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Almost totally disconnected minimal systems

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Abstract. A space X is said to be almost totally disconnected if the set of its degenerate components is dense in X. We prove that an almost totally disconnected compact metric space admits a minimal map if and only if either it is a finite set or it has no isolated point. As a consequence we obtain a characterization of minimal sets on dendrites and local dendrites.

1. *Introduction*

By a *dynamical system* we mean a pair (X, f) where X is a compact metric space and f is a continuous map $X \to X$ (we write $f \in C(X, X)$). The map f is in general non-invertible and so the *orbit* of a point $x \in X$ under f will mean its forward orbit $\mathrm{Orb}_f(x) = \{x, f(x), f^2(x), \ldots\}$.

The most fundamental dynamical systems are the minimal ones. These are systems which have no non-trivial subsystems. More precisely, a system (X, f) is called *minimal* if there is no proper subset $M \subseteq X$ which is non-empty, closed and f-invariant (i.e. $f(M) \subseteq M$). In such a case we also say that the map f itself is minimal. Clearly, the system (X, f) is minimal if and only if the orbit of every point $x \in X$ is dense in X.

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Due to the compactness of X, the minimality of f implies its surjectivity. A system (X, f) is *totally minimal* if (X, f^n) is minimal for all $n = 1, 2, \ldots$ (here f^n is the nth iterate of f). For example, if X is connected then any minimal map f on X is totally minimal (see [**Ban97**]). To consider only compact metric spaces is a bit less restrictive than it could seem—recall the classical Gottschalk's result [**Got44**] saying that a metric space which is locally compact but not compact does not admit any minimal map.

Probably the best known examples of minimal systems are periodic orbits, irrational rotations on the tori and odometers (adding machines) on the Cantor set. However, in general it is difficult to construct a minimal map, see for instance the examples of minimal homeomorphisms on the Klein bottle in [**Ell65**] and [**Par74**]. On the other hand, many spaces do not admit minimal maps at all (as a trivial example let us mention non-degenerate spaces with the fixed point property).

The classification, i.e. the full topological characterization of compact metric spaces admitting minimal maps is a well-known open problem in topological dynamics, solved only in few particular cases. When attacking this problem, one observation is immediate: the number of components of X plays a certain role here. (By a component of X we always mean a maximal connected subset of X.)

Let $\mathcal{C}$ denote the decomposition of a compact metric space X into the components (many of them may be single points). By collapsing them to points we get a quotient space $X/\mathcal{C}$ which is a totally disconnected compact metrizable space (see §2 for a more detailed discussion). Denote by $\pi_{\mathcal{C}}$ the corresponding quotient map.

If $f : X \to X$ is continuous then a component of X is mapped by f into, but not necessarily onto, a component. So the map $f : X \to X$ naturally induces a map $f/\mathcal{C} : X/\mathcal{C} \to X/\mathcal{C}$ on the space of components. Obviously, $f/\mathcal{C}$ is continuous and $(X/\mathcal{C}, f/\mathcal{C})$ is a factor of (X, f). That is, the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & X \\ {\scriptstyle \pi_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{C}}} \\ X/\mathcal{C} & \xrightarrow{\quad f/\mathcal{C} \quad} & X/\mathcal{C} \end{array}$$

If we additionally assume that (X, f) is minimal then the following holds.

(C-1) The dynamical system $(X/\mathcal{C}, f/\mathcal{C})$, being a factor of the minimal system (X, f), is again minimal.

If a minimal system (X, f) contains an isolated point x then the (forward) orbit of $f(x)$ enters the open set $\{x\}$. Then x is a periodic point and so by minimality its finite orbit is the entire space X. This, together with the fact that $X/\mathcal{C}$ is always totally disconnected, gives the following for the minimal system (X, f).

(C-2) If X has finitely many components, then they are cyclically permuted by f and the restriction of f^k to each of them is totally minimal, k being the number of components.

(C-3) If $X/\mathcal{C}$ contains an isolated point then it is finite, otherwise it is a Cantor set.

Now let us come back to the *problem of the classification of spaces admitting minimal maps*. By (C-3), a *necessary condition* for a compact metric space X to admit a minimal

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map is that it either has finitely many components or it has $\mathfrak{c}$ (the cardinality of a continuum) components and none of them is isolated (in the space $X/\mathcal{C}$). Of course, this condition is far from being sufficient. What about *sufficient conditions* or even necessary and sufficient conditions? *If X has finitely many components*, even if X is connected, then it is probably hopeless to try to find a reasonable necessary and sufficient condition for X to admit a minimal map. Nevertheless, some partial results are known. One of them says that among the connected 2-manifolds with or without boundary only the torus and the Klein bottle (only the finite union of tori and the finite union of Klein bottles, if we get rid of connectedness) admit minimal maps, see [**BOT05**].

Consider the second case, *when $X/\mathcal{C}$ is a Cantor set*. Distinguish three subcases.

(i) *All the components of X are degenerate.* Then X is a Cantor set and so it admits minimal maps.

(ii) *All the components of X are non-degenerate.* Among such spaces there are both the spaces admitting minimal maps and the spaces which do not admit minimal maps—the product of the Cantor set and the circle admits a minimal map (the direct product of an odometer and an irrational rotation) and the product of the Cantor set and a compact real interval does not admit any minimal map (see [**KST07**]).

(iii) *There are both singletons and non-degenerate sets among the components of X.* Again, similarly, as in the previous subcase, both possibilities can occur. Some of such spaces admit minimal maps; Floyd–Auslander minimal homeomorphisms, see e.g. [**HJ97**], can serve as examples. On the other hand the space $X = (C_1 \times [0, 1]) \cup (C_2 \times \{0\})$, where C_1, C_2 are disjoint Cantor sets in $[0, 1]$, does not admit any minimal map. For if this set admitted a minimal map then it could be extended to a triangular map in the space $(C_1 \cup C_2) \times [0, 1]$ with X being its minimal set with non-empty interior in $(C_1 \cup C_2) \times [0, 1]$, which contradicts a result from [**KST07**].

In this paper we prove that, within subcase (iii), there is a large class of spaces admitting minimal maps. We are in fact able to *characterize* spaces admitting minimal maps among all almost totally disconnected compact metric spaces. We will consider the set

$$X^{\deg} = \{x \in X \mid \{x\} \text{ is a component of } X\}$$

of all degenerate components of X, considered as a subset of X.

Definition 1. A space X is said to be *almost totally disconnected* if the set $X^{\deg}$ is dense in X.

Notice that if C is the standard middle-third Cantor set in the unit interval then $(\{0\} \times [0, 1]) \cup (C \times \{0\})$ is *not* almost totally disconnected though all its components except one are singletons. If we say that a compact metric space X is almost totally disconnected, we refer to the fact that its degenerate components form a topologically large subset of X. The definition speaks about the *density* of this subset. Nevertheless it could equivalently speak about the *residuality* of this subset because it is always of type G_δ in X (see §2). Thus, if X is an almost totally disconnected compact metric space then the union of degenerate components of X is a G_δ-dense subset of X and consequently the union of non-degenerate components of X is F_σ-first category set in X. Notice that a totally disconnected space is obviously almost totally disconnected.

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Definition 2. A compact metric space X is said to be a *cantoroid* if it is almost totally disconnected and has no isolated point.

From what was said above we get that for every cantoroid X, $X/\mathcal{C}$ is a Cantor set. Any cantoroid (as well as any uncountable almost totally disconnected compact metric space) has $\mathfrak{c}$ degenerate components and either finitely many (possibly zero) or $\aleph_0$ or $\mathfrak{c}$ non-degenerate ones. Notice that a Cantor set is obviously a cantoroid.

Our main result is as follows.

THEOREM A. *An almost totally disconnected compact metric space admits a minimal map if and only if it is either a finite set or a cantoroid.*

Moreover, we prove that every cantoroid admits a minimal map with arbitrary topological entropy (see Theorem 25).

Theorem A illustrates the importance of cantoroids in the theory of minimal dynamical systems. To discuss properties of cantoroids, first recall that a continuous map $\pi : X \to Y$ between compact metric spaces is called *irreducible* if it is surjective and $f(A) \neq Y$ for every proper closed subset $A \subset X$. (Note that some authors do not require the surjectivity but then every non-surjective continuous map between compact metric spaces would be irreducible.) Any minimal map, even when it is non-invertible, is irreducible (see [**KST01**, Lemma 2.1]). Since a composition of surjections is irreducible if and only if every factor is irreducible we get that every iterate of a minimal map is irreducible.

We list here some *properties of cantoroids* to emphasize how broad the class of such spaces is.

(a) If X is a cantoroid and $\emptyset \neq A \subseteq X$ is a regular closed set (i.e. A is the closure of its interior) then A is a cantoroid. In particular, every non-empty clopen subset of a cantoroid is a cantoroid.

(b) Every compact metric space X which can be written as a countable union of cantoroids $(A_n)_n$ is a cantoroid.

Proof. To see this, first realize that X has no isolated point. Further, let U be the union of the interiors $\operatorname{int} A_n$. By the Baire category theorem U is dense in X. By (a) each closure $\overline{U \cap A_n}$ is a cantoroid, so $U \cap \operatorname{int} A_n$ (if non-empty) contains a dense set of points which are degenerate components of X. Hence X is almost totally disconnected. □

(c) Every compact metric space which contains a countable system of clopen cantoroids whose union is dense in X is a cantoroid.

Proof. The proof is analogous as in (b) since $U = \bigcup_n \operatorname{int} A_n = \bigcup_n A_n$. □

(d) A countable product of cantoroids is a cantoroid.

Proof. If $X = \prod_n X_n$ and X_n are cantoroids then X is compact, metric and has no isolated point. Moreover, $X^{\deg} = \prod_n X_n^{\deg}$ is dense in X since $X_n^{\deg}$ is dense in X_n for every n. □

(e) If $\pi : X \to Y$ is irreducible and Y is a cantoroid then X is a cantoroid.

Proof. The set $X_1 = \{x \in X \mid \pi^{-1}(\pi(x)) = \{x\}\}$ is G_δ. The irreducibility of π is equivalent to the fact that X_1 is dense—see §6 for details. So X_1 is residual. For

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irreducible maps it holds that the preimage of a dense set is dense. It follows that $\pi^{-1}(Y^{\deg}) \cap X_1 \subseteq X^{\deg}$ is residual in X. Hence X is almost totally disconnected. Suppose that X has an isolated point x_0; put $y_0 = \pi(x_0)$. Then $X \setminus \{x_0\}$ is closed in X. Since $\pi(X) = Y$ and π is irreducible, $\pi(X \setminus \{x_0\}) = Y \setminus \{y_0\}$. Since this set is closed, y_0 is isolated in Y, a contradiction. □

(f) The following are equivalent for a compact metric space X.
   (i) X is a cantoroid.
   (ii) $X/\mathcal{C}$ is Cantor and $\pi_{\mathcal{C}}$ is irreducible.
   (iii) X has no isolated points and $\pi_{\mathcal{C}}$ is irreducible.

   Proof. Use (e) and the equivalent definition of irreducibility mentioned there. □

(g) X is a cantoroid if and only if it is an irreducible lift of a Cantor set.

   Proof. Use (e) and (f). □

(h) Every compact metric space Y can be embedded in a cantoroid.

   Proof. Let $\alpha : C \to Y$ be a surjection from a Cantor set C. In the space $C \times \{0, 1, 1/2, 1/3, \ldots\}$ identify points $(c, 0)$ and $(c', 0)$ if and only if $\alpha(c) = \alpha(c')$. The corresponding quotient space X is a cantoroid containing a homeomorphic copy of Y. □

(i) Let (X, f) be a minimal system and let A be a closed subset of X with non-empty interior. If A is a cantoroid then X is a cantoroid.

   Proof. Let U be the interior of A; by (a), $\overline{U}$ is a cantoroid. Put $V_n = f^{-n}(U)$. Since f^n is irreducible, each $f^n|_{\overline{V_n}} : \overline{V_n} \to \overline{U}$ is an irreducible map. By (e), each $\overline{V_n}$ is a cantoroid and, by minimality of f, $X = \bigcup_n V_n = \bigcup_n \overline{V_n}$. Hence, by (b), X is a cantoroid. □

(j) Disjoint union of a cantoroid Y and compact metric space X admits a minimal map if and only if also X is a cantoroid.

   Proof. Use (i), Theorem A and the fact that a disjoint union of two cantoroids is a cantoroid. □

Let us go to the *consequences of Theorem A*. The first observation is that it shows that there are many *non-homogeneous* minimal systems (a system is *homogeneous* if for any points $x, y \in X$ there is a homeomorphism $h : X \to X$ with $h(x) = y$). In the literature there were few explicit examples of this kind so far. The authors are aware of non-homogeneous minimal systems on the pinched torus and on the Sierpinski curve on the torus (see [**BKS03**]) and of examples, scattered in the literature, which are almost one-to-one extensions of other minimal systems—among them there are Floyd–Auslander systems which were historically the first of such examples (see e.g. [**HJ97**] and references therein), some generalizations of Floyd–Auslander systems (see [**SS06**]) and some others.

