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ARTICLE

Continuous-time Mallows processes

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Abstract

In this article, we introduce *Mallows processes*, defined to be continuous-time cadlag processes with Mallows distributed marginals. We show that such processes exist and that they can be restricted to have certain natural properties. In particular, we prove that there exist *regular* Mallows processes, defined to have their inversion numbers $\mathrm{Inv}_j(\sigma) = |\{i \in [j-1] : \sigma(i) > \sigma(j)\}|$ be independent increasing stochastic processes with jumps of size 1. We further show that there exists a unique Markov process which is a regular Mallows process. Finally, we study properties of regular Mallows processes and show various results on the structure of these objects. Among others, we prove that the graph structure related to regular Mallows processes looks like an *expanded hypercube* where we stacked k hypercubes on the dimension $k \in [n]$; we also prove that the first jumping times of regular Mallows processes converge to a Poisson point process; we eventually provide results regarding the Markov property of the discrete jumping chain corresponding to a Mallows process.

Keywords: Mallows permutations; stochastic processes; time-inhomogeneous processes; continuous-time Markov chains
2020 MSC Codes: Primary: 60J27; Secondary: 60C05, 60J74, 05A05

1. Introduction

The Mallows distribution of permutations, first introduced by Mallows [31], has started attracting a lot of attention in the recent years, mostly due to its connection to the uniform model on permutations and its strong regenerative properties [35]. Many interesting results regarding its structural properties have been produced in the last fifteen years, with a particular interest in the length of its longest increasing subsequences [8, 10, 27, 32] as well as the cycle and subsequence structure [13, 21, 25, 33, 34]. Other works have also started using Mallows permutations in broader context, studying the possibility to extend this model to permutations on $\mathbb{N}$ [22] and $\mathbb{Z}$ [23], its connection to random matchings [6] and colourings [26], or properties of the corresponding binary search trees [3, 11]. This variety of results has led to interesting insights on properties of these random permutations, whose applications extend to convergence to stationarity [9, 16], statistical physics [37, 38], or even statistical learning [39] and genomics [20]. Finally, it is worth noting that this work already led to a second project studying the local and global limit of Mallows processes [2], making the connection between these processes and the permuton limit of Mallows permutations [37] as well as the two-sided infinite Mallows permutations [23].

In this work, we propose a novel approach on the study of Mallows permutations by considering *Mallows processes*, that is processes with Mallows distributed marginals. These processes naturally arise from the fact that many statistics of the Mallows model seem to be monotone; for example, the asymptotic formulas for the longest increasing sequence, the number of cycles, and the height of the corresponding binary search tree all decrease as q increases within $[0, 1]$.

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Whether these statistics are actually stochastically monotone is not known, and Mallows processes are meant to provide a framework for studying such properties.

As an introductory study, we focus here on providing a set of monotone properties for Mallows processes which naturally arise from the original properties of Mallows permutations. Most notably, Mallows permutations are known to have an independent inversion sequence, and each term of this inversion sequence is stochastically monotone with the parameter of the permutation. Thus, we extend these two properties by considering Mallows processes with monotone and independent inversion sequences, not simply seen as fixed random variables, but as multi-dimensional processes; we later refer to these processes as regular. What is most striking with this approach is that regular Mallows processes perfectly interact with the Markov property, leading to a uniquely defined distribution (see Theorem 1.1).

Following this main result, we study other properties of regular Mallows processes, such as the graph structure of the process (Theorem 1.2) and the distribution of its first jumps (Theorem 1.3). Further using the jumps of the (continuous-time) Mallows process, we define a discrete-time chain on the set of permutations corresponding to the sequence of jumps operated by the continuous-time process. We then study properties of this discrete chain, in particular whether or not it satisfies the (discrete) Markov property. This leads to an interesting dichotomy for Mallows process, where we conjecture that it is impossible for them to be regular and satisfy both the discrete and the continuous Markov property (Conjecture 1.4), but it is possible when we drop one of the regularity assumptions (Theorem 1.5).

On top of general properties of Mallows processes, this work provides three different but specifically defined processes: the unique regular and Markovian one, referred to as the *birth Mallows process* (see Section 3), a similarly yet more easily sampled one based on n uniforms, referred to as the *uniform Mallows process* (see Section 4), and a non-regular but Markovian in both continuous and discrete time one (see Section 3.4). For the latter process, we mostly refer to [1] for a better understanding of its properties, since the most technical part of the proof arises from their Theorem 2.3. This both continuous and discrete Markovian but not regular process is particularly interesting in comparison to the birth Mallows process, being the unique regular Mallows process which also has the (continuous-time) Markov property (Theorem 1.1). In particular, an extensive study of the birth Mallows process allowed us to characterize some of its properties (on top of the fact that it satisfies both Theorems 1.2 and 1.3): its jumping rate (Lemma 3.1) and its corresponding jumping process (Proposition 3.3). Finally, based on computations and simulations, we conjecture that it is not doubly Markovian (Conjecture 1.4), although a formal proof still eludes us.

To conclude this work, we consider properties of the uniform Mallows process, a process based on a common method to sample Mallows permutations using only n uniforms. Since this process is based on a fixed amount of randomness, our focus here is the amount of information we can extract from its evolution. More precisely, we consider both the amount of time it takes to be able to infer the values of all the 'hidden' uniforms (Proposition 4.3) as well as the number of uniforms for which we know the value by a given time (Theorem 4.4).

We now provide more detailed definitions of Mallows processes as well as the main objects and properties considered in this work.

1.1 Regular Mallows processes

Write $\mathbb{N} = \{1, 2, \ldots\}$ for the set of positive integers and let $\mathcal{S}_n$ be the set of permutations on $[n] = \{1, 2, \ldots, n\}$ and $\mathrm{id} = \mathrm{id}_n$ be the identity function. We typically represent a permutation $\sigma \in \mathcal{S}_n$ as a sequence of integers $(\sigma(1), \ldots, \sigma(n))$, so for example, the permutation (3, 1, 4, 2) sends 1–3, 2–1, 3–4 and 4–2. It is also useful to have more succinct notations for transpositions, so for $i \neq j$, we write $(i\ j)$ for the transposition sending i to j and j to i. Given two permutations $\sigma, \sigma' \in \mathcal{S}_n$, write $\sigma \cdot \sigma'$ for the composed permutation defined by $\sigma \cdot \sigma'(i) = \sigma(\sigma'(i))$.

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For $q \in [0, \infty)$, the Mallows distribution [31] with parameters n and q is the probability measure $\pi_{n,q}$ on $\mathcal{S}_n$ defined by

$$\pi_{n,q}(\sigma) = \frac{q^{\mathrm{Inv}(\sigma)}}{Z_{n,q}},$$

where $\mathrm{Inv}(\sigma) = |\{i < j : \sigma(i) > \sigma(j)\}|$ is the number of inversions of the permutation σ and $Z_{n,q} = \prod_{k=1}^{n} (\sum_{\ell=0}^{k-1} q^{\ell})$ is a normalizing constant. Furthermore, write $\mathrm{Inv}_j(\sigma) = |\{i \in [j-1] : \sigma(i) > \sigma(j)\}|$ for the number of inversions of σ created by $\sigma(j)$, implying that $\mathrm{Inv}(\sigma) = \sum_{j=1}^{n} \mathrm{Inv}_j(\sigma)$ and that $0 \le \mathrm{Inv}_j(\sigma) \le j-1$. Finally, if σ is $\pi_{n,q}$-distributed and σ' is $\pi_{n,q'}$-distributed with $q \le q'$, note that $\mathrm{Inv}(\sigma)$ is stochastically dominated by $\mathrm{Inv}(\sigma')$, and so is $\mathrm{Inv}_j(\sigma)$ by $\mathrm{Inv}_j(\sigma)$ for all $j \in [n]$.

We define a *continuous-time Mallows process* (or simply *Mallows process*) to be an $\mathcal{S}_n$-valued cadlag stochastic process $(\mathcal{M}_t^n)_{t\in[0,\infty)}$ such that, for all $t \in [0, \infty)$, $\mathcal{M}_t^n$ is $\pi_{n,t}$-distributed. For simplification, and since it is always clear from context, we drop the superscript n and write $(\mathcal{M}_t)_{t\in[0,\infty)}$. For any $t > 0$, we also denote $\mathcal{M}_{t-}$ for the left limit of the process at time t, which always exists thanks to its cadlag property.

Given a Mallows process $(\mathcal{M}_t)_{t\in[0,\infty)}$ and using the stochastic monotony of $\mathrm{Inv}(\cdot)$ and $\mathrm{Inv}_j(\cdot)$, say that it is *monotone* if, for all $t < t'$, we have

$$\mathrm{Inv}(\mathcal{M}_t) \le \mathrm{Inv}(\mathcal{M}_{t'}),$$

and say that it is *strongly monotone* if furthermore

$$\mathrm{Inv}_j(\mathcal{M}_t) \le \mathrm{Inv}_j(\mathcal{M}_{t'})$$

for all $j \in [n]$. Moreover, say that $(\mathcal{M}_t)_{t\in[0,\infty)}$ is *smooth* if it is strongly monotone and for all $t \in (0, \infty)$, we have

$$\mathrm{Inv}(\mathcal{M}_t) \le \mathrm{Inv}(\mathcal{M}_{t-}) + 1,$$

i.e. if its number of inversions never jumps by more than 1 at the time. Finally say that it has *independent inversions* if the processes $(\mathrm{Inv}_j(\mathcal{M}_t))_{t\in[0,\infty)}$ are independent over $j \in [n]$. Note that, for any fixed t, the distribution of the Mallows permutation implies that the random variables $\mathrm{Inv}_1(\mathcal{M}_t), \mathrm{Inv}_2(\mathcal{M}_t), \ldots, \mathrm{Inv}_n(\mathcal{M}_t)$ are independent (see Proposition 2.1); thus the independent inversion property extends this point-wise independence to the independence of the processes.

In this work we are mainly interested in studying *regular* Mallows processes, defined to be smooth Mallows processes with independent inversions. Our first result states that regular Mallows processes exist and that there is a unique such process with the Markov property.

Theorem 1.1. *Fix $n \in \mathbb{N}$. There exists a unique cadlag Markov process $(\mathcal{M}_t)_{t\in[0,\infty)}$ on $\mathcal{S}_n$ which is a regular Mallows process.*

The proof of Theorem 1.1 can be found in Section 3.3; it boils down to showing that $(\mathcal{M}_t)_{t\in[0,\infty)}$ is characterized by the processes $(\mathrm{Inv}_j(\mathcal{M}_t))_{t\in[0,\infty)}$ for $j \in [n]$, and that such processes are time-inhomogeneous birth processes with a well-defined generator. This unique regular Markovian Mallows process is going to correspond to the *birth Mallows process*, more precisely defined in Section 3.1.

1.2 Graph structure of a smooth Mallows process

Given a Mallows process $\mathcal{M} = (\mathcal{M}_t)_{t\in[0,\infty)}$, let $\mathcal{G}_{\mathcal{M}} = (\mathcal{S}_n, E(\mathcal{G}_{\mathcal{M}}))$ be the undirected transition graph of $\mathcal{M}$ whose edge set is defined by

$$E(\mathcal{G}_{\mathcal{M}}) := \{(\sigma, \sigma') : \mathbb{P}(\exists t \in (0, \infty), \mathcal{M}_{t-} = \sigma, \mathcal{M}_t = \sigma') > 0\}.$$

In other words, $(\sigma, \sigma') \in E(\mathcal{G}_{\mathcal{M}})$ if and only if it is possible for $\mathcal{M}$ to directly transition from σ to σ'. Observe that the notations are slightly abusive here, as $\mathcal{G}_{\mathcal{M}}$ is not directly a function of

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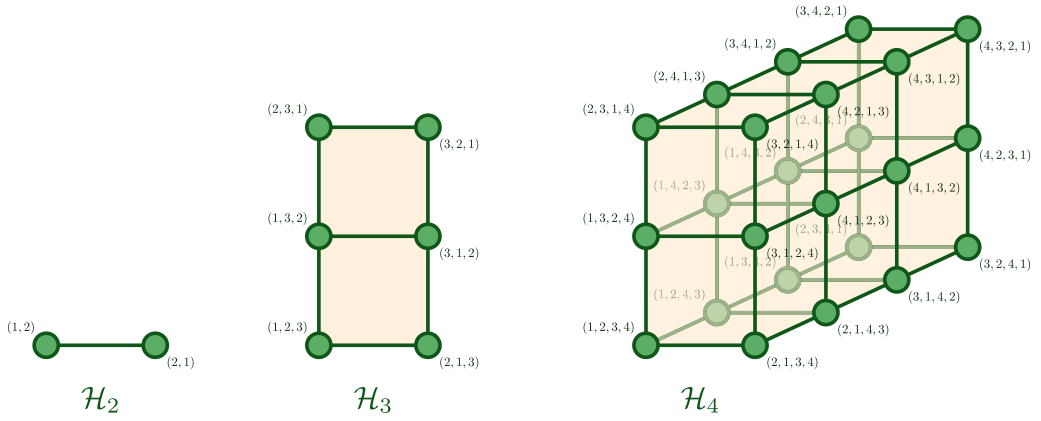


Figure 1. A visual representation of the expanded hypercubes $\mathcal{H}_2$, $\mathcal{H}_3$, and $\mathcal{H}_4$. Using these figures, $\mathcal{H}_n$ can be seen as a stack of hypercubes where each dimension, $k \in [n]$ has k hypercubes on top of each other.

the realisation of the process $\mathcal{M} = (\mathcal{M}_t)_{t\in[0,\infty)}$, but rather of its distribution. However, the link between the process and its distribution will always be clear from the context.

For any graph $\mathcal{G} = (\mathcal{S}_n, E(\mathcal{G}))$, consider its *generator* defined by

$$\langle \mathcal{G} \rangle := \left\{ \sigma^{-1} \cdot \sigma' : (\sigma, \sigma') \in E(\mathcal{G}) \right\},$$

so $\tau \in \langle \mathcal{G} \rangle$ if and only if $\mathcal{G}$ contains at least one edge of the form $(\sigma, \sigma \cdot \tau)$. We let $\mathcal{T} = \{(i\,j) : i \neq j\}$ be the set of transpositions and $\mathcal{T}^a = \{(i\,i+1) : i \in [n-1]\} \subset \mathcal{T}$ be the set of adjacent transpositions.

Our next theorem controls the structure of $\mathcal{G}_\mathcal{M}$, and in particular the permutations that belong to the generator $\langle \mathcal{G}_\mathcal{M} \rangle$ when $\mathcal{M}$ is a smooth Mallows process. Let $\mathcal{H}_n = (\mathcal{S}_n, E)$ be the *expanded hypercube* of size n defined by

$$E = \left\{ (\sigma, \sigma') : \sum_{j=1}^{n} \left| \mathrm{Inv}_j(\sigma) - \mathrm{Inv}_j(\sigma') \right| = 1 \right\}.$$

The reason for naming this graph the expanded hypercube comes from its structure. Indeed, this graph is isomorphic to $\mathcal{G}_n = (V, E)$ defined by

$$V = \{(x_1, \ldots, x_n) : \forall i \in [n], 0 \leq x_i < i\}$$

and

$$E = \left\{ (x, x') \in V^2 : \sum_{j=1}^{n} |x_j - x'_j| = 1 \right\};$$

the latter graph can be seen as a hypercube expanded k times in dimension k. A representation of $\mathcal{H}_n$ for $n \in \{2, 3, 4\}$ can be found in Figure 1. The fact that $\mathcal{H}_n$ and $\mathcal{G}_n$ are isomorphic follows by considering the image of $\mathcal{H}_n$ under the map Φ, defined in Section 2.1 and corresponding to the Lehmer code of the permutation [28, 30].

