Econometric Theory, **24**, 2008, 1093–1129. Printed in the United States of America.
doi:10.1017/S0266466608080420

TESTING FOR SEASONAL UNIT ROOTS IN PERIODIC INTEGRATED AUTOREGRESSIVE PROCESSES

TOMAS DEL BARRIO CASTRO
University of the Balearic Islands

DENISE R. OSBORN
University of Manchester

This paper examines the implications of applying the Hylleberg, Engle, Granger, and Yoo (1990, *Journal of Econometrics* 44, 215–238) (HEGY) seasonal root tests to a process that is periodically integrated. As an important special case, the random walk process is also considered, where the zero-frequency unit root *t*-statistic is shown to converge to the Dickey–Fuller distribution and all seasonal unit root statistics diverge. For periodically integrated processes and a sufficiently high order of augmentation, the HEGY *t*-statistics for unit roots at the zero and semiannual frequencies both converge to the same Dickey–Fuller distribution. Further, the HEGY joint test statistic for a unit root at the annual frequency and all joint test statistics across frequencies converge to the square of this distribution. Results are also derived for a fixed order of augmentation. Finite-sample Monte Carlo results indicate that, in practice, the zero-frequency HEGY statistic (with augmentation) captures the single unit root of the periodic integrated process, but there may be a high probability of incorrectly concluding that the process is seasonally integrated.

1. INTRODUCTION

Periodic autoregressive (PAR) models can arise naturally from the application of economic theory when the underlying economic driving forces, such as preferences or technologies, vary seasonally, as shown in Gersovitz and McKinnon (1978), Osborn (1988), and Hansen and Sargent (1993). These PAR models account for seasonality by allowing the parameters of the autoregressive (AR) process to change with the seasons of the year. As such, they are generaliza-

We are grateful to Paulo Rodrigues, two referees, and a co-editor of this journal for comments on previous versions of this paper. In particular, the detailed comments of one referee have resulted in substantial improvements in our analysis. Much of this research was undertaken while the first author was a visitor to the University of Manchester. Financial assistance from the Agència de Gestió d'Ajusts Universitaris I de Recerca (Generalitat de Catalunya) under grant 2002BEAI4000 and Comisión Interministerial de Ciencia y Tecnología SEJ2005-07781/ECON is gratefully acknowledged. Address correspondence to Denise R. Osborn, Economics, School of Social Sciences, University of Manchester, Manchester M13 9PL, United Kingdom; e-mail: denise.osborn@manchester.ac.uk.

© 2008 Cambridge University Press 0266-4666/08 $15.00

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

tions of the dummy variable approach that is widely applied in empirical analyses for seasonal economic data. In recent years a number of papers have contributed to the development of a statistical tool kit for inference in PAR models and also to the exploration of their usefulness for the analysis of observed macroeconomic time series (mainly at the quarterly frequency); see Ghysels and Osborn (2001) and Franses (1996) for surveys.

Despite the attraction of PAR models from the perspective of economic decision making in a seasonal context, the more prominent approach of empirical workers is to assume that the AR coefficients, except for the intercept, are constant over the seasons of the year. These can be referred to as nonperiodic models, in contrast with the periodic case. Tiao and Grupe (1980) and Osborn (1991) study the properties of stationary PAR processes when analyzed as conventional nonperiodic autoregressive moving average (ARMA) processes, showing that a PAR process is (in general) converted into a process with seasonal AR dynamics and a high-order moving average (MA) component.

However, observed macroeconomic time series are typically nonstationary. Following the dominant use of the nonperiodic approach for empirical analyses, a stream of important research has examined the nature of nonstationarity in a seasonal context through the so-called seasonal unit root tests; see, for example, Dickey, Hasza, and Fuller (1984) (DHF), Hylleberg, Engle, Granger, and Yoo (1990) (HEGY), and Osborn, Chui, Smith, and Birchenhall (1988). The preeminent seasonal unit root test approach is now that of HEGY, with its properties studied by Smith and Taylor (1998), Rodrigues (2001), and others. Under the null hypothesis of these tests, nonstationary unit root behavior exists not only at the long-run (or zero) frequency but also at all the seasonal frequencies. Although this is not always discussed, the implication of these seasonal unit roots is that the seasons of the year are not cointegrated with each other and hence "summer may become winter"; see Osborn (1991) and Ghysels and Osborn (2001). From an economic perspective, this implication may be unattractive.

An alternative type of seasonal nonstationary process is the so-called periodically integrated (PI) process. This may be more plausible than the seasonally integrated (SI) process studied by HEGY and others, because it allows nonstationarity in conjunction with cointegration between the seasons of the year (Osborn, 1991; Franses, 1994). There is, however, an important gap in the literature, as little attention has been paid to the implications of testing for seasonal and zero-frequency unit roots when the underlying data generating process (DGP) is of the PI type. To our knowledge, only Boswijk and Franses (1996), Franses (1994, 1996), and Sanso, Artis, and Surinyach (1997) partially consider this issue. Boswijk and Franses (1996) derive the distributions of the DHF test statistics for a PI(1) process, and Franses (1994, 1996) and Sanso et al. (1997) present some Monte Carlo results for the HEGY test when applied to PAR processes.

The primary purpose of this paper is to analytically study the implications of applying the HEGY procedure to a PI(1) DGP. Indeed, we find that the single

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

unit root in the PI process plays a crucial role, with all HEGY seasonal unit root test statistics depending on the same Dickey–Fuller distribution. As a particular but distinctive special case, the distributions of these statistics are also obtained for a random walk process, where only the zero-frequency test statistic has a Dickey–Fuller distribution. Although both Boswijk and Franses (1996) and Taylor (2003) obtain the analytical distribution of the DHF test statistic for a random walk, we believe that our analysis is the first to study the HEGY statistics for this case. In addition, we provide finite-sample Monte Carlo results to complement our analytical findings.

The paper is organized as follows. Section 2 introduces preliminaries concerning PI(1) processes and the HEGY test needed for our analysis. The distributions of the statistics are obtained in Section 3, and Section 4 provides Monte Carlo results. Finally, Section 5 concludes.

2. PERIODIC PROCESSES AND THE HEGY TEST

Ghysels and Osborn (2001) survey developments relating to seasonal unit roots and PAR models. Further, Osborn and Rodrigues (2002) provide a unifying approach for the asymptotics of seasonal unit root tests. This section does not repeat this background but instead focuses on the main points needed for the subsequent analysis. However, as the misspecified constant parameter representation is particularly important for the results in Section 3, our discussion of this representation in Section 2.2 provides some development of previous analyses.

2.1. Periodic Processes

For simplicity of exposition, we assume a quarterly series with mean zero. We also focus on the PAR(1) case, because this keeps the analysis as simple as possible without losing any essential features. The PAR(1) process is given by

$$y_{s\tau} = \alpha_s y_{s-1,\tau} + \varepsilon_{s\tau}, \qquad s = 1,2,3,4, \qquad \tau = 1,2,\ldots, \tag{1}$$

where, for observation $y_{s\tau}$, the first subscript refers to the season (s) and the second subscript to the year (τ). When $s = 1$, it is understood that $y_{s-1,\tau} = y_{4,\tau-1}$. Also for ease of exposition, we assume that observations are available for precisely N years, so that the total sample size is $T = 4N$, with initial value $y_{04} = 0$. The PAR disturbance process $\varepsilon_{s\tau}$ is a zero-mean independent and identically distributed (i.i.d.) process that may be heteroskedastic over seasons, but to concentrate on the effects of periodic coefficients we assume homoskedasticity with $\mathrm{E}(\varepsilon_{s\tau}^2) = \sigma^2$.

The stationarity properties of the PAR(1) process are determined by the product $\alpha_1\alpha_2\alpha_3\alpha_4$. Specifically, the process is stationary when $|\alpha_1\alpha_2\alpha_3\alpha_4| < 1$ and is periodically integrated when $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$. A periodic process that is

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

integrated of order 1 is referred to as a PI(1) process (Osborn et al., 1988). The random walk is a special case of (1) with $\alpha_s = 1$ ($s = 1,2,3,4$), and in this case $y_{s\tau} \sim \mathrm{I}(1)$.

Franses (1994) exploits the vector autoregressive (VAR) representation of a periodic process, referring to it as the vector of quarters (VQ) representation. The VQ representation of the PAR(1) in (1) is

$$\Phi_0 Y_\tau = \Phi_1 Y_{\tau-1} + U_\tau, \tag{2}$$

where $Y_\tau = [y_{1\tau}, y_{2\tau}, y_{3\tau}, y_{4\tau}]'$, $U_\tau = [\varepsilon_{1\tau}, \varepsilon_{2\tau}, \varepsilon_{3\tau}, \varepsilon_{4\tau}]'$, $\mathrm{E}(U_\tau U_\tau') = \sigma^2 I$ and

$$\Phi_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\alpha_2 & 1 & 0 & 0 \\ 0 & -\alpha_3 & 1 & 0 \\ 0 & 0 & -\alpha_4 & 1 \end{bmatrix}, \qquad \Phi_1 = \begin{bmatrix} 0 & 0 & 0 & \alpha_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

As discussed by Ghysels and Osborn (2001), the key to the type of integration exhibited by a PAR process is the final equation representation

$$|\Phi_0 - \Phi_1 B| Y_\tau = adj(\Phi_0 - \Phi_1 B) U_\tau, \tag{3}$$

where $adj(\cdot)$ indicates the adjoint matrix for the expression in parentheses, $|\cdot|$ indicates the determinant, and B is the annual backshift (or lag) operator such that $BY_\tau = Y_{\tau-1}$. For quarter s, (3) is

$$\begin{aligned} y_{s\tau} = {} & \alpha_s \alpha_{s-1} \alpha_{s-2} \alpha_{s-3} y_{s,\tau-1} + \varepsilon_{s\tau} + \alpha_s \varepsilon_{s-1,\tau} + \alpha_s \alpha_{s-1} \varepsilon_{s-2,\tau} \\ & + \alpha_s \alpha_{s-1} \alpha_{s-2} \varepsilon_{s-3,\tau}, \end{aligned} \tag{4}$$

which can also be obtained directly from (1) by repeated substitution. Note that (4) has the (annual lag) AR coefficient $\alpha_1 \alpha_2 \alpha_3 \alpha_4$ for all $s = 1,2,3,4$ but the MA(3) term is seasonally varying. However, for the random walk special case, factors cancel across (4) to yield $y_{s\tau} = y_{s-1,\tau} + \varepsilon_{s\tau}$.

When the PAR(1) is integrated of order 1, PI(1), (3) becomes a vector moving average (VMA) in the annual difference $(1 - B)Y_\tau = Y_\tau - Y_{\tau-1}$ of the form

$$(1 - B) Y_\tau = (\Theta_0 + \Theta_1 B) U_\tau. \tag{5}$$

This VMA process is noninvertible, because the matrix $D(B) = (\Theta_0 + \Theta_1 B)$ has three unit roots. Therefore, the rank of $D(1)$ is one, and it is possible to write

$$D(1) = \Theta_0 + \Theta_1 = ab', \tag{6}$$

where

$$a = [1, \alpha_2, \alpha_2 \alpha_3, \alpha_2 \alpha_3 \alpha_4]' \quad \text{and} \quad b = [1, \alpha_1 \alpha_3 \alpha_4, \alpha_1 \alpha_4, \alpha_1]'. \tag{7}$$

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

In the random walk case, $a = b = [1,1,1,1]'$. From (6), there are three cointegrating relationships between the four quarterly series of the PI(1) or I(1) process, or equivalently there is a single common trend between values corresponding to the four quarters of the year (Boswijk and Franses, 1996; Franses, 1994; Ghysels and Osborn, 2001; Osborn, 1991).

To summarize the main characteristics of a PI(1) process, we use a version of Lemma 1 in Boswijk and Franses (1996), which is proved in that paper.

LEMMA 1. *Consider Y_τ in (2) with $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$. Assuming that U_τ is vector white noise with $\mathrm{E}(U_\tau U_\tau') = \sigma^2 I_4$, then as $N = T/4 \to \infty$*

$$\frac{1}{\sqrt{N}} Y_{[rN]} \Rightarrow B(r) \qquad r \in [0,1], \tag{8}$$

where $[rN]$ is the integer part of rN. The 4×1 vector Brownian motion process $B(r) = [B_1(r), B_2(r), B_3(r), B_4(r)]'$ satisfies

$$B(r) = ab'W(r) = \omega a w(r), \tag{9}$$

where $W(r)$ is 4×1 vector Brownian motion with variance matrix $\sigma^2 I_4$, $\omega = \sigma\{b'b\}^{1/2}$, $w(r)$ is scalar standard Brownian motion, and a and b are defined in (7).

Throughout the paper, $\to$ indicates convergence in probability, and $\Rightarrow$ indicates convergence in distribution.

2.2. The Misspecified Constant Parameter Representation

The misspecified constant parameter representation of a periodic process is considered in Osborn (1991) based on results initially due to Tiao and Grupe (1980); see also Ghysels and Osborn (2001, pp. 148–152). This representation provides the conventional (nonperiodic) ARMA process that has identical autocovariance properties to those that result from analyzing a periodic process as a conventional (nonperiodic) one.

The constant parameter representation has AR component given by the final equation representation of (3). Therefore, for a PI(1) process with $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$, the AR component is the annual difference $\Delta_4 y_{s\tau} = y_{s\tau} - y_{s,\tau-1}$, as on the left-hand side of (5). Further, the order of the MA component is determined by the order of the MA terms of the final equation representation, which (4) shows to be an MA(3) for the PI(1) process of (1).

Because of the periodic nature of $\Delta_4 y_{s\tau}$ in (4) its autocovariances

$$\gamma_s(j) = \mathrm{E}(\Delta_4 y_{s\tau} \Delta_4 y_{s-j,\tau}), \qquad j = 0,1,2,3 \tag{10}$$

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

vary with both the season s and the lag j. However, when its periodic nature is ignored, $\Delta_4 y_{s\tau}$ has autocovariances

$$\gamma(j) = \frac{1}{4}\sum_{s=1}^{4} \gamma_s(j) \qquad j = 0,1,2,3 \tag{11}$$

and $\gamma(j) = 0, j > 3$. Although, in general, $\gamma_s(j) \neq \gamma_s(-j), j = 1,2,3$, because these relate to covariances between different season pairs, it is straightforward to see that averaging in (11) results in $\gamma(j) = \gamma(-j), j = 1,2,3$.