A non-trivial result which is a consequence of Theorem A is a characterization of minimal sets on dendrites. Before formulating it, recall that the basic and well-known fact discovered by G. D. Birkhoff is that any compact system (X, f) has minimal subsystems $(M, f|_M)$. Such sets M are called *minimal sets* of (X, f). The fundamental question is to characterize or at least partially describe the *topological structure* of minimal sets in a given space.

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First suppose that X is a compact Hausdorff *zero-dimensional* space, $f \in C(X, X)$ and $M \subseteq X$ is a minimal set of f. Then the natural projection $M \to M/\mathcal{C}$ is a homeomorphism and so by (C-3) M is either a finite set (a periodic orbit of f) or a Cantor set. This is in fact a characterization because also, conversely, whenever $M \subseteq X$ is a finite or a Cantor set then there is an $f \in C(X, X)$ such that M is a minimal set of f (note that any non-empty closed subset of X is a retract of X).

Among *one-dimensional* spaces, the characterization of minimal sets is known on the interval—these are finite sets and Cantor sets (see [**BC92**, pp. 92–93]). Further, it is folklore that on the circle there is a third possibility for a minimal set—the whole circle itself. These results can be generalized to *graphs*—minimal sets on graphs are characterized as finite sets, Cantor sets and unions of finitely many pairwise disjoint circles, see [**BHS03**] (this has been rediscovered in [**Mai05**]). Nevertheless, in the case of dendrites, which form the nearest in a sense class of curves containing trees, the problem is difficult and the full characterization has not been known so far, see [**BHS03**]. A *dendrite* is defined as a locally connected continuum which contains no simple closed curve; here a *continuum* is a compact connected metric space. Recall that dendrites appear in dynamics in several settings. For instance, they often appear as Julia sets in complex dynamics (see [**LS98, Mil00, Mor99**] and [**BCM05**]), as ω-limit sets in topological dynamics (see [**AC91a, AC91b**] and [**BG05**]) and as phase spaces of chaotic systems in one-dimensional dynamics (see [**AC04**]). Using our Theorem A we will be able to give a full topological characterization of minimal sets on dendrites. To formulate the result, we first introduce a special class of cantoroids by the following definition. (Recall that a family of sets is *null* if for any $\varepsilon > 0$ only finitely many of its members have diameters greater than ε. Thus, a null family has either finitely many non-degenerate members or countably infinitely many with diameters going to zero.)

Definition 3. By a *brain* we mean a cantoroid whose non-degenerate components are dendrites and form a null family.

The first picture of a brain we drew reminded us of the picture of a human brain (see Figure 1), hence the name.

THEOREM B. *Let D be a dendrite and let M be a subset of D. Then M is a minimal set for some dynamical system on D if and only if M is either a finite set or a brain.*

In a similar way one can even find a characterization of minimal sets on local dendrites. A *local dendrite* is a continuum having the property that every its point has a neighborhood whose closure is a dendrite.

Definition 4. By a *generalized brain* we mean a cantoroid whose non-degenerate components form a null family, they are local dendrites and only finitely many of them contain circles.

THEOREM C. *Let L be a local dendrite and let M be a subset of L. Then M is a minimal set for some dynamical system on L if and only if M is either a finite set or a finite union of disjoint circles or a generalized brain.*

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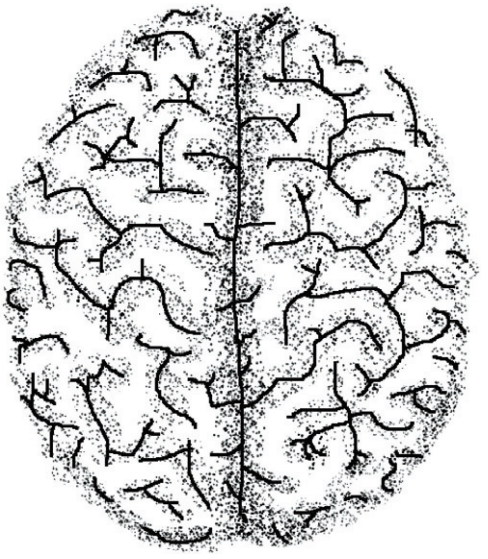


FIGURE 1. An example of a brain.

The organization of the paper is as follows. In §2 we give a short informal outline of the (long and complicated) proof of Theorem A. Then in §3 we summarize some facts on decompositions of compact metric spaces. Section 4 is devoted to cantoroids and to their exhausting sequences (this is a technical notion defined in Definition 10). In §5 we construct special surjections from a cantoroid onto another one. The main result of our paper, Theorem A, and some related results are proved in §6. Theorems B and C are proved in §7.

2. *Outline of the proof of Theorem A*

The proof of Theorem A, which is our main result, is long and complicated. It arrives at after a long string of lemmas and propositions. To make the life of the reader easier, we decided to write this short outline of the proof. (The notation here corresponds to that used in the detailed proof of the theorem.)

We want to prove that an almost totally disconnected compact metric space admits a minimal map if and only if it is either a finite set or a cantoroid. The implication from the left to the right is easy. Concerning the converse implication, only the case when the space is a cantoroid needs a proof. So let X be a cantoroid. We will construct a minimal map $\Psi : X \to X$ as an extension of a Cantor minimal homeomorphism $\varphi : Y \to Y$ (in the following diagram, π is a continuous surjection):

$$\begin{array}{ccc} X & \xrightarrow{\Psi\text{-minimal}} & X \\ \pi \downarrow & & \downarrow \pi \\ Y & \xrightarrow{\varphi} & Y \end{array}$$

The first reduction of the problem is that it is sufficient to construct a *continuous* extension Ψ provided the factor map π is *strongly almost one-to-one* (see Definition 18).

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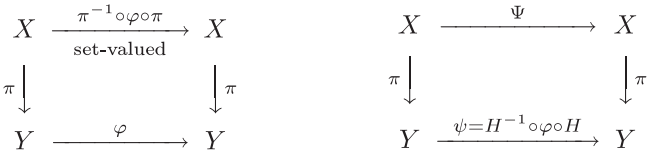


FIGURE 2.

In fact, Ψ is then automatically minimal (see Lemma 19). So we fix a Cantor minimal homeomorphism $\varphi : Y \to Y$ and a continuous surjection $\pi : X \to Y$ satisfying (among others) the following two conditions.

- Only countably many fibers $\pi^{-1}(y)$ $(y \in Y)$ are non-degenerate.
- Every (non-degenerate) fiber is nowhere dense.

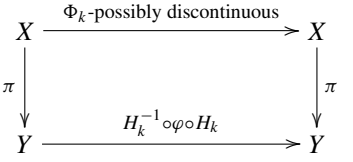
Then π is strongly almost one-to-one as we need. In fact, we will require that π satisfy four more conditions but it would be cumbersome to mention them here (all these conditions correspond to the ones from Definition 10). The fact that such π exists will be proved in Proposition 12.

So we have a cantoroid X, a Cantor minimal homeomorphism $\varphi : Y \to Y$, a factor map $\pi : X \to Y$ with 'good' properties and we want to find a continuous map $\Psi : X \to X$ such that $\pi \circ \Psi = \varphi \circ \pi$. Of course, the only candidates for Ψ are the maps 'compatible' with the set-valued map $\pi^{-1} \circ \varphi \circ \pi$ (see the left-hand diagram in Figure 2); by this we mean that $\Psi(x)$ belongs to the fiber $\pi^{-1} \circ \varphi \circ \pi(x)$ for every $x \in X$. It is easy to see that any such 'compatible' map Ψ, being factored onto the continuous map φ, is continuous at every point x for which $\Psi(x)$ belongs to a degenerate fiber.

Unfortunately, if $\Psi(x)$ belongs to a non-degenerate fiber then this argument does not work. For instance in such a case it can happen that x is in a degenerate fiber; then Ψ is not continuous at x. (Use the facts that the union of degenerate fibers is dense in X and φ is a homeomorphism.)

To overcome these problems we construct an appropriate homeomorphism $H : Y \to Y$ and then we conjugate φ via H, i.e. we consider the homeomorphism $\psi : Y \to Y$ defined by $\psi = H^{-1} \circ \varphi \circ H$. The homeomorphism H will be defined in such a way that there exists a *continuous surjective* map $\Psi : X \to X$ compatible with the set-valued map $\pi^{-1} \circ \psi \circ \pi$, i.e. the right-hand diagram in Figure 2 will commute.

The homeomorphism H and the map Ψ will be defined as uniform limits of a sequence of homeomorphisms $H_k : Y \to Y$ and a sequence of (not necessarily continuous) maps $\Phi_k : X \to X$ with the property that the following diagram commutes for every k:



Moreover, the maps Φ_k will be 'more and more' continuous as $k \to \infty$, namely, Φ_k will be continuous at every point which is mapped by it into the k 'largest' fibers. The inductive construction of H_k and Φ_k is based on Lemma 21 whose proof uses Proposition 17.

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We have thus briefly described how we construct a minimal map Ψ on a cantoroid starting with a minimal homeomorphism φ on a Cantor set. In fact we prove a bit more in our paper (see Proposition 22).

3. *Preliminaries*

Given a metric space (X, d) and subsets A, B of X we use the notation $\operatorname{diam}(A)$ for the diameter of A and $\operatorname{dist}(A, B)$ or, more precisely, $\operatorname{dist}_d(A, B)$ for the infimum of the distances $d(a, b)$, $a \in A$, $b \in B$.

Let $\mathcal{D}$ be a decomposition of a compact metric space X (i.e. a system of non-empty pairwise disjoint subsets of X whose union is X). For $x \in X$ denote by $[x] = [x]_{\mathcal{D}}$ the member of $\mathcal{D}$ containing x. On $X/\mathcal{D} = \{[x] \mid x \in X\}$ we consider the usual quotient topology (a set $U \subseteq X/\mathcal{D}$ is open in $X/\mathcal{D}$ if $\bigcup U$ is open in X). Let $\pi = \pi_{\mathcal{D}} : X \to X/\mathcal{D}$ be the quotient map sending x to $[x]$. The topological space $X/\mathcal{D}$ is compact, being a continuous image of the compact space X. A set $A \subseteq X$ is called $\mathcal{D}$-*saturated* if $[x] \subseteq A$ for every $x \in A$; i.e. it is the union of some subfamily of $\mathcal{D}$. Equivalently, A is $\mathcal{D}$-saturated if $A = \pi^{-1}(\pi(A))$. The union, intersection, difference and symmetric difference of any two $\mathcal{D}$-saturated sets is also a $\mathcal{D}$-saturated set. The decomposition $\mathcal{D}$ is called *upper semicontinuous (usc)* if for each $x \in X$ and each neighborhood U of $[x]$ in X there is an open $\mathcal{D}$-saturated set V such that $[x] \subseteq V \subseteq U$. The importance of usc decompositions lies in the fact that if X is a compact metric space then for the (compact) topological space $X/\mathcal{D}$ the following are equivalent (see [**Nad92**, Theorems 3.3, 3.9, Exercise 3.26], [**Vri93**, (A.2)3, p. 629]):

(a) $\mathcal{D}$ is usc;
(b) $X/\mathcal{D}$ is metrizable;
(c) $X/\mathcal{D}$ is Hausdorff;
(d) the relation corresponding to the decomposition $\mathcal{D}$ is closed as a subset of $X \times X$;
(e) the quotient map π is closed.

(The closedness of the classes $[x]_{\mathcal{D}}$ is necessary but not sufficient for the above conditions.)

We will work with an important example of a usc decomposition. Namely, let $\mathcal{C}$ denote the family of all components of X. Before going further recall the following basic facts.

LEMMA 5. *Let $\mathcal{D}$ be a decomposition of a compact metric space X and $\pi : X \to X/\mathcal{D}$ be the quotient map. If $A \subseteq X$ is an open (closed) $\mathcal{D}$-saturated set, then $\pi(A)$ is open (closed).*

Proof. Because A is $\mathcal{D}$-saturated we have $A = \pi^{-1}(\pi(A))$. Hence, if A is open (closed) then $\pi(A)$ is open (closed) since π is a quotient map. □

LEMMA 6. *Let X be a compact metric space. If C is a component of X and $U \subseteq X$ its neighborhood then there is a clopen set $V \subseteq X$ such that $C \subseteq V \subseteq U$.*

Proof. See [**HY88**, Theorem 2–15]. A direct instructive proof is as follows. In a compact Hausdorff space components and quasicomponents coincide (recall that the quasicomponent of a point x is the intersection of all clopen sets which contain the point x). Hence the component C can be written in the form $\bigcap\{L \text{ clopen} \mid L \supseteq C\}$. Take

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the open cover of X given by U and the sets $X \setminus L$ for all clopen $L \supseteq C$. Choose a finite open subcover by sets $U, X \setminus L_1, \ldots, X \setminus L_n$. Putting $V = L_1 \cap \cdots \cap L_n$ we finish the proof. □

Obviously, every clopen subset $V \subseteq X$ is $\mathcal{C}$-saturated and $\pi(V)$ is clopen in $X/\mathcal{C}$. So, by Lemma 6 (or, see [**Nad92**, Lemma 13.2]), the decomposition $\mathcal{C}$ of X into its components is upper semicontinuous. Hence $X/\mathcal{C}$ is a compact metric space. The space $X/\mathcal{C}$ is totally disconnected (given two components C_1 and C_2 of X, choose by Lemma 6 their disjoint clopen neighbourhoods V_1, V_2; then $\pi(V_1)$ and $\pi(V_2)$ are disjoint clopen sets in $X/\mathcal{C}$, whence $\pi(C_1)$ and $\pi(C_2)$ do not simultaneously belong to any connected set).