We now use $\mathcal{T}$, $\mathcal{T}^a$, and $\mathcal{H}_n$ to characterize the structure of $\mathcal{G}_\mathcal{M}$ and of $\langle \mathcal{G}_\mathcal{M} \rangle$ when $\mathcal{M}$ is a smooth Mallows process. The proof of the next theorem appears in Section 2.2.

Theorem 1.2. *Let $\mathcal{M} = (\mathcal{M}_t)_{t\in[0,\infty)}$ be a smooth Mallows process. Then, $\mathcal{G}_\mathcal{M}$ is a subgraph of $\mathcal{H}_n$ and*

$$\mathcal{T}^a \subset \langle \mathcal{G}_\mathcal{M} \rangle \subset \mathcal{T}.$$

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Moreover, if $\mathcal{M}$ is regular, then

$$\mathcal{G}_{\mathcal{M}} = \mathcal{H}_n\,,$$

and $\langle \mathcal{G}_{\mathcal{M}} \rangle = \langle \mathcal{H}_n \rangle = \mathcal{T}$.

1.3 Jumping times of a monotone Mallows process

Consider a monotone Mallows process $(\mathcal{M}_t)_{t\in[0,\infty)}$. Since $(\mathrm{Inv}(\mathcal{M}_t))_{t\in[0,\infty)}$ is a jump process, we will be considering its *jumping times* $T_1, \dots, T_N$ with $N = \binom{n}{2}$ defined as follows. For any $0 \le k \le N$, let

$$T_k = \inf\left\{t \in [0,\infty) : \mathrm{Inv}(\mathcal{M}_t) \ge k\right\} = \min\left\{t \in [0,\infty) : \mathrm{Inv}(\mathcal{M}_t) \ge k\right\},$$

the second equality following form the cadlag property of $(\mathcal{M}_t)_{t\in[0,\infty)}$. Note that, without the assumption of smoothness, it is possible to have $T_{k-1} = T_k$ for some $k \ge 1$. The next results characterizes the asymptotic behaviour of the early jumping times.

Theorem 1.3. *Let $(\mathcal{M}_t)_{t\in[0,\infty)}$ be a monotone Mallows process and $T_1, \dots, T_N$ be its jumping times. Then for any fixed $k \in \mathbb{N}$, as n tends to infinity, we have*

$$nT_k \xrightarrow{d} \mathrm{GAMMA}(k,1)\,.$$

Moreover, if $(\mathcal{M}_t)_{t\in[0,\infty)}$ is regular, then

$$(nT_1, \dots, nT_k) \xrightarrow{d} PP_k\,,$$

where PP_k is the distribution of the first k points of a homogeneous Poisson point process on $[0,\infty)$ with rate 1.

The proof of Theorem 1.3 can be found in Section 2.3 and boils down to first using the relation between number of jumps and inversion numbers in a monotone process, then showing that the first k jumps of a regular Mallows process occur for distinct inversion numbers, making them independent of each other. It is worth noting that Theorem 1.3 does not assume that $(\mathcal{M}_t)_{t\in[0,\infty)}$ is a Markov process.

1.4 Continuous and discrete Markov property

Given a monotone Mallows process $(\mathcal{M}_t)_{t\in[0,\infty)}$ with jumping times $T_1, \dots, T_N$ let $(M_k)_{0\le k\le N}$ be the corresponding discrete process defined by

$$M_k := \mathcal{M}_{T_k}$$

with the convention that $T_0 = 0$ and so $M_0 = \mathrm{id}$. It is straightforward to check that $(M_k)_{0\le k\le N}$ is a discrete process evolving on $\mathcal{G}_{\mathcal{M}}$ and that $\mathrm{Inv}(M_k) \ge k$ with equality when $(\mathcal{M}_t)_{t\in[0,\infty)}$ is smooth.

We are now interested in studying whether $(\mathcal{M}_t)_{t\in[0,\infty)}$ and $(M_k)_{0\le k\le N}$ both satisfy the Markov property, having in mind that the first one is a continuous process and the second one is a discrete chain. When both $(\mathcal{M}_t)_{t\in[0,\infty)}$ is a continuous Markov process and $(M_k)_{0\le k\le N}$ is a discrete Markov chain, we say that $(\mathcal{M}_t)_{t\in[0,\infty)}$ is *doubly-Markovian*.

Considering the statement of Theorem 1.1, we know that there exists a unique Markov process $(\mathcal{M}_t)_{t\in[0,\infty)}$ which is also a regular Mallows process, called the birth Mallows process. We conjecture here that this process is actually not doubly Markovian.

Conjecture 1.4. *For any $n \ge 4$, the unique process from Theorem 1.1 is not doubly Markovian. In other words, it is impossible to have a Mallows process which is smooth, with independent inversion, and doubly Markovian.*

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It is actually not too difficult to verify that the discrete version of any continuous-time Mallows process with $n \leq 3$ satisfies the Markov property. For $n = 1$ and $n = 2$, the discrete process is deterministic. For $n = 3$, since $N = \binom{n}{2} = 3$ and the discrete process only consists of 4 steps in total, it suffices to show that both M_2 and M_3 satisfy the Markov property. On one hand, since $M_0 = \mathrm{id} = (1, 2, 3)$, the distribution of M_2 conditioned on M_1 and M_0 only depends on the value of M_1. On the other hand, $M_3 = \mathrm{id}^{\mathrm{rev}} = (3, 2, 1)$, so its distribution does not depend on the previous values and thus satisfies the Markov property. This explains why Conjecture 1.4 assumes $n \geq 4$.

The previous conjecture states that the three properties, smooth, independent inversion, and doubly Markovian, cannot coexist. It is thus natural to wonder whether we can drop one of the first assumptions and obtain a doubly Markovian Mallows process. The following theorem states that it is possible to do so by dropping the independent inversion property.

Theorem 1.5. *For any $n \geq 1$, there exists a smooth and doubly Markovian Mallows process.*

The proof of this proposition can be found in Section 3.4 and strongly relies on a previously established result [1, Theorem 2.3] showing that it is possible to evolve over the expanded hypercube while remaining uniformly distributed over all permutations with a given number of inversions. Using this result, the proof of the theorem boils down to showing the existence of a jumping process whose marginal distribution is exactly the probability to have a given number of inversions in a Mallows permutation.

1.5 Organization of the paper

The rest of this work is organized as follows. In Section 2, we give useful tools to construct regular Mallows processes and use them to prove Theorems 1.2 and 1.3. In Section 3, we define the birth Mallows process, study some of its properties, and use these results to prove Theorems 1.1 and 1.5. In Section 4, we define the uniform Mallows process and study its properties, in particular regarding the amount of time for full information (Proposition 4.3) and the amount of information at a given time (Theorem 4.4). Finally, in Section 5, we provide some discussions regarding our results and following open problems.

2. Regular Mallows process

In this section, we provide a natural framework for constructing regular Mallows processes (in Section 2.1) and use it to prove Theorem 1.2 (in Section 2.2) and Theorem 1.3 (in Section 2.3).

2.1 An important bijection

Consider the set $\mathcal{E}_n$ of inversion sequences defined by

$$\mathcal{E}_n = \left\{(I_1, \ldots, I_n) : \forall j \in [n], 0 \leq I_j \leq j - 1\right\},$$

and let

$$\begin{aligned} \Phi : \mathcal{E}_n &\longrightarrow \mathcal{S}_n \\ I &\longmapsto \sigma\ , \end{aligned}$$

be the unique bijection between $\mathcal{E}_n$ and $\mathcal{S}_n$ such that $I_j = \mathrm{Inv}_j(\sigma)$. A simple way to generate Φ is by defining $[n] \setminus \{\sigma(n), \ldots, \sigma(k+1)\} = \{x_1 < \ldots < x_k\}$ and setting $\sigma(k) = x_{k-I_k}$ (see [12] for further details). This bijection shows to be an efficient way to easily generate Mallows permutations and Mallows processes, as shown in the next two propositions.

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Proposition 2.1 ([22, Proposition 3.2]). *Fix $n \in \mathbb{N}$ and $q \in [0, \infty)$. Let $I = (I_1, \ldots, I_n) \in \mathcal{E}_n$ be a random inversion sequence such that the different entries of I are independent and, for all $j \in [n]$, for all $0 \leq k \leq j-1$, we have*

$$\mathbb{P}(I_j = k) = \frac{q^k}{\sum_{\ell=0}^{j-1} q^\ell}.$$

Then $\Phi(I)$ is $\pi_{n,q}$-distributed.

With this proposition, it becomes easy to generate regular Mallows processes, as stated in the following proposition.

Proposition 2.2. *Fix $n \in \mathbb{N}$. Let $(I(t))_{t\in[0,\infty)} = (I_1(t), \ldots, I_n(t))_{t\in[0,\infty)}$ be a cadlag process taking values in $\mathcal{E}_n$ such that the coordinate processes $\{(I_j(t))_{t\in[0,\infty)}, j \in [n]\}$ are independent and, for all $j \in [n]$ and $0 \leq k \leq j-1$, we have*

$$\mathbb{P}\big(I_j(t) = k\big) = \frac{t^k}{\sum_{\ell=0}^{j-1} t^\ell}.$$

Then, $(\Phi(I(t)))_{t\in[0,\infty)}$ is a Mallows process with independent inversions. Moreover, if the processes $(I_j(t))_{t\in[0,\infty)}$ are increasing, then $(\Phi(I(t)))_{t\in[0,\infty)}$ is a strongly monotone Mallows process. Finally, if additionally the increments of the processes $(I_j(t))_{t\in[0,\infty)}$ are of size 1, then $(\Phi(I(t)))_{t\in[0,\infty)}$ is a regular Mallows process.

Proof. Write $(\mathcal{M}_t)_{t\in[0,\infty)} = (\Phi(I(t)))_{t\in[0,\infty)}$. The fact that $(\mathcal{M}_t)_{t\in[0,\infty)}$ is a Mallows process simply follows from applying Proposition 2.1 and the fact that it has independent inversions simply follows from the fact that $\mathrm{Inv}_j(\Phi(I(t))) = I_j(t)$ and the assumption on $(I(t))_{t\in[0,\infty)}$. Similarly, if the coordinates of $(I_j(t))_{t\in[0,\infty)}$ are increasing, then, by the definition of Φ, $(\mathcal{M}_t)_{t\in[0,\infty)}$ is strongly monotone. Assume now that the processes $(I_j(t))_{t\in[0,\infty)}$ are increasing with increments of size 1. To prove that $(\mathcal{M}_t)_{t\in[0,\infty)}$ is regular, it suffices to show that $(\mathrm{Inv}(\mathcal{M}_t))_{t\in[0,\infty)}$ has increments of size 1.

Let T_k^j by the k-th jumping time of I_j, that is,

$$T_k^j = \inf\big\{t : [0, \infty) : I_j(t) \geq k\big\}.$$

Using our assumption on $(I_j(t))_{t\in[0,\infty)}$, we know that $T_1^j < T_2^j < \ldots < T_{j-1}^j$. Moreover, the definition of T_k^j implies that

$$\mathbb{P}\big(T_k^j \leq t\big) = \mathbb{P}\big(I_j(t) \geq k\big) = \frac{1}{\sum_{\ell=0}^{j-1} t^\ell} \sum_{\ell=k}^{j-1} t^\ell,$$

so T_k^j is a continuous random variable. Since the coordinate processes $((I_j(t))_{t\in[0,\infty)}, j \in [n])$ are independent, it follows that for $j \neq j'$ almost surely $T_k^j \neq T_{k'}^{j'}$, which implies that all the values $\{T_k^j : j \in [n], k \in [j-1]\}$ are almost surely distinct. Using that

$$\mathrm{Inv}(\mathcal{M}_t) = \big|\big\{1 \leq k < j \leq n : T_k^j \leq t\big\}\big|,$$

it follows that $(\mathrm{Inv}(\mathcal{M}_t))_{t\in[0,\infty)}$ only has increments of size 1. □

Proposition 2.2 is a useful tool for building and studying regular Mallows processes. This is used to prove Theorem 1.2 in Section 2.2 and to prove Theorem 1.3 in Section 2.3, as well as to define two interesting regular Mallows processes, the birth Mallows process and the uniform Mallows process, respectively in Sections 3.1 and 4.1.

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2.2 Transition graph

In this section, we prove the results regarding the graph structure of smooth and regular Mallows processes. We start by providing a key lemma stating that increasing a single entry of the inversion sequence by 1 corresponds to applying a transposition on the permutation.