Using the autocovariances computed from (10) and (11), the nonperiodic parameters θ_1, θ_2, θ_3 and $\sigma_\eta^2 = \mathrm{E}[\eta_{s\tau}^2]$ in the misspecified constant parameter representation

$$\Delta_4 y_{st} = (1 + \theta_1 L + \theta_2 L^2 + \theta_3 L^3)\eta_{s\tau} \tag{12}$$

can be obtained, where $\Delta_4 = 1 - B = 1 - L^4$ is again the annual difference operator and the one period lag operator is denoted by L, so that $Ly_{s\tau} = y_{s-1,\tau}$. In other words, the constant parameter representation has identical autocovariances $\gamma(j), j = 0,1,2,3$ to those of (11) obtained from the true PI(1) process. As is well known (see, e.g., Hamilton, 1994, pp. 67–68), these $\gamma(j)$ generally lead to multiple solutions for the parameters of (12), but only one of these multiple solutions corresponds to an invertible MA.

Because of the key role played by (12) in our later analysis, Lemma 2 examines the nature of this constant parameter representation.

LEMMA 2. *Consider the DGP of (1) with* $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$, *all* $\alpha_s > 0$ ($s = 1, 2,3,4$) *but with the restriction* $\alpha_s = 1$ ($s = 1,2,3,4$) *not holding, and* $\mathrm{E}[\varepsilon_{s\tau}^2] = \sigma^2$. *Then the constant parameter representation (12) for this process has a unique invertible MA(3) component unless the PI coefficients satisfy one of the following sets of restrictions:*

(i) $\alpha_2 = \alpha_4 = \alpha_1^{-1} = \alpha_3^{-1}$;
(ii) $\alpha_4 = (1 - \alpha_3 + \alpha_2\alpha_3)^{-1}$.

Note that the restrictions in (i) imply the presence of two semiannual cycles with identical coefficients, one applying for quarters 1 and 2 and the other for quarters 3 and 4. The proof of this lemma, which is contained in the Appendix, uses the implicit aggregation underlying the constant parameter representation as analyzed by Osborn (1991). The two sets of restrictions in the lemma, together with the I(1) process where all $\alpha_s = 1$, specify the only cases where cancellation occurs across the AR and MA components of (12). In all other cases the MA(3) does not contain unit roots and hence admits an invertible representation.

In the later analysis, we consider the six specific quarterly DGPs of Table 1, each of which has the form of (1) with $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$. Note that DGP6 is a random walk, whereas the remaining five are PI(1) processes. All coeffi-

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

TABLE 1. Data generating processes considered

DGP	α_1	α_2	α_3	α_4
1	0.500	0.900	1.500	1.481
2	1.200	0.600	1.000	1.389
3	1.250	0.800	0.900	1.111
4	2.000	0.500	1.500	0.667
5	1.000	0.800	1.200	1.042
6	1.000	1.000	1.000	1.000

Note: α_j ($j = 1,2,3,4$) are coefficients of PI(1) processes of (1) satisfying $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ and used in subsequent analysis.

cients are positive, reflecting the positive autocorrelations anticipated for economic time series, and each has constant disturbance variance $E[\varepsilon_{st}^2] = \sigma^2 = 1$ ($s = 1,2,3,4$). For each PI(1) process, Table 2 collects the coefficients and factors of the implied invertible MA(3) polynomial of (12), together with σ_η^2. These are obtained from the DGPs of Table 1 using (10) and (11) and the theoretical relationships between the parameters of a (nonperiodic) MA(3) process and its autocovariances $\gamma(j)$, $j = 0,1,2,3$, selecting the unique invertible solution. The proximity of the MA roots in Table 2 to the seasonal AR roots -1, $\pm i$ indicates how close the factors of (12) are to canceling. For example, the implied MA factor $(1 + \theta_1 L) = (1 + 0.929L)$ of DGP3 is close to cancellation with the biannual seasonal AR unit root of $(1 + L)$. In all cases $\sigma_\eta^2 > 1$, so that the variance of the white noise disturbance associated with the (invert-

TABLE 2. Invertible Moving Averages in misspecified constant parameter representation

	MA coefficients			MA factors		
DGP	θ_1	θ_2	θ_3	Real factor	Complex factors	σ_η^2
1	0.835	0.548	0.417	$1 + 0.801L$	$1 + (0.017 \pm 0.721\, i)L$	2.266
2	0.920	0.666	0.572	$1 + 0.892L$	$1 + (0.014 \pm 0.801\, i)L$	1.843
3	0.939	0.795	0.731	$1 + 0.929L$	$1 + (0.005 \pm 0.887\, i)L$	1.389
4	0.569	0.777	0.447	$1 + 0.573L$	$1 - (0.002 \pm 0.883\, i)L$	2.612
5	0.925	0.864	0.787	$1 + 0.918L$	$1 + (0.004 \pm 0.926\, i)L$	1.285

Note: The DGPs considered are the periodic DGPs defined in Table 1 with $\sigma^2 = 1$; the misspecified constant parameter is representation given by (12). The invertible MA parameters are obtained analytically, by solving the four equations given by writing the autocovariances $\gamma(k)$ of (10) and (11) for $k = 0,1,2,3$ in terms of the parameters of (12). Of the eight solutions to these equations, the table shows the parameters for the unique invertible MA solution. All DGPs considered imply one real factor and one factor with a pair of complex roots in the MA component of this constant parameter representation.

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

ible) misspecified constant parameter process is larger than that of the true PI(1) process.

The unique invertible MA(3) process in the seasonal difference for the constant parameter representation of (12) is of particular interest in our analysis. It is well known that an invertible MA can be approximated by an autoregression of sufficiently high order. In the context of a (zero frequency) unit root test, Said and Dickey (1984) show that provided the order of AR augmentation p is selected in relation to the sample size T such that $T^{-1/3}p \to 0$ as $T \to \infty$, then the limiting Dickey–Fuller distribution holds when the true DGP is an integrated invertible ARMA process. In the PI(1) case, the constant parameter representation in (12) suggests the presence of four AR unit roots (i.e., the zero frequency and all three seasonal unit roots), because it has an annual difference in the AR operator with an invertible MA component. This is despite the presence of three noninvertible roots in the VMA of the true DGP in (5). However, if a univariate seasonal unit root analysis is conducted through the usual HEGY approach, the VMA is not explicitly considered.[1] The subsequent sections investigate the implications of this failure to recognize the periodic nature of the process when conducting HEGY seasonal unit root tests.

2.3. The HEGY Test

The regression for the HEGY test, with no deterministic terms and no augmentation, is

$$\Delta_4 y_{s\tau} = \pi_1 x_{s\tau}^{(1)} + \pi_2 x_{s\tau}^{(2)} + \pi_3 x_{s\tau}^{(3)} + \pi_4 x_{s\tau}^{(4)} + u_{s\tau}. \tag{13}$$

The regressors of (13) are defined as

$$\begin{aligned} x_{s\tau}^{(1)} &= L(1+L)(1+L^2)y_{s\tau}, \\ x_{s\tau}^{(2)} &= -L(1-L)(1+L^2)y_{s\tau}, \\ x_{s\tau}^{(3)} &= -L^2(1-L)(1+L)y_{s\tau}, \\ x_{s\tau}^{(4)} &= -L(1-L)(1+L)y_{s\tau}, \end{aligned} \tag{14}$$

where $x_{s\tau}^{(1)}, x_{s\tau}^{(2)}$ are associated with the presence of the factors $(1 - L)$ and $(1 + L)$, respectively, of $\Delta_4 = (1 - L)(1 + L)(1 + L^2)$, whereas $(1 + L^2)$ is associated with the pair of variables $x_{s\tau}^{(4)}$ and $x_{s\tau}^{(3)} = x_{s-1,\tau}^{(4)}$. In an analogous notation to that used in the VQ representation of (2), we define the vector of observations for year τ for the regressors in (13) by $X_\tau^{(i)} = [x_{1\tau}^{(i)}, x_{2\tau}^{(i)}, x_{3\tau}^{(i)}, x_{4\tau}^{(i)}]'$ for $i = 1,2,3,4$.

For simplicity, assume that $u_{s\tau}$ in (13) is white noise with constant variance. The overall HEGY null hypothesis of seasonal integration, $y_{s\tau} \sim \mathrm{SI}(1)$, implies the presence of unit roots at the zero frequency (captured through π_1) and at

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

the seasonal frequencies (captured through π_2, π_3, and π_4), so that $\pi_1 = \pi_2 = \pi_3 = \pi_4 = 0$. Under this overall null hypothesis, the asymptotic distribution of the vector Y_τ is given by $N^{-0.5}Y_{[rN]} \Rightarrow W(r)$, where $W(r)$ is a 4×1 vector of independent Brownian motions; see, for example, Ghysels and Osborn (2001). Thus, the three cointegrating relationships between the four quarters for the PI(1) process contrast with no cointegration between the quarters for the SI(1) process.

As discussed by HEGY, the regressors in (13) are, by construction, asymptotically orthogonal under the seasonal integration null hypothesis. Therefore, the associated asymptotic distributions of the HEGY test statistics can be obtained considering the three factors of Δ_4 one by one. For example, the regression

$$\Delta_4 y_{s\tau} = \pi_1 x_{s\tau}^{(1)} + u_{s\tau} \tag{15}$$

provides a test of the zero-frequency unit root, while maintaining the presence of all seasonal unit roots. The orthogonality of the HEGY regressors under seasonal integration implies that the same asymptotic distributions for testing $\pi_1 = 0$ apply whether (13) or (15) is used; see, for example, Ghysels and Osborn (2001).

The asymptotic distributions for the *t*-ratio statistics relating to the estimated coefficients of (13) under the seasonal integration null hypothesis can be found, for example, in Ghysels, Lee, and Noh (1994) and Osborn and Rodrigues (2002). As discussed by HEGY, $t_{\hat{\pi}_1}$ and $t_{\hat{\pi}_2}$ follow the well-known Dickey–Fuller distributions that apply when testing a conventional (zero-frequency) unit root in an appropriately augmented test regression. Further, $t_{\hat{\pi}_3}$ follows DHF distribution for testing seasonal integration for biannual data (see, e.g., Ghysels and Osborn, 2001, pp. 56–57). In addition, HEGY promote the use of an *F*-type statistic for $\pi_3 = \pi_4 = 0$,[2] and Ghysels et al. (1994) extend this to consider $\pi_2 = \pi_3 = \pi_4 = 0$ and $\pi_1 = \pi_2 = \pi_3 = \pi_4 = 0$, where these correspond to seasonal unit roots and zero-frequency and seasonal unit roots, respectively.

3. ASYMPTOTIC DISTRIBUTIONS

In this section we derive the asymptotic distributions of the HEGY test statistics applied to PI(1) processes. Section 3.1 includes some preliminary results on the behavior of the variables in the HEGY regression (13) for a PI(1). Section 3.2 then obtains the asymptotic distributions for the HEGY regression in the special case of an I(1) DGP, with Section 3.3 analyzing the PI(1) case. Where required, proofs can be found in the Appendix.

3.1. Preliminary Results

The results of Section 2.3 apply for a SI process, whereas the purpose of this paper is to analyze the HEGY test applied to a PI process. To this end, Lemma 3

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

examines the asymptotic nature of the variables used in the HEGY regression for a PI(1) DGP when the MA component of the constant parameter representation (12) is invertible.

LEMMA 3. *Consider the DGP of (1) with* $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ *but with neither the restriction* $\alpha_s = 1$ $(s = 1,2,3,4)$ *nor those of Lemma 2 holding and with* $\mathrm{E}[\varepsilon_{s\tau}^2] = \sigma^2$. *Then the following hold.*

(i) The HEGY regressors are nonstationary with asymptotic distributions

$$N^{-1/2}X_\tau^{(i)} = N^{-1/2}(x_{1\tau}^{(i)}, x_{2\tau}^{(i)}, x_{3\tau}^{(i)}, x_{4\tau}^{(i)})' \Rightarrow C_i B(r) = \omega C_i a w(r), \qquad i = 1,2,3,4, \tag{16}$$

where the HEGY regressors $x_{s\tau}^{(i)}$ *are defined in (14), the matrices* C_i *are circulant matrices (for details, see Davis, 1979) defined as* $C_1 = Circ[1,1,1,1]$, $C_2 = Circ[1,-1,1,-1]$, $C_3 = Circ[1,0,-1,0]$, *and* $C_4 = Circ[0,1,0,-1]$, $\omega = \sigma\{b'b\}^{1/2}$, $w(r)$ *is scalar standard Brownian motion, and a and b are defined in (7).*

(ii) The HEGY regressors satisfy

$$\begin{aligned}(4N)^{-2}\sum_{\tau=1}^{N} X_\tau^{(i)'}X_\tau^{(i)} &\Rightarrow \frac{\omega^2 a' C_i a}{4}\int w(r)^2\,dr, \qquad i = 1,2,\\ (4N)^{-2}\sum_{\tau=1}^{N} X_\tau^{(i)'}X_\tau^{(i)} &\Rightarrow \frac{\omega^2 a' C_3 a}{8}\int w(r)^2\,dr, \qquad i = 3,4,\end{aligned} \tag{17}$$

and

$$(4N)^{-2}\sum_{\tau=1}^{N} X_\tau^{(i)'}X_\tau^{(j)} \to 0, \qquad i \neq j; \quad i,j = 1,2,3,4. \tag{18}$$

(iii) The HEGY dependent variable vector $\Delta_4 Y_\tau = (\Delta_4 y_{1\tau}, \Delta_4 y_{2\tau}, \Delta_4 y_{3\tau}, \Delta_4 y_{4\tau})'$ *is stationary.*

Part (i) follows from Lemma 1 together with the definitions of the HEGY variables. Part (ii) uses the (easily verified) results that C_i satisfy $C_i' C_i = 4C_i$ for $i = 1,2$, $C_i' C_i = 2C_3$ for $i = 3,4$, and $C_i' C_j = 0$ $(i \neq j)$ with the exceptions $C_3' C_4 = 2C_4$ and $C_4' C_3 = -2C_4$, but in this latter case (18) still holds as $\omega^2 a' C_4 a = 0$. The final part follows in a straightforward way from (4).