The upper semicontinuity of the decomposition $\mathcal{C}$ (or Lemma 6) easily implies that for any positive ε the union of all the components of X with diameters at least ε is closed. Hence the set $X^{\deg}$ (the union of all degenerate components of X) is G_δ.

Rather than work with the decomposition $\mathcal{C}$ into the components of X, throughout the paper we will have to work with usc decompositions $\mathcal{D}$ into sets which are unions of components of X. We wish to emphasize that, contrary to $\mathcal{C}$, for such a decomposition $\mathcal{D}$ the quotient space $X/\mathcal{D}$ need not be totally disconnected. In fact, let X be the middle-third Cantor set and let $f : X \to [0, 1]$ be the Cantor stairs map restricted to X. This defines a usc decomposition $\mathcal{D}$ of X into f-preimages of singletons (trivially $\mathcal{C}$-saturated) but the factor space $X/\mathcal{D}$ is homeomorphic to the interval $f(X) = [0, 1]$ (see [**Eng89**, Corollary 2.4.8 and Proposition 2.4.3]). Notice that in this example the assertion of the next lemma does not hold.

LEMMA 7. *Let X be a compact metric space and let $\mathcal{D}$ be a decomposition of X such that the elements of $\mathcal{D}$ are $\mathcal{C}$-saturated (i.e. $\mathcal{C}$ is a refinement of $\mathcal{D}$). Assume that $\mathcal{D}$ is usc and $X/\mathcal{D}$ is totally disconnected.*

(a) *If D is a closed $\mathcal{D}$-saturated set in X and $U \subseteq X$ its neighborhood then there is a clopen $\mathcal{D}$-saturated set $V \subseteq X$ such that $D \subseteq V \subseteq U$ (in other words, clopen $\mathcal{D}$-saturated sets form a local basis of neighborhoods of any closed $\mathcal{D}$-saturated set D).*

(b) *If D is a closed $\mathcal{D}$-saturated set in X such that every member of $\mathcal{D}$ in D has diameter smaller than ε and $U \subseteq X$ is a neighborhood of D, then there are mutually disjoint clopen $\mathcal{D}$-saturated sets $V_1, \ldots, V_k$ of diameters smaller than ε such that $D \subseteq V_1 \cup \cdots \cup V_k \subseteq U$.*

Proof. (a) Since $\mathcal{D}$ is usc, by passing to a subset if necessary, we may assume that the open set U is $\mathcal{D}$-saturated. Hence $\pi(U)$ is open in $X/\mathcal{D}$. Further, $\pi(D)$ is a compact subset of $\pi(U)$. Therefore, since $X/\mathcal{D}$ is assumed to be totally disconnected, there exists a clopen set W with $\pi(D) \subseteq W \subseteq \pi(U)$. Then $V = \pi^{-1}(W)$ is a clopen set with $D \subseteq V \subseteq U$.

(b) We have $D = \bigcup_{\alpha \in \mathcal{A}} D_\alpha$, $\mathcal{A}$ being an index set and $D_\alpha \in \mathcal{D}$. Each D_α, having diameter less than ε, has an open neighborhood $U_\alpha \subseteq U$ with $\operatorname{diam} U_\alpha < \varepsilon$. Since every D_α is closed and $\mathcal{D}$-saturated, by (a) there is a clopen $\mathcal{D}$-saturated set V_α such that $D_\alpha \subseteq V_\alpha \subseteq U_\alpha$. By compactness finitely many of these V_α are sufficient to cover D. □

Notice that if, in a compact metric space X, $(D_n)_n$ is a sequence of closed sets with diameters tending to zero then the union of the diagonal with $\bigcup_n D_n \times D_n$ is closed

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in $X \times X$. Hence, the relation induced by a null decomposition of X is closed in $X \times X$. So we have the following lemma.

LEMMA 8. *Any null decomposition of a compact metric space into closed sets is upper semicontinuous.*

The above Lemmas 6, 7 and 8 as well as the following obvious lemma will be used throughout the paper, often without mentioning them explicitly.

LEMMA 9. *Let X, Y be compact metric spaces and $f : X \to Y$ be continuous. Let $A \subseteq X$, $B \subseteq Y$ be compact sets such that $f(A) = B$ and $A = f^{-1}(B)$. Then for any open set $U \supseteq A$ there is an open set $V \supseteq B$ with $f^{-1}(V) \subseteq U$.*

4. *Cantoroids*

Recall that a compact metric space X is called a *cantoroid* if it has no isolated point and is almost totally disconnected, i.e. $X^{\deg}$ is dense. Note that the property of being a cantoroid is hereditary with respect to clopen subsets, i.e. every non-empty clopen subset of a cantoroid is a cantoroid.

Let X be a compact metric space and let $\mathcal{D}$ be a decomposition of X. A set $A \subseteq X$ is said to be $\mathcal{D}$-saturated if every element $D \in \mathcal{D}$ is either contained in or disjoint from A. Recall that $\mathcal{C}$ denotes the decomposition of X into the components (many of them may be single points). Notice that every clopen set is $\mathcal{C}$-saturated.

Definition 10. Let X be a compact metric space and K_i $(i \in \mathbb{N})$ a sequence of mutually disjoint subsets of X. By $\mathcal{K}$ we denote the decomposition of X into the sets K_i $(i \in \mathbb{N})$ and the remaining points. The sequence K_i $(i \in \mathbb{N})$ is said to be an *exhausting sequence* in X if the following hold.

(1) Each K_i is nowhere dense.
(2) $\operatorname{diam}(K_i) \to 0$ (equivalently, $\mathcal{K}$ is a null family).
(3) $\bigcup_i K_i$ contains all non-degenerate components of X.
(4) $\bigcup_i K_i$ is dense in X.
(5) Each K_i contains a Cantor set C_i relatively clopen in K_i.
(6) Each K_i is a decreasing intersection of clopen $\mathcal{K}$-saturated sets $B_{i,j}$, $j \geq k_i$ for some $k_i \in \mathbb{N}$ (see Figure 3).

Notice the following.

- Property (5) implies that each K_i is non-empty, even has uncountably many components.
- Property (6) implies that for every i, $\operatorname{diam} B_{i,j} \to \operatorname{diam} K_i$ for $j \to \infty$.
- Property (6) implies that each K_i is closed and $\mathcal{C}$-saturated (because every $B_{i,j}$ is $\mathcal{C}$-saturated).
- Properties (3) and (6) imply that $\mathcal{C}$ is a refinement of $\mathcal{K}$ (i.e. every $\mathcal{K}$-saturated set is also $\mathcal{C}$-saturated).

PROPOSITION 11. *Let X be a compact metric space and let $\mathcal{K}$ be a decomposition corresponding to an exhausting sequence in X. Then $\mathcal{K}$ is usc, $X/\mathcal{K}$ is a Cantor set and the projection $\pi_{\mathcal{K}} : X \to X/\mathcal{K}$ is strongly almost one-to-one.*

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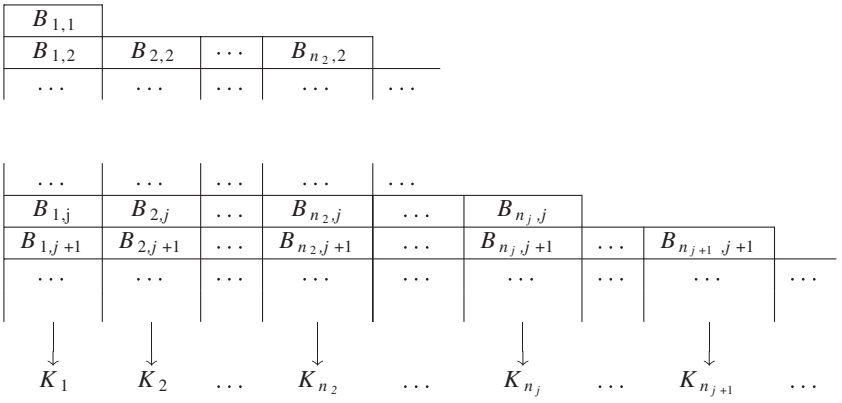


FIGURE 3. An illustration to the part (6) of Definition 10 (here $k_1 = 1$, $k_2 = \cdots = k_{n_2} = 2$, $k_{n_{j-1}+1} = \cdots = k_{n_j} = j$, $k_{n_j+1} = \cdots = k_{n_{j+1}} = j+1, \ldots$).

Proof. Let $\mathcal{K}$ be a decomposition corresponding to an exhausting sequence (K_i) in X and let $Y := X/\mathcal{K}$. Because $\mathcal{K}$ is a null family of closed sets the decomposition $\mathcal{K}$ is usc, see Lemma 8. So the quotient space Y is compact and metrizable. Denote the quotient map $\pi_{\mathcal{K}} : X \to Y$ by π. By (1) $X \setminus \bigcup_i K_i$ is dense and so π is strongly almost one-to-one. Finally to prove the total disconnectedness of Y we show that for every $y \in Y$ there is a clopen base of neighborhoods at y. To simplify the things we will use such metrics in X and Y that π is a contraction (recall that any continuous map $f : X \to Y$ can be regarded as a contraction by replacing $d_X(x, x')$ with $d_X(x, x') + 2 \cdot d_Y(f(x), f(x'))$).

Consider first the point $y = \pi(K_i) \in Y$ for some $i \in \mathbb{N}$. For every $j \geq k_i$ the set $B_{i,j}$ is clopen $\mathcal{K}$-saturated and so, by Lemma 5, $\pi(B_{i,j})$ is clopen in Y. Due to continuity of π the intersection over j of these images is $\{y\}$, so we are done.

If y is not of the form $\pi(K_i)$ then $y = \pi(x)$ for a unique $x \notin \bigcup_i K_i$. By (3), $\{x\}$ is a degenerate component of X. For fixed ε let U be a clopen set of diameter smaller than $\varepsilon/3$ around x, disjoint from all sets K_i of diameters larger than $\varepsilon/6$ (use Lemma 6). Let V be the union of U and the sets B_{i,j_i} over such indices i that K_i intersects U, and where j_i is such that $\operatorname{diam}(B_{i,j_i}) < 2 \cdot \operatorname{diam}(K_i) \leq \varepsilon/3$. Clearly, V has diameter smaller than ε. V is obviously open, but also closed, because every convergent sequence of its elements is either eventually contained in U (then its limit is in U) or in some B_{i,j_i} (then its limit is in B_{i,j_i}), or it visits infinitely many of the sets B_{i,j_i} and then, by (2), it approaches U, so its limit again belongs to U. Finally, the set V is $\mathcal{K}$-saturated, because every K_i contained in or intersecting U is included in V, while any other set $K_{i'}$ intersecting V intersects some B_{i,j_i} which is $\mathcal{K}$-saturated by (6), so it contains $K_{i'}$ entirely. Thus, by Lemma 5, the image $\pi(V)$ is a clopen neighborhood of y in Y, and it has diameter smaller than ε (π is a contraction).

Finally suppose that y is an isolated point in Y. Then $\pi^{-1}(y)$ is open in X. If $\pi^{-1}(y)$ is a point, we have a contradiction with (4), otherwise a contradiction with (1). □

PROPOSITION 12. *A compact metric space X admits an exhausting sequence if and only if it is a cantoroid.*

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Proof. One implication is easy. Let X admit an exhausting sequence. By (1), $\bigcup_i K_i$ is of first category, so, by (3) and the Baire theorem, the degenerate components are dense, hence X is almost totally disconnected. Further, X has no isolated point: the points outside $\bigcup_i K_i$ are not isolated by (4) and any point in any K_i is not isolated by (1). Thus X is a cantoroid. (For this implication we have used only (1), (3) and (4).)

To prove the converse implication we will construct inductively the sets $B_{i,j}$ from condition (6). The pairs of indices (i, j) will form a staircase matrix $1 \le i \le n_j$, $j \in \mathbb{N}$ (see Figure 3). In the step $j+1$ of the induction we will define all the sets $B_{i,j+1}$ in the $(j+1)$st row, i.e. with i ranging from 1 to n_{j+1}. We will do this in such a way that the following hold.

