prove Theorem 2, we take $\Pi = \Pi_1$, any polygon with non-parallel sides. To prove Theorem 3, we take $\Pi = \Pi_2$, a rectangle $|x_1| \leqslant A_1$, $|x_2| \leqslant A_2$.

An immediate application of Lemma 2 gives Theorem 2; it gives also the result of Theorem 3, except for those points whose x_1 and x_2 coordinates are both zero. But for these a similar argument applies for sections parallel to a plane containing any two of the coordinate axes. Hence every point except $(0, 0, \ldots, 0)$ is an N-point for the cuboid formed from Π_2.

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4. *Problem* 2. Ulam writes:

'Let C be a star-shaped closed plane curve, i.e. a polar curve given by $\rho = \rho(\theta)$, and suppose that $\rho(\theta)$ has a continuous derivative except at a finite number of points. It can be shown that there exists a constant $k > 0$ such that the curve given by $\rho = \rho(\theta) + k$ is convex. An analogous remark applies to surfaces in n-dimensions. Suppose C is a curve $(\theta(t), \phi(t))$ in three-space contained in the surface S of a star-shaped region including the origin. Under what conditions is it true that the surface S can be expanded by adding a constant to each radius so that the curve which results from the given curve by the expansion can be obtained as an intersection of convex bodies?'

His statement of the plane case is not quite correct. In fact, we have:

THEOREM 4 (i). *If the plane curve $\rho = \rho(\theta)$ has a continuous second derivative everywhere in the range $0 \leqslant \theta \leqslant 2\pi$, then $\rho = \rho(\theta) + k$ is convex for some constant $k > 0$.*

(ii) *The above conclusion is not necessarily true if we allow an exceptional point to the condition of continuity of the second derivative (even if a second derivative exists at this point).*†

In three dimensions, the situation is as follows:

THEOREM 5 (i). *Let S be a surface, star-shaped with respect to the origin O, i.e. given by $\rho = \rho(\theta, \phi)$, where $\rho(\theta, \phi)$ is a one-valued function of the two spherical polar coordinates θ, ϕ in the ranges $0 \leqslant \theta \leqslant \pi$; $0 \leqslant \phi \leqslant 2\pi$; further, suppose $\rho(\theta, \phi)$ has continuous second derivatives (i.e. continuous in both variables, and with the obvious conventions at $\phi = 0, 2\pi$ and $\theta = 0, \pi$); let C be a curve lying on S, by which we mean a set of points characterized by $\theta = \theta(t)$, $\phi = \phi(t)$, where $\theta(t)$, $\phi(t)$ are continuous functions of t, which lies in the closed interval $[0, 1]$. Then there exists a $k > 0$, such that, if the surface is expanded in the manner described, the resulting curve C' is the intersection of two convex surfaces.*

(ii) *The above conclusion is not necessarily true if there is one point on S not satisfying the condition of continuity of the second derivatives.*

5. *Proof of Theorem* 4. (i) We first observe that, since $\rho''(\theta)$ is continuous in $0 \leqslant \theta \leqslant 2\pi$ (a closed set),

$$|\rho''(\theta)| < A, \tag{1}$$

where A is a constant.

We shall show that for $k = 10 \max\{A, \max \rho(\theta)\}$, the resulting curve C' is convex. We suppose that this is not so, and deduce a contradiction. If C' is not convex, we can find a chord which joins two points X', Y', on C', and which lies wholly outside C'; so then the internal bisector of $X'OY'$ (O is the origin) cuts $X'Y'$ in a point Z'', which lies outside C' (see Fig. 3); let this bisector cut C' in Z'; and let $X'\hat{O}Z' = Z'\hat{O}Y' = \theta$. Our supposition implies then that OZ'' is longer than OZ'. By trigonometry, we have

$$OZ'' = 2OX'.OY' \cos\theta/(OX' + OY'). \tag{2}$$

Now OX', OY', OZ' are equal to $k+\rho_1$, $k+\rho_2$, $k+\rho_3$, where OX, OY, OZ are ρ_1, ρ_2, ρ_3 (X, Y, Z being the points on C corresponding to X', Y', Z' on C').

† As will be seen, here and below in Theorem 5, the conclusion is true if the second derivative is uniformly bounded; the proof is very similar to that given.