For a PI(1) DGP, Lemma 3 establishes that all regressors in (13) are nonstationary, with elements that asymptotically behave as scalar multiples of the single Brownian motion $w(r)$ that underlies the PI process. Although the annual summation involved in the definition of $x_{s\tau}^{(1)}$ implies that this variable asymptotically displays no seasonality over $s = 1,2,3,4$, the scalar multiples of $w(r)$ that apply for the remaining HEGY variables do exhibit a seasonal pattern. This Brownian motion again appears in the distribution of the (scaled) sum of squares corresponding to the HEGY variables in (17). Intuitively, these results are due to the single unit root process underlying the PI(1) case, so that all

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

HEGY regressors derive their properties from the same nonstationary process. This contrasts with a quarterly SI process, which contains four distinct unit root processes. Nevertheless, (18) implies that the HEGY regressors in the PI(1) case retain the same asymptotic orthogonality as under the seasonal integration null hypothesis. This last result is, perhaps, surprising when a single Brownian motion underlies all variables, but it is a consequence of the cross-product in (18) resulting in the effective cancellation of the different seasonal patterns in the scalar multiples of $w(r)$ that apply for the individual HEGY variables.

By assumption, Lemma 3 rules out the random walk case, which is considered separately in Lemma 4, which follows. The results in this lemma are a direct consequence of $\alpha_s = 1$ and the definitions of (14).

LEMMA 4. *For the random walk DGP of (1) with* $\alpha_s = 1$ $(s = 1,2,3,4)$*:*

(i) $x_{s\tau}^{(1)} \sim I(1)$*, and* $\Delta_4 y_{s\tau}, x_{s\tau}^{(2)}, x_{s\tau}^{(3)}, x_{s\tau}^{(4)}$ *are stationary.*

(ii) The sample covariances between $x_{s\tau}^{(1)}$ *and each of* $x_{s\tau}^{(2)}, x_{s\tau}^{(3)}, x_{s\tau}^{(4)}$ *satisfy*

$$(4N)^{-1} \sum_{s=1}^{4} \sum_{\tau=1}^{N} x_{s\tau}^{(1)} x_{s\tau}^{(i)} = O_p(1), \qquad i = 2,3,4. \tag{19}$$

Lemma 3 also rules out the cases where the conditions of Lemma 2 apply and hence where the constant parameter representation does not contain the full set of AR unit roots -1, $\pm i$. If the restrictions of Lemma 2 apply, then either $x_{s\tau}^{(2)}$ or $(x_{s\tau}^{(3)}, x_{s\tau}^{(4)})$ are stationary. However, we do not separately consider these cases, as the PI(1) process without these restrictions and the I(1) process cover the cases of principal interest in empirical analyses.

It is clear from Lemmas 3 and 4 that although both the I(1) and PI(1) processes have a single unit root, the stationarity properties of the HEGY regressors differ across these two processes. Consequently, as our later analysis shows, different asymptotic distributions result when the HEGY test is applied to these two cases.

The final preliminary result in contained in Lemma 5, which provides some relationships between the parameters of the PI(1) process and the autocovariances $\gamma(j)$ of $\Delta_4 y_{s\tau}$ in the misspecified constant parameter representation (12). Although these results apply for all PI(1) DGPs, including the I(1) special case, they will be particularly useful for our analysis of the PI(1) case when no cancellations occur in (12).

LEMMA 5. *Consider the DGP of (1) with* $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ *and* $\mathrm{E}[\varepsilon_{s\tau}^2] = \sigma^2$*. Then*

$$\frac{\omega^2}{4} a'a = \gamma(0),$$

$$\frac{\omega^2}{4} a'C_1 a = \Psi_1(4) = \gamma(0) + 2\gamma(1) + 2\gamma(2) + 2\gamma(3),$$

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

$$\frac{\omega^2}{4}\, a' C_2 a = \Psi_2(4) = \gamma(0) - 2\gamma(1) + 2\gamma(2) - 2\gamma(3),$$
$$\frac{\omega^2}{8}\, a' C_3 a = \frac{1}{2}\,\Psi_3(4) = \frac{1}{2}\,[\gamma(0) - 2\gamma(2)], \tag{20}$$

where $\omega = \sigma^2 b'b$ with a and b as defined in (7), and $\gamma(j)$ is the autocovariance at lag j of the constant parameter MA process for $\Delta_4 y_{s\tau}$ in (12). The notation $\Psi_j(i)$ is defined in Theorem 2 in Section 3.3.

From (20), it is clear that the scalings of the Brownian motion distributions in (17) depend on the autocovariances in (11) of the annually differenced process. These autocovariances are identical to those of the misspecified constant parameter process (12).

3.2. The Random Walk DGP

As just pointed out, the properties of the HEGY regressors differ for the PI(1) and I(1) cases. We first consider the latter and then turn to the periodic case in Section 3.3.

Because of the I(1) property of $x_{s\tau}^{(1)}$ and the stationarity of the vector $x_{s\tau}^{(*)} = [x_{s\tau}^{(2)}, x_{s\tau}^{(3)}, x_{s\tau}^{(4)}]'$, established in Lemma 4, the coefficients corresponding to these variables converge at different rates when (13) is estimated for a random walk DGP. To reflect this, define the 4×4 diagonal scaling matrix M, with first element T and remaining diagonal elements $T^{1/2}$. It is then straightforward to see that the scaled ordinary least squares (OLS) estimators for the HEGY regression (13) are given by

$$M\hat{\Pi} = \begin{bmatrix} T\hat{\pi}_1 \\ T^{1/2}\hat{\Pi}_2 \end{bmatrix}$$
$$= \begin{bmatrix} T^{-2}\sum_{s=1}^{4}\sum_{\tau=1}^{N}(x_{s\tau}^{(1)})^2 & T^{-3/2}\sum_{s=1}^{4}\sum_{\tau=1}^{N} x_{s\tau}^{(1)}(x_{s\tau}^{(*)})' \\ T^{-3/2}\sum_{s=1}^{4}\sum_{\tau=1}^{N} x_{s\tau}^{(1)} x_{s\tau}^{(*)} & T^{-1}\sum_{s=1}^{4}\sum_{\tau=1}^{N} x_{s\tau}^{(*)}(x_{s\tau}^{(*)})' \end{bmatrix}^{-1}$$
$$\times \begin{bmatrix} T^{-1}\sum_{s=1}^{4}\sum_{\tau=1}^{N} x_{s\tau}^{(1)}\Delta_4 y_{s\tau} \\ T^{-1/2}\sum_{s=1}^{4}\sum_{\tau=1}^{N} x_{s\tau}^{(*)}\Delta_4 y_{s\tau} \end{bmatrix}, \tag{21}$$

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

where $\hat{\Pi}_2 = [\hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_4]'$. Our results relating to the distribution of the OLS estimators for the HEGY regression applied to a random walk derive from (21) and are contained in Theorem 1, which follows.

THEOREM 1. *Assume that* $\alpha_s = 1$ $(s = 1,2,3,4)$ *in (1), so that* $y_{s\tau} = y_{s-1,\tau} + \varepsilon_{s\tau}$ *with* $\mathrm{E}[\varepsilon_{s\tau}^2] = \sigma^2$. *Then, for the HEGY regression of (13) the following hold.*

(i) The appropriately scaled OLS estimator $\hat{\pi}_1$ *is asymptotically orthogonal to* $\hat{\Pi}_2 = [\hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_4]'$.

(ii) The asymptotic distribution of $T\hat{\pi}_1$ *is given by*

$$T\hat{\pi}_1 \Rightarrow \frac{1}{4} \frac{\int w(r)\, dw(r)}{\int w(r)^2\, dr}. \tag{22}$$

(iii) The asymptotic distribution of the t-ratio for $\hat{\pi}_1$ *is given by*

$$t_{\hat{\pi}_1} \Rightarrow t^{DF} = \frac{\int w(r)\, dw(r)}{\sqrt{\int w(r)^2\, dr}}. \tag{23}$$

(iv) The OLS estimator of Π_2 *converges to* $(-0.5, -0.5, -0.5)'$.

Note from Theorem 1, specifically equation (22), that the distribution of the HEGY normalized bias statistic for the zero-frequency unit root leads to a scaling of the distribution of Dickey and Fuller (1979) when the process is a random walk. Thus, the usual Dickey–Fuller distribution for the normalized bias does not apply in this case. However, the distribution in (23) is the usual asymptotic Dickey–Fuller distribution for the *t*-ratio, implying that this distribution remains valid for testing a zero-frequency unit root within the context of the HEGY regression, even though no seasonal unit roots are present in the DGP. Notice that although the dependent variable is autocorrelated, the stationary regressors $x_{s\tau}^{(i)}$, $i = 2,3,4$ in (13) effectively act in the same way as the augmentation in a conventional augmented Dickey–Fuller (ADF) test and the disturbances are white noise. This result implies that the zero-frequency unit root test embedded in the HEGY approach remains meaningful even if the data under examination are nonseasonal and hence the overall null hypothesis of seasonal integration is not valid.

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

Part (iv) of Theorem 1 implies that the scaled estimator $T\hat{\pi}_j$ for $j = 2,3,4$ diverges to $-\infty$ as $T \to \infty$ when the HEGY regression (13) is applied to a random walk process. This divergence has also been obtained by Rodrigues (2001) and Taylor (2002), when they consider local alternatives to the HEGY seasonal integration null hypothesis.

In summary, therefore, the DF t-ratio distribution continues to apply for testing the zero-frequency unit root in the HEGY regression applied to the random walk. No augmentation is required,[3] because the remaining regressors are stationary and these account for the autocorrelation in the dependent variable. Corresponding to these stationary regressors, the distributions for AR unit root t-statistics at the seasonal frequencies diverge as $T \to \infty$, and, hence, we may expect good performance in terms of power for these statistics.

3.3. The Periodically Integrated DGP

Having examined the I(1) special case, we now turn to the case of primary interest, namely, the application of the HEGY regression to a PI(1) process in which the special cases in Lemma 2 do not apply. That is, we consider cases where the misspecified constant parameter process of (12) has an invertible MA(3) component and, hence, as shown in Lemma 3, the HEGY regressors in (13) are nonstationary.

To reflect practice in applied research, and to account for the autocorrelation that usually applies, we here consider the generalization of (13) with AR augmentation, namely,

$$\Delta_4 y_{s\tau} = \pi_1 x_{s\tau}^{(1)} + \pi_2 x_{s\tau}^{(2)} + \pi_3 x_{s\tau}^{(3)} + \pi_4 x_{s\tau}^{(4)} + \sum_{i=1}^{p} \phi_i \Delta_4 y_{s-i,\tau} + u_{s\tau}. \qquad \textbf{(24)}$$

Clearly, results corresponding to no augmentation can be obtained as a special case of the following discussion with $p = 0$.

Now, denote the regressors in (24) by

$$X_{s\tau}^{H} = [x_{s\tau}^{(1)}, x_{s\tau}^{(2)}, x_{s\tau}^{(3)}, x_{s\tau}^{(4)}]', \qquad \Delta_4 Y_{s\tau}^{(p)} = [\Delta_4 y_{s-1,\tau}, \ldots, \Delta_4 y_{s-p,\tau}]'$$

and the corresponding estimated coefficient vectors as

$$\hat{\Pi} = [\hat{\pi}_1, \hat{\pi}_2, \hat{\pi}_3, \hat{\pi}_4]', \qquad \hat{\Phi}^{(p)} = [\hat{\phi}_1, \ldots, \hat{\phi}_p]'.$$

Then, appropriately scaling the cross-product matrices to reflect the different rates of convergence of the nonstationary and stationary regressors, we have

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

$$\begin{bmatrix} T\hat{\Pi} \\ T^{1/2}\hat{\Phi}^{(p)} \end{bmatrix} = \begin{bmatrix} T^{-2}\sum_{\tau=1}^{N}\sum_{s=1}^{4} X_{s\tau}^{H}(X_{s\tau}^{H})' & T^{-3/2}\sum_{\tau=1}^{N}\sum_{s=1}^{4} X_{s\tau}^{H}(\Delta_4 Y_{s\tau}^{(p)})' \\ T^{-3/2}\sum_{\tau=1}^{N}\sum_{s=1}^{4} \Delta_4 Y_{s\tau}^{(p)}(X_{s\tau}^{H})' & T^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} \Delta_4 Y_{s\tau}^{(p)}(\Delta_4 Y_{s\tau}^{(p)})' \end{bmatrix}^{-1} \times \begin{bmatrix} T^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} \Delta_4 y_{s\tau} X_{s\tau}^{H} \\ T^{-1/2}\sum_{\tau=1}^{N}\sum_{s=1}^{4} \Delta_4 y_{s\tau}\Delta_4 Y_{s\tau}^{(p)} \end{bmatrix}. \quad (25)$$

Because, as $T \to \infty$, the first matrix on the right-hand side of (25) is block diagonal, the scaled estimators $T\hat{\Pi}$ and $T^{1/2}\hat{\Phi}^{(p)}$ are asymptotically orthogonal. Therefore,

$$\hat{\Phi}^{(p)} \to \Gamma^{-1} g = \Phi^{(p)}, \quad (26)$$

where the $p \times p$ symmetric nonperiodic autocovariance matrix Γ has (i,j)th element given by $\gamma(i-j)$ for $i \geq j$ and the $p \times 1$ vector g has jth element equal to $\gamma(j)$, with $\gamma(j)$ as defined in (11). Note, in particular, that all elements of $\Phi^{(p)}$ depend on the order of augmentation, p.

Now, $T\hat{\Pi}$ in (25) can be expressed as

$$T\hat{\Pi} = [T^{-2}(X^H)'QX^H]^{-1}T^{-1}(X^H)'Q\Delta_4 Y, \quad (27)$$

where X^H and $\Delta_4 Y$ are $T \times 4$ and $T \times 1$, respectively, with $(X_{s\tau}^H)'$ being a generic row of X^H and $\Delta_4 y_{s\tau}$ a generic element of $\Delta_4 Y$. Further, Q is the $T \times T$ matrix $Q = I - Z(Z'Z)^{-1}Z'$, with $Z = [\Delta_4 Y_{-1}, \ldots, \Delta_4 Y_{-p}]$ and $\Delta_4 Y_{-j}$ being the jth lag of $\Delta_4 Y$. As in Phillips and Oularis (1990),

$$T^{-2}X^{H\prime}QX^H = T^{-2}X^{H\prime}X^H + o_p(1), \quad (28)$$

so that the denominator expressions for the scaled HEGY estimators are asymptotically unaffected by augmentation. Further, the HEGY variables retain the asymptotic orthogonality noted in Section 3.1 for the SI case, with this essentially being due to the constant parameter representation (12) being an invertible MA in the seasonal difference of $y_{s\tau}$.