Lemma 2.3. *For any $I = (I_1, \dots, I_n) \in \mathcal{E}_n$ and any $\ell \in [n]$ such that $I_\ell < \ell - 1$, let $I' = (I'_1, \dots, I'_n) \in \mathcal{E}_n$ be defined by $I'_j = I_j$ for $j \neq \ell$, and $I'_\ell = I_\ell + 1$. Then $\Phi(I') \cdot (\Phi(I))^{-1}$ is a transposition. In words, increasing an entry of the inversion sequence by 1 only transposes two elements of the corresponding permutation.*

Proof. We let $\sigma = \Phi(I)$ and $\sigma' = \Phi(I')$ and want to understand the relation between σ and σ'. Using the characterization of Φ we gave in Section 2.1, we verify that $\sigma(j) = \sigma'(j)$ for all $j > \ell$, since $I_j = I'_j$. Now, at step ℓ, we have

$$[n] \setminus \{\sigma(\ell+1), \dots, \sigma(n)\} = [n] \setminus \{\sigma'(\ell+1), \dots, \sigma'(n)\} = \{x_1 < \dots < x_\ell\}.$$

By definition, we then have $\sigma(\ell) = x_{\ell - I_\ell}$ and $\sigma'(\ell) = x_{\ell - I'_\ell} = x_{\ell - I_\ell - 1}$. From this, it follows that

$$\begin{aligned}[n] \setminus \{\sigma(\ell), \dots, \sigma(n)\} &= ([n] \setminus \{\sigma(\ell+1), \dots, \sigma(n)\}) \setminus \{x_{\ell - I_\ell}\} \\ &= \{x_1 < \dots < x_{\ell - I_\ell - 2} < x_{\ell - I_\ell - 1} < x_{\ell - I_\ell + 1} < \dots < x_\ell\}\end{aligned}$$

and

$$\begin{aligned}[n] \setminus \{\sigma'(\ell), \dots, \sigma'(n)\} &= ([n] \setminus \{\sigma(\ell+1), \dots, \sigma(n)\}) \setminus \{x_{\ell - I_\ell - 1}\} \\ &= \{x_1 < \dots < x_{\ell - I_\ell - 2} < x_{\ell - I_\ell} < x_{\ell - I_\ell + 1} < \dots < x_\ell\}.\end{aligned}$$

Since $I_{\ell-1} = I'_{\ell-1}$ two cases follow. Either $I_{\ell-1} = I'_{\ell-1} = I_\ell$ and then $\sigma(\ell-1) = x_{\ell - I_\ell - 1}$ and $\sigma'(\ell-1) = x_{\ell - I_\ell}$, meaning that $\sigma(\ell-1) = \sigma'(\ell)$ and $\sigma'(\ell-1) = \sigma(\ell)$; or $I_{\ell-1} = I'_{\ell-1} \neq I_\ell$, and then $\sigma(\ell-1) = \sigma'(\ell-1)$. In this second case, we have that

$$[n] \setminus \{\sigma(\ell-1), \dots, \sigma(n)\} = \{x_1 < \dots < x_{\ell - I_\ell - 1} < x_{\ell - I_\ell + 1} < \dots < x_\ell\} \setminus \{\sigma(\ell-1)\}$$

and that

$$[n] \setminus \{\sigma'(\ell-1), \dots, \sigma'(n)\} = \{x_1 < \dots < x_{\ell - I_\ell - 2} < x_{\ell - I_\ell} < \dots < x_\ell\} \setminus \{\sigma(\ell-1)\}$$

By applying the same argument as before, depending on the value of $I_{\ell-2} = I'_{\ell-2}$, we obtain that either $\sigma(\ell-2) = x_{\ell - I_\ell - 1} = \sigma'(\ell)$ and $\sigma'(\ell-2) = x_{\ell - I_\ell} = \sigma(\ell)$, or $\sigma(\ell-2) = \sigma'(\ell-2) \notin \{x_{\ell - I_\ell}, x_{\ell - I_\ell - 1}\}$. By repeating this argument, it follows that there exists $i \in [\ell - 1]$ such that $\sigma(\ell - i) = \sigma'(\ell)$, $\sigma'(\ell - i) = \sigma(\ell)$, and for all $\ell - i < j < \ell$, $\sigma(j) = \sigma'(j)$. Note that we necessarily have that $i \leq \ell - 1$ since, in the case where $i > \ell - 2$, then $\sigma(j) = \sigma'(j)$ for all $\ell - i < j < \ell$. Now, using that $\ell - i \leq 1$, it follows that $\sigma(j) = \sigma'(j)$ for all $2 \leq j < \ell$; but we also know that this equality holds for all $j > \ell$ and then, since σ and σ' are permutation and since $\sigma(\ell) \neq \sigma'(\ell)$, it follows that $\sigma(1) = \sigma'(\ell)$ and $\sigma(\ell) = \sigma'(1)$, hence $i = \ell - 1$. Write now $k = \ell - i$ and note that

$$[n] \setminus \{\sigma(k), \dots, \sigma(n)\} = [n] \setminus \{\sigma'(k), \dots, \sigma'(n)\},$$

and that, for all $j < k$, we have $I_j = I'_j$. These two statement along with the description of Φ given in Section 2.1 imply that $\sigma(j) = \sigma'(j)$ for all $j < k$. This shows that, for all $j \notin \{k, \ell\}$, $\sigma(j) = \sigma'(j)$, and that $\sigma(\ell) = \sigma'(k)$ and $\sigma(k) = \sigma'(\ell)$. This is equivalent to saying that σ' is obtained from σ by transposing k and ℓ, or in other words, that $\sigma' = \sigma \cdot (k\,\ell)$. □

While the previous lemma has likely been studied before, we were unable to find a proper reference, as all relevant literature only showed that adjacent transpositions change a single entry by 1, but not that changing a single entry by 1 corresponds to applying a non-necessarily adjacent transposition. In fact, an even stronger statement holds, stating that all transpositions can be obtained via the previous jumping process, as shown below.

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Lemma 2.4. *Any transposition can be obtained as the transition between two adjacent permutations on the expanded hypercube.*

Proof. To see that the lemma holds, fix a pair $i<j$ and let $I=(I_k)_{k\in[n]}\in\mathcal{E}_n$ be the inversion sequence such that $I_j=j-i-1<j-1$ and $I_k=0$ for $k\neq j$. Further let $I'=(I'_k)_{k\in[n]}\in\mathcal{E}_n$ be such that $I'_k=I_k=0$ for $k\neq j$ and $I'_j=I_j+1=j-i\leq j-1$. Then $(\Phi(I),\Phi(I'))\in E(\mathcal{H}_n)$ and by observing that the corresponding permutations are given by

$$\Phi(I)(k)=\begin{cases}k & \text{if } k\leq i \text{ or } k>j\\ i+1 & \text{if } k=j\\ k+1 & \text{otherwise}\end{cases}$$
$$=(1,\ldots,i-1,\boldsymbol{i},i+2,i+3,\ldots,j-1,j,\boldsymbol{i+1},j+1,\ldots,n)$$

and

$$\Phi(I')(k)=\begin{cases}k & \text{if } k<i \text{ or } k>j\\ i & \text{if } k=j\\ k+1 & \text{otherwise}.\end{cases}$$
$$=(1,\ldots,i-1,\boldsymbol{i+1},i+2,i+3,\ldots,j-1,j,\boldsymbol{i},j+1,\ldots,n),$$

we see that $\Phi(I')=\Phi(I)\cdot(i\,j)$. □

With the previous two lemmas, we now have all the tools to proof Theorem 1.2.

Proof of Theorem 1.2. The first part of the proof addresses the case where $\mathcal{M}$ is smooth; the second part proves the assertions of the theorem for regular $\mathcal{M}$. Recall from Section 2.1 that for any $I\in\mathcal{E}_n$ and for $\sigma=\Phi(I)$, the values $\sigma(n),\ldots,\sigma(1)$ can be recursively constructed by letting

$$[n]\setminus\{\sigma(n),\ldots,\sigma(j+1)\}=\{x_1<\ldots<x_j\}$$

and by setting $\sigma(j)=x_{j-I_j}$.

For the first part of this proof, assume that $\mathcal{M}=(\mathcal{M}_t)_{t\in[0,\infty)}$ is a smooth Mallows process. Note that the definition of the expanded hypercube directly implies that $\mathcal{G}_{\mathcal{M}}$ is a subgraph of $\mathcal{H}$, since every jump of $(\mathcal{M}_t)_{t\in[0,\infty)}$ corresponds to an increment of exactly one of its inversion numbers by 1, and by definition every such increment corresponds to an edge of $\mathcal{H}_n$.

To see that $\langle\mathcal{G}_{\mathcal{M}}\rangle\subset\mathcal{T}$, simply use that $\mathcal{G}_{\mathcal{M}}$ is a subgraph of $\mathcal{H}_n$ along with Lemma 2.3 to find that $\langle\mathcal{G}_{\mathcal{M}}\rangle\subset\langle\mathcal{H}_n\rangle\subset\mathcal{T}$.

We finally prove that $\mathcal{T}^a\subset\langle\mathcal{G}_{\mathcal{M}}\rangle$. Note first that $\mathcal{T}^a=\{\sigma\in\mathcal{S}_n:\mathrm{Inv}(\sigma)=1\}$. Suppose that $\mathcal{T}^a\setminus\langle\mathcal{G}_{\mathcal{M}}\rangle\neq\emptyset$ and consider $\tau\in\mathcal{T}^a\setminus\langle\mathcal{G}_{\mathcal{M}}\rangle$. We note that

$$\mathbb{P}\big(\mathcal{M}_t=\tau\big)=\mathbb{P}\big(\mathcal{M}_t=\tau,\exists u>0:\mathcal{M}_{u-}=\mathrm{id},\mathcal{M}_u=\tau\big)$$
$$\leq\mathbb{P}\big(\exists u>0:\mathcal{M}_{u-}=\mathrm{id},\mathcal{M}_u=\tau\big).$$

Since the leftmost probability is strictly positive whenever $t>0$, it follows that $(\mathrm{id},\tau)\in\mathcal{G}_{\mathcal{M}}$ and so $\tau\in\langle\mathcal{G}_{\mathcal{M}}\rangle$. Hence, $\mathcal{T}^a\subset\langle\mathcal{G}_{\mathcal{M}}\rangle$.

We now move on to the second part of the proof and prove that if $\mathcal{M}$ is a regular Mallows process, then $\mathcal{G}_{\mathcal{M}}=\mathcal{H}_n$ and $\langle\mathcal{G}_{\mathcal{M}}\rangle=\langle\mathcal{H}_n\rangle=\mathcal{T}$. We start by proving that $\mathcal{G}_{\mathcal{M}}=\mathcal{H}_n$. Since we already proved that $\mathcal{G}_{\mathcal{M}}$ is a subgraph of $\mathcal{H}_n$, it suffices to show that $\mathcal{H}_n$ is a subgraph of $\mathcal{G}_{\mathcal{M}}$.

Consider $(\sigma,\sigma')\in E(\mathcal{H}_n)$ and order σ and σ' so that $\mathrm{Inv}(\sigma')=\mathrm{Inv}(\sigma)+1$; write $I=(I_1,\ldots,I_n)=\Phi^{-1}(\sigma)$ and $I'=(I'_1,\ldots,I'_n)=\Phi^{-1}(I')$. Using the definition of $\mathcal{H}_n$, we know that there exists $\ell\in[n]$ such that $I'_j=I_j$ for all $j\neq\ell$ and $I'_\ell=I_\ell+1$. For $j\in[n]$, write $(T^j_k)_{k\in[j-1]}$ for

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the jumping times of $(\mathrm{Inv}_j(\mathcal{M}_t))_{t\in[0,\infty)}$, so

$$T_k^j = \inf\left\{t \in [0,\infty) : \mathrm{Inv}_j(\mathcal{M}_t) \geq k\right\}.$$

For convenience, set $T_0^j = 0$ and $T_k^j = \infty$ for $k \geq j$. Note that the processes $((T_k^j)_{k\in[j-1]})_{j\in[n]}$ are mutually independent (over $j \in [n]$) and all have continuous distributions. It follows that the event

$$E = \left\{\max\left\{T_{I_j}^j : j \in [n]\right\} < T_{I_\ell+1}^\ell < \min\left\{T_{I_j+1}^j : j \neq \ell\right\}\right\},$$

which specifies the ordering of the I_j-th and (I_j+1)-th jumps of the j-th entry of the inversion sequence, occurs with non-zero probability. Under the event E, for any $t > 0$ such that

$$\max\left\{T_{I_j}^j : j \in [n]\right\} < t < T_{I_\ell+1}^\ell < \min\left\{T_{I_j+1}^j : j \neq \ell\right\},$$

we have that $\mathrm{Inv}_j(\mathcal{M}_t) = I_j$ for all $j \in [n]$, implying that $\mathcal{M}_t = \Phi(I) = \sigma$. Then, at time $T_{I_\ell+1}^\ell$, the ℓ-th inversion number of $(\mathcal{M}_t)_{t\in[0,\infty)}$ jumps to $I_j + 1$, while no other coordinate jumps, which yields that $\mathcal{M}_{T_{I_\ell+1}^\ell} = \Phi(I') = \sigma'$. Since there is no jump between t and $T_{I_\ell+1}^\ell$, this implies that

$$\mathbb{P}(\exists t \in (0,\infty) : \mathcal{M}_{t-} = \sigma, \mathcal{M}_t = \sigma') \geq \mathbb{P}(E) > 0.$$

Hence $(\sigma, \sigma') \in E(\mathcal{G}_\mathcal{M})$, as desired.

To prove the last statement of the theorem, that is $\langle \mathcal{G}_M \rangle = \langle \mathcal{H}_n \rangle = \mathcal{T}$, simply use that $\mathcal{G}_\mathcal{M} = \mathcal{H}_n$ along with Lemmas 2.3 and 2.4, implying that $\langle \mathcal{H}_n \rangle = \mathcal{T}$. This concludes the proof of the theorem. □

2.3 *Jumping times*

In this section, we prove the results regarding the structure of the jumps of Mallows processes. The proof is divided into two parts: first we derive the distributional limit of a single jumping time, then we show that the full distribution is as claimed.

Proof of Theorem 1.3. Fix $k \in \mathbb{N}$. In the case of a monotone Mallows process $(\mathcal{M}_t)_{t\in[0,\infty)}$, since $\mathrm{Inv}(\mathcal{M}_t)$ is increasing, we have that

$$\mathbb{P}(T_k > t) = \mathbb{P}(\mathrm{Inv}(\mathcal{M}_t) \leq k-1).$$

Using the definition of Mallows permutations, it follows that

$$\mathbb{P}(T_k > t) = \frac{1}{\prod_{k=1}^n \left(\sum_{\ell=0}^{k-1} t^\ell\right)} \sum_{\ell=0}^{k-1} |\{\sigma \in \mathcal{S}_n : \mathrm{Inv}(\sigma) = \ell\}| t^\ell$$

which means that

$$\mathbb{P}(nT_k > t) = \frac{1}{\prod_{k=1}^n \left(\sum_{\ell=0}^{k-1} \left(\frac{t}{n}\right)^\ell\right)} \sum_{\ell=0}^{k-1} \frac{|\{\sigma \in \mathcal{S}_n : \mathrm{Inv}(\sigma) = \ell\}|}{n^\ell} t^\ell.$$

For the denominator, note that, for $n > t$, we have

$$\prod_{k=1}^n \left(\sum_{\ell=0}^{k-1} \left(\frac{t}{n}\right)^\ell\right) = \prod_{k=1}^n \left(\frac{1-\left(\frac{t}{n}\right)^k}{1-\frac{t}{n}}\right) = \left(1-\frac{t}{n}\right)^{-n} \prod_{k=1}^n \left(1-\left(\frac{t}{n}\right)^k\right).$$

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Now, as n tends to infinity, the first term converges to e^t and the second term can be written as

$$\prod_{k=1}^{n}\left(1-\left(\frac{t}{n}\right)^k\right)=\exp\left(\sum_{k=1}^{n}\log\left(1-\left(\frac{t}{n}\right)^k\right)\right).$$

Since $(t/n)^k$ is decreasing in k, we can apply a uniform bound over all the logarithmic terms and find that the previous term converges to 1. Combining the previous results imply that

$$\prod_{k=1}^{n}\left(\sum_{\ell=0}^{k-1}\left(\frac{t}{n}\right)^\ell\right)\longrightarrow e^t$$

as n tends to infinity.