- In each column the sets form a decreasing sequence.
- In each row the sets are mutually disjoint.
- Each of the sets is either disjoint from or contained in any of the sets in the previous rows.

Notice that, at the first coordinate, every index $i \in [n_j + 1, n_{j+1}]$ appears for the first time in the row $j+1$. The definition of $B_{i,j+1}$ for such indices i will be different from that for smaller i. In particular, the sets C_i from condition (5) for these indices i will be defined here. For each i we will denote by k_i the smallest index j in the ith column, i.e. the number of the construction step (row) in which the index i appears for the first time. After the induction is completed, each set K_i of the desired exhausting sequence is simply defined as the intersection over $j \ge k_i$ of the sets $B_{i,j}$. We will verify the properties (1)–(6) at the end.

Fix a decreasing-to-zero sequence ε_j $(j \in \mathbb{N})$. Also fix in X two dense, mutually disjoint, one-to-one sequences of degenerate components $\mathbf{x} = (x_j)$ and $\mathbf{z} = (z_j)$. Let D_j be the union of $\{z_1, \dots, z_j\}$ and of all components of X of diameters larger than or equal to ε_j. The sets D_j are increasing, closed and $\mathcal{C}$-saturated, and their union covers all non-degenerate components and the sequence $\mathbf{z}$ (so it is dense and notice that it is disjoint from $\mathbf{x}$).

We are going to describe step 1 of the construction (see Figure 4). By Lemma 7 find a clopen neighborhood W_1 of D_1 not containing x_1. Define $B_{1,1} = W_1$ and let $n_1 = k_1 = 1$. Inside $B_{1,1}$ fix a Cantor set C_1 consisting of degenerate components, disjoint from the entire sequences $\mathbf{x}$ and $\mathbf{z}$ (the union of all non-degenerate components and of these two sequences is of first category, so in the clopen set $B_{1,1}$ the rest is an uncountable G_δ-set, hence it contains a Cantor set). Notice that C_1 and D_1 are disjoint closed subsets of $B_{1,1}$. By Lemma 7 choose a decreasing sequence of clopen neighborhoods $V_{1,j'}$ $(j' \ge 1)$ of C_1 contained in $B_{1,1}$ disjoint from D_1 whose intersection over j' equals C_1. This concludes the step 1 of the induction.

In step 2 (see Figure 5), by Lemma 7 find a clopen set W_2 containing D_2 (hence also D_1) and C_1, but not containing x_1 nor x_2. Let $B_{1,2} = (B_{1,1} \cap W_2) \setminus (V_{1,1} \setminus V_{1,2})$. This clopen set covers C_1. Note that $B_{1,2}$ also covers a closed subset of D_2 containing D_1 (both $B_{1,1}$ and W_2 contain D_1, while $V_{1,1}$ is disjoint from D_1). Because $B_{1,2}$ is clopen, the uncovered (by it) part of D_2 is closed as well. Because all the uncovered components of D_2 are not in D_1, their diameters are smaller than ε_1. Thus, by Lemma 7, we can cover them by mutually disjoint clopen sets $B_{2,2}, B_{3,2}, \dots, B_{n_2,2}$ of diameters smaller than ε_1 which are contained

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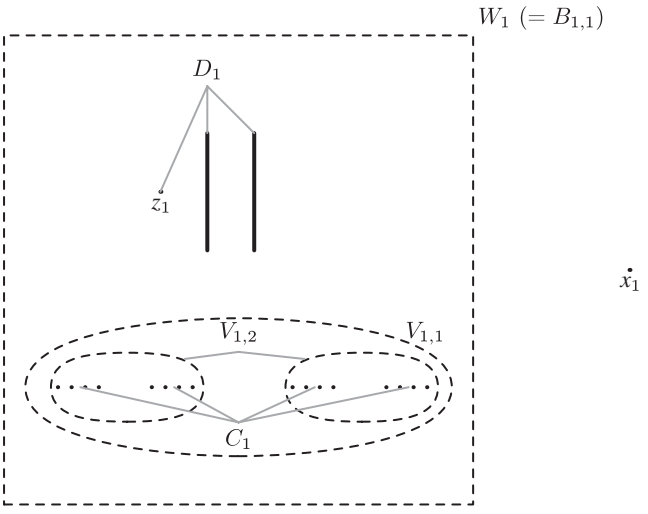


FIGURE 4. Step 1: $W_1 \supseteq D_1$, $W_1 \not\ni x_1$, $B_{1,1} = W_1$. Choose Cantor C_1 and clopen $V_{1,j'}$ ($j' \geq 1$).

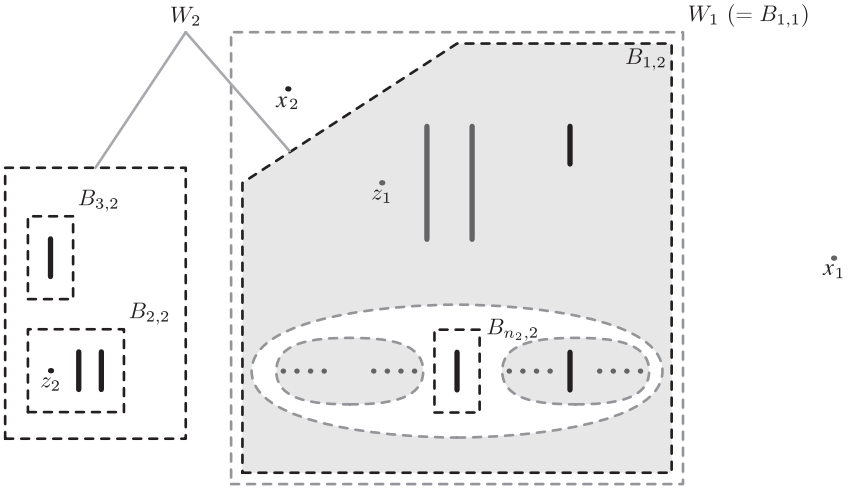


FIGURE 5. Step 2: $W_2 \supseteq D_2$, $W_2 \supseteq C_1$, $W_2 \not\ni x_1, x_2$, $B_{1,2} = (B_{1,1} \cap W_2) \setminus (V_{1,1} \setminus V_{1,2})$. Further, for $i \in [2, n_2]$, in $B_{i,2}$ choose Cantor C_i and clopen $V_{i,j'}$ ($j' \geq 2$) as in Step 1.

in W_2 and disjoint from $B_{1,2}$; moreover, we can suppose that each of these sets is either disjoint from or contained in $B_{1,1}$ (indeed, if necessary, we may intersect them with $B_{1,1}$ and with its complement). For every $i \in [2, n_2]$ let $k_i = 2$. By an analogous procedure as in step 1, for every such i find a Cantor set C_i inside $B_{i,2}$ consisting of degenerate components and disjoint from the sequences $\mathbf{x}$ and $\mathbf{z}$, and then fix a decreasing sequence of clopen neighborhoods $V_{i,j'}$ ($j' \geq 2$) of C_i, contained in $B_{i,2}$, disjoint from D_2 and whose intersection over j' is C_i. This concludes the step 2.

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And so on: in step $j+1$ we find a clopen set W_{j+1} containing D_{j+1} (hence also $D_1, D_2, \ldots, D_j$) and $C_1, C_2, \ldots, C_{n_j}$, but disjoint from $\{x_1, x_2, \ldots, x_j\}$. For $i \in [1, n_j]$ let $B_{i,j+1} = (B_{i,j} \cap W_{j+1}) \setminus (V_{i,k_i} \setminus V_{i,j+1})$; notice that this clopen set covers C_i. Further, the sets $B_{i,j+1}$ ($i \in [1, n_j]$) cover a closed part of D_{j+1} containing D_j. The uncovered part of D_{j+1} is also closed, and the uncovered components of D_{j+1} have diameters smaller than ε_j. Thus, by Lemma 7, we can cover them by mutually disjoint clopen sets $B_{n_j+1,j+1}, \ldots, B_{n_{j+1},j+1}$ of diameters smaller than ε_j which are contained in W_{j+1} and disjoint from the sets $B_{1,j+1}, \ldots, B_{n_j,j+1}$. Moreover, we can suppose that the sets $B_{n_j+1,j+1}, \ldots, B_{n_{j+1},j+1}$ are either disjoint from or contained in any of the sets $B_{i',j'}$ defined so far ($j' \le j$, $i' \le n_{j'}$). Let $k_i = j+1$ for each new index i ($i \in [n_j+1, n_{j+1}]$). Finally, for every such i, inside $B_{i,j+1}$ find a Cantor set C_i consisting of degenerate components and disjoint from the sequences $\mathbf{x}$ and $\mathbf{z}$, and then fix a decreasing sequence of clopen neighborhoods $V_{i,j'}$ ($j' \ge k_i$) of C_i, contained in $B_{i,j+1}$, disjoint from D_{j+1} and whose intersection is C_i. This concludes the step $j+1$.

After the induction is completed, define

$$K_i = \bigcap_{j \ge k_i} B_{i,j}.$$

Clearly, the sets K_i are mutually disjoint. It now remains to check the properties (1)–(6).

The property (1) follows from the facts that each K_i is closed and disjoint from the dense sequence $\mathbf{x}$. The property (2) is also evident as $\operatorname{diam}(B_{i,k_i})$ is smaller than ε_{k_i-1} for $i \ge 2$. For (3) and (4) it is sufficient to show that every component of every D_j is contained in some K_i (since $\bigcup_j D_j$ contains every non-degenerate component and the dense sequence $\mathbf{z}$). Note that each component F of D_j is covered by $B_{i,j}$ for some $i \in [1, n_j]$. If $k_i = j$ (i.e. if $i > n_{j-1}$), then, because V_{i,k_i} is disjoint from D_j, the component F remains covered by $B_{i,j'}$ for each $j' > j$, and thus it is contained in K_i. If $k_i < j$ then it may happen that F will be removed from $B_{i,j'}$ in some step $j' > j$. But in such case F, as a component of $D_{j'}$, will be immediately (in the same step j') included in some $B_{i',j'}$ with $k_{i'} = j'$. Hence F will be contained in $K_{i'}$ by the previous argument.

Clearly, each K_i contains the Cantor set C_i and nothing more within V_{i,k_i}, so (5) is satisfied. Finally, we check (6). By the construction, for every i the sets $B_{i,j}$ ($j \ge k_i$) form a decreasing sequence of clopen sets. It remains to show that every $B_{i,j}$ is $\mathcal{K}$-saturated. Fix some $B_{i,j}$ ($i \le n_j$) and $K_{i'}$. If $i' = i$ then $K_{i'} \subseteq B_{i,j}$. If $i' < i$ then $i' < n_j$ hence there exists $B_{i',j}$ and it contains $K_{i'}$. Since $B_{i',j}$ and $B_{i,j}$ are disjoint, $K_{i'}$ is disjoint from $B_{i,j}$. Finally, if $i' > i$ then $K_{i'} \subseteq B_{i',j'}$ for a suitable $j' > j$. In this case $B_{i',j'}$ is constructed later than $B_{i,j}$, hence it is either contained in or disjoint from this last set, and so is $K_{i'}$. This completes the proof of (6) and of Proposition 12. □

We end this section with a proposition, which will be used in §6 to simplify the proof of Proposition 22.

PROPOSITION 13. *Let (X, d) be a cantoroid and let $\mathcal{K}$ be its decomposition corresponding to an exhausting sequence $(K_i)_{i\in\mathbb{N}}$. Then there is a metric d' on X equivalent to d such that*

$$\sum_{i\in\mathbb{N}} \operatorname{diam}_{d'} K_i < \infty.$$

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The equivalent metric d' will be constructed using a uniformly convergent sequence of metrics equivalent to d. In the inductive step we will use the following.

LEMMA 14. *Let (X, d) be a compact metric space. Let $U, V_1, V_2, \ldots, V_k$ be pairwise disjoint non-empty clopen subsets of X covering X. Then for any $c > 0$ there is a metric d' equivalent to d such that:*

(a) $\operatorname{diam}_{d'} V_i \le c$ *for every* $i = 1, \ldots, k$;

(b) $d'|_{U\times U} = d|_{U\times U}$ *and* $d'|_{V_i\times V_i} \le d|_{V_i\times V_i}$ *for every* $i = 1, \ldots, k$;

(c) $\sup_{X\times X}|d' - d| \le 4 \cdot \max_{i=1,\ldots,k} \operatorname{diam}_d V_i$.