These results lead to Theorem 2. To conserve space, asymptotic distributions are presented here only for the *t*-ratio statistics, with those corresponding to the normalized bias shown in the Appendix.

THEOREM 2. *Assume* $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ *in (1), with neither the restriction* $\alpha_s = 1$ $(s = 1,2,3,4)$ *nor those of Lemma 2 holding, and* $\mathrm{E}[\varepsilon_{s\tau}^2] = \sigma^2$. *Then for the HEGY regression (24) augmented with p lags, the asymptotic distributions are as follows.*

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

(i) The t-ratio test statistics for $\pi_j = 0$, $j = 1,2$, satisfy

$$t_{\hat{\pi}_j} \Rightarrow \frac{\Psi_j(4)\varphi_j^p(1)\int w(r)\,dw(r) + \left(\Psi_j(0) - \sum_{i=1}^{p}\phi_i^p\Psi_j(i)\right)}{\sqrt{\Psi_j(4)[\gamma(0) - \Phi^{(p)\prime}\Gamma\Phi^{(p)}]}\sqrt{\int [w(r)]^2\,dr}}$$

$$= t^{DF} \times \frac{\varphi_j^p(1)\sqrt{\Psi_j(4)}}{\sqrt{\gamma(0) - \Phi^{(p)\prime}\Gamma\Phi^{(p)}}}$$

$$+ \frac{\Psi_j(0) - \sum_{i=1}^{p}\phi_i^p\Psi_j(i)}{\sqrt{\Psi_j(4)[\gamma(0) - \Phi^{(p)\prime}\Gamma\Phi^{(p)}]}\sqrt{\int [w(r)]^2\,dr}}, \qquad j = 1,2. \quad \textbf{(29)}$$

(ii) The t-ratio test statistics for $\pi_j = 0$, $j = 3,4$, satisfy

$$t_{\hat{\pi}_j} \Rightarrow \frac{\Psi_3(4)\varphi_j^p(1)\int w(r)\,dw(r) + \left(\Psi_j(0) - \sum_{i=1}^{p}\phi_i^p\Psi_j(i)\right)}{\sqrt{\frac{1}{2}\Psi_3(4)[\gamma(0) - \Phi^{(p)\prime}\Gamma\Phi^{(p)}]}\sqrt{\int [w(r)]^2\,dr}}$$

$$= t^{DF} \times \frac{\varphi_j^p(1)\sqrt{2\Psi_3(4)}}{\sqrt{\gamma(0) - \Phi^{(p)\prime}\Gamma\Phi^{(p)}}}$$

$$+ \frac{\sqrt{2}\left[\Psi_j(0) - \sum_{i=1}^{p}\phi_i^p\Psi_j(i)\right]}{\sqrt{\Psi_3(4)[\gamma(0) - \Phi^{(p)\prime}\Gamma\Phi^{(p)}]}\sqrt{\int [w(r)]^2\,dr}}, \qquad j = 3,4, \quad \textbf{(30)}$$

where $w(r)$ is scalar standard Brownian motion, t^{DF} is the asymptotic Dickey–Fuller t-distribution, $\gamma(j)$ is defined in (10) and (11); $\Phi^{(p)} = [1, \phi_1^p, \ldots, \phi_p^p]' = \Gamma^{-1}g$, where the symmetric $p \times p$ matrix Γ has (i,j)th element $\gamma(i-j)$ and the $p \times 1$ vector g has $\gamma(j)$ as its jth element; $\Psi_1(i) = \sum_{k=-3}^{i-1}\gamma(k), \Psi_2(i) = \sum_{k=-3}^{i-1}(-1)^{k+i}\gamma(k)$, $i = 0,1,2,\ldots$; $\Psi_3(i) = \sum_{k=-1}^{(i/2)-1}(-1)^{k+(i/2)}\gamma(2k)$, $i = 0,2,4,\ldots$, $\Psi_3(1) = -\gamma(1) + \gamma(3)$, $\Psi_3(3) = -\gamma(3)$, $\Psi_3(i) = 0$, $i > 3$ and odd; $\Psi_4(i) = \Psi_3(i+1)$, $i = 0,1,2,\ldots$; $\varphi_1^p(1) = 1 - \sum_{i=1}^{p}\phi_i^p$, $\varphi_2^p(1) = 1 - \sum_{i=1}^{p}(-1)^i\phi_i^p$, $\varphi_3^p(1) = 1 - \sum_{i=1}^{[p/2]}(-1)^i\phi_{2i}^p$, and $\varphi_4^p(1) = \sum_{i=1}^{[(p+1)/2]}(-1)^{i+1}\phi_{2i-1}^p$, where $[\cdot]$ indicates the integer part of the expression in parentheses.

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

Note that, because $\Delta_4 y_{s\tau}$ is a (periodic) MA(3) process, $\gamma(k) = 0$, $k > 3$, and hence $\Psi_j(i) = \Psi_j(4)$ for $i > 4$ and $j = 1,2,3$.

Theorem 2 shows that the distribution of each of the four t-ratio tests is a function of a single Dickey–Fuller distribution. This is a consequence of the fact that the four quarters of the series are linked through the presence of three cointegration relationships, or equivalently that the long-run behavior of the four quarters is driven by the same common stochastic trend. Therefore, the limiting distributions of the HEGY test statistics applied at the zero, annual, and semiannual frequencies for a PI process, using a level of augmentation p, are not independent.

Although the distributions of (29) and (30) are informative, it is also relevant to consider the behavior of these distributions as the order of augmentation p increases. The existence of a limiting distribution for each normalized bias unit root test statistic for a PI(1) DGP implies that $\hat{\pi}_j \to 0$ ($j = 1,2,3,4$) in (24). This indicates that the behavior of the distributions in (29) and (30) as p increases depends on the extent to which the augmentation in (24) accounts for the MA process in the misspecified constant parameter representation (12). As shown in Lemma 2, this misspecified process is (in general) an invertible MA(3) in $\Delta_4 y_{s\tau}$ and hence can be approximated by an AR of sufficiently high order. Theorem 3 examines the consequences.

THEOREM 3. *Assume $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$ in (1), with neither the restriction $\alpha_s = 1 (s = 1,2,3,4)$ nor those of Lemma 2 holding, and $\mathrm{E}[\varepsilon_{s\tau}^2] = \sigma^2$. Then for the HEGY regression (24) estimated using T observations and augmented with p lags such that $p = O(\delta \ln N)$, $\delta > 0$, the asymptotic distributions are given as follows.*

(i) The t-ratio test statistics for $\pi_j = 0$, $j = 1,2$, satisfy

$$t_{\hat{\pi}_j} \Rightarrow \frac{\int w(r)\, dw(r)}{\sqrt{\int [w(r)]^2\, dr}} = t^{DF}, \qquad j = 1,2. \tag{31}$$

(ii) The F-type statistics for $\pi_3 = \pi_4 = 0$, $\pi_2 = \pi_3 = \pi_4 = 0$, and $\pi_1 = \pi_2 = \pi_3 = \pi_4 = 0$ satisfy, respectively,

$$F_{34} = \frac{1}{2}\sum_{j=3}^{4} t_{\hat{\pi}_j}^2 + o_p(1) \Rightarrow \left(\frac{\int w(r)\, dw(r)}{\sqrt{\int [w(r)]^2\, dr}} \right)^2 = (t^{DF})^2,$$

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

$$F_{234} = \frac{1}{3}\sum_{j=2}^{4} t^2_{\hat{\pi}_j} + o_p(1) \Rightarrow \left(\frac{\int w(r)\, dw(r)}{\sqrt{\int [w(r)]^2\, dr}} \right)^2 = (t^{DF})^2,$$

$$F_{1234} = \frac{1}{4}\sum_{j=1}^{4} t^2_{\hat{\pi}_j} + o_p(1) \Rightarrow \left(\frac{\int w(r)\, dw(r)}{\sqrt{\int [w(r)]^2\, dr}} \right)^2 = (t^{DF})^2, \quad \textbf{(32)}$$

where $w(r)$ is scalar standard Brownian motion and t^{DF} is the asymptotic Dickey–Fuller t-distribution.

Therefore, the individual tests applied for the zero- and semiannual-frequency unit roots both converge to the same Dickey–Fuller distribution when the process is PI(1), whereas all joint *F*-type statistics converge to the square of this distribution, as the order of augmentation increases. Although HEGY find that the Dickey–Fuller distribution asymptotically applies for $t_{\hat{\pi}_j}$, $j = 1, 2$, our result in (31) differs from the HEGY result, because the latter involves two independent Dickey–Fuller distributions. Similarly, the distributions in (32) differ from those reported by HEGY and Ghysels et al. (1994) that apply when the process is SI.

The results in (31) and (32) are established in the Appendix from the arguments of Galbraith and Zinde-Walsh (1999), who study the asymptotic distribution of a conventional ADF test in the presence of MA disturbances. The Appendix discussion includes the behavior of the individual *t*-ratio statistics of (30) for the pair of complex unit roots at the annual frequency as *p* increases. Here, in common with the result of Burridge and Taylor (2001) that in the presence of AR serial correlation the distributions of these statistics are not pivotal for the SI(1) case, the distributions of (30) are also not pivotal as *p* increases. However, the *F*-type statistic associated with this frequency is pivotal, as shown in (32).

These findings are supported by results collected in Tables 3 to 5.[4] These analyze the five PI processes of Table 1, with Tables 3 and 4 presenting analytical results for the shift and scale factors for the *t*-ratio tests in (29) at the zero- and semiannual-frequency π, with Table 5 showing corresponding factors for the joint *F*-type test at the annual frequency $\pi/2$, for orders of augmentation $p = 0, 1, \ldots, 8, 12, 24, 36, 60, 100$ lags. In each case, the shift and scale factors are expressed in relation to the asymptotic distributions of (31) and (32) that apply for $p = O(\delta \ln N)$.

Four points are worthy of note from Tables 3 and 4. First, the speed with which the shift factors tend to zero and the scale factors tend to one as the order of augmentation increases varies over the illustrative DGPs. Not surprisingly, this convergence is slowest for DGP3 and DGP5, which are the processes with roots of the constant parameter representation closest to the noninvertibility region in

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

TABLE 3. Scale and shift factors in the asymptotic distribution for zero-frequency test statistic

		Augmentation													
DGP	Factor	0	1	2	3	4	5	6	7	8	12	24	36	60	100
1	Scale	1.900	0.797	0.931	0.996	1.137	0.926	0.994	0.995	1.040	1.013	1.001	1.000	1.000	1.000
	Shift	0.687	−0.228	−0.072	−0.004	0.129	−0.077	−0.006	−0.005	0.039	0.013	0.001	0.000	0.000	0.000
2	Scale	1.952	0.769	0.913	1.013	1.188	0.889	0.985	1.002	1.076	1.036	1.006	1.001	1.000	1.000
	Shift	0.720	−0.265	−0.091	0.013	0.173	−0.118	−0.016	0.002	0.073	0.035	0.006	0.001	0.000	0.000
3	Scale	1.985	0.761	0.885	1.027	1.236	0.867	0.953	1.020	1.120	1.071	1.019	1.006	1.001	1.000
	Shift	0.740	−0.277	−0.122	0.026	0.214	−0.143	−0.048	0.020	0.113	0.068	0.019	0.006	0.001	0.000
4	Scale	1.915	0.899	0.789	1.013	1.209	0.970	0.882	1.016	1.101	1.057	1.012	1.003	1.000	1.000
	Shift	0.696	−0.106	−0.239	0.013	0.191	−0.030	−0.125	0.016	0.097	0.056	0.012	0.003	0.000	0.000
5	Scale	1.992	0.759	0.876	1.033	1.249	0.862	0.939	1.031	1.134	1.085	1.028	1.010	1.001	1.000
	Shift	0.742	−0.279	−0.133	0.033	0.224	−0.149	−0.063	0.031	0.126	0.081	0.028	0.010	0.001	0.000

Note: The test statistic is $t_{\hat{\pi}_1}$ in the augmented HEGY regression (24) for the periodic DGPs defined in Table 1 with $\sigma^2 = 1$. The scale and shift factors are defined by $A_1 = \sqrt{\Psi_1(4)}(1 - \sum_{i=1}^{p} \phi_i^p)/\sqrt{[\gamma(0) - \Phi^{(p)\prime}\Gamma\Phi^{(p)}]}$ and $B_1 = (\Psi_1(0) - \sum_{i=1}^{p} \phi_i^p \Psi_1(i))/\sqrt{\Psi_1(4)}\sqrt{[\gamma(0) - \Phi^{(p)\prime}\Gamma\Phi^{(p)}]}$, respectively, with $t_{\hat{\pi}_1} \Rightarrow A_1 \times t_{\hat{\pi}_1}^{DF} + B_1/\sqrt{\int [w(r)]^2\, dr}$; see (29). All notation is defined in the text.

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

TABLE 4. Scale and shift factors in the asymptotic distribution for semiannual-frequency statistic

		Augmentation													
DGP	Factor	0	1	2	3	4	5	6	7	8	12	24	36	60	100
1	Scale	0.199	0.474	0.554	0.517	0.591	0.725	0.778	0.778	0.814	0.921	0.994	1.000	1.000	1.000
	Shift	−2.412	−0.817	−0.626	−0.708	−0.551	−0.327	−0.253	−0.253	−0.207	−0.083	−0.006	0.000	0.000	0.000
2	Scale	0.107	0.272	0.323	0.291	0.341	0.456	0.505	0.496	0.533	0.681	0.911	0.977	0.998	1.000
	Shift	−4.608	−1.701	−1.385	−1.571	−1.294	−0.869	−0.738	−0.759	−0.671	−0.393	−0.094	−0.024	−0.002	0.000
3	Scale	0.072	0.187	0.218	0.188	0.226	0.322	0.354	0.331	0.363	0.485	0.752	0.890	0.980	0.999
	Shift	−6.934	−2.579	−2.188	−2.570	−2.100	−1.391	−1.235	−1.347	−1.197	−0.788	−0.288	−0.116	−0.020	−0.001
4	Scale	0.522	1.112	0.975	0.760	0.907	1.131	1.029	0.893	0.968	0.984	0.997	0.999	1.000	1.000
	Shift	−0.696	0.106	−0.025	−0.278	−0.098	0.123	0.028	−0.113	−0.033	−0.017	−0.003	−0.001	0.000	0.000
5	Scale	0.085	0.222	0.257	0.218	0.263	0.381	0.415	0.378	0.416	0.547	0.809	0.926	0.990	1.000
	Shift	−5.855	−2.137	−1.821	−2.190	−1.770	−1.122	−0.997	−1.134	−0.996	−0.642	−0.214	−0.077	−0.010	0.000

Note: The test statistic is $t_{\hat{\pi}_2}$ in the augmented HEGY regression (24) for the periodic DGPs defined in Table 1 with $\sigma^2 = 1$. The scale and shift factors are defined by $A_2 = \sqrt{\Psi_2(4)}(1 - \Sigma_{i=1}^{p}(-1)^i \phi_i^p)/\sqrt{[\gamma(0) - \Phi^{(p)\prime}\Gamma\Phi^{(p)}]}$ and $B_2 = (\Psi_2(0) - \Sigma_{i=1}^{p} \phi_i^p \Psi_2(i))/\sqrt{\Psi_2(4)[\gamma(0) - \Phi^{(p)\prime}\Gamma\Phi^{(p)}]}$, respectively, with $t_{\hat{\pi}_2} \Rightarrow A_2 \times t_{\hat{\pi}_1}^{DF} + B_2/\sqrt{\int [w(r)]^2\, dr}$; see (29). All notation is defined in the text.