Consider now $S_n^\ell=|\{\sigma\in\mathcal{S}_n:\mathrm{Inv}(\sigma)=\ell\}|$. Using the bijection Φ from Section 2.1, we have that

$$S_n^\ell=|\{(I_1,\ldots,I_n)\in\mathcal{E}_n:I_1+\ldots+I_n=\ell\}|.$$

By changing the condition that $0\le I_j\le j-1$ in the definition of $\mathcal{E}_n$ to $I_j\ge 0$ for an upper bound and to $0\le I_j\le\min(1,j-1)$ for a lower bound, we obtain

$$S_n^\ell\ge|\{(I_1,\ldots,I_n):I_1+\ldots+I_n=\ell\text{ and }0\le I_j\le\min(1,j-1)\}|=\binom{n-1}{\ell}$$

and

$$S_n^\ell\le|\{(I_1,\ldots,I_n):I_1+\ldots+I_n=\ell\text{ and }I_j\ge 0\}|=\binom{n+\ell-1}{\ell}.$$

It follows that, for any fixed ℓ and as n goes to infinity, we have $S_n^\ell/m^\ell\to 1/\ell!$. Since k is fixed, this implies that

$$\sum_{\ell=0}^{k-1}\frac{|\{\sigma\in\mathcal{S}_n:\mathrm{Inv}(\sigma)=\ell\}|}{n^\ell}t^\ell\longrightarrow\sum_{\ell=0}^{k-1}\frac{t^\ell}{\ell!}.$$

Combining the previous results, when n tends to infinity, we have that

$$\mathbb{P}\big(nT_k>t\big)\longrightarrow e^{-t}\sum_{\ell=0}^{k-1}\frac{t^\ell}{\ell!}=\int_t^\infty\frac{u^{k-1}e^{-u}}{(k-1)!}du,$$

which proves that $nT_k\xrightarrow{d}\mathrm{GAMMA}(k,1)$.

We now prove the second part of the theorem, stating that for any $k\in\mathbb{N}$, the first k inter-jumping times of regular Mallows processes converge to a Poisson point process. Let $t\in[0,\infty)$ and $s>0$. Also, for any $\sigma\in\mathcal{S}_n$ let $I=(I_j)_{j\in[n]}=\Phi^{-1}(\sigma)$. Since the process $(\mathcal{M}_t)_{t\in[0,\infty)}$ is regular, it has independent inversions, which implies that

$$\begin{aligned}
&\mathbb{P}\Big(\mathcal{M}_{\frac{t+s}{n}}=\sigma\;\Big|\;\mathcal{M}_{\frac{t}{n}}=\sigma,(\mathcal{M}_u)_{0\le u<\frac{t}{n}}\Big)\\
&\quad=\mathbb{P}\Big(\forall j\in[n],\ \mathrm{Inv}_j(\mathcal{M}_{\frac{t+s}{n}})=I_j\;\Big|\;\forall j\in[n],\ \mathrm{Inv}_j\big(\mathcal{M}_{\frac{t}{n}}\big)=I_j,\big(\mathrm{Inv}_j(\mathcal{M}_u)\big)_{0\le u<\frac{t}{n}}\Big)\\
&\quad=\prod_{j=1}^{n}\mathbb{P}\Big(\mathrm{Inv}_j(\mathcal{M}_{\frac{t+s}{n}})=I_j\;\Big|\;\mathrm{Inv}_j\big(\mathcal{M}_{\frac{t}{n}}\big)=I_j,\big(\mathrm{Inv}_j(\mathcal{M}_u)\big)_{0\le u<\frac{t}{n}}\Big).
\end{aligned}\tag{1}$$

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Note that, whatever the value of $I_j \geq 0$ is, we have that

$$\begin{aligned}&\mathbb{P}\Big(\mathrm{Inv}_j(\mathcal{M}_{\frac{t+s}{n}}) = I_j \;\Big|\; \mathrm{Inv}_j\big(\mathcal{M}_{\frac{t}{n}}\big) = I_j, \big(\mathrm{Inv}_j(\mathcal{M}_u)\big)_{0\leq u<\frac{t}{n}}\Big)\\ &\quad = 1 - \mathbb{P}\Big(\mathrm{Inv}_j(\mathcal{M}_{\frac{t+s}{n}}) > I_j \;\Big|\; \mathrm{Inv}_j\big(\mathcal{M}_{\frac{t}{n}}\big) = I_j, \big(\mathrm{Inv}_j(\mathcal{M}_u)\big)_{0\leq u<\frac{t}{n}}\Big)\end{aligned}$$

Moreover, by definition, we have

$$\begin{aligned}&\mathbb{E}\Big[\mathbb{P}\Big(\mathrm{Inv}_j(\mathcal{M}_{\frac{t+s}{n}}) > I_j \;\Big|\; \mathrm{Inv}_j\big(\mathcal{M}_{\frac{t}{n}}\big) = I_j, \big(\mathrm{Inv}_j(\mathcal{M}_u)\big)_{0\leq u<\frac{t}{n}}\Big)\Big]\\ &\quad = \mathbb{P}\Big(\mathrm{Inv}_j(\mathcal{M}_{\frac{t+s}{n}}) > I_j \;\Big|\; \mathrm{Inv}_j\big(\mathcal{M}_{\frac{t}{n}}\big) = I_j\Big)\\ &\quad \leq \frac{\mathbb{P}\big(\mathrm{Inv}_j(\mathcal{M}_{\frac{t+s}{n}}) > I_j\big)}{\mathbb{P}\big(\mathrm{Inv}_j\big(\mathcal{M}_{\frac{t}{n}}\big) = I_j\big)},\end{aligned}$$

and by the definition of $(\mathcal{M}_t)_{t\in[0,\infty)}$, it follows that

$$\frac{\mathbb{P}\big(\mathrm{Inv}_j(\mathcal{M}_{\frac{t+s}{n}}) > I_j\big)}{\mathbb{P}\big(\mathrm{Inv}_j\big(\mathcal{M}_{\frac{t}{n}}\big) = I_j\big)} = \frac{1-\frac{t}{n}}{\big(\frac{t}{n}\big)^{I_j} - \big(\frac{t}{n}\big)^j} \cdot \frac{\big(\frac{t+s}{n}\big)^{I_j+1} - \big(\frac{t+s}{n}\big)^j}{1-\frac{t+s}{n}} \underset{n\to\infty}{\longrightarrow} 0\,.$$

The previous results imply that

$$\mathbb{P}\Big(\mathrm{Inv}_j(\mathcal{M}_{\frac{t+s}{n}}) = I_j \;\Big|\; \mathrm{Inv}_j\big(\mathcal{M}_{\frac{t}{n}}\big) = I_j, \big(\mathrm{Inv}_j(\mathcal{M}_u)\big)_{0\leq u<\frac{t}{n}}\Big) \longrightarrow 1 \tag{2}$$

in probability, as n goes to infinity. Now, recall that, by assumption, for any $j \in [n]$, the process $(\mathrm{Inv}_j(\mathcal{M}_t))_{t\in[0,\infty)}$ is increasing. This implies that, for $I_j = 0$, we have

$$\begin{aligned}&\mathbb{P}\Big(\mathrm{Inv}_j(\mathcal{M}_{\frac{t+s}{n}}) = 0 \;\Big|\; \mathrm{Inv}_j\big(\mathcal{M}_{\frac{t}{n}}\big) = 0, \big(\mathrm{Inv}_j(\mathcal{M}_u)\big)_{0\leq u<\frac{t}{n}}\Big)\\ &\quad = \mathbb{P}\Big(\mathrm{Inv}_j(\mathcal{M}_{\frac{t+s}{n}}) = 0 \;\Big|\; \mathrm{Inv}_j\big(\mathcal{M}_{\frac{t}{n}}\big) = 0\Big)\\ &\quad = \frac{\mathbb{P}\big(\mathrm{Inv}_j(\mathcal{M}_{\frac{t+s}{n}}) = 0\big)}{\mathbb{P}\big(\mathrm{Inv}_j\big(\mathcal{M}_{\frac{t}{n}}\big) = 0\big)}\\ &\quad = \frac{1-\frac{t+s}{n}}{1-\big(\frac{t+s}{n}\big)^j} \cdot \frac{1-\big(\frac{t}{n}\big)^j}{1-\frac{t}{n}}\,.\end{aligned} \tag{3}$$

We will now combine (2) and (3) with (1) to obtain the convergence of the sequence of the first jumping times of $(\mathcal{M}_t)_{t\in[0,\infty)}$. Fix $k \geq 0$. For any $\sigma \in \mathcal{S}_n$ such that $\mathrm{Inv}(\sigma) \leq k$, by separating according to whether $I_j = 0$ or $I_j > 0$, (1) and (3) tell us that

$$\begin{aligned}&\mathbb{P}\Big(\mathcal{M}_{\frac{t+s}{n}} = \sigma \;\Big|\; \mathcal{M}_{\frac{t}{n}} = \sigma, (\mathcal{M}_u)_{0\leq u<\frac{t}{n}}\Big)\\ &\quad = \prod_{j\,:\,I_j>0} \mathbb{P}\Big(\mathrm{Inv}_j(\mathcal{M}_{\frac{t+s}{n}}) = I_j \;\Big|\; \mathrm{Inv}_j\big(\mathcal{M}_{\frac{t}{n}}\big) = I_j, \big(\mathrm{Inv}_j(\mathcal{M}_u)\big)_{0\leq u<\frac{t}{n}}\Big)\\ &\qquad \times \prod_{j\,:\,I_j=0} \frac{1-\frac{t+s}{n}}{1-\big(\frac{t+s}{n}\big)^j} \cdot \frac{1-\big(\frac{t}{n}\big)^j}{1-\frac{t}{n}}\,.\end{aligned}$$

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Note that $|\{j : I_j > 0\}| \leq \mathrm{Inv}(\sigma) \leq k$. Combining this fact with the result of (2) and the assumption that the different inversion numbers are independent processes, it follows that

$$\prod_{j:I_j>0} \frac{\mathbb{P}\Big(\mathrm{Inv}_j\big(\mathcal{M}_{\frac{t+s}{n}}\big) = I_j \,\Big|\, \mathrm{Inv}_j\big(\mathcal{M}_{\frac{t}{n}}\big) = I_j, \big(\mathrm{Inv}_j(\mathcal{M}_u)\big)_{0\leq u<\frac{t}{n}}\Big)}{\frac{1-\frac{t+s}{n}}{1-\left(\frac{t+s}{n}\right)^j} \cdot \frac{1-\left(\frac{t}{n}\right)^j}{1-\frac{t}{n}}} = 1 + o_{\mathbb{P}}(1)\,.$$

This means that the previous result is

$$\mathbb{P}\Big(\mathcal{M}_{\frac{t+s}{n}} = \sigma \,\Big|\, \mathcal{M}_{\frac{t}{n}} = \sigma, (\mathcal{M}_u)_{u<\frac{t}{n}}\Big) = \big(1 + o_{\mathbb{P}}(1)\big) \prod_{j=1}^{n} \frac{1-\frac{t+s}{n}}{1-\left(\frac{t+s}{n}\right)^j} \cdot \frac{1-\left(\frac{t}{n}\right)^j}{1-\frac{t}{n}}$$

as $n \to \infty$, where for any fixed $k \in \mathbb{N}$ the $o_{\mathbb{P}}(1)$ term is uniform over all permutations with $\mathrm{Inv}(\sigma) \leq k$. Using the convergence result stated in the previous case, we know that

$$\prod_{j=1}^{n} \frac{1-\left(\frac{t}{n}\right)^j}{1-\left(\frac{t+s}{n}\right)^j} \longrightarrow 1\,,$$

from which it follows that, for any $k \in \mathbb{N}$, uniformly over permutations σ with $\mathrm{Inv}(\sigma) \leq k$,

$$\mathbb{P}\Big(\mathcal{M}_{\frac{t+s}{n}} = \sigma \,\Big|\, \mathcal{M}_{\frac{t}{n}} = \sigma, (\mathcal{M}_u)_{u<\frac{t}{n}}\Big) = \big(1 + o_{\mathbb{P}}(1)\big)\left(\frac{1-\frac{t+s}{n}}{1-\frac{t}{n}}\right)^n = \big(1 + o_{\mathbb{P}}(1)\big)e^{-s}\,.$$

This exactly corresponds to saying that the first inter jumping times of the process are asymptotically distributed as independent EXPONENTIAL(1) random variables, proving the desired result. □

3. The birth Mallows process

In this section, we construct and study the birth Mallows process, a regular Mallows process with the Markov property. In Section 3.1, we start by defining it using an appropriate family of processes $(I(t))_{t\in[0,\infty)}$ and applying Proposition 2.2. In Section 3.2, we then study the marginals of its corresponding discrete chain. In Section 3.3, we show that it is the unique process from Theorem 1.1. Finally, in Section 3.4, we show the existence of a smooth and doubly Markovian Mallows process as stated in Theorem 1.5. For the rest of this section, fix $n \in \mathbb{N}$.

3.1 Definition

The construction of the birth Mallows process is based on the time-inhomogeneous birth processes defined in the following lemma.

Lemma 3.1. *Fix $j \in \mathbb{N}$. For any $0 \leq k \leq j-1$ and $t \in [0,\infty)$, let*

$$p_j(t,k) := \frac{1}{\sum_{\ell=0}^{j-1} t^\ell}\left[(k+1)\sum_{\ell=0}^{j-2}(\ell+1)t^\ell - j\sum_{\ell=j-k-2}^{j-2}(\ell-j+k+2)t^\ell\right].$$

Define now the time-inhomogenous birth process $(B_j(t))_{t\in[0,\infty)}$ by $B_j(0) = 0$ and infinitesimal rates given by

$$\mathbb{P}\big(B_j(t+h) = k+\ell \,\big|\, B_j(t) = k\big) = \begin{cases} 1 - hp_j(t,k) + o(h) & \text{if } \ell = 0 \\ hp_j(t,k) + o(h) & \text{if } \ell = 1 \\ o(h) & \text{otherwise}\,. \end{cases}$$

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Then for all $0 \le k \le j-1$ *and all* $t \in [0,\infty)$*, we have*

$$\mathbb{P}(B_j(t)=k)=\frac{t^k}{\sum_{\ell=0}^{j-1} t^\ell}\,.$$

Proof. First, note that, when $k=j-1$

$$p_j(t,j-1)=\frac{1}{\sum_{\ell=0}^{j-1} t^\ell}\left[j\sum_{\ell=0}^{j-2}(\ell+1)t^\ell - j\sum_{\ell=-1}^{j-2}(\ell+1)t^\ell\right]=0\,,$$

which supports the fact that $B_j(t)\le j-1$. Now, before proving the result, we verify that p_j is non-negative, a necessary assumption for this birth process to be well-defined. For $0\le k\le j-2$ and $t\in[0,\infty)$, we have

$$\begin{aligned}&(k+1)\sum_{\ell=0}^{j-2}(\ell+1)t^\ell - j\sum_{\ell=j-k-2}^{j-2}(\ell-j+k+2)t^\ell\\ &\quad=(k+1)\sum_{\ell=0}^{j-k-3}(\ell+1)t^\ell+\sum_{\ell=j-k-2}^{j-2}[(k+1)(\ell+1)-j(\ell-j+k+2)]t^\ell\,.\end{aligned}$$

The first sum is non-negative, and by re-organizing the terms inside the second sum, we have

$$(k+1)(\ell+1)-j(\ell-j+k+2)=k+1-j(k+2-j)-\ell(j-k-1)\,.$$

Since $j-k-1\ge 1>0$, the previous equation is decreasing in ℓ. Using that $\ell\le j-2$, it follows that

$$\begin{aligned}(k+1)(\ell+1)-j(\ell-j+k+2)&\ge(k+1)(j-2+1)-j(j-2-j+k+2)\\&=j-k-1>0\,,\end{aligned}$$

proving that $p_j(t,k)\ge 0$.