Proof. Without loss of generality we may assume that every V_i contains at least two points, so $\operatorname{diam}_d V_i > 0$ for every i. Put $\alpha = \min\{1, c/\operatorname{diam}_d V_1, \ldots, c/\operatorname{diam}_d V_k\}$ and for $i = 1, \ldots, k$ fix $y_i \in V_i$. Define $d' : X \times X \to [0, \infty)$ by

$$d'(x, y) = \begin{cases} d(x, y) & \text{if } x, y \in U, \\ \alpha \cdot d(x, y) & \text{if } x, y \in V_i \text{ for some } i, \\ d(x, y_i) + \alpha \cdot d(y_i, y) & \text{if } x \in U,\ y \in V_i \text{ for some } i, \\ \alpha \cdot d(x, y_i) + d(y_i, y) & \text{if } x \in V_i \text{ for some } i,\ y \in U, \\ \alpha \cdot d(x, y_i) + d(y_i, y_j) + \alpha \cdot d(y_j, y) & \text{if } x \in V_i,\ y \in V_j \text{ for some } i \ne j. \end{cases}$$

It is an easy but tedious exercise to show that d' is a metric.

Since U and all V_i are clopen sets in both (X, d) and (X, d') (indeed, $\operatorname{dist}_{d'}(U, V_i) \ge \operatorname{dist}_d(U, V_i)$ and $\operatorname{dist}_{d'}(V_i, V_j) \ge \operatorname{dist}_d(V_i, V_j)$) we easily get that d' is equivalent to d. Immediately from the definition of d' it follows that (a) and (b) hold. Finally, (c) follows from the following estimate:

$$|d'(x, y) - d(x, y)| \le \begin{cases} 0 & \text{if } x, y \in U, \\ (1 - \alpha) \cdot \operatorname{diam}_d V_i & \text{if } x, y \in V_i \text{ for some } i, \\ (1 + \alpha) \cdot \operatorname{diam}_d V_i & \text{if } x \in U,\ y \in V_i \text{ for some } i, \\ (1 + \alpha) \cdot \operatorname{diam}_d V_i & \text{if } x \in V_i \text{ for some } i,\ y \in U, \\ (1 + \alpha) \cdot (\operatorname{diam}_d V_i + \operatorname{diam}_d V_j) & \text{if } x \in V_i,\ y \in V_j \text{ for some } i \ne j. \end{cases}$$

□

Proof of Proposition 13. Fix a sequence of positive reals $(\alpha_k)_{k\in\mathbb{N}}$ with $\alpha_1 > \operatorname{diam}_d X$ and $\sum_k \alpha_k < \infty$. Put $n_0 = 0$ and fix an increasing sequence of integers $1 \le n_1 < n_2 < \cdots$ such that $\operatorname{diam}_d K_i < \alpha_k$ for every $n_{k-1} < i \le n_k$ and every $k \in \mathbb{N}$. Finally fix a sequence $(c_k)_{k\in\mathbb{N}}$ of positive reals satisfying $\sum_k (n_k - n_{k-1})c_k < \infty$.

We will define an auxiliary pseudo-metric d^* as the uniform limit of metrics $d_1, d_2, \ldots$ constructed inductively using Lemma 14. Put $d_1 = d$ and suppose that d_{k-1} is defined for some $k \ge 2$ in such a way that $\operatorname{diam}_{d_{k-1}} K_i \le \operatorname{diam}_d K_i$ for every $i \in \mathbb{N}$. We are going to define d_k. Using Lemma 7 we can write (X, d_{k-1}) as a disjoint union of non-empty clopen $\mathcal{K}$-saturated sets

$$X = U^k \cup V^k_{n_{k-1}+1} \cup \cdots \cup V^k_{n_k}$$

in such a way that U^k contains all K_i with $i \le n_{k-1}$ and for $i = n_{k-1} + 1, \ldots, n_k$ the set V^k_i contains K_i and d_{k-1}-diameter of V^k_i is less than α_k. Let d_k be a metric obtained by Lemma 14 for this decomposition and for $c = c_k$. Hence $\operatorname{diam}_{d_k} K_i \le \operatorname{diam}_{d_k} V^k_i \le c_k$ for $i = n_{k-1} + 1, \ldots, n_k$ and $d_k|_{K_i\times K_i} = d_{k-1}|_{K_i\times K_i}$ for every $i \le n_{k-1}$. Note that

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$\operatorname{diam}_{d_k} K_i \le \operatorname{diam}_{d_{k-1}} K_i$ for every $i \in \mathbb{N}$. Hence again $\operatorname{diam}_{d_k} K_i \le \operatorname{diam}_{d} K_i$ for every $i \in \mathbb{N}$ and the induction step is finished.

In this way we obtain a sequence of metrics $d_1, d_2, \ldots$ equivalent to d such that $\sup_{X\times X}|d_k - d_{k-1}| \le 4\alpha_k$ for every $k \ge 2$. Hence we can define $d^* : X \times X \to [0, \infty)$, $d^* = \lim_{k\to\infty} d_k$. This is trivially a pseudo-metric on X. Moreover, we have

$$0 < \operatorname{diam}_{d^*} K_i \le c_k \quad \text{for any } n_{k-1} < i \le n_k.$$

Hence immediately $\sum_i \operatorname{diam}_{d^*} K_i \le \sum_k (n_k - n_{k-1})c_k < \infty$.

Now define $d' : X \times X \to [0, \infty)$ by

$$d'(x, y) = d^*(x, y) + \rho(\pi x, \pi y),$$

where ρ is a metric on $X/\mathcal{K}$ and $\pi : X \to X/\mathcal{K}$ is the natural projection. This is obviously a pseudo-metric on X. To show that it is a metric assume that $d'(x, y) = 0$. Then $d^*(x, y) = 0$ and $\pi x = \pi y$, i.e. x and y belong to the same element of $\mathcal{K}$. If this element is degenerate we have $x = y$. Otherwise $x, y \in K_i$ for some i. For any fixed integer k satisfying $n_k \ge i$ we then have $d_k(x, y) = d_{k+1}(x, y) = \cdots = d^*(x, y) = 0$. Here d_k is a metric and so $x = y$.

Since $\operatorname{diam}_{d'} K_i = \operatorname{diam}_{d^*} K_i$ for every i, we have $\sum_i \operatorname{diam}_{d'} K_i < \infty$. It remains to show that d' is equivalent to d. The identity map from (X, d) to (X, d') is continuous as a uniform limit of continuous maps, hence d' is weaker than d. Due to compactness of (X, d) these two metrics are then equivalent. □

5. *Continuous surjections on cantoroids*

In this section we prove that given two cantoroids X, X' and their exhausting sequences $\mathcal{K}$, $\mathcal{K}'$ one can find a map from X onto X' which induces a homeomorphism of corresponding factor spaces $X/\mathcal{K}$ and $X'/\mathcal{K}'$.

LEMMA 15. *Let X be a cantoroid and let $\mathcal{K}$ be a decomposition corresponding to an exhausting sequence in X. Let $A \ne \emptyset$ be a nowhere-dense closed $\mathcal{K}$-saturated subset of X. Then there is an infinite decomposition $(U_n)_{n\in\mathbb{N}}$ of $X \setminus A$ into non-empty clopen $\mathcal{K}$-saturated sets such that* $\operatorname{diam} U_n \to 0$ *and* $\operatorname{dist}(U_n, A) \to 0$. *Moreover, the diameter of each U_n can be made smaller than twice the largest diameter of $K \in \mathcal{K}$, $K \subseteq X \setminus A$.*

Proof. Put $V_0 = X$. By Lemma 7 for each $i \in \mathbb{N}$ we can find a clopen $\mathcal{K}$-saturated neighborhood V_i of A which is contained in the $1/i$-neighborhood of A. We may assume that $V_i \supseteq V_{i+1}$, $i \in \mathbb{N}$. Put $W_i = V_{i-1} \setminus V_i$ $(i \in \mathbb{N})$. These sets are clopen $\mathcal{K}$-saturated. Without loss of generality we can suppose that all W_i are non-empty. (This follows from the fact that A, being nowhere dense, is not clopen; hence the sequence $V_0, V_1, \ldots$ cannot be eventually constant.) If d_i is the supremum of the diameters of all $K \in \mathcal{K}$, $K \subseteq W_i$, then $d_i \to 0$ by (2).

Fix $i \in \mathbb{N}$ and apply Lemma 7 to the space W_i, its decomposition $\mathcal{K}_i := \mathcal{K} \cap W_i$, closed $\mathcal{K}_i$-saturated set W_i and $\varepsilon = 2d_i$. We obtain a finite decomposition of W_i into clopen $\mathcal{K}_i$-saturated sets with diameters smaller than $2d_i$. Having done this for every $i \in \mathbb{N}$ we get the desired decomposition $(U_n)_{n\in\mathbb{N}}$ of $X \setminus A$. □

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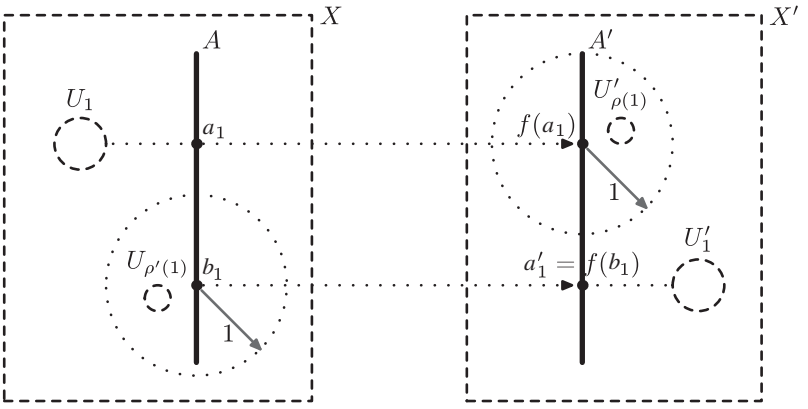


FIGURE 6. Construction of $\varrho(1)$, $\varrho'(1)$.

Let us remark that if U is any open set which meets the set A from Lemma 15 then $U_n \subseteq U$ for infinitely many n. (To see this, let $a \in U \cap A$ and $(x_n)_n$ be a sequence in $U \setminus A$ which converges to a (it exists because U is open and A is nowhere dense). Let $x_k \in U_{n_k}$. By passing to a subsequence we can assume that the n_k are distinct, because the U_n are closed and disjoint from A. For sufficiently large k, $U_{n_k} \subseteq U$ since $x_k \to a$ and $\operatorname{diam} U_{n_k} \to 0$.) This simple observation will be used in the proof of the following lemma.

LEMMA 16. *Let X and X' be cantoroids and let $\mathcal{K}$ and $\mathcal{K}'$ be decompositions corresponding to exhausting sequences in X and X', respectively. Let A (A') be a non-empty nowhere-dense closed $\mathcal{K}$-saturated ($\mathcal{K}'$-saturated) subset of X (X'). Let $(U_n)_{n\in\mathbb{N}}$ and $(U'_m)_{m\in\mathbb{N}}$ be as in the preceding lemma (for X and X', respectively). Then for any continuous surjective map $f : A \to A'$ there is a permutation $\varrho = \varrho_f$ of $\mathbb{N}$ such that every extension $F : X \to X'$ of f satisfying*

$$F(U_n) \subseteq U'_{\varrho(n)} \quad \textit{for each } n \in \mathbb{N}$$

is continuous at every point of A.

Proof. For each n, m choose $a_n \in A$, $a'_m \in A'$ such that

$$\operatorname{dist}(a_n, U_n) = \operatorname{dist}(A, U_n), \quad \operatorname{dist}(a'_m, U'_m) = \operatorname{dist}(A', U'_m).$$

By Lemma 15, $d_n := \operatorname{dist}(a_n, U_n) \to 0$ and $d'_m := \operatorname{dist}(a'_m, U'_m) \to 0$.

Fix a continuous surjective map $f : A \to A'$. In the following we will inductively construct the required permutation ϱ of $\mathbb{N}$ and its inverse $\varrho' = \varrho^{-1}$. Let $\varrho(1)$ be such that $U'_{\varrho(1)} \subseteq B(f(a_1), 1)$. If $\varrho(1) = 1$, put $\varrho'(1) = 1$; otherwise choose any $b_1 \in f^{-1}(a'_1)$ and let $\varrho'(1) > 1$ be such that $U_{\varrho'(1)} \subseteq B(b_1, 1)$ (see Figure 6).

Fix an integer $k \geq 2$ and let us suppose that we have defined $\varrho(1), \ldots, \varrho(k-1)$, $\varrho'(1), \ldots, \varrho'(k-1)$; we want to define $\varrho(k)$, $\varrho'(k)$. If k is equal to some $\varrho'(j)$ with $j < k$, put $\varrho(k) = j$; otherwise let $\varrho(k) \geq k$ be different from $\varrho(1), \ldots, \varrho(k-1)$ and such that $U'_{\varrho(k)} \subseteq B(f(a_k), 1/k)$. Analogously, if k is equal to some $\varrho(i)$ with $i \leq k$,

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put $\varrho'(k) = i$; otherwise choose any $b_k \in f^{-1}(a'_k)$ and let $\varrho'(k) > k$ be different from $\varrho'(1), \dots, \varrho'(k-1)$ and such that $U_{\varrho'(k)} \subseteq B(b_k, 1/k)$.