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

TABLE 5. Scale and shift factors in the asymptotic distribution for joint annual frequency test statistic

		Augmentation													
DGP	Factor	0	1	2	3	4	8	9	10	11	12	24	36	60	100
1	Scale	0.698	1.022	0.962	1.102	1.153	1.700	1.940	1.928	1.898	1.903	1.996	2.000	2.000	2.000
	Shift	4.910	0.283	0.288	0.193	0.178	0.015	0.002	0.001	0.001	0.002	0.000	0.000	0.000	0.000
2	Scale	0.355	0.586	0.559	0.688	0.717	1.296	1.687	1.683	1.652	1.651	1.958	1.993	2.000	2.000
	Shift	10.592	0.950	0.950	0.648	0.625	0.104	0.021	0.015	0.019	0.021	0.000	0.000	0.000	0.000
3	Scale	0.112	0.195	0.186	0.251	0.268	0.610	0.906	0.900	0.961	0.966	1.674	1.911	1.993	2.000
	Shift	34.994	4.341	4.495	3.162	2.948	0.815	0.341	0.337	0.282	0.283	0.017	0.001	0.000	0.000
4	Scale	0.121	0.144	0.165	0.254	0.307	0.677	0.817	0.868	0.998	1.047	1.732	1.937	1.997	2.000
	Shift	27.640	6.076	5.165	3.096	2.402	0.651	0.432	0.371	0.254	0.218	0.010	0.001	0.000	0.000
5	Scale	0.047	0.082	0.079	0.109	0.118	0.291	0.451	0.447	0.507	0.511	1.186	1.629	1.938	1.997
	Shift	84.969	11.512	11.922	8.443	7.867	2.554	1.349	1.350	1.103	1.100	0.141	0.022	0.001	0.000

Note: The test statistic is F_{34} in the augmented HEGY regression (24) for the periodic DGPs defined in Table 1 with $\sigma^2 = 1$. The scale and shift factors are $A_3^2 + A_4^2$ and $B_3^2 + B_4^2$, respectively, where $A_3 = \sqrt{2\Psi_3(4)}(1 - \sum_{i=1}^{[p/2]}(-1)^i\phi_{2i}^p)/\sqrt{[\gamma(0) - \Phi^{(p)\prime}\Gamma\Phi^{(p)}]}$ and $B_3 = \sqrt{2}[\Psi_3(0) - \sum_{i=1}^{p}\phi_i^p\Psi_3(i)]/\sqrt{\Psi_3(4)[\gamma(0) - \Phi^{(p)\prime}\Gamma\Phi^{(p)}]}$ are the shift and scale factors in the asymptotic distribution for $t_{\hat{\pi}_3}$, whereas $A_4 = \sqrt{2\Psi_3(4)}(\sum_{i=1}^{[p/2]}(-1)^{i+1}\phi_{2i-1}^p)/\sqrt{[\gamma(0) - \Phi^{(p)\prime}\Gamma\Phi^{(p)}]}$ and $B_4 = \sqrt{2}[\Psi_4(0) - \sum_{i=1}^{p}\phi_i^p\Psi_4(i)]/\sqrt{\Psi_3(4)[\gamma(0) - \Phi^{(p)\prime}\Gamma\Phi^{(p)}]}$ are the shift and scale factors in the asymptotic distribution for $t_{\hat{\pi}_4}$; see (30). All notation is defined in the text.

Table 2. Second, convergence to the respective limits is not monotonic for $p = 0,1,\ldots,8$ as a result of the effect of seasonality. Third, the shift and scale factors operate in opposite directions in the two tables, so that for lags that are multiples of the seasonal frequency of four, the zero-frequency distribution is inflated and shifted to the right, whereas the semiannual distribution is scaled down and shifted to the left, in relation to the asymptotic Dickey–Fuller distribution. Finally, the factors approach their limits much more quickly for the zero-frequency statistic in Table 3 as p increases than for the semiannual test statistic shown in Table 4. For example, $p = 12$ results in relatively modest distortion in relation to the t^{DF} distribution for all DGPs in the zero-frequency case, whereas substantial distortions are implied for the semiannual test statistic with this augmentation for DGPs 2, 4, and 5.

Empirical seasonal unit root analyses commonly use the joint F_{34} statistic, proposed by HEGY, for testing the pair of complex unit roots that exist at frequency $\pi/2$ under the SI(1) null hypothesis, with the results of Burridge and Taylor (2001) underlining the importance of this choice because of the nonpivotal nature of the individual t-ratio tests in the presence of (AR) serial correlation. From Table 5 it is clear that the shift factors of the two t-ratio tests tend to zero. Although the scale factors for the individual t-ratio tests (not shown) do not individually tend to zero, Table 5 shows that the sum of the squared scale factors tends to 2, which is precisely what is required for the F-type statistic to follow the square of the Dickey–Fuller distribution as in (32). Once again, DGP3 and (more particularly) DGP5 exhibit the slowest convergence to this asymptotic distribution as p increases.

4. MONTE CARLO ANALYSIS

To examine how the asymptotic results of Section 3 impact on finite-sample tests for seasonal integration when the true process is PI(1) or I(1), we conduct a Monte Carlo experiment using the DGPs of Table 1. For each DGP, we generate 5,000 replications of 100 observations (corresponding to $N = 25$ years of data) and obtain the HEGY test statistics using the test regression (24). More specifically, we separately consider the null three hypotheses $\pi_1 = 0$, $\pi_2 = 0$, and $\pi_3 = \pi_4 = 0$, corresponding to the presence of unit roots at the zero, semiannual, and annual frequencies, respectively. The first two hypothesis tests are conducted using the relevant t-ratios ($t_{\hat{\pi}_1}$ and $t_{\hat{\pi}_2}$), whereas the last uses the joint F_{34} statistic proposed by HEGY. In addition, we compute joint F-type statistics for the hypotheses of unit roots at all seasonal frequencies and of seasonal integration using the F_{234} and F_{1234} statistics, respectively. All results are presented as the proportion of times that the relevant null hypothesis is rejected, using a nominal significance level of 5% for an SI(1) process.[5] Results are shown in Table 6. Augmentation is considered with $p = 0$, 4, 8, 12 lags (up to three years) of the dependent variable. In addition, the Akaike information cri-

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

TABLE 6. Finite-sample proportion of rejections for HEGY null hypotheses

DGP	AUG	$t_{\hat{\pi}_1}$	$t_{\hat{\pi}_2}$	F_{34}	F_{234}	F_{1234}
1	0	0.02	0.89	0.73	0.86	0.93
	4	0.04	0.20	0.15	0.24	0.22
	8	0.05	0.07	0.08	0.12	0.13
	12	0.05	0.04	0.07	0.09	0.12
	NP	0.05	0.09	0.11	0.14	0.16
	AIC	0.06	0.29	0.20	0.28	0.29
2	0	0.02	1.00	0.89	0.99	1.00
	4	0.04	0.50	0.25	0.45	0.41
	8	0.04	0.20	0.11	0.21	0.19
	12	0.04	0.09	0.07	0.12	0.13
	NP	0.05	0.17	0.12	0.18	0.19
	AIC	0.06	0.52	0.33	0.47	0.45
3	0	0.04	1.00	1.00	1.00	1.00
	4	0.04	0.73	0.53	0.71	0.68
	8	0.04	0.37	0.24	0.39	0.35
	12	0.04	0.17	0.12	0.21	0.19
	NP	0.05	0.25	0.19	0.27	0.26
	AIC	0.05	0.66	0.57	0.64	0.63
4	0	0.03	0.20	0.99	0.98	1.00
	4	0.04	0.06	0.51	0.45	0.41
	8	0.04	0.04	0.24	0.23	0.21
	12	0.04	0.03	0.13	0.14	0.14
	NP	0.05	0.05	0.21	0.21	0.21
	AIC	0.07	0.10	0.58	0.55	0.53
5	0	0.04	1.00	1.00	1.00	1.00
	4	0.04	0.62	0.79	0.82	0.80
	8	0.04	0.28	0.44	0.50	0.45
	12	0.03	0.13	0.22	0.27	0.24
	NP	0.05	0.21	0.31	0.34	0.32
	AIC	0.04	0.65	0.72	0.73	0.71
6	0	0.05	1.00	1.00	1.00	1.00
	4	0.05	1.00	1.00	1.00	1.00
	8	0.04	0.90	0.99	1.00	1.00
	12	0.03	0.66	0.89	0.98	0.96
	NP	0.05	0.78	0.93	0.98	0.97
	AIC	0.05	1.00	1.00	1.00	1.00

Note: Based on 5,000 replications, with $T = 100$ ($N = 25$ years) for the DGPs of Table 1 with zero starting values. Here $t_{\hat{\pi}_1}$ and $t_{\hat{\pi}_2}$ are tests for $\pi_1 = 0$ and $\pi_2 = 0$, respectively, F_{34}, F_{234}, and F_{1234} are joint tests for $\pi_3 = \pi_4 = 0$, $\pi_2 = \pi_3 = \pi_4 = 0$, and $\pi_1 = \pi_2 = \pi_3 = \pi_4 = 0$, respectively, all in the HEGY regression (24) with a nominal test size of 0.05. AUG is the order of augmentation, where NP indicates use of the Ng and Perron (1995) sequential procedure and AIC the Akaike information criterion (both to a maximum 12 lags).

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

terion (AIC) and the sequential method recently proposed by Ng and Perron (1995) are used for lag selection (up to a maximum of 12 lags).

Although the statistic $t_{\hat{\pi}_1}$ is undersized (compared to an ADF unit root test) when no augmentation is undertaken, augmentation fairly quickly leads to reasonable size for all cases in Table 6. The exception is for the random walk DGP (namely, DGP6) where, in line with the analysis of Section 3.2, the correct size is obtained with no augmentation. Both the Ng–Perron criterion and AIC generally work well in selecting the lag length to deliver an empirical size of 5% for this test in PI(1) processes, with the former being particularly reliable in this respect. Thus, the asymptotic result that yields the Dickey–Fuller distribution for $t_{\hat{\pi}_1}$ apparently carries over to the finite-sample case, provided reasonable orders of augmentation are applied.

Turning to the seasonal unit roots tests, the rejection rates are close to unity with no augmentation. The only notable exception is $t_{\hat{\pi}_2}$ for DGP4, where the coefficient of the real factor implies that the semiannual root is relatively far from unity (see Table 2). With the unsurprising exception of the random walk DGP6, increasing the order of augmentation leads to dramatic declines in the proportion of rejections of seasonal unit roots by all test statistics ($t_{\hat{\pi}_2}$, F_{34}, F_{234}, F_{1234}). Although the nominal size of the test using $t_{\hat{\pi}_2}$ moves closer to 0.05 as the order of augmentation increases for the PI(1) DGPs, an order of 12 is sometimes insufficient for the asymptotic Dickey–Fuller distribution to apply; see particularly DGP3 and DGP5. Indeed, when AIC is used for lag selection, the test is very badly oversized in relation to the DF distribution. Although the Ng–Perron criterion is again better in this respect, it is clear that the optimal lag order is typically selected too low. For the random walk DGP6, the statistic has a rejection rate of unity for low orders of augmentation, although this decreases as p increases, indicating some loss of power.

The size analysis is more complicated for the joint F-type statistics, because each of these follows the asymptotic distribution of a squared Dickey–Fuller t-ratio statistic as p increases but this is not the distribution to which the statistics are compared when testing an SI(1) null hypothesis. Indeed, because the critical values, tabulated by HEGY and Ghysels et al. (1994), for joint tests relating to an SI(1) null hypothesis are smaller than the squared Dickey–Fuller critical value,[6] the asymptotic result in (32) implies that such tests applied to a PI(1) process will substantially overreject compared to an SI(1) DGP even when the augmentation is sufficient to account for the presence of autocorrelation. This applies also in Table 6, which finds substantial empirical rejection rates for DGPs 1–5. However, the joint hypotheses are correctly rejected in almost all cases when the DGP is a random walk.

It is not surprising that the probability of finding evidence of unit roots in these periodic processes depends on the specific parameter values α_j and the corresponding constant parameter MA components shown in Table 2. Nevertheless, it is also clear that the HEGY test procedure will frequently lead to the conclusion that the PI(1) process is SI. For example, basing lag selection on

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

AIC, Table 6 shows that the F_{1234} statistic rejects the seasonal integration null hypothesis only 29% of the time for DGP1. This rejection frequency rises to 71% for DGP5, although the use of the Ng–Perron lag selection procedure would reduce the rejection frequency for this DGP to 32%. Indeed, one striking feature is how the dependence of the asymptotic distributions for the HEGY test statistics on the order of augmentation p is evidenced in Table 6 by the different finite-sample rejection frequencies resulting from the use of AIC and the Ng–Perron criterion.

5. CONCLUDING REMARKS

This paper has tackled the consequences of testing for seasonal unit roots in a process that is, in fact, either a random walk or a PI process. The true process under consideration therefore has a single unit root, with no distinct unit roots at the seasonal frequencies. We derive the asymptotic distributions of the unit root test statistics, showing that all commonly used t-ratio and F-type statistics converge to the same Dickey–Fuller distribution (or its square) for sufficiently high orders of augmentation when the DGP is a PI process. For a given order of augmentation, however, the distributions are subject to shift and scaling effects, which is a consequence of the MA autocorrelation present in the HEGY test regression.