We now want to prove that $B_j(t)$ has the desired distribution. To do so, we prove by induction on k that

$$\mathbb{P}(B_j(t)=k)=\frac{t^k}{\sum_{\ell=0}^{j-1} t^\ell}\,.$$

For $k=0$, using that

$$\mathbb{P}\big(B_j(t+h)=0\,\big|\,B_j(t)=0\big)=1-hp_j(t,0)+o(h)\,,$$

we have that

$$\frac{\partial}{\partial t}(\mathbb{P}(B_j(t)=0))=-p_j(t,0)\mathbb{P}(B_j(t)=0)\,.$$

We solve this equation using that $\mathbb{P}(B_j(0)=0)=1$ and that

$$p_j(t,0)=\frac{1}{\sum_{\ell=0}^{j-1} t^\ell}\sum_{\ell=0}^{j-2}(\ell+1)t^\ell=\frac{\partial}{\partial t}\left(\log\left(\sum_{\ell=0}^{j-1}t^\ell\right)\right)\,,$$

to obtain

$$\mathbb{P}(B_j(t)=0)=\frac{1}{\sum_{\ell=0}^{j-1} t^\ell}\,.$$

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Assume now that for some $0 \le k \le j-2$, we have

$$\mathbb{P}(B_j(t)=k) = \frac{t^k}{\sum_{\ell=0}^{j-1} t^\ell}\,.$$

Then, the forward equation tells us that

$$\begin{aligned}\mathbb{P}(B_j(t+h)=k+1) &= \sum_{\ell=0}^{k+1} \mathbb{P}(B_j(t+h)=k+1 \mid B_j(t)=\ell)\mathbb{P}(B_j(t)=\ell)\\ &= [1-hp_j(t,k+1)]\mathbb{P}(B_j(t)=k+1) + hp_j(t,k)\frac{t^k}{\sum_{\ell=0}^{j-1} t^\ell} + o(h)\,,\end{aligned}$$

from which it follows that

$$\frac{\partial}{\partial t}(\mathbb{P}(B_j(t)=k+1)) = -p_j(t,k+1)\mathbb{P}(B_j(t)=k+1) + p_j(t,k)\frac{t^k}{\sum_{\ell=0}^{j-1} t^\ell}\,. \tag{4}$$

Consider the target function for $\mathbb{P}(B_j(t)=k+1)$:

$$f(t) = \frac{t^{k+1}}{\sum_{\ell=0}^{j-1} t^\ell}\,. \tag{5}$$

Note that

$$-p_j(t,k+1)\,f(t) + p_j(t,k)\frac{t^k}{\sum_{\ell=0}^{j-1} t^\ell} = \frac{t^k}{\sum_{\ell=0}^{j-1} t^\ell}[p_j(t,k) - tp_j(t,k+1)]\,, \tag{6}$$

from which, using the definition of p_j, we obtain

$$\begin{aligned}p_j(t,k) - tp_j(t,k+1) = \frac{1}{\sum_{\ell=0}^{j-1} t^\ell}&\left[(k+1)\sum_{\ell=0}^{j-2}(\ell+1)t^\ell - j\sum_{\ell=j-k-2}^{j-2}(\ell-j+k+2)t^\ell\right]\\ &- \frac{1}{\sum_{\ell=0}^{j-1} t^\ell}\left[(k+2)\sum_{\ell=0}^{j-2}(\ell+1)t^{\ell+1} - j\sum_{\ell=j-k-3}^{j-2}(\ell-j+k+3)t^{\ell+1}\right].\end{aligned}$$

Using that

$$(k+1)\sum_{\ell=0}^{j-2}(\ell+1)t^\ell - (k+2)\sum_{\ell=0}^{j-2}(\ell+1)t^{\ell+1} = -(k+2)(j-1)t^{j-1} + \sum_{\ell=0}^{j-2}(k-\ell+1)t^\ell\,,$$

and that

$$-j\sum_{\ell=j-k-2}^{j-2}(\ell-j+k+2)t^\ell + j\sum_{\ell=j-k-3}^{j-2}(\ell-j+k+3)t^{\ell+1} = j(k+1)t^{j-1}\,,$$

we obtain

$$p_j(t,k) - tp_j(t,k+1) = \frac{1}{\sum_{\ell=0}^{j-1} t^\ell}\sum_{\ell=0}^{j-1}(k-\ell+1)t^\ell\,.$$

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Plug the previous formula into (6) and use the definition of f from (5) to obtain that

$$-p_j(t,k+1)f(t)+p_j(t,k)\frac{t^k}{\sum_{\ell=0}^{j-1}t^\ell}=\frac{t^k}{\left(\sum_{\ell=0}^{j-1}t^\ell\right)^2}\left[\sum_{\ell=0}^{j-1}(k+1-\ell)t^\ell\right]=f'(t)\,.$$

Note now that this last equation is the same as (4) where the probability $\mathbb{P}(B_j(t)=k+1)$ is replaced by $f(t)$. Using that $f(0)=\mathbb{P}(B_j(0)=k+1)=0$, it follows that $\mathbb{P}(B_j(t)=k+1)=f(t)$, which completes the inductive step. This concludes the proof of the lemma. □

Using Lemma 3.1, we can now define the birth Mallows process.

Proposition 3.2. *Let $(B(t))_{t\in[0,\infty)}=(B_1(t),\dots,B_n(t))_{t\in[0,\infty)}$, where $(B_j(t))_{t\in[0,\infty)}$ is defined as in Lemma3.1, and Φ be the bijection from Section 2.1. Define the birth Mallows process by*

$$(\mathcal{M}_t^B)_{t\in[0,\infty)}:=(\Phi(B(t)))_{t\in[0,\infty)}\,.$$

Then, $(\mathcal{M}_t^B)_{t\in[0,\infty)}$ is a regular Mallows process and has the Markov property.

Proof. By the definition of $B_1,\dots,B_n$ from Lemma 3.1, we know that $(B(t))_{t\in[,\infty)}$ is a cadlag process of $\mathcal{E}_n$ satisfying all conditions of Proposition 2.2. This implies that $(\mathcal{M}_t^B)_{t\in[0,\infty)}$ is a regular Mallows process. To see that it is also a Markov process, recall that Φ is a bijection, meaning that

$$\left\{\mathcal{M}_t^B=\sigma\right\}=\left\{B(t)=\Phi^{-1}(\sigma)\right\}=\left\{(B_1(t),\dots,B_n(t))=\Phi^{-1}(\sigma)\right\}.$$

Now, since all the processes $(B_1(t))_{t\in[0,\infty)},\dots,(B_n(t))_{t\in[0,\infty)}$ are time-inhomogeneous birth processes, in particular they are Markov processes. This implies that, for any $t_1<\ldots<t_k<t$ and $\sigma_1,\dots,\sigma_k,\sigma\in\mathcal{S}_n$, we have

$$\begin{aligned}&\mathbb{P}\big(\mathcal{M}_t^B=\sigma\,\big|\,\mathcal{M}_{t_1}^B=\sigma_1,\dots,\mathcal{M}_{t_k}^B=\sigma_k\big)\\&\quad=\mathbb{P}\big(B(t)=\Phi^{-1}(\sigma)\,\big|\,B(t_1)=\Phi^{-1}(\sigma_1),\dots,B(t_k)=\Phi^{-1}(\sigma_k)\big)\\&\quad=\prod_{j=1}^n\mathbb{P}\big(B_j(t)=\Phi^{-1}(\sigma)_j\,\big|\,B_j(t_1)=\Phi^{-1}(\sigma_1)_j,\dots,B_j(t_k)=\Phi^{-1}(\sigma_k)_j\big)\,,\end{aligned}$$

where the second inequality follows from the independence of the different coordinates of the processes $(B(t))_{t\in[0,\infty)}$. But then, using that these are Markov processes, we obtain

$$\begin{aligned}\mathbb{P}\big(\mathcal{M}_t^B=\sigma\,\big|\,\mathcal{M}_{t_1}^B=\sigma_1,\dots,\mathcal{M}_{t_k}^B=\sigma_k\big)&=\prod_{j=1}^n\mathbb{P}\big(B_j(t)=\Phi^{-1}(\sigma)_j\,\big|\,B_j(t_k)=\Phi^{-1}(\sigma_k)_j\big)\\&=\mathbb{P}\big(\mathcal{M}_t^B=\sigma\,\big|\,\mathcal{M}_{t_k}^B=\sigma_k\big)\,,\end{aligned}$$

which exactly corresponds to the fact that $(\mathcal{M}_t^B)_{t\in[0,\infty)}$ is a Markov process. □

3.2 Discrete jumps

In this section, we provide a formula for the distribution of the discrete process $(M_k^B)_{0\le k\le N}$ corresponding to the birth Mallows process. This formula could be used to prove that Conjecture 1.4 holds, although the computations proved to be more difficult than expected and prevented any substantial progress on the conjecture. For clarity, we drop the superscript B on both continuous and discrete time birth Mallows processes for the rest of this section.

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Recall that, since $(\mathcal{M}_t)_{t\in[0,\infty)}$ is smooth, then $\mathrm{Inv}(M_k) = k$ and $(M_k)_{0\le k\le N}$ evolves on the expanded hypercube (Theorem 1.2), meaning that

$$\sum_{j=1}^{n} \left|\mathrm{Inv}_j(M_k) - \mathrm{Inv}(M_{k+1})\right| = 1\,.$$

Moreover, given a permutation $\sigma \in \mathcal{S}_n$, define

$$P(t,\sigma) = \sum_{j=1}^{n} p_j\big(t, \mathrm{Inv}_j(\sigma)\big) \tag{7}$$

for the sum of the rates of the birth Mallows process defined in Lemma 3.1, when the current permutation is σ. Then, we have the following formula for sequences of values of $(M_k)_{0\le k\le N}$.

Proposition 3.3. *Let $(\mathcal{M}_t)_{t\in[0,\infty)}$ be the birth Mallows process defined in Section 3.1 and $(M_k)_{0\le k\le N}$ be its corresponding jumping process. Let $(\sigma_0,\dots,\sigma_m)$ be a sequence of permutations forming a path on the expanded hypercube and such that $\mathrm{Inv}(\sigma_k) = k$. By this definition, there exists $j_1,\dots,j_m$ such that, for all $1 \le k \le m$*

$$\mathrm{Inv}_j(\sigma_k) = \begin{cases} \mathrm{Inv}_j(\sigma_{k-1}) + 1 & \text{if } j = j_k \\ \mathrm{Inv}_j(\sigma_{k-1}) & \text{if } j \neq j_k\,. \end{cases}$$

Then, we have

$$\begin{aligned}&\mathbb{P}(M_0 = \sigma_0, M_1 = \sigma_1, \dots, M_m = \sigma_m)\\ &= \int_{0=u_0<u_1<\dots<u_m} \prod_{k=1}^{m} \left(p_{j_k}(u_k, \mathrm{Inv}_{j_k}(\sigma_{k-1}))e^{-\int_{u_{k-1}}^{u_k} P(u,\sigma_{k-1})du} \right) du_1\cdots du_m\,.\end{aligned}$$

Proof. Start by fixing $t \in [0,\infty)$ and let $T = \inf\{s : \mathcal{M}_s \neq \mathcal{M}_t\}$ be the first jumping time after t. Let $I = (I_j)_{j\in[n]} \in \mathcal{E}_n$ and $\sigma = \Phi(I)$ be as defined in Section 2.1 and such that $\mathrm{Inv}(\sigma) < \binom{n}{2}$ or equivalently there exists j such that $I_j < j-1$. Finally, let $T^{(j)} = \inf\{s : B_j(s) > B_j(t)\}$ be the next jumping time of B_j, as defined in Lemma 3.1. Note that $T = \min\{T^{(1)},\dots,T^{(n)}\}$. We now want to characterize the distribution of $(T^{(1)},\dots,T^{(n)})$.

Since $(\mathcal{M}_t)_{t\in[0,\infty)}$ has independent inversions, $T^{(1)},\dots,T^{(n)}$ are independent of each other and it suffices to characterize their distribution individually. Fix $j \in [n]$. Using the rate of B_j, we observe that

$$\begin{aligned}&\mathbb{P}\big(B_j(t+s+h) = I_j \,\big|\, B_j(t) = I_j\big)\\ &\quad= \mathbb{P}\big(B_j(t+s+h) = I_j \,\big|\, B_j(t+s) = I_j\big)\mathbb{P}\big(B_j(t+s) = I_j \,\big|\, B_j(t) = I_j\big)\\ &\quad= (1 - hp_j(t, I_j) = o(h))\mathbb{P}\big(B_j(t+s) = I_j \,\big|\, B_j(t) = I_j\big)\end{aligned}$$

which means that

$$\frac{\partial}{\partial s}\mathbb{P}\big(B_j(t+s) = I_j \,\big|\, B_j(t) = I_j\big) = -p_j(t, I_j)\mathbb{P}\big(B_j(t+s) = I_j \,\big|\, B_j(t) = I_j\big)\,.$$

By solving this equation and using that $\mathbb{P}(B_j(t) = I_j \mid B_j(t) = I_j) = 1$, we obtain

$$\mathbb{P}\big(B_j(t+s) = I_j \,\big|\, B_j(t) = I_j\big) = \exp\left(-\int_t^{t+s} p_j(u, I_j)du\right)\,.$$

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Finally, observe that $\{T^{(j)} > t+s \mid B_j(t) = I_j\} = \{B_j(t+s) = I_j \mid B_j(t) = I_j\}$ so that the probability density function of $T^{(j)}$ conditionally given that $B_j(t) = I_j$ is

$$\frac{\partial}{\partial s}\mathbb{P}\big(T^{(j)} > t+s \,\big|\, B_j(t) = I_j\big) = p_j(t+s, I_j)e^{-\int_t^{t+s} p_j(u,I_j)du}.$$

This provides us the distribution of the vector $(T^{(1)}, \dots, T^{(n)})$. Note that this formula remains true when $I_j = j-1$ since, in that case $p_j(u, I_j) = 0$.

Use the previous formula to obtain the joint distribution of the next jump and the time of the jump as

$$\begin{aligned}
&\mathbb{P}\big(T = T^{(J)}, T \le t+r \,\big|\, \mathcal{M}_t = \sigma\big)\\
&\quad = \mathbb{E}\big[\mathbb{P}\big(T = T^{(J)}, T^{(J)} \le t+r \,\big|\, T^{(J)}, \mathcal{M}_t = \sigma\big)\big]\\
&\quad = \int_0^r \prod_{j\neq J} \mathbb{P}\big(T^{(j)} > t+s \,\big|\, B_j(t) = I_j\big) p_J(t+s, I_J)e^{-\int_t^{t+s} p_J(u,I_J)du} ds\\
&\quad = \int_0^r p_J(t+s, I_J)e^{-\int_t^{t+s} P(u,\sigma)du} ds\,.
\end{aligned}$$

Note that $\{T = T^{(J)} \mid \mathcal{M}_t = \sigma\} = \{\mathcal{M}_T = \sigma^{(J)} \mid \mathcal{M}_t = \sigma\}$ where $\sigma^{(J)}$ is obtain by increasing the J-th inversion number of σ by 1. Thus, the previous formula can be re-written as

$$\mathbb{P}\big(\mathcal{M}_T = \sigma^{(J)}, T \le t' \,\big|\, \mathcal{M}_t = \sigma\big) = \int_t^{t'} p_J(s, I_J)e^{-\int_t^s P(u,\sigma)du} ds\,. \tag{8}$$

We conclude the proof by using this formula to show the desired result.