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Using the mean value theorem, and (1), we have

$$\left.\begin{aligned} \rho_1 &= \rho_3 - \theta\rho_3' + \alpha\theta^2, \quad |\alpha| \leqslant \tfrac{1}{2}A, \\ \rho_2 &= \rho_3 + \theta\rho_3' + \beta\theta^2, \quad |\beta| \leqslant \tfrac{1}{2}A; \\ \text{hence} \qquad 2\rho_3 &= \rho_1 + \rho_2 + \gamma\theta^2, \quad |\gamma| \leqslant A. \end{aligned}\right\} \tag{3}$$

Using (2), our hypothesis that $OZ'' > OZ'$ is then equivalent to

$$k + \tfrac{1}{2}(\rho_1 + \rho_2 + \gamma\theta^2) < 2(k+\rho_1)(k+\rho_2)\cos\theta/(2k+\rho_1+\rho_2).$$

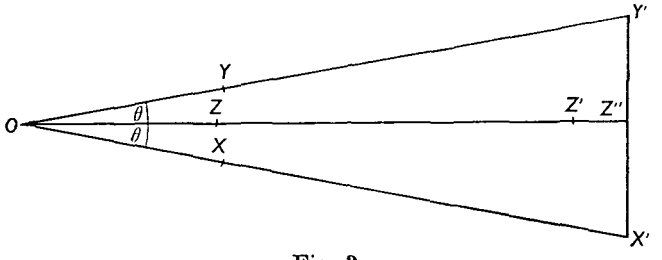


Fig. 3

Multiplying up, we obtain

$$2k^2 + 2k(\rho_1+\rho_2) + \tfrac{1}{2}\gamma\theta^2(2k+\rho_1+\rho_2) + \tfrac{1}{2}(\rho_1+\rho_2)^2 < 2(k^2 + k(\rho_1+\rho_2) + \rho_1\rho_2)\cos\theta.$$

Because $2k(\rho_1+\rho_2) > 2k(\rho_1+\rho_2)\cos\theta$, and $\frac{1}{2}(\rho_1+\rho_2)^2 > 2\rho_1\rho_2\cos\theta$, this equation implies

$$2k^2(1-\cos\theta) < -\tfrac{1}{2}\gamma(2k+\rho_1+\rho_2)\theta^2. \tag{4}$$

Using $k \geqslant 10\max\rho(\theta)$, and $\rho > 0$, and $1-\cos\theta > \theta^2/\pi$ (we may assume $\theta < \frac{1}{2}\pi$), (4) implies

$$2k^2\theta^2/\pi < \tfrac{1}{2}A \,.\, 3k\theta^2.$$

Finally, since $k \geqslant 10A$, we obtain

$$k^2\theta^2 < \tfrac{1}{2}k^2\theta^2.$$

The contradiction is established, and Theorem 4 (i) proved.

(ii) Consider $C \equiv \rho(\theta) = 100 + |\theta|^{\frac{7}{2}}\sin(\theta^{-1})$, $(-\frac{1}{2}\pi \leqslant \theta \leqslant \pi)$, and defined to be infinitely differentiable for the rest of the range. Then $\rho(\theta)$ is (positive and) infinitely differentiable except at $\theta = 0$, where a second derivative exists (but is not continuous). We consider the points X, Y, Z, on C, given by

$$\theta = \theta_1 = \{\tfrac{1}{2}(r-1)\pi\}^{-1}, \quad \theta = \theta_2 = \{\tfrac{1}{2}(r+1)\pi\}^{-1}, \quad \theta = \theta_3 = \tfrac{1}{2}(\theta_1+\theta_2),$$

where r is a large positive integer and $r \equiv 3 \pmod 4$. Then on the expanded curve $C' \equiv \rho = K + |\theta|^{\frac{7}{2}}\sin(\theta^{-1})$, where $K = k + 100$, we have

$$OX' = OY' = K, \quad OZ' = K - \{\tfrac{1}{2}r\pi + o(1)\}^{-\frac{7}{2}},$$

where $o(1)$ refers to the operation $r \to \infty$. The arc $X'Z'Y'$ can form part of a convex curve with O as an interior point only if (see Fig. 4)

$$K - \{\tfrac{1}{2}r\pi + o(1)\}^{-\frac{7}{2}} > K\cos\phi,$$

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where we find $\phi = K^{-1}(2/\pi r^2) + o(1)$; but this implies

$$r^{-\frac{7}{2}} < \{(\tfrac{1}{2}K4/\pi^2)(\tfrac{1}{2}\pi)^{\frac{7}{2}} + o(1)\}r^{-4}.$$

But this equation cannot be true for any one fixed K (however large), and all large integers r (that satisfy $r \equiv 3 \pmod 4$). The theorem is therefore proved.

6. *Proof of Theorem* 5 (i). We first need to prove a 'straight' analogue of Theorem 4 (i) for three dimensions, namely: a surface $S \equiv \rho = \rho(\theta, \phi)$ in polar coordinates, having partial second derivatives, each continuous everywhere in both θ and ϕ together, can be expanded, in the sense described, to a convex surface.