The map ϱ is a permutation of $\mathbb{N}$ since both ϱ and ϱ' are defined on the whole $\mathbb{N}$ and ϱ' is the inverse of ϱ. We claim that ϱ satisfies the condition from the lemma. To see this, fix any extension $F : X \to X'$ of f such that $F(U_n) \subseteq U'_{\varrho(n)}$ for every n. To prove the continuity of F at every point of A, take any $a \in A$ and any sequence $(x_i)_{i\in\mathbb{N}}$ from X converging to a. Since f is continuous on A we may assume that none of the points x_i belongs to A; hence $x_i \in U_{n_i}$ for some $n_i \in \mathbb{N}$. Since $\operatorname{dist}(x_i, A) \to 0$ we have $n_i \to \infty$ and then $\varrho(n_i) \to \infty$. Further, $d_X(a_{n_i}, a) \le d_X(a_{n_i}, x_i) + d_X(x_i, a) \le d_{n_i} + \operatorname{diam} U_{n_i} + d_X(x_i, a)$ and so we have $a_{n_i} \to a$ for $i \to \infty$.

From the construction we have that for every i

$$U'_{\varrho(n_i)} \subseteq B(f(a_{n_i}), 1/n_i) \quad \text{or} \quad U_{n_i} \subseteq B(b_{\varrho(n_i)}, 1/\varrho(n_i)).$$

Let $\mathbb{N}_1$ or $\mathbb{N}_2$ denote the set of all i for which the first or the second of these two cases happens, respectively. Notice that $\mathbb{N}_1$ and $\mathbb{N}_2$ may not be disjoint but their union is $\mathbb{N}$. Denote by d_X and $d_{X'}$ the metrics in X and X', respectively.

Assume that $\mathbb{N}_1$ is infinite. If $i \in \mathbb{N}_1$ then $d_{X'}(F(x_i), f(a_{n_i})) < 1/n_i$. Hence

$$\lim_{i\in\mathbb{N}_1} F(x_i) = \lim_{i\in\mathbb{N}_1} f(a_{n_i}) = f(a).$$

Assume that $\mathbb{N}_2$ is infinite. For $i \in \mathbb{N}_2$ we have $d_X(b_{\varrho(n_i)}, a) \le d_X(b_{\varrho(n_i)}, x_i) + d_X(x_i, a) < 1/\varrho(n_i) + d_X(x_i, a)$. Thus $\lim_{i\in\mathbb{N}_2} b_{\varrho(n_i)} = a$. Further, $F(x_i) \in U'_{\varrho(n_i)}$ whence $d_{X'}(F(x_i), a'_{\varrho(n_i)}) \le \operatorname{diam} U'_{\varrho(n_i)} + d'_{\varrho(n_i)}$. So we get

$$\lim_{i\in\mathbb{N}_2} F(x_i) = \lim_{i\in\mathbb{N}_2} a'_{\varrho(n_i)} = \lim_{i\in\mathbb{N}_2} f(b_{\varrho(n_i)}) = f(a).$$

We conclude that $\lim_{i\to\infty} F(x_i) = f(a) = F(a)$ and the continuity of F at a is proved. □

In what follows a (not necessarily continuous) map $F : X \to Y$ is said to be a *C-map* if it collapses each connected component of X to a point or, equivalently, if it factors through the projection $\pi_{\mathcal{C}} : X \to X/\mathcal{C}$ (where $\mathcal{C}$ is the system of all connected components of X).

PROPOSITION 17. *Let X and X' be cantoroids and let $\mathcal{K}$ and $\mathcal{K}'$ be decompositions corresponding to exhausting sequences in X and X', respectively. Then there is a continuous surjective C-map $F : X \to X'$ which induces a homeomorphism $h : X/\mathcal{K} \to X'/\mathcal{K}'$.*

More precisely, if we put $\pi(x) = [x]_{\mathcal{K}}$ and $\pi'(x') = [x']_{\mathcal{K}'}$, the proposition says that there are a continuous surjective C-map $F : X \to X'$ and a homeomorphism $h : X/\mathcal{K} \to X'/\mathcal{K}'$ such that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\quad F \quad} & X' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ X/\mathcal{K} & \xrightarrow{\quad h \quad} & X'/\mathcal{K}' \end{array}$$

Obviously h is defined by $h([x]_{\mathcal{K}}) = [F(x)]_{\mathcal{K}'}$.

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Proof. We are going to define F in infinitely many steps.

Step 0 of the construction. Let A, A' be sets from $\mathcal{K}$, $\mathcal{K}'$ with maximal diameters (if there are several such members we fix one). Take any continuous surjective C-map $f : A \to A'$ (such an f exists because A contains a Cantor set as its clopen subset—see Definition 10(5)).

Step 1 of the construction. Let $(U_n)_n$, $(U'_n)_n$ be clopen $\mathcal{K}$, $\mathcal{K}'$-saturated decompositions of $X \setminus A$, $X' \setminus A'$ from Lemma 15. Lemma 16 gives a permutation ϱ of $\mathbb{N}$ such that every extension $X \to X'$ of f sending U_n into $U'_{\varrho(n)}$ is continuous on A. By renumbering $(U'_n)_n$ we may assume that ϱ is the identity (note that the conditions $\operatorname{diam} U'_n \to 0$ and $\operatorname{dist}(U'_n, A') \to 0$ are not affected). Fix any $n \in \mathbb{N}$. Denote by A_n (A'_n) a member of $\mathcal{K} \cap U_n$ ($\mathcal{K}' \cap U'_n$) with maximal diameter. Let f_n be a continuous surjective C-map $A_n \to A'_n$.

Step 2 of the construction. Recall that Step 1 was applied to 7-tuple $(X, X', \mathcal{K}, \mathcal{K}', A, A', f)$ and we obtained infinitely many 7-tuples $(U_n, U'_n, \mathcal{K} \cap U_n, \mathcal{K}' \cap U'_n, A_n, A'_n, f_n)$, $n \in \mathbb{N}$. Now we apply Step 1 to each of these 7-tuples. For every n we obtain infinitely many 7-tuples $(U_{n,m}, U'_{n,m}, \mathcal{K} \cap U_{n,m}, \mathcal{K}' \cap U'_{n,m}, A_{n,m}, A'_{n,m}, f_{n,m})$, $m \in \mathbb{N}$.

Step k of the construction. If for $k \geq 3$ we have already defined all 7-tuples

$$(U_{n_1,\dots,n_{k-1}}, U'_{n_1,\dots,n_{k-1}}, \mathcal{K} \cap U_{n_1,\dots,n_{k-1}}, \mathcal{K}' \cap U'_{n_1,\dots,n_{k-1}}, A_{n_1,\dots,n_{k-1}}, A'_{n_1,\dots,n_{k-1}}, f_{n_1,\dots,n_{k-1}})$$

($n_1, \dots, n_{k-1} \in \mathbb{N}$), by applying Step 1 to each of them we obtain all 7-tuples

$$(U_{n_1,\dots,n_k}, U'_{n_1,\dots,n_k}, \mathcal{K} \cap U_{n_1,\dots,n_k}, \mathcal{K}' \cap U'_{n_1,\dots,n_k}, A_{n_1,\dots,n_k}, A'_{n_1,\dots,n_k}, f_{n_1,\dots,n_k})$$

($n_1, \dots, n_k \in \mathbb{N}$). Notice that for every sequence $(n_k)_k$ the diameters of $U_{n_1,\dots,n_k}$, $U'_{n_1,\dots,n_k}$ tend to zero. (This follows from the facts that $\mathcal{K}$, $\mathcal{K}'$ are null and from the last statement of Lemma 15.)

We are ready to define F. For $x \in A$ put $F(x) = f(x)$ and for $x \in A_{n_1,\dots,n_k}$ put $F(x) = f_{n_1,\dots,n_k}(x)$ whenever k and $n_1, \dots, n_k$ are positive integers. For every $x \notin B$, where B is the union of A and all $A_{n_1,\dots,n_k}$, there is a unique sequence $(n_k)_k$ of integers such that $x \in U_{n_1,\dots,n_k}$. This enables us to define $F(x)$ for such an x as the unique point belonging to $\bigcap_k U'_{n_1,\dots,n_k}$. Obviously F is a surjective C-map.

Notice that the map F is continuous on A due to Lemma 16 and analogously F is continuous on every $A_{n_1,\dots,n_k}$. Fix $x \notin B$ and take the unique sequence $(n_k)_k$ with $x \in \bigcap_k U_{n_1,\dots,n_k}$. The continuity of F at x follows from the facts that $F(U_{n_1,\dots,n_k}) \subseteq U'_{n_1,\dots,n_k}$ for every k and that the diameters of $U'_{n_1,\dots,n_k}$ tend to 0 for $k \to \infty$.

From the construction of F one can see that for every non-degenerate $K \in \mathcal{K}$ there exists a non-degenerate $K' \in \mathcal{K}'$ with $F(K) = K'$ and, conversely, for every non-degenerate $K' \in \mathcal{K}'$ there exists a unique non-degenerate $K \in \mathcal{K}$ with $F(K) = K'$. In this way F is a bijection between non-degenerate members of $\mathcal{K}$ and non-degenerate members of $\mathcal{K}'$. Similarly one can see that F is also a bijection between degenerate members of $\mathcal{K}$ and degenerate members of $\mathcal{K}'$. Thus we can naturally define a bijection $h : X/\mathcal{K} \to X'/\mathcal{K}'$ sending every $[x]_{\mathcal{K}}$ to $[F(x)]_{\mathcal{K}'}$. The fact that h is a homeomorphism is now an immediate consequence of the fact that F, being continuous and closed, is a quotient map. □

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6. *Construction of minimal systems on cantoroids*

We are going to prove Theorem 24 which will have important consequences, namely Theorems A, B and C.

Let $\pi : X \to Y$ be a continuous map of compact metric spaces. Let Y_1 denote the set of points $y \in Y$ whose fibers $\pi^{-1}(y)$ are singletons. Let $X_1 = \pi^{-1}(Y_1) = \{x \in X : \pi^{-1}(\pi(x)) = x\}$. Note that Y_1 is a G_δ-set since $Y_1 = \bigcap_{n=1}^{\infty}\{y \in Y : \operatorname{diam} \pi^{-1}(y) < 1/n\}$ and the sets in the intersection are open due to compactness of X. So, in the case when Y_1 is dense, it is automatically residual. The same is true for X_1 since this set, being a continuous preimage of Y_1, is also G_δ.

A continuous map $\pi : X \to Y$ with X_1 dense is usually called *almost one-to-one*, see e.g. [**AG01, BOT06**]. However, some other authors call π almost one-to-one if Y_1 is dense, see e.g. [**Dow05**]. These two definitions are not equivalent. Therefore we do not use the notion of an almost one-to-one map in this paper. Instead, we introduce the following definition.

Definition 18. A continuous map $\pi : X \to Y$ between compact metric spaces is called *strongly almost one-to-one* if it is surjective and X_1 is dense in X.

Note that a continuous map $\pi : X \to Y$ between compact metric spaces is strongly almost one-to-one if and only if it is irreducible (see e.g. [**AG01**, Lemma 1.1(c)]). Of course, for such a map also the set Y_1 is dense in Y.

Recall that given two dynamical systems, (X, Φ) and (Y, φ), a continuous surjection $\pi : X \to Y$ is called a *factor map* if $\pi \circ \Phi = \varphi \circ \pi$. In such a case (X, Φ) is called an *extension* of (Y, φ). If, moreover, π is strongly almost one-to-one, then the system (X, Φ) is said to be a *strongly almost one-to-one extension* of (Y, φ).

Note that if $\pi : (X, \Phi) \to (Y, \varphi)$ is a factor map and the system (Y, φ) is minimal then the assumption that Y_1 is dense in Y is not sufficient for (X, Φ) to be minimal. Nevertheless, the density of X_1 in X is sufficient (this is trivial if we use the fact that density of X_1 is equivalent with irreducibility of π). For completeness we wish to give a complete proof of the following lemma in the language of strongly almost one-to-one extensions.

LEMMA 19. *Let (Y, φ) be a minimal system and let $\pi : (X, \Phi) \to (Y, \varphi)$ be a factor map with Y_1 dense in Y. Then (X, Φ) is minimal if and only if π is strongly almost one-to-one.*

Proof. Let (X, Φ) be minimal. The set Y_1 of points with singleton preimages is residual. Because the map φ is minimal, it preserves residuality by preimage (see [**KST01**, Theorem 2.5]). So, the preimages of Y_1 under powers of φ are all residual and thus the set $W = \bigcap_{n=0}^{\infty} \varphi^{-n}(Y_1)$ is non-empty. Take a point $x \in \pi^{-1}(W)$. The orbit of this point lies entirely inside X_1. Since Φ is minimal, the set X_1 is dense in X.