In order to prove the formula of the proposition, we prove by induction on m that

$$\begin{aligned}
&\mathbb{P}(M_0 = \sigma_0, M_1 = \sigma_1, \dots, M_m = \sigma_m, T_m \le t)\\
&\quad = \int_{0=u_0<u_1<\dots<u_m<t} \prod_{k=1}^m \left(p_{j_k}(u_k, \mathrm{Inv}_{j_k}(\sigma_{k-1}))e^{-\int_{u_{k-1}}^{u_k} P(u,\sigma_{k-1})du}\right) du_1 \cdots du_m\,.
\end{aligned}$$

For $m = 1$, this directly follows from (8) as

$$\begin{aligned}
\mathbb{P}(M_0 = \sigma_0, M_1 = \sigma_1, T_1 \le t) &= \mathbb{P}\big(\mathcal{M}_{T_1} = \sigma_1, T_1 \le t \,\big|\, \mathcal{M}_0 = \sigma_0\big)\\
&= \int_0^t p_{j_1}(u_1, \mathrm{Inv}_{j_1}(\sigma_0))e^{-\int_0^{u_1} P(u,\sigma)du} du_1\,.
\end{aligned}$$

Now, assuming this holds for some $m \ge 1$, we have that

$$\begin{aligned}
&\mathbb{P}(M_0 = \sigma_0, M_1 = \sigma_1, \dots, M_m = \sigma_m, M_{m+1} = \sigma_{m+1}, T_{m+1} \le t)\\
&\quad = \mathbb{E}\big[\mathbb{P}\big(M_{m+1} = \sigma_{m+1}, T_{m+1} \le t \,\big|\, T_m, M_0 = \sigma_0, M_1 = \sigma_1, \dots, M_m = \sigma_m\big)\big]\\
&\quad = \int_0^t \big[\mathbb{P}\big(M_{m+1} = \sigma_{m+1}, T_{m+1} \le t \,\big|\, T_m = s, M_0 = \sigma_0, M_1 = \sigma_1, \dots, M_m = \sigma_m\big)\\
&\qquad \times \frac{\partial}{\partial t}\mathbb{P}(M_0 = \sigma_0, M_1 = \sigma_1, \dots, M_m = \sigma_m, T_m \le t)\big|_{t=s}\big] ds\,.
\end{aligned}$$

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On one hand, by the induction hypothesis, we have that

$$\frac{\partial}{\partial t}\mathbb{P}(M_0=\sigma_0, M_1=\sigma_1, \dots, M_m=\sigma_m, T_m\le t)\big|_{t=s}$$
$$=\int_{0=u_0<u_1<\dots<u_{m-1}<u_m=s}\prod_{k=1}^{m}\left(p_{j_k}(u_k, \mathrm{Inv}_{j_k}(\sigma_{k-1}))e^{-\int_{u_{k-1}}^{u_k}P(u,\sigma_{k-1})du}\right)du_1\cdots du_{m-1}\,.$$

On the other hand, by the Markov property and thanks to (8), we obtain

$$\begin{aligned}&\mathbb{P}\big(M_{m+1}=\sigma_{m+1}, T_{m+1}\le t \,\big|\, T_m=s, M_0=\sigma_0, M_1=\sigma_1, \dots, M_m=\sigma_m\big)\\ &\quad=\mathbb{P}\big(M_{m+1}=\sigma_{m+1}, T_{m+1}\le t \,\big|\, \mathcal{M}_s=\sigma_m\big)\\ &\quad=\int_s^t p_{j_{m+1}}(u_{m+1}, \mathrm{Inv}_{j_{m+1}}(\sigma_m))e^{-\int_s^{u_{m+1}}P(u,\sigma_m)du}du_{m+1}\,.\end{aligned}$$

Thus, the three previous equations combined lead to

$$\mathbb{P}(M_0=\sigma_0, M_1=\sigma_1, \dots, M_m=\sigma_m, M_{m+1}=\sigma_{m+1}, T_{m+1}\le t)$$
$$=\int_{0=u_0<u_1<\dots<u_{m+1}<t}\prod_{k=1}^{m+1}\left(p_{j_k}(u_k, \mathrm{Inv}_{j_k}(\sigma_{k-1}))e^{-\int_{u_{k-1}}^{u_k}P(u,\sigma_{k-1})du}\right)du_1\cdots du_m du_{m+1}\,.$$

This proves that the induction hypothesis holds and the statement of the proposition follows by taking the limit as t goes to infinity. □

3.3 Uniqueness

Using the definition of the birth Mallows process in Section 3.1, in particular its properties from Proposition 3.2, it only remains to prove that it is the unique regular Mallows process which is also a Markov process in order to conclude the proof of Theorem 1.1.

Proof of Theorem 1.1. Thanks to Proposition 3.2, we know that there exists a cadlag Markov process which is a regular Mallows process. Conversely, using Φ as defined in Section 2.1, regular Mallows processes are fully characterized by their corresponds inversion sequence processes $(\mathrm{Inv}_j(\mathcal{M}_t))_{t\in[0,\infty)}, (\mathrm{Inv}_2(\mathcal{M}_t))_{t\in[0,\infty)}, \dots, (\mathrm{Inv}_n(\mathcal{M}_t))_{t\in[0,\infty)}$. Since regular Mallows processes have the independent inversion property, these processes are independent jump processes. Moreover, using that a regular Mallows process is by definition smooth, we know that these processes will only jump by increments of 1. This means that each $(\mathrm{Inv}_j(\mathcal{M}_t))_{t\in[0,\infty)}$ for $j\in[n]$ is a time-inhomogeneous birth process, for which properties of time-homogenous birth process apply by adapting the formulas to incorporate the time parameter (see [24, Section 6.8] for the properties of time-homogeneous birth processes, which extend to time-inhomogenous birth processes). In particular, for any $j\in[n]$, since the time-inhomogenous birth process $(\mathrm{Inv}_j(\mathcal{M}_t))_{t\in[0,\infty)}$ has a continuous distribution, it is fully characterized by its infinitesimal generator

$$g_j(t,k)=\lim_{h\downarrow 0}\frac{1-\mathbb{P}\big(\mathrm{Inv}_j(\mathcal{M}_{t+h})=k \,\big|\, \mathrm{Inv}_j(\mathcal{M}_t)=k\big)}{h}\,,$$

just like in the case of time-homogeneous processes. Furthermore, these incremental generators g_j have to satisfy the forward equation

$$\frac{\partial}{\partial t}(\mathbb{P}(\mathrm{Inv}_j(\mathcal{M}_t)=k+1))=g_j(t,k)\mathbb{P}(\mathrm{Inv}_j(\mathcal{M}_t)=k)-g_j(t,k+1)\mathbb{P}(\mathrm{Inv}_j(\mathcal{M}_t)=k+1)\,.$$

Since the previous equation is the same as (4) with p_j replaced by g_j, it follows that $g_j=p_j$ for all $j\in[n]$. This proves the uniqueness of such process. □

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3.4 Doubly Markovian property

In this section, we focus on the doubly Markovian property of Mallows processes. The goal of this section is to prove the existence of a smooth doubly Markovian Mallows process, as stated in Theorem 1.5. The proof relies on one key lemma stated below.

Write $L_0, \dots, L_N$ with $N = \binom{n}{2}$ for the *levels* of the expanded hypercube, that is the set of permutations with constant inversion number:

$$L_i = \{\sigma \in \mathcal{S}_n : \mathrm{Inv}(\sigma) = i\}\,.$$

We now use these levels to provide a method for generating smooth doubly Markovian Mallows processes by splitting its continuous jumping times and its transitions probability on the levels of the expanded hypercube.

Lemma 3.4. *Let $(M_k)_{0\le k\le N}$ be a Markov chain on $\mathcal{H}_n$ such that*

$$\mathbb{P}(M_i = \sigma) = \frac{1}{|L_i|}\mathbb{1}_{\sigma\in L_i}$$

and $(X_t)_{t\in[0,\infty)}$ be an independent cadlag Markov process such that $X_t - X_{t-} \in \{0,1\}$ and, for all $0 \le i \le N$,

$$\mathbb{P}(X_t = i) = \frac{|L_i| t^i}{Z_{n,t}}\,.$$

Then the process $(\mathcal{M}_t)_{t\in[0,\infty)}$ defined by $\mathcal{M}_t = M_{X_t}$ is a smooth doubly Markovian Mallows process.

Proof. We start by observing that $\mathrm{Inv}(M_i) = i$ and that $(X_t)_{t\in[0,\infty)}$ only increases, so that $(\mathcal{M}_t)_{t\in[0,\infty)}$ is monotone. Moreover, since $(M_k)_{0\le k\le N}$ evolves on $\mathcal{H}_n$ and $(X_t)_{t\in[0,\infty)}$ is cadlag and only has jumps of size 1, it also follows that $(\mathcal{M}_t)_{t\in[0,\infty)}$ is cadlag and smooth (while not necessarily being a Mallows process yet). We can also check that $(M_k)_{0\le k\le N}$ is the discrete jumping process corresponding to $(\mathcal{M}_t)_{t\in[0,\infty)}$ and so $(\mathcal{M}_t)_{t\in[0,\infty)}$ has the discrete Markov property. We now prove that $(\mathcal{M}_t)_{t\in[0,\infty)}$ is itself a Markov process.

Fix $t < t'$ and $\sigma, \sigma' \in \mathcal{S}_n$ with $i = \mathrm{Inv}(\sigma)$ and $i' = \mathrm{Inv}(\sigma')$. We first observe that

$$\{\mathcal{M}_t = \sigma\} = \{X_t = i, M_i = \sigma\}$$

and that the same formula holds for t', σ', and i'. Moreover, $\{\mathcal{M}_t = \sigma, (\mathcal{M}_s)_{s<t}\}$ is a function of $\{X_t = i, (X_s)_{s<t}\}$ and $\{M_i = \sigma, (M_k)_{0\le k\le i}\}$. Combining the previous two arguments, we see that

$$\begin{aligned}
&\mathbb{P}\big(\mathcal{M}_{t'} = \sigma' \,\big|\, \mathcal{M}_t = \sigma, (\mathcal{M}_s)_{s<t}\big)\\
&\quad= \mathbb{P}\big(M_{i'} = \sigma' \,\big|\, M_i = \sigma, (M_k)_{0\le k<i}\big)\mathbb{P}\big(X_{t'} = i' \,\big|\, X_t = i, (X_s)_{s<t}\big)\\
&\quad= \mathbb{P}\big(M_{i'} = \sigma' \,\big|\, M_i = \sigma\big)\mathbb{P}\big(X_{t'} = i' \,\big|\, X_t = i\big)\\
&\quad= \mathbb{P}\big(\mathcal{M}_{t'} = \sigma' \,\big|\, \mathcal{M}_t = \sigma\big)\,,
\end{aligned}$$

thus proving that $(\mathcal{M}_t)_{t\in[0,\infty)}$ is a Markov process.

We now conclude the proof by showing that $\mathcal{M}_t$ is $\pi_{n,t}$-distributed. Using again that $\{\mathcal{M}_t = \sigma\} = \{X_t = \mathrm{Inv}(\sigma), M_{\mathrm{Inv}(\sigma)} = \sigma\}$, and by independence of $(M_k)_{0\le k\le N}$ and $(X_t)_{t\in[0,\infty)}$, we see that

$$\mathbb{P}(\mathcal{M}_t = \sigma) = \mathbb{P}(X_t = \mathrm{Inv}(\sigma))\mathbb{P}(M_{\mathrm{Inv}(\sigma)} = \sigma)\,.$$

But now, using the given distribution of $(X_t)_{t\in[0,\infty)}$ and $(M_k)_{0\le k\le N}$, we obtain

$$\mathbb{P}(\mathcal{M}_t = \sigma) = \frac{|L_{\mathrm{Inv}(\sigma)}| t^{\mathrm{Inv}(\sigma)}}{Z_{n,t}} \cdot \frac{1}{|L_{\mathrm{Inv}(\sigma)}|} = \frac{t^{\mathrm{Inv}(\sigma)}}{Z_{n,t}}\,.$$

This is exactly the Mallows distribution, which concludes the proof that $(\mathcal{M}_t)_{t\in[0,\infty)}$ is a smooth doubly Markovian Mallows process. □

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With the previous proposition, we now can prove the existence of a smooth and doubly Markovian Mallows process.

Proof of Theorem 1.5. Using Lemma 3.4, it suffices to show the existence of $(M_k)_{0\le k\le N}$ and $(X_t)_{t\in[0,\infty)}$ as stated. The existence of $(M_k)_{0\le k\le N}$ is proven in [1, Theorem 2.3], so we now show that we can define $(X_t)_{t\in[0,\infty)}$ as desired.