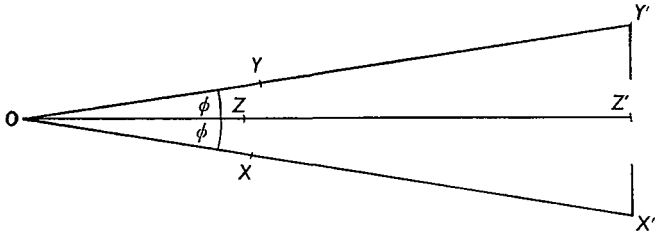


Fig. 4

To prove this, we first observe the following: if

$$\max_{\theta,\phi}\left(\left|\frac{\partial^2\rho}{\partial\theta^2}\right|, \left|\frac{\partial^2\rho}{\partial\theta\,\partial\phi}\right|, \left|\frac{\partial^2\rho}{\partial\phi\,\partial\theta}\right|, \left|\frac{\partial^2\rho}{\partial\phi^2}\right|\right) = d < \infty,$$

and if we take a plane section P of S through O, and let its bounding curve in P be given in polar coordinates (from O) as $\rho = \rho(\alpha)$, then $|d^2\rho/da^2| \leqslant 4d$. For if the plane be at angle β to the ray $\theta = 0$, then

$$\frac{d\rho}{d\alpha} = \frac{\partial\rho}{\partial\theta}\cos\beta + \frac{\partial\rho}{\partial\phi}\sin\beta,$$

$$\frac{d^2\rho}{d\alpha^2} = \frac{\partial^2\rho}{\partial\theta^2}\cos^2\beta + \left(\frac{\partial^2\rho}{\partial\theta\,\partial\phi} + \frac{\partial^2\rho}{\partial\phi\,\partial\theta}\right)\sin\beta\cos\beta + \frac{\partial^2\rho}{\partial\phi^2}\sin^2\beta$$

and so

$$\left|\frac{d^2\rho}{d\alpha^2}\right| \leqslant 4d. \tag{5}$$

(Actually, our conditions imply $\partial^2\rho/\partial\theta\,\partial\rho = \partial^2\rho/\partial\phi\,\partial\theta$, but we do not need this.)

Hence by Theorem 4 (i), we may expand each section through O, until it becomes convex; moreover, the constant of expansion k needed, as is seen from the proof of Theorem 4 (i), depends merely on the maximum of the second derivative; that is, we can expand S so that each section through O becomes convex, with k depending on $\max\rho$ and

$$M \equiv \max_{\alpha,\rho}\left|\frac{d^2\rho}{d\alpha^2}\right|$$

which, by (5), is finite. So every plane section of S' through O is convex; a little consideration will show that necessarily S' is a convex surface. Moreover, if we take k *strictly* greater than M, it can be seen that S' will contain no line segments. We need this remark below, and assume k chosen thus.

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Thus we have proved the 'straight' analogue mentioned above, and also partially proved Theorem 5 (i), for S' is a convex surface, and we know C' lies on it. Consider S'', the surface of the convex cover of all points of C'. We shall prove that the intersection of S' and S'' is just C'; this will prove the theorem, as it establishes C' as the intersection of two convex surfaces, namely S', S''.

We shall prove this by showing (a) if p is a point of S' lying on C', then p lies on S'', and (b) if p is a point of S' not lying on C', then p does not lies on S''. Define $t(p)$ to be a supporting plane to S' at p. By our remark above, we may suppose $t(p)$ to contain no other points of S' than p. First, if p lies on C', then it must lie on or inside the surface S'', by the definition of S'' as the surface of the convex cover of C'. On the other hand, C' lies (not strictly) on one side of $t(p)$, for C' is contained in S' and $t(p)$ is a supporting plane of S'. So the convex cover of C' lies on one side of $t(p)$. It follows that p cannot lie inside S'', but must lie on it. This proves (a).

Again, if p lies on S', but not on C', we see that, because C' is a closed set of points, lying *strictly* on one side of $t(p)$ (for $t(p)$ contains no points of S', *a fortiori* of C', other than p), its convex cover has no point in common with $t(p)$. In particular p does not lie on S''. This proves (b).

(ii) Take $S \equiv \rho = \rho(\theta, \phi) = \rho(\theta)$ as defined in Theorem 4 (ii). Then S is a surface of revolution, infinitely differentiable except at the point $\theta = 0$, where it has a (not continuous) second derivative. This contains a curve C, namely the plane section of S given by $\phi = 0$, which, however much it is expanded in the sense described, never becomes a plane convex curve (by Theorem 4 (ii)), and so C' can never lie on any convex surface, for every plane section of a convex surface is a convex curve.

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