Conversely, let $\pi : (X, \Phi) \to (Y, \varphi)$ be a strongly almost one-to-one factor map. Fix $x \in X$. Choose arbitrary $x_1 \in X_1$ and put $y = \pi(x)$, $y_1 = \pi(x_1)$. By minimality of φ, y_1 is a limit point of the trajectory of y. By compactness of X there exists a point z in the fiber $\pi^{-1}(y_1)$ which is a limit point of the trajectory of x. Since $\pi^{-1}(y_1) = \{x_1\}$ we have $z = x_1$. Hence the ω-limit set of x contains the whole set X_1. Now minimality of Φ follows from the fact that X_1 is dense in X. □

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This lemma together with some simple arguments yields the following observation on (in general non-invertible) dynamical systems on cantoroids and Cantor sets.

PROPOSITION 20. *Let (X, Φ) be a (minimal) dynamical system. Then X is a cantoroid if and only if (X, Φ) is a strongly almost one-to-one extension of a Cantor (minimal) system (C, φ).*

Proof. First we prove the version without minimality. A system on a cantoroid X yields, by collapsing components to points, a system on a Cantor set. This Cantor system is obviously a strongly almost one-to-one factor of the system on X since degenerate components are dense in X. Conversely, let a system (X, Φ) be a strongly almost one-to-one extension of a Cantor system (C, φ). Denote the corresponding factor map by π. We claim that X is a cantoroid. Indeed, the system X_1 of degenerate fibers is dense in X by definition. Since C is totally disconnected, each of these degenerate fibers is a degenerate component of X. Hence X is almost totally disconnected. To see that X has no isolated point suppose on the contrary that x is isolated in X. Since X_1 is dense in X, this point x belongs to X_1, i.e. $\{x\}$ is a degenerate fiber. Take a sequence $(y_n)_n$ in C of points different from $\pi(x)$ and converging to $\pi(x)$. Choose points $x_n \in \pi^{-1}(y_n)$. By compactness of X a subsequence of $(x_n)_n$ converges to a point in $\pi^{-1}(\pi(x)) = \{x\}$. Hence x is not isolated, a contradiction.

To prove the version with minimality, use the version without minimality, Lemma 19 and the fact that a factor of a minimal system is a minimal system. □

We are going to prove Lemma 21 mentioned in the outline in §2. To formulate it we first introduce some notation.

Let X be a compact metric space with a metric d_X. Let $B(X)$ be the space of all (not necessarily continuous) self-maps of X with the supremum metric $d_{B(X)}(f, g) = \sup_{x\in X} d_X(f(x), g(x))$. Further, denote by $H(X)$ the set of all homeomorphisms of X endowed with the metric

$$d_{H(X)}(\varphi, \psi) = \max\{d_{B(X)}(\varphi, \psi), d_{B(X)}(\varphi^{-1}, \psi^{-1})\}.$$

Recall that both $B(X)$ and $H(X)$ are complete metric spaces. If a sequence (φ_n) converges to φ in the space $H(X)$ we will write $\varphi = \mathrm{Lim}_{n\to\infty} \varphi_n$. The symbol $f = \lim_{n\to\infty} f_n$ will mean that (f_n) converges to f in $B(X)$.

Denote by $B_C(X)$ the set of all (not necessarily continuous) C-maps from X to X. Obviously this is a closed subspace of $B(X)$.

Till the end of this section let $\mathcal{K}$ be the decomposition corresponding to a fixed exhausting sequence $\{K_i \mid i \in \mathbb{N}\}$ in X, let $Y = X/\mathcal{K}$, let $\pi : X \to Y$ be the natural projection and for every $i \in \mathbb{N}$ put $y_i = \pi(K_i) \in Y$ (note that we identify a singleton set with its only point). Let $S : Y \to X$ be a (usually discontinuous) selector of the set-valued map π^{-1}. That is, each $S(y_i)$ is a point chosen in K_i and for the other $y \in Y$ (i.e. when $\pi^{-1}(y)$ is a singleton) $S(y)$ is the unique point from $\pi^{-1}(y)$.

Let $\Phi \in B_C(X)$, $\varphi \in H(Y)$ and let $k \geq 0$. We say that the couple (Φ, φ) is *k-admissible* provided the following conditions hold.

(A) $\pi \circ \Phi = \varphi \circ \pi$.

(B) φ has no periodic point.

(C) Φ is continuous at every point of $\Phi^{-1}(K_1 \cup \cdots \cup K_k)$.

(Note that for $k = 0$ the condition (C) is trivially satisfied by every Φ.)

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LEMMA 21. *Let $k \geq 1$ and let (Φ, φ) be $(k-1)$-admissible. Let $\varepsilon > \max_{i \geq k} \operatorname{diam} K_i$ and let $\delta > 0$. Then there are a homeomorphism $h \in H(Y)$ and a map $\Psi \in B_C(X)$ such that the following hold.*

(i) *(Ψ, ψ) is k-admissible, where $\psi = h^{-1} \circ \varphi \circ h$.*

(ii) *$\Psi(x) = \Phi(x)$ for every $x \in \Phi^{-1}(K_1 \cup \cdots \cup K_{k-1})$.*

(iii) *$d_{H(Y)}(h, \mathrm{id}_Y) < \delta$, $d_{H(Y)}(\psi, \varphi) < \delta$, $d_{B(X)}(\Psi, \Phi) < \varepsilon$.*

We will say that the triple (h, ψ, Ψ) corresponds *to the 5-tuple $(\varphi, \Phi, k, \varepsilon, \delta)$.*

Proof. It is easy to show that there is $0 < \eta < \delta$ such that $d_{H(Y)}(h^{-1} \circ \varphi \circ h, \varphi) < \delta$ whenever $h \in H(Y)$ is such that $d_{H(Y)}(h, \mathrm{id}_Y) < \eta$.

Let $l \geq 1$ be an integer such that

$$\varphi^{-l}(y_k) \notin \{y_1, \ldots, y_k\}.$$

Trivially such $l \leq k$ exists since φ has no periodic point.

For any set $V \subseteq Y$ and for any integer $i \geq 0$ put $V^{-i} = \varphi^{-i}(V)$ and $U_V^{-i} = \pi^{-1}(V^{-i})$. Using the choice of l and the facts that φ is continuous, has no periodic point and $\varepsilon > \max_{i \geq k} \operatorname{diam} K_i$, we can find a subset V of Y satisfying the following conditions (we write U^{-i} instead of U_V^{-i} and U instead of U_V^{-0}).

(a) V is a clopen neighborhood of y_k.

(b) $V, V^{-1}, \ldots, V^{-(l+1)}$ are pairwise disjoint.

(c) V and V^{-l} are disjoint from $\{y_1, \ldots, y_{k-1}\}$.

(d) $\operatorname{diam} V^{-i} < \eta$ for every $i = 0, 1, \ldots, l+1$.

(e) $\operatorname{diam} U < \varepsilon$ and $\operatorname{diam} U^{-l} < \varepsilon$.

By Proposition 17 applied to the cantoroids U^{-1} and U with exhausting sequences inherited from $\mathcal{K}$ there are a continuous surjection $F : U^{-1} \to U$ and a homeomorphism $f : V^{-1} \to V$ such that $\pi|_U \circ F = f \circ \pi|_{U^{-1}}$. Define $h : Y \to Y$ by

$$h(y) = \begin{cases} \varphi^{-i} \circ f \circ \varphi^{i-1}(y) & \text{if } y \in V^{-i} \text{ for } i = 1, 2, \ldots, l, \\ y & \text{otherwise.} \end{cases}$$

Notice that h maps homeomorphically the clopen set V^{-i} onto itself for every $i = 1, \ldots, l$. So h is a homeomorphism of Y and its inverse is given by

$$h^{-1}(y) = \begin{cases} \varphi^{-(i-1)} \circ f^{-1} \circ \varphi^{i}(y) & \text{if } y \in V^{-i} \text{ for } i = 1, 2, \ldots, l, \\ y & \text{otherwise.} \end{cases}$$

Put $\psi = h^{-1} \circ \varphi \circ h$. This is a homeomorphism of Y conjugate to φ. Moreover, ψ may differ from φ only on V^{-1}, where it coincides with f, and on $V^{-(l+1)}$. Nevertheless, still $\psi(V^{-1}) = \varphi(V^{-1}) = V$ and $\psi(V^{-(l+1)}) = \varphi(V^{-(l+1)}) = V^{-l}$. In fact, a tedious but easy calculation which uses only formulas for h and h^{-1}, the definition of V^{-i} and the property (b) above, gives

$$\psi(y) = \begin{cases} f(y) & \text{if } x \in V^{-1}, \\ \varphi^{-(l-1)} \circ f^{-1} \circ \varphi^{l+1}(y) & \text{if } x \in V^{-(l+1)}, \\ \varphi(y) & \text{otherwise.} \end{cases}$$

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Consider the map $\Psi : X \to X$ satisfying the following.

- $\Psi(x) = F(x)$ for $x \in U^{-1}$.
- $\Psi(x) = S \circ \psi \circ \pi(x)$ for $x \in U^{-(l+1)}$.
- $\Psi(x) = \Phi(x)$ for $x \in X \setminus (U^{-1} \cup U^{-(l+1)})$.

We show that h, ψ and Ψ satisfy the conditions from the lemma. Obviously Ψ defined by the above three conditions is a C-map since F and Φ are C-maps and for a component lying in $U^{-(l+1)}$ already its π-image is a singleton. Further, (ii) follows from the fact that $V^{-1} \cup V^{-(l+1)}$ contains no point from $\varphi^{-1}\{y_1, \ldots, y_{k-1}\}$ by (c). The first two inequalities from (iii) follow from (d) and the choice of η; the last one follows from (e) and the fact that $\Psi = \Phi$ outside $U^{-1} \cup U^{-(l+1)}$.

To finish the proof it suffices to show (i). The conditions (A) and (B) of k-admissibility of (Ψ, ψ) are trivially satisfied. By (c), the set $X \setminus (U^{-1} \cup U^{-(l+1)})$ is a neighborhood of $\Phi^{-1}(K_1 \cup \cdots \cup K_{k-1})$. Since $\Psi(x) = \Phi(x)$ on this neighborhood, the condition (C) from the definition of $(k-1)$-admissibility of (Φ, φ) gives that Ψ is continuous on $\Phi^{-1}(K_1 \cup \cdots \cup K_{k-1})$. Moreover, $\Phi^{-1}(K_1 \cup \cdots \cup K_{k-1}) = \pi^{-1} \circ \varphi^{-1}\{y_1, \ldots, y_k\}$. Similarly, $\Psi^{-1}(K_1 \cup \cdots \cup K_{k-1}) = \pi^{-1} \circ \psi^{-1}\{y_1, \ldots, y_k\}$. Since φ^{-1} and ψ^{-1} are equal outside $V \cup V^{-l}$, from the condition (c) we get that $\Phi^{-1}(K_1 \cup \cdots \cup K_{k-1}) = \Psi^{-1}(K_1 \cup \cdots \cup K_{k-1})$. Thus to prove k-admissibility of (Ψ, ψ) it remains to show that Ψ is continuous on $\Psi^{-1}(K_k)$. This is obvious because $\Psi(x) = F(x)$ on the neighborhood U^{-1} of $\Psi^{-1}(K_k) = \pi^{-1} \circ \psi^{-1}(y_k)$ (note that $y_k \in V$) and F is continuous. □

Now we are ready to prove the following proposition, which plays the key role in our paper.

PROPOSITION 22. *Let X be a cantoroid and let φ be any Cantor homeomorphism with no periodic point. Then there is a continuous surjective C-map of X which is a (strongly) almost one-to-one extension of φ.*

Proof. Using Proposition 13 we can assume that $\sum_i \operatorname{diam} K_i < \infty$; by relabeling the sets K_i we can also assume that $(\operatorname{diam} K_i)_{i \in \mathbb{N}}$ is a non-increasing sequence. Hence $\max_{i \geq k} \operatorname{diam} K_i = \operatorname{diam} K_k$ for every k and obviously there is a sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ of positive reals such that $\operatorname{diam} K_k < \varepsilon_k$ for every k and still $\sum_k \varepsilon_k < \infty$.

When proving the proposition we are in the 'starting position' described by the left-hand side diagram in Figure 2 and we wish to get the 'final position' described by the right-hand side diagram in Figure 2. The required continuous surjection $\Psi : X \to X$ and the homeomorphism $H : Y \to Y$ will be obtained as the uniform limit of a sequence of (not necessarily continuous and not necessarily surjective) maps $\Phi_0, \Phi_1, \ldots$ and, respectively, as the uniform limit of homeomorphisms $H_0, H_1, \ldots$. These sequences will be constructed by induction based on Lemma 21. The main trick will be that in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\Phi_k} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow{\varphi_k = H_k^{-1} \circ \varphi \circ H_k} & Y \end{array}$$

the pair (Φ_k, φ_k) will be k-admissible for every $k = 0, 1, \ldots$.

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For $k = 0$ the construction of such a diagram is immediate; we put $H_0 = \mathrm{id}_Y$ and so $\varphi_0 = \varphi$ and fix any C-map $\Phi_0 : X \to X$ such that $\pi \circ \Phi_0 = \varphi_0 \circ \pi$. Note that (Φ_0, φ_0) is 0-admissible.