A simple interpretation of our results is that a PI process behaves in the same way as the misspecified constant parameter process in the annual difference of the series. This interpretation is correct in the sense that the autocorrelation properties of the HEGY test regression are the same as those of this misspecified process. However, the interpretation does not apply when considering unit roots, because the SI process of interest to HEGY contains four unit roots, whereas a PI process has only one.

Our Monte Carlo analysis indicates that the empirical finite-sample size of the zero-frequency unit root test statistic for a PI process is relatively close to the nominal size when the regression is sufficiently augmented. In practice, therefore, the zero-frequency unit root test detects not only a unit root at this frequency in a nonperiodic process but also the single unit root in a PI process.

PI processes do not contain seasonal unit roots. However, the transformed variables used in the HEGY seasonal unit root test do not remove the nonstationarity in a periodic process. Consequently, the use of these variables in a seasonal unit root test regression may lead to the conclusion that seasonal unit roots are present in the process. Indeed, depending on the specific parameter values of the DGP and the criterion used for selecting the order of augmentation, the conclusion of seasonal integration can have a high probability.

In addition to the analysis of PI processes, we show that the asymptotic Dickey–Fuller distribution continues to apply for the zero-frequency unit root test statistic in the important special case of a random walk. Further, the seasonal unit root test statistics diverge to infinity in this case. Therefore, although the HEGY seasonal unit root test regression has been developed under the null

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

hypothesis of an SI process, it continues to be applicable when only the zero-frequency unit root is present.

Although our analysis considers a zero mean PI(1) process with no deterministic components in the HEGY test regression, the implications of our results extend to more general cases. More specifically, with the inclusion of deterministic terms (such as seasonal dummies or seasonal dummies and a linear trend) in the test regression, the results will carry over when expressed using demeaned and detrended standard Brownian motions, as in Boswijk and Franses (1996). Further, as shown by Smith and Taylor (1998), the inclusion of seasonal dummies in the HEGY regression makes the test statistics invariant to starting values, and seasonal trends further yield invariance to seasonal drifts. These results can be expected to apply also in our context.

Although we consider only first-order periodic processes, the results carry over to higher order PI processes with a single unit root, provided that the HEGY test regression is sufficiently augmented. This follows as the stationary periodic component of the PI(1) process has a misspecified stationary ARMA representation, which can be approximated by AR augmentation. Finally, although we consider the case of a nonperiodic disturbance variance, Osborn (1991) derives the misspecified constant parameter representation of a PAR process with periodic disturbance variance. Because this representation provides the basis of our analysis, we anticipate that the results obtained here will also apply in this more general context.

NOTES

1. See Franses (1994) for further discussion of the relationship between the VMA representation and the HEGY seasonal unit root tests.

2. Burridge and Taylor (2001) emphasize the use of this F_{34} statistic rather than separate t-ratios on these coefficients, because the asymptotic distributions of these t-ratio statistics under the unit root null hypothesis are affected by the presence of stationary serial correlation, whereas the F-statistic is not.

3. Of course, if the DGP is a more general I(1) process, then augmentation is required to account for serial correlation.

4. Although not included here, figures showing the evolution of the empirical distribution of $t_{\tilde{\pi}_1}$, $t_{\tilde{\pi}_2}$, and F_{34} for processes in Table 1 as the order of augmentation increases are available upon request from the authors.

5. Critical values for tests of $\pi_1 = 0$ and $\pi_2 = 0$ are taken from the Dickey–Fuller critical values presented for a sample size of 25 observations in Hamilton (1994). The joint test of $\pi_3 = \pi_4 = 0$ uses the critical values for this sample size in HEGY. Finally, the remaining joint tests use critical values presented by Ghysels et al. (1994).

6. In making this comparison, we use the case corresponding to the theoretical analysis of Section 3, with no intercept, trend, or seasonal dummies included in the test regression.

REFERENCES

Boswijk, H.P. & P.H. Franses (1996) Unit roots in periodic autoregressions. *Journal of Time Series Analysis* 17, 221–245.

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

Burridge, P. & A.M.R. Taylor (2001) On the properties of regression-based tests for seasonal unit roots in the presence of higher-order serial correlation. *Journal of Business & Economic Statistics* 19, 374–379.

Darroch, J., M. Jiřich, & J. McDonald (1986) The sum of finite order moving average processes. *Journal of Time Series Analysis* 7, 21–25.

Davis, P.J. (1979) *Circulant Matrices*. Wiley-Interscience.

Dickey, D.A. & W.A. Fuller (1979) Distribution of the estimators for autoregressive time series with a unit root. *Journal of the American Statistical Association* 74, 427–431.

Dickey, D.A., D.P. Hasza, & W.A. Fuller (1984) Testing for unit roots in seasonal time series. *Journal of the American Statistical Association* 79, 355–367.

Franses, P.H. (1994) A multivariate approach to modelling univariate seasonal time series. *Journal of Econometrics* 63, 133–151.

Franses, P.H. (1996) *Periodicity and Stochastic Trends in Economic Time Series*. Oxford University Press.

Galbraith, J.W. & V. Zinde-Walsh (1999) On the distributions of augmented Dickey–Fuller statistics in processes with moving average components. *Journal of Econometrics* 93, 25–47.

Gersovitz, M. & J.G. McKinnon (1978) Seasonality in regression: An application of smoothness priors. *Journal of the American Statistical Association* 73, 264–273.

Ghysels, E., H.S. Lee, & J. Noh (1994) Testing for unit roots in seasonal time series: Some theoretical extensions and a Monte Carlo investigation. *Journal of Econometrics* 62, 415–442.

Ghysels, E. & D.R. Osborn (2001) *The Econometric Analysis of Seasonal Time Series*. Cambridge University Press.

Hamilton, J.D. (1994) *Time Series Analysis*. Princeton University Press.

Hansen, L.P. & T.J. Sargent (1993) Seasonality and approximation errors in rational expectations models. *Journal of Econometrics* 55, 21–56.

Hylleberg, S., R.F. Engle, C.W.J. Granger, & B.S. Yoo (1990) Seasonal integration and cointegration. *Journal of Econometrics* 44, 215–238.

Ng, S. & P. Perron (1995) Unit root tests in ARMA models with data dependent methods for the truncation lag. *Journal of the American Statistical Association* 90, 268–281.

Osborn, D.R. (1988) Seasonality and habit persistence in a life cycle model of consumption. *Journal of Applied Econometrics* 3, 255–266.

Osborn, D.R. (1991) The implications of periodically varying coefficients for seasonal time series. *Journal of Econometrics* 28, 373–384.

Osborn, D.R., A.P.L. Chui, J.P. Smith, & C.R. Birchenhall (1988) Seasonality and the order of integration for consumption. *Oxford Bulletin of Economics and Statistics* 50, 361–377.

Osborn, D.R. & P.M.M. Rodrigues (2002) Asymptotic distributions of seasonal unit root tests: A unifying approach. *Econometric Reviews* 21, 221–241.

Phillips, P.C.B. (1988) Weak convergence of sample covariance matrices to stochastic integrals via martingale approximations. *Econometric Theory* 4, 528–533.

Phillips, P.C.B. & S. Oularis (1990) Asymptotic properties of residual based tests for cointegration. *Econometrica* 58, 165–193.

Rodrigues, P.M.M. (2001) Near seasonal integration. *Econometric Theory* 17, 70–86.

Said, E.S. & D.A. Dickey (1984) Testing for unit roots in autoregressive-moving average models of unknown order. *Biometrika* 71, 599–607.

Sanso, A., M. Artis, & J. Surinyach (1997) Comportamiento en muestra finita de los contrastes de integración estacional para datos mensuales. *Estadística Espanola* 39, 141–184.

Smith, R.J. & A.M.R. Taylor (1998) Additional critical values and asymptotic representations for seasonal unit root tests. *Journal of Econometrics* 85, 269–288.

Taylor, A.M.R. (2003) On the asymptotic properties of some seasonal unit root tests. *Econometric Theory* 19, 311–321.

Taylor, A.M.R. (2002) Regression-based unit root tests with recursive mean adjustment for seasonal and nonseasonal time series. *Journal of Business & Economic Statistics* 20, 269–281.

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

Tiao, G. & M. Grupe (1980) Autoregressive-moving average models in time series data. *Biometrika* 67, 365–373.

APPENDIX: Proofs

Proof of Lemma 2. All MAs have an invertible representation unless they contain one or more roots on the unit circle. For the constant parameter representation of the PI process, given in (12), Osborn (1991) shows that the MA(3) component is equivalent to the aggregate of four independent hypothetical processes with coefficients

$$1 + \alpha_s L + \alpha_s \alpha_{s-1} L^2 + \alpha_s \alpha_{s-1} \alpha_{s-2} L^3, \qquad s = 1,2,3,4 \tag{A.1}$$

with, in our case, equal disturbance variances of $\sigma^2/4$. In (A.1), a subscript $s - j < 0$ should be understood to refer to the quarter $s - j + 4$.

Because all $\alpha_s > 0$ in (A.1), the aggregated MA of (12) has $\theta_i > 0$ $(i = 1,2,3)$. By considering the factorization $(1 + \theta_1 L + \theta_2 L^2 + \theta_3 L^3) = (1 - \lambda L)(1 + \theta_1^* L + \theta_2^* L^2)$, it can be seen that a real root satisfies $\lambda^{-1} < 0$ and hence the only possible real MA unit root is -1. Also, as $-1 \leq \lambda < 0$, the requirement of positive MA coefficients in (12) further implies that the only possible complex unit roots are the pair $\pm i$. Therefore, any MA unit roots in (12) are one or more of the seasonal unit roots $-1, \pm i$.

Now, Darroch, Jiřich, and McDonald (1986) show that the aggregate of MA processes contains a root on the unit circle only if each component MA process contains this root. For the noninvertible complex root $-i$, substituting $L = -i^{-1} = i$ in (A.1) implies that the presence of this unit root requires each component process to satisfy $1 - \alpha_s \alpha_{s-1} + \alpha_s (1 - \alpha_{s-1} \alpha_{s-2}) i = 0$. With real PAR coefficients, this in turn requires $\alpha_s \alpha_{s-1} = 1$ $(s = 1,2,3,4)$, leading to the condition in part (i) of Lemma 2. It is straightforward to see that the same condition is also implied by the complex root i.

In terms of (A.1), the root -1 implies $1 - \alpha_s + \alpha_s \alpha_{s-1} - \alpha_s \alpha_{s-1} \alpha_{s-2} = 0$ $(s = 1,2,3,4)$. By substituting the PI restriction $\alpha_s \alpha_{s-1} \alpha_{s-2} = (\alpha_{s+1})^{-1}$, it can be seen that if this condition holds for any s then it must also hold for $s + 1$. Therefore, an MA unit root of -1 in (12) implies a single additional restriction for the PI coefficients, which can be written as in Lemma 2(ii). ■

Proof of Lemma 5. From (5) and (6), it can be seen that each column of $D(1)$ gives the lag coefficients relating to one of the hypothetical uncorrelated MA(3) processes whose aggregate is the constant parameter process, as discussed by Ghysels and Osborn (2001, p. 150). Specifically, the elements of this matrix are

$$d_{s+j,s} = \begin{cases} 1 & j = 0 \\ \prod_{i=1}^{j} \alpha_{s+i} & j = 1,2,3, \end{cases} \tag{A.2}$$

where it is understood that $s + j = s + j - 4$ when $s + j > 4$.

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

Using (A.2), and using (10) and (11), it is then relatively straightforward to see that the constant parameter process corresponding to the PI(1) has

$$\text{Var}(\Delta_4 y_{s\tau}) = \gamma(0) = \frac{\sigma^2}{4}\,\text{tr}[D(1)D(1)'] = \frac{\omega^2 a'a}{4}, \tag{A.3}$$

where $\omega^2 = \sigma^2 b'b$. The other results in (20) can also be established using (A.2) in conjunction with (10) and (11). For example, the sum of all nonzero autocovariances of the constant parameter process is given by

$$\sum_{k=-3}^{3} \gamma(k) = \Psi_1(4) = \frac{\sigma^2}{4}\,\text{tr}[D(1)C_1 D(1)'] = \frac{\sigma^2}{4}(b'b)a'C_1 a = \omega^2 a'C_1 a. \tag{A.4}$$

■

Proof of Theorem 1. The asymptotic orthogonality result in part (i), when the estimators are scaled by M, follows directly from part (ii) of Lemma 4.

To prove part (ii), note that the scaled estimator $T\hat{\pi}_1$ of π_1 in (13) can be expressed as

$$T\hat{\pi}_1 = \frac{(4N)^{-1}X^{(1)\prime}Q\Delta_4 Y}{(4N)^{-2}X^{(1)\prime}QX^{(1)}}, \tag{A.5}$$

where $X^{(1)}$ and $\Delta_4 Y$ are $4N \times 1$ vectors with generic elements $x_{s\tau}^{(1)}$ and $\Delta_4 y_{s\tau}$, respectively, and Q is a $4N \times 4N$ matrix $Q = I - Z(Z'Z)^{-1}Z'$, with the columns of Z having generic elements $x_{s\tau}^{(2)}$, $x_{s\tau}^{(3)}$, and $x_{s\tau}^{(4)}$. As $(4N)^{-2}X^{(1)\prime}QX^{(1)} = (4N)^{-2}X^{(1)\prime}X^{(1)} + o_p(1)$ (as in Phillips and Oularis, 1990), the denominator of (A.5) is

$$(4N)^{-2}X^{(1)'}QX^{(1)} = (4N)^{-2}X^{(1)'}X^{(1)} + o_p(1) = (4N)^{-2}\sum_{\tau=1}^{N}\sum_{s=1}^{4}[x_{s\tau}^{(1)}]^2 + o_p(1)$$

$$\Rightarrow \frac{\omega^2 a'C_1 a}{4}\int [w(r)]^2\,dr = 16\sigma^2\int [w(r)]^2\,dr. \tag{A.6}$$

Note that, as $x_{s\tau}^{(1)}$ is nonstationary for the random walk DGP, the first result in part (ii) of Lemma 3 applies in this case, with $\omega^2 a'C_1 a = 64\sigma^2$.