Define the time-inhomogeneous birth process $(X_t)_{t\in[0,\infty)}$ with infinitesimal rates given by

$$\mathbb{P}(X_{t+h}=k+\ell \mid X_t=k)=\begin{cases}1-hr_k(t)+o(h) & \text{if } \ell=0\\ hr_k(t)+o(h) & \text{if } \ell=1\\ o(h) & \text{otherwise}\,,\end{cases}$$

where

$$r_k(t)=-\frac{Z_{n,t}}{|L_k|t^k}\frac{d}{dt}\left(\sum_{j=0}^{k}\frac{|L_j|t^j}{Z_{n,t}}\right)=-\frac{1}{\mathbb{P}(\mathrm{Inv}(\sigma_{n,t})=k)}\frac{d}{dt}\mathbb{P}(\mathrm{Inv}(\sigma_{n,t})\le k)$$

and $\sigma_{n,t}$ is $\pi_{n,t}$-distributed. We prove that $(X_t)_{t\in[0,\infty)}$ has the desired distribution by proving by induction on k that

$$\mathbb{P}(X_t=k)=\frac{|L_k|t^k}{Z_{n,t}}=\mathbb{P}(\mathrm{Inv}(\sigma_{n,t})=k)\,.$$

We also note that all rate functions r_k are positive, since $t\mapsto\mathbb{P}(\mathrm{Inv}(\sigma_{n,t})\le k)$ is decreasing. For $k=0$, we use the rate function r_0 to see that

$$\mathbb{P}(X_{t+h}=0)=\mathbb{P}(X_{t+h}=0,X_t=0)=(1-hr_0(t)+o(h))\mathbb{P}(X_t=0)\,.$$

This implies that

$$\frac{d}{dt}\mathbb{P}(X_t=0)=-r_0(t)\mathbb{P}(X_t=0)=\frac{d}{dt}\mathbb{P}(\mathrm{Inv}(\sigma_{n,t})=0)$$

which proves the induction hypothesis for $k=0$ since $\mathbb{P}(X_0=0)=\mathbb{P}(\mathrm{Inv}(\sigma_{n,0})=0)=1$. Assume now that

$$\mathbb{P}(X_t=k)=\frac{|L_k|t^k}{Z_{n,t}}=\mathbb{P}(\mathrm{Inv}(\sigma_{n,t})=k)\,.$$

for some $k<N$ and use the forward equation to see that

$$\begin{aligned}\mathbb{P}(X_{t+h}=k+1)&=\sum_{\ell=0}^{k+1}\mathbb{P}(X_{t+h}=k\mid X_t=\ell)\mathbb{P}(X_t=\ell)\\&=(1-hr_{k+1}(t))\mathbb{P}(X_t=k+1)+hr_k(t)\mathbb{P}(X_t=k)+o(h)\,,\end{aligned}$$

which leads to

$$\frac{d}{dt}\mathbb{P}(X_t=k+1)=r_k(t)\mathbb{P}(X_t=k)-r_{k+1}(t)\mathbb{P}(X_t=k+1)\,.$$

Now, using the induction hypothesis on $\mathbb{P}(X_t=k)$ and the definition of r_k and r_{k+1}, we obtain

$$\frac{d}{dt}\mathbb{P}(X_t=k+1)=\frac{\mathbb{P}(X_t=k+1)}{\mathbb{P}(\mathrm{Inv}(\sigma_{n,t})=k+1)}\frac{d}{dt}\mathbb{P}(\mathrm{Inv}(\sigma_{n,t})\le k+1)-\frac{d}{dt}\mathbb{P}(\mathrm{Inv}(\sigma_{n,t})\le k)\,.$$

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We observe that the function $t \mapsto \mathbb{P}(\mathrm{Inv}(\sigma_{n,t}) = k+1)$ satisfies the same equation, that is

$$\frac{d}{dt}\mathbb{P}(\mathrm{Inv}(\sigma_{n,t}) = k+1) = \frac{\mathbb{P}(\mathrm{Inv}(\sigma_{n,t}) = k+1)}{\mathbb{P}(\mathrm{Inv}(\sigma_{n,t}) = k+1)} \frac{d}{dt}\mathbb{P}(\mathrm{Inv}(\sigma_{n,t}) \leq k+1) - \frac{d}{dt}\mathbb{P}(\mathrm{Inv}(\sigma_{n,t}) \leq k)\,,$$

and furthermore that $\mathbb{P}(X_0 = k+1) = \mathbb{P}(\mathrm{Inv}(\sigma_{n,0}) = k+1) = 0$. By uniqueness of the dimension of the solutions to a differential equation of the first order, this proves the induction hypothesis. The theorem thus follows from defining such $(M_k)_{0\leq k\leq N}$ and $(X_t)_{t\in[0,\infty)}$ independently and by applying Lemma 3.4. □

4. The uniform Mallows process

In this section, we construct the uniform Mallows process, a regular Mallows process defined using only n uniforms and thus easily sampled. The strategy is similar as for the case of the birth Mallows process: define an appropriate process $(I(t))_{t\in[0,\infty)}$ and apply Proposition 2.2. We provide its formal definition in Section 4.1 and study two interesting properties regarding the relation between the amount of time we observed the process and the number of uniforms we can infer, in Sections 4.2.1 and 4.2.2.

4.1 Definition

We provide here the background necessary to construct the *uniform Mallows process*, written $(\mathcal{M}_t^U)_{t\in[0,\infty)}$, based on using a set of n uniform random variables along with the following lemma.

Lemma 4.1. *Fix $j \in \mathbb{N}$ and $u \in (0,1)$, and define a function $f_j(\,\cdot\,;u) : [0,\infty) \mapsto \{0,1,\ldots,j-1\}$ by*

$$f_j(t;u) = \begin{cases} \left\lfloor \frac{\log(1-u(1-t^j))}{\log t} \right\rfloor & \text{if } t \notin \{0,1\} \\ 0 & \text{if } t = 0 \\ \lfloor ju \rfloor & \text{if } t = 1 \end{cases}.$$

Then, $f_j(\,\cdot\,;u)$ is cadlag and increasing. Moreover, if U is a UNIFORM$([0,1])$ *random variable, then for all $0 \leq k \leq j-1$ and all $t \in [0,\infty)$, we have*

$$\mathbb{P}(f_j(t;U) = k) = \frac{t^k}{\sum_{\ell=0}^{j-1} t^\ell}\,.$$

Proof. First of all, consider $\phi(t) = \frac{\log(1-u(1-t^j))}{\log t}$. Then, we have

$$\begin{aligned}\phi'(t) &= \frac{1}{(\log t)^2}\left[\frac{jut^{j-1}}{1-u(1-t^j)}\log t - \frac{1}{t}\log\left(1-u\left(1-t^j\right)\right)\right] \\ &= \frac{1}{t(\log t)^2}\left[\frac{ut^j}{1-u(1-t^j)}\log\left(t^j\right) - \log\left(1-u\left(1-t^j\right)\right)\right].\end{aligned}$$

Now, by the concavity of log, we know that

$$\lambda \log\frac{1}{x} + (1-\lambda)\log\frac{1}{y} \geq \log\left(\frac{1}{\lambda x + (1-\lambda)y}\right).$$

Apply this formula with $\lambda = \frac{ut^j}{1-u(1-t^j)}$, $x = \frac{1}{t^j}$ and $y = 1$ to obtain

$$\frac{ut^j}{1-u(1-t^j)}\log\left(t^j\right) \geq \log\left(\frac{1}{\frac{u}{1-u(1-t^j)} + \frac{1-u}{1-u(1-t^j)}}\right) = \log\left(1-u\left(1-t^j\right)\right),$$

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which proves that $\phi'(t) \geq 0$. This proves that $f_j(\,\cdot\,; u)$ is increasing on $(0,1)$ and on $(1,\infty)$. To conclude the proof of the first claim of the lemma, note that ϕ can continuously be extended to 0 and 1 and corresponds to the previous definition, thus $f_j(\,\cdot\,; u)$ is cadlag.

For the second part of the lemma, first note that the statement holds for $t = 0$ and $t = 1$. Then, for $t > 0$ with $t \neq 1$, and for all $0 \leq k \leq j-1$, we have

$$\begin{aligned}\mathbb{P}\Big(f_j(t;U) = k\Big) &= \mathbb{P}\left(k \leq \frac{\log\left(1 - U(1-t^j)\right)}{\log t} < k+1\right)\\ &= \mathbb{P}\left(\frac{t^k - 1}{t^j - 1} \leq U < \frac{t^{k+1}-1}{t^j-1}\right)\\ &= \frac{t^k(t-1)}{t^j - 1},\end{aligned}$$

which is the desired formula. This concludes the proof of the proposition. □

Using this lemma, we now define the uniform Mallows process.

Definition 4.2. *Let $U_1, \ldots, U_n$ be a set of n independent* UNIFORM($[0,1]$) *random variables and let $I(t) = (f_1(t;U_1), \ldots, f_n(t;U_n))$, where $f_1, \ldots, f_n$ are as in Lemma 4.1. Then the uniform Mallows process is defined by $(\mathcal{M}_t^U)_{t\in[0,\infty)} = (\Phi(I(t)))_{t\in[0,\infty)}$.*

Thanks to Lemma 4.1 and Proposition 2.2, the process $(\mathcal{M}_t^U)_{t\in[0,\infty)}$ is a regular Mallows process. However, this process is not a Markov process, since it is not random anymore once all coordinates of its inversion process have jumped at least once. Even more so, the uniform Mallows process is defined using a set of n uniform random variables. This means that this process is generated from a well-controlled amount of randomness and it is naturally interesting to wonder when all the information can be retrieved from the evolution of the process. For the rest of this section, we drop the superscript U on $\mathcal{M}^U$ and write UMP instead of uniform Mallows process.

4.2 Properties

Given an occurrence of the UMP $(\mathcal{M}_t)_{t\in[0,\infty)}$ and a random time T, say that U_j is *retrievable* with respect to $(\mathcal{M}_t)_{t\in[0,\infty)}$ and T if the random variable U_j is measurable with respect to the σ-algebra $\sigma(T, (\mathcal{M}_t)_{t\in[0,T]})$. Note that, once one of the component processes $(I_j(t))_{t\in[0,\infty)}$ jumps, it is possible to deduce the value of U_j that generated this process (except for $j = 1$ since $I_1(t) = 0$ for all t). This remark implies that $U_2, \ldots, U_n$ are retrievable with respect to $(\mathcal{M}_t)_{t\in[0,\infty)}$ and $T_{\binom{n}{2}} = \inf\{t > 0 : \mathrm{Inv}(\mathcal{M}_t) = \binom{n}{2}\}$. We are now interested in answering the following two questions:

- What is the minimal time such that $U_2, \ldots, U_n$ are all retrievable?
- Given a deterministic time $t \in [0,\infty)$, how many of the variables $U_2, \ldots, U_n$ are retrievable?

We answer these two questions respectively in Sections 4.2.1 and 4.2.2 below.

4.2.1 Time of full information

In this section, we characterize the minimal time necessary for all uniforms to be retrievable and study some of its properties. Interestingly, we show that the time it takes to retrieve all uniforms has infinite mean.

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Proposition 4.3. *Let $(\mathcal{M}_t)_{t\in[0,\infty)}$ be the UMP. Then there exists an $\mathcal{M}_t$-stopping time T_U such that $U_2,\dots,U_n$ are retrievable with respect to $(\mathcal{M}_t)_{t\in[0,\infty)}$ and T_U, and such that, for any T such that $T<T_U$, not all $U_2,\dots,U_n$ are retrievable with respect to $(\mathcal{M}_t)_{t\in[0,\infty)}$ and T. Moreover, there exists a subset $A_U\subset\mathcal{S}_n$ such that*

$$T_U=\inf\{t\in[0,\infty):\mathcal{M}_t\in A_U\}\,.$$

Finally, we have that

$$\lim_{t\to\infty} t\mathbb{P}(T_U>t)=1\,,$$

which implies that $\mathbb{E}[T_U]=\infty$.

Proof. By using that U_i is retrievable if and only if $f_i(t;U_i)\geq 1$, the definition of A_U simply corresponds to

$$\begin{aligned}A_U&=\left\{\sigma:\Phi^{-1}(\sigma)_j\geq 1,\ \forall 2\leq j\leq n\right\}\\&=\Phi\Big(\left\{I=(I_1,\dots,I_n)\in\mathcal{E}_n:I_j\geq 1,\ \forall 2\leq j\leq n\right\}\Big)\,.\end{aligned}$$

This proves the first two statements of the proposition.

For the third and last statement, by the definition of A_U we have

$$\begin{aligned}\mathbb{P}(T_U<t)&=\mathbb{P}(f_j(t;U_j)\geq 1,\forall 2\leq j\leq n)\\&=\prod_{j=2}^{n}\left(1-\frac{1}{\sum_{\ell=0}^{j-1}t^{\ell}}\right)\,.\end{aligned}$$

It follows that, as t tends to infinity (with n being fixed), we have

$$\mathbb{P}\big(T_U<t\big)=\prod_{j=2}^{n}\left(1-(1+o(1))\frac{1}{t^{j-1}}\right)=1-\frac{1}{t}+o\left(\frac{1}{t}\right)\,.$$

This proves that $t\mathbb{P}(T_U>t)=1+o(1)$ and concludes the proof of this proposition. □

4.2.2 Amount of information

In the previous section, we consider when does the UMP become completely deterministic and we show that there exists a specific time at which we have all the information on this process. However, we also prove that the expected time to have all the information on this process is infinite. In this section, we consider the converse approach and ask what amount of information we have access to by a given time.

Fix $t\in(0,\infty)$ and let $\mathcal{I}_t$ be the (random) amount of information of the UMP at time t defined by

$$\mathcal{I}_t=\frac{1}{n}\times\left|\{j\in[n]:U_j\text{ is retrievable with respect to }(\mathcal{M}_s)_{s\in[0,\infty)}\text{ and }t\}\right|\,.$$

The fraction $\frac{1}{n}$ is used here so that $\mathcal{I}_t\in[0,1]$. The following theorem characterizes the behaviour of $\mathcal{I}_t$ for different regimes and, in particular, allows us to see a strong change of behaviour at $t=1$.

Theorem 4.4. *For any fixed $t\in[0,\infty)$, as n tends to infinity, we have*

$$\mathcal{I}_t\longrightarrow t\wedge 1\,,$$

almost surely. Moreover, its fluctuations around its asymptotic value are given by the following formulas. For $t\in[0,1)$, we have that

$$\left(\sqrt{n}(\mathcal{I}_t-t)\right)_{t\in[0,1)}\xrightarrow{d}(W_t)_{t\in[0,1)}\,,$$

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where $(W_t)_{t\in[0,1)}$ *is the restriction to* $[0,1)$ *of a standard Brownian bridge and the convergence is with respect to the topology of uniform convergence on* $\mathcal{C}([0,1))$. *For* $t=1$, *the following central limit theorem holds*

$$\frac{n}{\sqrt{\log n}}\left(\mathcal{I}_1 - 1 + \frac{\log n}{n}\right) \xrightarrow{d} \text{NORMAL}(0,1)\,.$$

Finally, for $t>1$, *then*

$$(n(1-\mathcal{I}_t))_{t>1} \xrightarrow{d} (X_t)_{t>1}\,,$$

where $(X_t)_{t>1}$ *is a decreasing process with moment generating function*

$$\mathbb{E}[u^{X_t}] = \prod_{k=1}^{\infty}\left(1+(u-1)\frac{t-1}{t^k-1}\right)\,.$$

Before proving Theorem 4.4, we provide a lemma later used to prove the convergence to the standard Brownian bridge.

Lemma 4.5. *Let* $(U_j)_{j\in[n]}$ *be a set of n independent* UNIFORM$([0,1])$ *random variables and set*

$$\mathcal{I}_t = \mathcal{I}_t^n = \frac{1}{n}\sum_{j=1}^{n}\min\left\{1, \left\lfloor U_j\frac{1-t^j}{1-t}\right\rfloor\right\}\,.$$

Then, we have the convergence in distribution

$$\left(\sqrt{n}(\mathcal{I}_t - t)\right)_{t\in[0,1)} \xrightarrow{d} (W_t)_{t\in[0,1)}\,,$$

where $(W_t)_{t\in[0,1)}$ *is the restriction to* $[0,1)$ *of a standard Brownian bridge and the convergence is with respect to the topology of uniform convergence on* $\mathcal{C}([0,1))$.

Proof. For the rest of the proof, fix $s\in(0,1)$. Consider first the process

$$(\mathcal{J}_t)_{t\in[0,s]} := \left(\frac{1}{n}\sum_{j=1}^{n}\min\left\{1, \left\lfloor U_j\frac{1}{1-t}\right\rfloor\right\}\right)_{t\in[0,s]}\,.$$

and note that, since $t\le s<1$, we have

$$\min\left\{1, \left\lfloor U_j\frac{1}{1-t}\right\rfloor\right\} = \mathbb{1}_{1-U_j\le t}\,.$$

This means that $(\mathcal{J}_t)_{t\in[0,s]}$ is the empirical distribution function of $(1-U_1,\dots,1-U_n)$ restricted to $[0,s]$, and from [4, Corollary 20.14], we know that $(\sqrt{n}(\mathcal{J}_t-t))_{t\in[0,s]}$ converges to the restriction to $[0,s]$ of a standard Brownian bridge (in fact the convergence occurs on $[0,1]$, but we are only interested in its behaviour on $[0,s]$). Hence, we now simply need to relate $\mathcal{I}$ and $\mathcal{J}$.