By Lemma 21 there is a triple (h_1, φ_1, Φ_1) corresponding to the 5-tuple $(\varphi_0, \Phi_0, 1, \varepsilon_1, \delta_1 = \varepsilon_1)$. Note that Φ_1 is a C-map. Since $\varphi_1 = h_1^{-1} \circ \varphi_0 \circ h_1$, by denoting $H_1 = H_0 \circ h_1$ and substituting $\varphi_0 = H_0^{-1} \circ \varphi \circ H_0$ from the diagram already constructed we get that $\varphi_1 = H_1^{-1} \circ \varphi \circ H_1$. So we have the diagram for $k = 1$. Moreover, notice that $d_{H(Y)}(H_1, H_0) < \varepsilon_1$.

In the next step, take $0 < \delta_2 < \varepsilon_2$ small enough to get $d_Y(H_1(y), H_1(y')) < \varepsilon_2$ whenever $d_Y(y, y') < \delta_2$. Let (h_2, φ_2, Φ_2) be a triple corresponding to the 5-tuple $(\varphi_1, \Phi_1, 2, \varepsilon_2, \delta_2)$. Putting $H_2 = H_1 \circ h_2$ we get the diagram for $k = 2$. Moreover, notice that $d_{H(Y)}(H_2, H_1) < \varepsilon_2$.

Proceeding in this way, we get sequences $(H_k)_{k\geq 0}$, $(\varphi_k)_{k\geq 0}$, $(\Phi_k)_{k\geq 0}$ such that for every $k \geq 1$ we have the following.

(a) $\varphi_k = H_k^{-1} \circ \varphi \circ H_k$.

(b) Φ_k is a C-map and (Φ_k, φ_k) is k-admissible.

(c) $\Phi_k = \Phi_{k-1}$ on $\Phi_{k-1}^{-1}(K_1 \cup \cdots \cup K_{k-1})$.

(d) $d_{H(Y)}(H_k, H_{k-1}) < \varepsilon_k$, $d_{H(Y)}(\varphi_k, \varphi_{k-1}) < \delta_k < \varepsilon_k$, $d_{B(X)}(\Phi_k, \Phi_{k-1}) < \varepsilon_k$.

By (d) and completeness of the spaces $H(Y)$ and $B_C(X)$ there exist limits $H = \mathrm{Lim}_{k\to\infty} H_k$, $\psi = \mathrm{Lim}_{k\to\infty} \varphi_k$ and $\Psi = \lim_{k\to\infty} \Phi_k$. Here H and ψ are homeomorphisms of Y and Ψ is a C-map of X.

Now consider the following diagram:

$$\begin{array}{ccc} X & \xrightarrow{\Psi} & X \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ Y & \xrightarrow{\psi} & Y \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\ Y & \xrightarrow{\varphi} & Y \end{array}$$

The diagram commutes. In fact, the equality $\pi \circ \Psi = \psi \circ \pi$ is obtained by taking the limit in $\pi \circ \Phi_k = \varphi_k \circ \pi$ (see (b)) and the equality $\psi = H^{-1} \circ \varphi \circ H$ is obtained by taking the limit in (a).

Further, we show that Ψ is continuous. The continuity of Ψ is obvious at every point $x \in X$ with $\Psi(x) \in X \setminus \bigcup_i K_i$. Now consider $x \in X$ with $\Psi(x) \in K_j$ for some j. Denote $A = \Phi_j^{-1}(K_j) = \pi^{-1} \circ \varphi_j^{-1}(y_j)$, so A is the *whole* fiber over $\varphi_j^{-1}(y_j)$. On the other hand, by (c) the maps $\Phi_j, \Phi_{j+1}, \ldots$ coincide on A and hence coincide on A also with Ψ. So $\Psi(A) = \Phi_j(A) \subseteq K_j$ and recall that also $\Psi(x) \in K_j$. Since ψ is a homeomorphism, Ψ maps (different) fibers to (different) fibers and so we get that A and x lie in the same fiber. Thus A, being a whole fiber, contains x. Since $\Phi_j, \Phi_{j+1}, \ldots$ are continuous on A (see (b)) and hence continuous at x, their uniform limit Ψ is also continuous at x.

The (continuous) C-map Ψ is surjective because its range $\Psi(X)$ is dense in X (in fact, it contains every point outside $\bigcup_i K_i$ due to the surjectivity of ψ and commutativity of the diagram).

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Finally, since π is strongly almost one-to-one and H is a homeomorphism, the (continuous) C-map Ψ is a strongly almost one-to-one extension of φ. □

LEMMA 23. *Let (X, Φ) be an extension of (Y, φ) under a factor map π. Assume that there are no periodic orbits in (Y, φ) and that the fibers $\pi^{-1}(y)$ $(y \in Y)$ form a null system in X. Then the adjacent map π^* is a one-to-one correspondence between invariant measures of (X, Φ) and of (Y, φ) and for every μ on X, π is a measure-theoretic isomorphism between μ and $\pi^*(\mu)$. In other words, (X, Φ) and (Y, φ) have the same simplex of invariant measures, hence, in particular, the same topological entropy.*

Proof. The set on which π is not injective is a union A of countably many fibers (because fibers form a null family, only countably many of them may not be singletons). Since, by aperiodicity, single points in Y have measure zero for every invariant measure on Y, the set A has measure zero for every invariant measure on X. So, π is invertible except on a set ignorable for any invariant measure. This implies the assertion. □

From Proposition 22 and Lemmas 19 and 23 we immediately obtain the following result.

THEOREM 24. *Let X be a cantoroid and let φ be any Cantor minimal homeomorphism. Then there is a minimal system on X which is a (strongly) almost one-to-one extension of φ and has the same topological entropy as φ.*

Since there is a Cantor minimal homeomorphism with arbitrary entropy (see e.g. [**Wal82**, p. 178–179]), this theorem implies the following result.

THEOREM 25. *Any cantoroid admits a minimal system with arbitrary entropy.*

Theorem 24 asserts that every cantoroid (i.e. every almost totally disconnected compact metric space with no isolated point) admits a minimal system. If an almost totally disconnected compact metric space has an isolated point then it admits a minimal system if and only if it is finite. So we get the main result of the present paper.

THEOREM A. *An almost totally disconnected compact metric space admits a minimal system if and only if it is either a finite set or a cantoroid.*

7. *Minimal sets on dendrites and local dendrites*

As an application of our results, in this section we give a full topological characterization of minimal sets on dendrites and local dendrites. Recall that a brain is a cantoroid whose non-degenerate components are dendrites and form a null family.

THEOREM B. *Let D be a dendrite and let M be a subset of D. Then M is a minimal set for some dynamical system on D if and only if M is either a finite set or a brain.*

Proof. Let (M, f) be a minimal system with $M \subseteq D$. By (C-3) from §1 either M is a periodic orbit (and we are done) or M has uncountably many components, none of them being isolated. Assume that we are in the latter case. We want to show that M is a brain.

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First recall that any subcontinuum of a dendrite is a dendrite. Thus, all the non-degenerate components of M are dendrites and D is a hereditarily locally connected continuum. Hence, by [**Kur68**, Theorem 9, p. 272], there are only countably many non-degenerate components of M and they form a null family. Since we already know that M has no isolated point, it remains to prove that M is almost totally disconnected. To see that the union of the degenerate components of M is really dense in M, proceed as follows (when we speak about components in the rest of this paragraph, we always mean the components of M). The forward orbit of each of countably many non-degenerate components intersects only countably many components. Therefore there are uncountably many degenerate components which are not covered by the forward orbits of the non-degenerate components. Take any of these degenerate components and any of its backward orbits. It necessarily consists only of degenerate components. Now it suffices to use the folklore fact that every backward orbit in a minimal system is dense. (In fact, the set of all limit points of a backward orbit taken as a sequence is a non-empty closed invariant set. Hence the minimality implies that it coincides with the whole phase space.)

Conversely, if M is a finite set, it admits a minimal system trivially (a periodic orbit) and if M is a brain, it admits a minimal system by Theorem 24. This system can be then extended to the whole dendrite by [**BHS03**, Lemma 2.4]. □

In connection with Theorem B it is natural to ask whether every brain can be obtained as a minimal set of a dynamical system on an appropriately chosen dendrite. In other words we ask whether any brain can be imbedded into a dendrite (recall that a dynamical system on a closed subset of a dendrite can be extended to the whole dendrite). The answer is positive and known since 1933 when the paper [**Zip33**] appeared. In fact the following theorem is proved there.

THEOREM 26. [**Zip33**] *Let B be at most one-dimensional closed subset of a locally connected continuum X such that the components of B are all locally connected and form a null family. Then there is a locally connected subcontinuum D of X such that $B \subseteq D \subseteq X$ and D is irreducible about B (i.e. no proper locally connected subcontinuum of D contains B).*

Note that although the version of this theorem in [**Zip33**] explicitly speaks about one-dimensional B, it is in fact proved there for at most one-dimensional B. Further, in a special case when B in this theorem contains no simple closed curve, the locally connected continuum D that is obtained contains no simple closed curve either (see [**Zip33**] or [**War73**]); hence D is a dendrite.

It is an interesting story that, much later, several authors proved that for any zero-dimensional closed subset B of a locally connected continuum X there is a dendrite D such that $B \subseteq D \subseteq X$. This was proved by Bing in [**Bin49**] (nevertheless, some additional properties are required for D and so his result does not follow from Theorem 26). Later an elementary proof was given by Armentrout and Singh in [**AS83**] and recently Charatonik and Charatonik, not being aware of this story, gave another proof (ascribed to Illanes) in [**CC01**].

From the above Zippin's theorem and the discussion after it we get that, in particular, if B is a brain lying in a locally connected continuum X then there is a dendrite D such

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that $B \subseteq D \subseteq X$. To see this we need only to show that brains are at most one-dimensional. Let us start with the observation that any system of non-degenerate mutually disjoint subcontinua of a brain B is countable (the reason is that non-degenerate components of B are dendrites, there are only countably many of them and any dendrite has this property). Fix any point x of B and take all the open balls $B(x, r)$, $r > 0$. The boundaries of these balls are mutually disjoint and since only countably many of them may contain a *non-degenerate* subcontinuum of B, there are arbitrarily small r such that the boundary of $B(x, r)$ is (compact) totally disconnected hence zero-dimensional. So, by the definition, $\dim B \leq 1$.

Thus Theorem 26 together with the fact that by the Menger–Noebeling theorem any brain, being at most one-dimensional, can be imbedded into a three-dimensional unit cube leads to the following result.

COROLLARY 27. *Any brain can be imbedded into a dendrite.*

So, using [**BHS03**, Lemma 2.4], any dynamical system on a brain can be realized as a subsystem of a system on a dendrite.

Recall that a continuum having the property that every its point has a neighborhood whose closure is a dendrite is called a *local dendrite*. Equivalently, a local dendrite is a locally connected continuum containing only finitely many circles (see [**Kur68**, Theorem 4, p. 303]). From this and the fact that local dendrites are hereditarily locally connected (see [**Kur68**, Theorem 1, p. 303 and Theorem 2, p. 283]) we get that any subcontinuum of a local dendrite is again a local dendrite.

In [**BOT05**] the authors proved the following.

PROPOSITION 28. [**BOT05**] *Suppose that L is a local dendrite. If $f : L \to L$ is minimal then L is a circle and f is conjugate to an irrational rotation.*

Let us repeat that by a *generalized brain* we mean a cantoroid whose non-degenerate components form a null family, these components being local dendrites and only finitely many of them containing circles.

THEOREM C. *Let L be a local dendrite and let M be a subset of L. Then M is a minimal set for some dynamical system on L if and only if M is either a finite set or a finite union of disjoint circles or a generalized brain.*

Proof. One can proceed analogously as in the proof of Theorem B. There are only two differences. First, if $M \subseteq L$ is a minimal set with finitely many components then apart from the possibility that M is a finite set it may happen that M is a finite union of disjoint circles (use Proposition 28). Second, if $M \subseteq L$ is either a finite set or a finite union of disjoint circles or a generalized brain, then any dynamical system on it can be extended to the whole L. To see that this is true, let M be any closed subset of L and let $f : M \to L$ be continuous. Analogously as in [**BHS03**, Lemma 2.4] one can extend f to a continuous map $g : M \cup G \to L$ where G is a graph in L containing all the circles in L. Then using ideas from [**Nad92**, 8.38] one can extend g to a continuous map $F : L \to L$. (Alternatively, one may use an extension lemma from [**Spi08**].) □

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Again, just as in the dendrite case, whenever a set is of one of the types mentioned in the previous theorem, it can be obtained as a minimal set of a dynamical system on an appropriately chosen local dendrite. In fact, as in §6, we get from Zippin's Theorem 26 the following.

COROLLARY 29. *Any generalized brain can be imbedded into a local dendrite.*

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