For the numerator, note first that under the null hypothesis $\pi_1 = 0$ in (13), $Q\Delta_4 Y = QU$ where U is a $4N \times 1$ vector with generic element $u_{s\tau}$. Therefore, the numerator of (A.5) is

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

$$(4N)^{-1}X^{(1)\prime}Q\Delta_4 Y = (4N)^{-1}X^{(1)\prime}QU = (4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} x_{s\tau}^{(1)}u_{s\tau} + o_p(1).$$

Because in this case, $u_{s\tau} = \varepsilon_{s\tau}$ is white noise, we have

$$(4N)^{-1}X^{(1)\prime}Q\Delta_4 Y = (4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} x_{s\tau}^{(1)}\varepsilon_{s\tau} + o_p(1) = (4N)^{-1}\sum_{\tau=1}^{N} Y'_{\tau-1}C_1 U_\tau + o_p(1)$$

$$\Rightarrow \frac{1}{4}\int B'(r)C_1\, dW(r) = \int W'(r)C_1\, dW(r),$$

where $U_\tau = (\varepsilon_{1\tau}, \varepsilon_{2\tau}, \varepsilon_{3\tau}, \varepsilon_{4\tau})'$. In the last step we use $B'(r) = W'(r)ba'$ from the definition of $B(r)$ in Lemma 1, and as $a = b = [1,1,1,1]'$, we have $b'a = C_1$ and $b'aC_1 = 4C_1$. Furthermore as $\int W'(r)C_1\, dW(r) = 4\sigma^2\int w(r)\, dw(r)$, then

$$(4N)^{-1}X^{(1)\prime}Q\Delta_4 Y \Rightarrow 4\sigma^2\int w(r)\, dw(r). \tag{A.7}$$

Therefore, from (A.6) and (A.7),

$$T\hat{\pi}_1 \Rightarrow \frac{4\sigma^2\int w(r)\, dw(r)}{16\sigma^2\int [w(r)]^2\, dr} = \frac{1}{4}\frac{\int w(r)\, dw(r)}{\int [w(r)]^2\, dr}. \tag{A.8}$$

For part (iii), consider the asymptotic limit of the residual variance $\hat{\sigma}_u^2$ when (13) is estimated for a random walk DGP. The definitions of (14), together with the identity $\Delta_4 = \Delta(1 + L + L^2 + L^3)$ and the DGP $(1 - L)y_{s\tau} = \varepsilon_{s\tau}$, imply that

$$\Delta_4 y_{s\tau} + 0.5x_{s\tau}^{(2)} + 0.5x_{s\tau}^{(3)} + 0.5x_{s\tau}^{(4)} = \varepsilon_{s\tau}.$$

Because, from part (iv) established in what follows, $\hat{\Pi}_2$ converges to $(-0.5, -0.5, -0.5)'$, it follows that $\hat{\sigma}_u^2 \to \mathrm{Var}(\varepsilon_{s\tau})$. Consequently, again using (A.6) and (A.7),

$$t_{\hat{\pi}_1} = \frac{T^{-1}X^{(1)\prime}Q\Delta_4 Y}{\sqrt{\hat{\sigma}^2 T^{-2}X^{(1)\prime}QX^{(1)}}} \Rightarrow \frac{4\sigma^2\int w(r)\, dw(r)}{\sqrt{\sigma^2 \times 16\sigma^2\int [w(r)]^2\, dr}} = \frac{\int w(r)\, dw(r)}{\sqrt{\int [w(r)]^2\, dr}}.$$

For part (iv), asymptotic orthogonality between the (scaled) estimators of π_1 and Π_2 implies that $\hat{\Pi}_2$ satisfies

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

$$\begin{bmatrix} \sum_{s=1}^{4}\sum_{\tau=1}^{N}[x_{s\tau}^{(2)}]^2/N & \sum_{s=1}^{4}\sum_{\tau=1}^{N}x_{s\tau}^{(2)}x_{s\tau}^{(3)}/N & \sum_{s=1}^{4}\sum_{\tau=1}^{N}x_{s\tau}^{(2)}x_{s\tau}^{(4)}/N \\ & \sum_{s=1}^{4}\sum_{\tau=1}^{N}[x_{s\tau}^{(3)}]^2/N & \sum_{s=1}^{4}\sum_{\tau=1}^{N}x_{s\tau}^{(3)}x_{s\tau}^{(4)}/N \\ & & \sum_{s=1}^{4}\sum_{\tau=1}^{N}[x_{s\tau}^{(4)}]^2/N \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} \sum_{s=1}^{4}\sum_{\tau=1}^{N}x_{s\tau}^{(2)}\Delta_4 y_{s\tau}/N \\ \sum_{s=1}^{4}\sum_{\tau=1}^{N}x_{s\tau}^{(3)}\Delta_4 y_{s\tau}/N \\ \sum_{s=1}^{4}\sum_{\tau=1}^{N}x_{s\tau}^{(4)}\Delta_4 y_{s\tau}/N \end{bmatrix}$$

$$\to \begin{bmatrix} \mathrm{Var}(x_{s\tau}^{(2)}) & \mathrm{Cov}(x_{s\tau}^{(2)},x_{s\tau}^{(3)}) & \mathrm{Cov}(x_{s\tau}^{(2)},x_{s\tau}^{(3)}) \\ & \mathrm{Var}(x_{s\tau}^{(3)}) & \mathrm{Cov}(x_{s\tau}^{(3)},x_{s\tau}^{(4)}) \\ & & \mathrm{Var}(x_{s\tau}^{(4)}) \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} \mathrm{Cov}(x_{s\tau}^{(2)},\Delta_4 y_{s\tau}) \\ \mathrm{Cov}(x_{s\tau}^{(3)},\Delta_4 y_{s\tau}) \\ \mathrm{Cov}(x_{s\tau}^{(4)},\Delta_4 y_{s\tau}) \end{bmatrix}$$

$$= \begin{bmatrix} 2\sigma^2 & \sigma^2 & \sigma^2 \\ & 2\sigma^2 & \sigma^2 \\ & & 2\sigma^2 \end{bmatrix}^{-1} \begin{bmatrix} -2\sigma^2 \\ -2\sigma^2 \\ -2\sigma^2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}.$$

■

Proof of Theorem 2. Because of the block diagonality between the nonstationary and stationary regressors in (24), we can consider the coefficients of the nonstationary HEGY regressors separately from the augmentation. Then the asymptotic orthogonality of the HEGY (scaled) regressors for a general PI(1) process, established in Lemma 3, allows each to be considered individually, so that, from (25)–(28),

$$T\hat{\pi}_j = \frac{(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4}x_{s\tau}^{(j)}\left(\Delta_4 y_{s\tau} - \sum_{i=1}^{p}\phi_i^p \Delta_4 y_{s-i,\tau}\right)}{(4N)^{-2}\sum_{\tau=1}^{N}\sum_{s=1}^{4}[x_{s\tau}^{(j)}]^2} + o_p(1), \qquad j = 1,2,3,4 \tag{A.9}$$

and

$$t_{\hat{\pi}_j} = \hat{\sigma}_u^{-1} T\hat{\pi}_j \times \sqrt{T^{-2}\sum_{\tau=1}^{N}\sum_{s=1}^{4}(x_{s\tau}^{(j)})^2} + o_p(1), \tag{A.10}$$

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

where

$$\hat{\sigma}_u^2 = \mathrm{E}([\Delta_4 y_{s\tau} - \Phi^{(p)\prime}\Delta_4 Y_{s\tau}^{(p)}][\Delta_4 y_{s\tau} - \Phi^{(p)\prime}\Delta_4 Y_{s\tau}^{(p)}]')$$

$$= \mathrm{E}[\Delta_4 y_{s\tau}]^2 - 2\Phi^{(p)\prime}\mathrm{E}[\Delta_4 Y_{s\tau}^{(p)}\Delta_4 y_{s\tau}] + \Phi^{(p)\prime}\mathrm{E}[\Delta_4 Y_{s\tau}^{(p)}\Delta_4 Y_{s\tau}^{(p)\prime}]\Phi^{(p)\prime}$$

$$= \gamma(0) - \Phi^{(p)\prime}\Gamma\Phi^{(p)}. \qquad \textbf{(A.11)}$$

Define $\Delta_4 y_{s\tau} = \nu_{s\tau}$ and consider first the HEGY variable corresponding to the conventional unit root 1, for which, using the definition of (14),

$$\sum_{s=1}^{4} x_{s\tau}^{(1)}\Delta_4 y_{s\tau} = \sum_{s=1}^{4}(y_{s-1,\tau} + y_{s-2,\tau} + y_{s-3,\tau} + y_{s,\tau-1})\nu_{s\tau}$$

$$= Y_{\tau-1}' C_1 V_\tau + \sum_{s=2}^{4}\sum_{k=1}^{s-1} v_{s\tau}v_{s-k,\tau},$$

where C_1 is defined as $C_1 = Circ[1,1,1,1]$. Noting that V_τ is autocorrelated, the theorem of Phillips (1988) implies that

$$(4N)^{-1}\sum_{\tau=1}^{N} Y_{\tau-1}' C_1 V_\tau \Rightarrow \frac{1}{4}\int B'(r)C_1\, dB(r) + \Psi_{12},$$

where

$$\Psi_{12} = \frac{1}{4}\mathrm{E}(V_{\tau-1}' C_1 V_\tau) = \frac{1}{4}\sum_{s=1}^{3}\sum_{j=s+1}^{4}\mathrm{E}(\Delta_4 y_{s\tau}\Delta_4 y_{j,\tau-1}) = \frac{1}{4}\sum_{s=1}^{3}\sum_{k=s}^{3}\gamma_s(k).$$

Further,

$$(4N)^{-1}\sum_{\tau=1}^{N}\left\{\sum_{s=2}^{4}\sum_{k=1}^{s-1} v_{s\tau}v_{s-k,\tau}\right\} \to \Psi_{11} = \frac{1}{4}\sum_{s=2}^{4}\sum_{k=1}^{s-1}\mathrm{E}(v_{s\tau}v_{s-k,\tau}) = \frac{1}{4}\sum_{s=2}^{4}\sum_{k=1}^{s-1}\gamma_s(k).$$

Combining these, and using Lemma 1, leads to

$$(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} x_{s\tau}^{(1)}\Delta_4 y_{s\tau} \Rightarrow \frac{1}{4}\int B'(r)C_1\, dB(r) + \Psi_1(0)$$

$$= \frac{\omega^2 a' C_1 a}{4}\int w(r)\, dw(r) + \Psi_1(0), \qquad \textbf{(A.12)}$$

where $\Psi_1(0) = \sum_{k=1}^{3}\gamma(k) = \Psi_{11} + \Psi_{12}$. Notice that Ψ_{11} is equal to the sum of the intra-year covariances between the elements of V_τ, whereas Ψ_{12} captures the corresponding interyear covariances, so that $\Psi_1(0)$ arises because of the covariance between the regressor $x_{s\tau}^{(1)}$ and the dependent variable $\Delta_4 y_{s\tau}$ and plays a similar role to Λ in the theorem of Phillips (1988).

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

For the other numerator terms in (A.9) with $j = 1$, note that the definition of the corresponding HEGY variable in (14) implies that

$$x_{s\tau}^{(1)}\nu_{s-i,\tau} = [x_{s-1,\tau}^{(1)} + \nu_{s-1,\tau}]\nu_{s-i,\tau} = \left[x_{s'\tau'}^{(1)} + \sum_{k=0}^{i-1}\nu_{s'+k,\tau'}\right]\nu_{s'\tau'} \qquad i > 0, \tag{A.13}$$

where s' and τ' indicate the quarter and year to which $\nu_{s-i,\tau}$ refers. Therefore, using (A.13) and also (20),

$$\begin{aligned}(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4}x_{s\tau}^{(1)}\Delta_4 y_{s-i,\tau} &= (4N)^{-1}\sum_{\tau'=1}^{N'}\sum_{s'=1}^{4}\left\{x_{s'\tau'}^{(1)}\nu_{s'\tau'} + \sum_{k=0}^{i-1}\nu_{s'\tau'}\nu_{s'+k,\tau'}\right\} \\ &\Rightarrow \frac{\omega^2 a' C_1 a}{4}\int w(r)\,dw(r) + \Psi_1(0) + \frac{1}{4}\sum_{k=0}^{i-1}\sum_{s=1}^{4}\gamma_s(k) \\ &= \Psi_1(4)\int w(r)\,dw(r) + \Psi_1(i), \qquad i = 1, 2, \ldots, \end{aligned} \tag{A.14}$$

in which $N' = N - [(4 + i - s)/4]$, where $[\cdot]$ indicates the integer part of the expression in brackets, whereas

$$\begin{aligned}\Psi_1(i) &= \sum_{k=-3}^{i-1}\gamma(k) = \frac{1}{4}\sum_{k=-3}^{i-1}\sum_{s=1}^{4}\gamma_s(k) \\ &= \Psi_1(0) + \sum_{k=0}^{i-1}\gamma(k).\end{aligned} \tag{A.15}$$

Notice that, because $v_{s\tau}$ is a (periodic) MA(3) process, $\gamma_s(k) = \gamma(k) = 0$ for $|k| > 3$, and hence $\Psi_1(i) = \Psi_1(4)$ for $i > 4$ is the sum of all nonzero autocovariances for the misspecified constant parameter process, as in (20).

Using the limiting distribution for the denominator given in (17) of Lemma 3, and substituting from (A.12), (A.14), and (A.15) into (A.9) with $j = 1$, yields

$$T\hat{\pi}_1 \Rightarrow \frac{\left(1 - \sum_{i=1}^{p}\phi_i^p\right)\int w(r)\,dw(r) + \left[\Psi_1(0) - \sum_{i=1}^{p}\phi_i^p\Psi_1(i)\right]\Big/\Psi_1(4)}{\int [w(r)]^2\,dr}. \tag{A.16}$$

To establish (29) for $j = 2$, we proceed from (A.9) in a similar way. In this case, however, the identity analogous to (A.13) is

$$\begin{aligned}x_{s\tau}^{(2)}\Delta_4 y_{s-i,\tau} &= x_{s\tau}^{(2)}\nu_{s-i,\tau} = [-x_{s-1,\tau}^{(2)} - \nu_{s-1,\tau}]\nu_{s-i,\tau} \\ &= \left[(-1)^i x_{s'\tau'}^{(2)} + \sum_{k=0}^{i-1}(-1)^{k+i}\nu_{s'+k,\tau'}\right]\nu_{s'\tau'} \qquad i > 0.\end{aligned}$$

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

From this, and also using Lemmas 3 and 5, it is possible to write

$$
\begin{aligned}
&(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} x_{s\tau}^{(2)}\Delta_4 y_{s-i,\tau} \\
&\quad = (-1)^i(4N)^{-1}\sum_{\tau'=1}^{N'}\sum_{s'=1}^{4}\left\{x_{s'\tau'}^{(2)}\nu_{s'\tau'} + \sum_{k=0}^{i-1}(-1)^k\nu_{s'+k,\tau}\nu_{s'\tau'}\right\} \\
&\quad \Rightarrow (-1)^i\left(\frac{\omega^2 a' C_2 a}{4}\int w(r)\,dw(r) + \Psi_2(0)\right) + \frac{1}{4}\sum_{k=0}^{i-1}\sum_{s=1}^{4}(-1)^{k+i}\gamma_s(k) \\
&\quad = (-1)^i\Psi_2(4)\int w(r)\,dw(r) + \Psi_2(i), \qquad i = 0,1,2,\ldots, \qquad \textbf{(A.17)}
\end{aligned}
$$

where $\Psi_2(i) = (-1)^i\Psi_2(0) + \sum_{k=0}^{i-1}(-1)^{k+i}\gamma(k) = \sum_{k=-3}^{i-1}(-1)^{k+i}\gamma(k)$, $i = 0,1,2,\ldots$.

Substituting in (A.9), using (A.17), (20), and the asymptotic distribution for $j = 2$ in (17), we have

$$
T\hat{\pi}_2 \Rightarrow \frac{\left(1 - \sum_{i=1}^{p}(-1)^i\phi_i^p\right)\int w(r)\,dw(r) + \left[\Psi_2(0) - \sum_{i=1}^{p}\phi_i^p\Psi_2(i)\right]\Big/\Psi_2(4)}{\int [w(r)]^2\,dr}. \qquad \textbf{(A.18)}
$$

The distribution given for $j = 2$ in (29) is then easily established.

To establish (30), note first that the relationship between $x_{s\tau}^{(3)}, x_{s\tau}^{(4)}$ implied by the definitions of (14) is such that

$$
x_{s\tau}^{(3)} = x_{s-1,\tau}^{(4)} = -x_{s-2,\tau}^{(3)} - \nu_{s-2,\tau}. \qquad \textbf{(A.19)}
$$

One consequence of (A.19) is that $x_{s\tau}^{(3)}$ is correlated with $\Delta_4 y_{s-1,\tau} = \nu_{s-1,\tau}$ only through the MA component in the latter. Using arguments analogous to those that lead to (A.14) and (A.17), it then follows that

$$
\begin{aligned}
&(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} x_{s\tau}^{(3)}\Delta_4 y_{s\tau} \Rightarrow \frac{\omega^2 a' C_3 a}{4}\int w(r)\,dw(r) + \Psi_3(0), \\
&(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} x_{s\tau}^{(3)}\Delta_4 y_{s-1,\tau} \to \Psi_3(1), \qquad \textbf{(A.20)}
\end{aligned}
$$

in which $\Psi_3(0) = -\gamma(2)$ and $\Psi_3(1) = \Psi_4(0) = -\gamma(1) + \gamma(3)$. Further, for even lags

$$
\begin{aligned}
&(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} x_{s\tau}^{(3)}\Delta_4 y_{s-i,\tau} \\
&\quad = (-1)^{i/2}(4N)^{-1}\sum_{\tau'=1}^{N'}\sum_{s'=1}^{4}\left\{x_{s'\tau'}^{(3)}\nu_{s'\tau'} + \sum_{k=0}^{(i/2)-1}(-1)^k\nu_{s'+2k,\tau}\nu_{s'\tau'}\right\} \\
&\quad \Rightarrow (-1)^{i/2}\left(\frac{\omega^2 a' C_3 a}{4}\int w(r)\,dw(r) + \Psi_3(0)\right) + \frac{1}{4}\sum_{k=0}^{(i/2)-1}\sum_{s=1}^{4}(-1)^{k+(i/2)}\gamma_s(2k) \\
&\quad = (-1)^{i/2}\Psi_3(4)\int w(r)\,dw(r) + \Psi_3(i), \qquad i = 2,4,6,\ldots, \qquad \textbf{(A.21)}
\end{aligned}
$$

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

where $\Psi_3(i) = (-1)^{i/2}\Psi_3(0) + \sum_{k=0}^{(i/2)-1}(-1)^{k+(i/2)}\gamma(2k) = \sum_{k=-1}^{(i/2)-1}(-1)^{k+(i/2)}\gamma(2k)$, $i = 0,2,4,\ldots$. For odd lags,

$$
\begin{aligned}
&(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} x_{s\tau}^{(3)}\Delta_4 y_{s-i,\tau} \\
&\quad = (-1)^{(i-1)/2}(4N)^{-1}\sum_{\tau'=1}^{N'}\sum_{s'=1}^{4}\left\{x_{s'\tau'}^{(3)}\nu_{s'+1,\tau'} + \sum_{k=0}^{(i-1)/2-1}(-1)^k\nu_{s'+2k,\tau}\nu_{s'+1,\tau'}\right\} \\
&\quad \to (-1)^{(i-1)/2}\Psi_3(1) + \frac{1}{4}\sum_{k=0}^{(i-1)/2-1}\sum_{s=1}^{4}(-1)^{k+(i-1)/2}\gamma_s(4k-1) \\
&\quad = \Psi_3(i), \qquad i = 3,5,7,\ldots,
\end{aligned}
\tag{A.22}
$$

where $\Psi_3(3) = -\gamma(3)$ and $\Psi_3(i) = 0$ for $i > 3$ and odd. Together with the asymptotic distribution for the denominator given in (17), (A.20)–(A.22) yield the asymptotic distribution for the normalized bias statistic of (A.9) with $j = 3$.

The asymptotic distribution for the normalized bias statistic corresponding to π_4 is then easily obtained by noting that (A.19) implies that

$$
(4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} x_{s\tau}^{(4)}\Delta_4 y_{s-i,\tau} = (4N)^{-1}\sum_{\tau=1}^{N}\sum_{s=1}^{4} x_{s\tau}^{(3)}\Delta_4 y_{s+1-i,\tau} + o_p(1) \tag{A.23}
$$

so that the asymptotic distributions applying in (A.20)–(A.22) for lag $i > 0$ apply in relation to the left-hand-side variable of (A.23) at lag $i - 1$. Finally, the asymptotic distributions of (30) can then obtained from (A.10). ■

Proof of Theorem 3. First note that the denominator scale factor in (29) and (30) can be expressed as $\gamma(0) - \Phi^{(p)\prime}\Gamma\Phi^{(p)} = \sigma_\eta^2(1 + \sum_{i=1}^{q}\beta_i^2)$, where β_i $(i = 1,\ldots,q)$ are the coefficients of the MA process of order $q = p + 3$ for the innovation in the augmented HEGY regression (24) and $\sigma_\eta^2 = \mathrm{E}[\eta_{s\tau}^2]$ in (12). Note also that using the definitions of $\Psi_1(4)$ and $\Psi_2(4)$ in (20) it is possible to establish the following identities:

$$
\begin{aligned}
\left(1 - \sum_{i=1}^{p}\phi_i^p\right)\sqrt{\Psi_1(4)} &= \sqrt{\sigma_\eta^2(1+\theta_1+\theta_2+\theta_3)^2}\,\phi^p(1) = \sigma_\eta\theta(1)\phi^p(1) = \sigma_\eta\beta(1), \\
\left(1 - \sum_{i=1}^{p}(-1)^i\phi_i^p\right)\sqrt{\Psi_2(4)} &= \sqrt{\sigma_\eta^2(1-\theta_1+\theta_2-\theta_3)^2}\,\phi^p(-1) \\
&= \sigma_\eta\theta(-1)\phi^p(-1) = \sigma_\eta\beta(-1),
\end{aligned}
\tag{A.24}
$$

where $\beta(L) = 1 + \beta_1 L + \cdots + \beta_q L^q$, with corresponding definitions for $\theta(L)$ and $\phi^p(L)$. Finally note that the numerator shift factors of (29) can be written

$$
\begin{aligned}
\Psi_1(0) - \sum_{i=1}^{p}\phi_i^p\Psi_1(i) &= \sigma_\eta^2\sum_{i=0}^{3}\theta_i\sum_{l=1+i}^{q}\beta_l, \\
\Psi_2(0) - \sum_{i=1}^{p}\phi_i^p\Psi_2(i) &= \sigma_\eta^2\sum_{i=0}^{3}(-1)^i\theta_i\sum_{l=1+i}^{q}(-1)^l\beta_l.
\end{aligned}
\tag{A.25}
$$

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

The distributions of (29) differ from t^{DF} by scale and shift factors, which (A.24) and (A.25) show are equivalent to the corresponding factors in Proposition 1 of Galbraith and Zinde-Walsh (1999), who analyze the performance of the augmented Dickey–Fuller test in the presence of an MA process. Their Propositions 2 and 3 establish that if p is chosen such that $p = O(\delta \ln N)$, $\delta > 0$, the scale factor of (30) tends to one and the shift factor to zero, which therefore establishes (31).

Using the notation of (A.24), the numerator of the shift factors in (30) can be established using the identities

$$\begin{aligned}\Psi_3(0) - \sum_{i=1}^{p} \phi_i^p \Psi_3(i) &= -\sigma_\eta^2 \sum_{i=0}^{3} \theta_i \sum_{l=2+i}^{q} \cos[(l-(i-2))\pi/2]\beta_l, \\ \Psi_4(0) - \sum_{i=1}^{p} \phi_i^p \Psi_4(i) &= -\sigma_\eta^2 \sum_{i=0}^{3} \theta_i \sum_{l=1+i}^{q} \sin[(l-i)\pi/2]\beta_l.\end{aligned} \tag{A.26}$$

Note that expressions in (A.26) are analogous to those in (A.25), once the HEGY transformations (14) are taken into account. Hence it is possible to see that the asymptotic distributions in (30) for the seasonal unit root test statistics at the annual frequency can be written for $p = O(\delta \ln N)$ as

$$\begin{aligned}t_{\hat{\pi}_3} &\Rightarrow \frac{\int w(r)\,dw(r)}{\sqrt{\int [w(r)]^2\,dr}} \times \frac{\left(\frac{1}{2}[\phi^p(i) + \phi^p(-i)]\right)\sqrt{2\theta(i)\theta(-i)}}{\sqrt{\left(1 + \sum_{k=1}^{q} \beta_k^2\right)}}, \\ t_{\hat{\pi}_4} &\Rightarrow \frac{\int w(r)\,dw(r)}{\sqrt{\int [w(r)]^2\,dr}} \times \frac{\left(-\frac{1}{2}[\phi^p(i) - \phi^p(-i)]\right)\sqrt{2\theta(i)\theta(-i)}}{\sqrt{\left(1 + \sum_{k=1}^{q} \beta_k^2\right)}},\end{aligned} \tag{A.27}$$

where we have also used the following identities:

$$\left(1 - \sum_{k=1}^{[p/2]} (-1)^k \phi_{2k}^p\right) = \frac{1}{2}[\phi^p(i) + \phi^p(-i)],$$

$$\left(\sum_{k=1}^{[(p+1)/2]} (-1)^{k+1} \phi_{2k-1}^p\right) = -\frac{i}{2}[\phi^p(i) - \phi^p(-i)],$$

$$\begin{aligned}\Psi_3(4) &= \left(\frac{1}{2}[\theta(i) + \theta(-i)]\right)^2 + \left(-\frac{i}{2}[\theta(i) - \theta(-i)]\right)^2 \\ &= \theta(i)\theta(-i),\end{aligned}$$

and $i = \sqrt{-1}$. The result in (A.27) is also a consequence of the shift term in (30) tending to zero, because the AR augmentation asymptotically accounts for the invertible MA disturbance in (24). The distributions in (A.27) are analogous to those obtained by Burridge and Taylor (2001) in their analysis of the asymptotic distribution of the HEGY

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press

t-test statistics at frequency $\pi/2$ in the presence of AR serial correlation. Similarly, in an analogous way to the corresponding result in Burridge and Taylor (2001), the F-type test associated with this frequency $\pi/2$ is pivotal and can be written as

$$F_{34} = \frac{1}{2}\left((t_{\hat{\pi}_3})^2 + (t_{\hat{\pi}_4})^2\right) + o_p(1) \Rightarrow \frac{\beta(i)\beta(-1)}{\left(1 + \sum_{k=1}^{q} \beta_k^2\right)} \times \left(\frac{\int w(r)\, dw(r)}{\sqrt{\int [w(r)]^2\, dr}} \right)^2. \qquad \textbf{(A.28)}$$

To obtain (A.28) we use (A.27) and the identities

$$\left(\frac{1}{2}[\phi^p(i) + \phi^p(-i)]\right)^2 + \left(-\frac{i}{2}[\phi^p(i) - \phi^p(-i)]\right)^2 = \phi^p(i)\phi^p(-i),$$

$$\theta(i)\phi^p(i)\theta(-i)\phi^p(-i) = \beta(i)\beta(-i).$$

It is easy to see that the numerator of the scale factor in (A.28) is

$$\beta(i)\beta(-i) = \sum_{k=0}^{q} \beta_k^2 + 2\sum_{k=0}^{q} \beta_k \sum_{j=1}^{[(q-k)/2]} (-1)^j \beta_{k+2j},$$

where $\beta_0 = 1$. Proposition 3 in Galbraith and Zinde-Walsh (1999) establishes that, by choosing p such that $p = O(\delta \ln N)$ as $N \to \infty$, both $\sum_{k=0}^{q} \beta_k^2 = 1 + \sum_{k=1}^{q} \beta_k^2$ and $\sum_{k=0}^{q} \beta_k = 1 + \sum_{k=1}^{q} \beta_k$ tend to one. Following similar lines to Galbraith and Zinde-Walsh (1999), it is possible to establish that $\sum_{k=0}^{q} \beta_k \sum_{j=1}^{[(q-k)/2]} (-1)^j \beta_{k+2j}$ tends to zero and hence the scale factor $\beta(i)\beta(-i)/[1 + \sum_{k=1}^{q} \beta_k^2]$ in (A.28) tends to one.

The remaining asymptotic distributions in (32) follow from (31) and (A.27) in combination with the asymptotic orthogonality of the scaled HEGY regressors for the PI(1) process, established by (18). ■

https://doi.org/10.1017/S0266466608080420 Published online by Cambridge University Press