Using that $\frac{1-t^j}{1-t}\le\frac{1}{1-t}$, we have

$$\begin{aligned}
&\min\left\{1, \left\lfloor U_j\frac{1-t^j}{1-t}\right\rfloor\right\} \neq \min\left\{1, \left\lfloor U_j\frac{1}{1-t}\right\rfloor\right\}\\
\iff &\min\left\{1, \left\lfloor U_j\frac{1-t^j}{1-t}\right\rfloor\right\} = 0 \text{ and } \min\left\{1, \left\lfloor U_j\frac{1}{1-t}\right\rfloor\right\} = 1\\
\iff &U_j\frac{1-t^j}{1-t} < 1 \text{ and } U_j\frac{1}{1-t}\ge 1\\
\iff &U_j\in\left[1-t, \frac{1-t}{1-t^j}\right)\,.
\end{aligned}$$

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Now, let $(N_t)_{t\in[0,s]}$ be the random process defined by

$$N_t := \left|\left\{j\in[n]:\ \min\left\{1,\left\lfloor U_j\frac{1-t^j}{1-t}\right\rfloor\right\}\neq\min\left\{1,\left\lfloor U_j\frac{1}{1-t}\right\rfloor\right\}\right\}\right|$$
$$= \left|\left\{j\in[n]: U_j\in\left[1-t,\frac{1-t}{1-t^j}\right)\right\}\right|$$

and let $N := \max_{t\in[0,s]} N_t$. The definition of N_t along with the definition of $(\mathcal{I}_t)_{t\in[0,s]}$ and $(\mathcal{J}_t)_{t\in[0,s]}$ immediately imply that

$$\left|\mathcal{I}_t-\mathcal{J}_t\right|\le\frac{N_t}{n}, \tag{9}$$

which also means that N is a uniform bound over the difference between the two processes $(\mathcal{I}_t)_{t\in[0,s]}$ and $(\mathcal{J}_t)_{t\in[0,s]}$. It thus suffices to prove an upper tail bound on N.

Before providing the upper tail bound on N, we claim that

$$N=\max_{i\in[n]:1-U_i\le s}\left\{N_{1-U_i}\right\}. \tag{10}$$

To see this, fix $t\in[0,s]$ and let $\delta>0$ be such that $[1-(t+\delta),1-t)\cap\{U_1,\dots,U_n\}=\emptyset$. Since the map $u\mapsto(1-u)/(1-u^j)$ is decreasing for any $j\in[n]$, if $U_j\in[1-(t+\delta),(1-(t+\delta))/(1-(t+\delta)^j))$, then $U_j\in[1-t,(1-t)/(1-t^j)]$. Thus,

$$\left\{j\in[n]:U_j\in\left[1-(t+\delta),\frac{1-(t+\delta)}{1-(t+\delta)^j}\right)\right\}\subseteq\left\{j\in[n]:U_j\in\left[1-t,\frac{1-t}{1-t^j}\right)\right\}$$

and so $N_{t+\delta}\le N_t$. Since $t\in[0,s]$ was arbitrary, this proves that $t\mapsto N_t$ only has increments at times $\{1-U_1,\dots,1-U_n\}$, establishing (10).

Thanks to (10) and the definition of N_t, we now have

$$N=\max_{i\in[n]:1-U_i\le s}\left|\left\{j\in[n]:U_j\in\left[U_i,\frac{U_i}{1-(1-U_i)^j}\right)\right\}\right|.$$

Using that

$$\frac{1-t}{1-t^j}=1-t+\frac{t^j(1-t)}{1-t^j}\le 1-t+s^j$$

for any $t\le s<1$, it follows that

$$\frac{U_i}{1-(1-U_i)^j}\le U_i+s^j$$

for any $1-U_i\le s$, and then

$$N\le\max_{i\in[n]:1-U_i\le s}\left|\left\{j\in[n]:U_j\in\left[U_i,U_i+s^j\right)\right\}\right|\le\max_{i\in[n]}\left|\left\{j\in[n]:U_j\in\left[U_i,U_i+s^j\right)\right\}\right|.$$

Fix now $k\in\mathbb{N}$ and use the previous bound to see that

$$\mathbb{P}(N\ge k)\le\sum_{i=1}^n\mathbb{P}\left(\sum_{j\in[n]:j\neq i}\mathbb{1}_{\{U_j\in[U_i,U_i+s^j)\}}\ge k-1\right)$$
$$=\sum_{i=1}^n\mathbb{P}\left(\sum_{j\in[n]:j\neq i}\mathbb{1}_{\{U_j<s^j\}}\ge k-1\right), \tag{11}$$

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where the second inequality uses that the variables $(U_j)_{j\in[n]}$ are independent UNIFORM([0, 1]). Moreover, for any $i \in [n]$, by Markov's inequality we have

$$\begin{aligned}\mathbb{P}\left(\sum_{j\in[n]:j\neq i} \mathbb{1}_{\{U_j<s^j\}} \geq k-1\right) &= \mathbb{P}\left(\exp\left(\sum_{j\in[n]:j\neq i} \mathbb{1}_{\{U_j<s^j\}}\right) \geq e^{k-1}\right) \\ &\leq e^{-k+1}\mathbb{E}\left[\exp\left(\sum_{j\in[n]:j\neq i} \mathbb{1}_{\{U_j<s^j\}}\right)\right] \\ &= e^{-k+1}\prod_{j\in[n]:j\neq i}(1+(e-1)s^j) \\ &\leq \exp\left(-k+1+(e-1)\frac{1}{1-s}\right),\end{aligned} \tag{12}$$

where the last inequality follows from convexity of the exponential and by taking the product over all $j \geq 0$ rather than over $j \in [n] \setminus \{i\}$. Combining (11) and (12), we obtain

$$\mathbb{P}(N \geq k) \leq \exp\left(\log n - k + \left[1+(e-1)\frac{1}{1-s}\right]\right), \tag{13}$$

which we now use to conclude the convergence of $(\sqrt{n}(\mathcal{I}_t - t))_{t\in[0,s]}$ to the Brownian bridge.

Using the definition of $\mathcal{I}$ and $\mathcal{J}$ along with (9), we have that

$$\sup_{t\in[0,s]} \left|\mathcal{I}_t - \mathcal{J}_t\right| = \frac{1}{n}N,$$

from which (13) tells us that

$$\mathbb{P}\left(\sup_{t\in[0,s]} \left|\mathcal{I}_t - \mathcal{J}_t\right| \geq \frac{2\log n}{n}\right) \leq \frac{C_s}{n},$$

where $C_s = \exp(1+(e-1)/(1-s))$. Moreover, we already stated that $(\sqrt{n}(\mathcal{J}_t - t))_{t\in[0,s]}$ converges to the restriction to $[0, s]$ of a standard Brownian bridge. The desired convergence then follows. □

We now prove Theorem 4.4, relying on the previous lemma to show the convergence towards the Brownian bridge.

Proof of Theorem 4.4. All the results of this proposition will strongly depend on the decomposition

$$\mathcal{I}_t = \frac{1}{n}\sum_{j=1}^{n} X_j(t),$$

where $X_j(t)$ is the indicator that U_j is retrievable from $(\mathcal{M}_t)_{t\in[0,\infty)}$ and t. Note that the different random variables $(X_j(t))_{j\in[n]}$ are independent Bernoulli random variables with

$$\mathbb{P}\big(X_j(t)=0\big) = 1 - \mathbb{P}\big(X_j(t)=1\big) = \frac{1}{\sum_{\ell=0}^{j-1} t^\ell}. \tag{14}$$

For any $t \in [0, \infty)$, since $(X_j(t))_{j\in[n]}$ are all bounded by 1, the strong law of large numbers applies [19] and yields that $\mathcal{I}_t$ almost surely converges to its asymptotic expected value.

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Furthermore, we have that

$$\mathbb{E}[\mathcal{I}_t] = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[X_j(t)] = \frac{1}{n}\sum_{j=1}^{n}\left[1 - \frac{1}{\sum_{\ell=0}^{j-1} t^\ell}\right].$$

For $t = 1$, this gives us that

$$\mathbb{E}[\mathcal{I}_1] = 1 - \frac{1}{n}\sum_{j=1}^{n}\frac{1}{j} = 1 - \frac{\log n}{n} + O\left(\frac{1}{n}\right) = 1 - o(1)\,, \tag{15}$$

Moreover, since $(\mathcal{I}_t)_{t\in[0,\infty)}$ is increasing in t and takes values in $[0, 1]$, it follows that $\mathbb{E}[\mathcal{I}_t] = 1 - o(1)$ for $t > 1$. Finally, for $t < 1$, we have

$$\mathbb{E}[\mathcal{I}_t] = \frac{1}{n}\sum_{j=1}^{n}\left[1 - \frac{1-t}{1-t^j}\right] = \frac{1}{n}\left[nt - \sum_{j=1}^{n}\frac{t^j(1-t)}{1-t^j}\right] = t + O\left(\frac{1}{n}\right) = t + o(1)\,;$$

together, these facts establish the first statement of the proposition.

For the second statement, we first use that $t \mapsto X_j(t)$ is increasing, along with (14), to see that

$$(X_j(t))_{t\in[0,1)} \overset{d}{=} \left(\min\left\{1, \left\lfloor U\frac{1-t^j}{1-t}\right\rfloor\right\}\right)_{t\in[0,1)},$$

where U is a UNIFORM($[0, 1]$) random variable. Since the variables $(X_j(t))_{t\in[0,s]}$ are independent over $j \in [n]$, fixing a set of n independent UNIFORM($[0, 1]$) random variables $(\tilde{U}_1, \dots, \tilde{U}_n)$, we then have

$$(\mathcal{I}_t)_{t\in[0,1)} \overset{d}{=} \left(\frac{1}{n}\sum_{j=1}^{n}\min\left\{1, \left\lfloor \tilde{U}_j\frac{1-t^j}{1-t}\right\rfloor\right\}\right)_{t\in[0,1)}.$$

The relation between $(U_1, \dots, U_n)$ in the definition of the UMP and $(\tilde{U}_1, \dots, \tilde{U}_n)$ here does not need to be specified; however, it is worth noting that they are not the same variables. The desired convergence towards the Brownian bridge then simply follows from Lemma 4.5.

Consider now the case $t = 1$. We already showed that $\mathbb{E}[\mathcal{I}_1] = 1 - \log n/n + O(1/n)$. Moreover, we have

$$\mathrm{Var}\,[\mathcal{I}_1] = \frac{1}{n^2}\mathrm{Var}\left[\sum_{j=1}^{n}X_j(1)\right] = \frac{1}{n^2}\sum_{j=1}^{n}\mathrm{Var}[X_j(1)]\,,$$

and using that $X_j(1)$ is a BERNOULLI($(j-1)/j$), it follows that

$$\mathrm{Var}\,[\mathcal{I}_1] = \frac{1}{n^2}\sum_{j=1}^{n}\frac{j-1}{j^2} = \big(1 + o(1)\big)\frac{\log n}{n^2}$$

Since $X_j(t) \le 1$, the conditions of the Lindeberg–Feller theorem [17, Theorem 3.4.10] are met, and it follows that

$$\frac{\mathcal{I}_1 - \mathbb{E}[\mathcal{I}_1]}{\sqrt{\mathrm{Var}(\mathcal{I}_1)}} \xrightarrow{d} \mathrm{NORMAL}(0, 1)\,.$$

This convergence combined with the previous asymptotic approximation of $\mathrm{Var}[\mathcal{I}_1]$ and (15) exactly corresponds to the desired result.

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For the last assertion of the proposition, using once again the distribution of $X_j(t)$, for $t > 1$ we have that

$$\begin{aligned}\mathbb{E}\left[e^{iun(1-\mathcal{I}_t)}\right] &= e^{iun}\prod_{j=1}^{n}\left[1+(e^{-iu}-1)\left(1-\frac{t-1}{t^j-1}\right)\right]\\ &= \prod_{j=1}^{n}\left[e^{iu}+(1-e^{iu})\left(1-\frac{t-1}{t^j-1}\right)\right]\\ &= \prod_{j=1}^{n}\left[1+(e^{iu}-1)\frac{t-1}{t^j-1}\right],\end{aligned}$$

and the final product converges to the desired function as n goes to infinity. To conclude this proof, simply note that $(\mathcal{I}_t)_{t>1}$ is increasing to see that $(X_t)_{t>1}$ is decreasing, corresponding to the last statement of the proposition. □

5. Conclusion and open questions

This work introduces and studies Mallows processes, both in a general way and on three specific examples. However, many properties of these processes remain to be proven; we now discuss some possible directions for future work on Mallows processes and their properties.

5.1 Graph structure

Theorem 1.2 states that $\mathcal{T}^a \subseteq \langle \mathcal{G}_{\mathcal{M}} \rangle \subseteq \mathcal{T}$ for any smooth Mallows process $\mathcal{M}$, and that $\langle \mathcal{G}_{\mathcal{M}} \rangle = \mathcal{T}$ whenever $\mathcal{M}$ is regular. The proof of the second statement strongly relies on the independent inversion property of $\mathcal{M}$ when it is regular, and we do not necessarily expect to have $\langle \mathcal{G}_{\mathcal{M}} \rangle = \mathcal{T}$ if we drop this assumption.

Conjecture 5.1. *For any $n \geq 3$, there exists a smooth Mallows process $\mathcal{M} = (\mathcal{M}_t)_{t\in[0,\infty)}$ such that*

$$\langle \mathcal{G}_{\mathcal{M}} \rangle \subsetneq \mathcal{T}\,.$$

Furthermore, when $n = 3$ (the first non-trivial case), it is not hard to see that it is even possible to define $\mathcal{M}$ so that $\langle \mathcal{G}_{\mathcal{M}} \rangle = \mathcal{T}^a$. Whether this result can be generalized to any $n \geq 3$ or not is unclear, leading to the following question.

Question 5.2. *What is the value $N \in \mathbb{N} \cup \{\infty\}$ such that, for any $n < N$, there exists a smooth Mallows process $\mathcal{M} = (\mathcal{M}_t)_{t\in[0,\infty)}$ taking values in $\mathcal{S}_n$ with*

$$\langle \mathcal{G}_{\mathcal{M}} \rangle = \mathcal{T}^a\,,$$

and for any $n \geq N$, any smooth Mallows process $\mathcal{M} = (\mathcal{M}_t)_{t\in[0,\infty)}$ taking values in $\mathcal{S}_n$ has the property that

$$\mathcal{T}^a \subsetneq \langle \mathcal{G}_{\mathcal{M}} \rangle\,.$$

Note that, in this question, we allow for $N = \infty$, which would mean that for any n, there exists a smooth Mallows process $\mathcal{M}$ with $\langle \mathcal{G}_{\mathcal{M}} \rangle = \mathcal{T}^a$. The fact that such N exists can be verified as follows. Fix $n \in \mathbb{N}$ and consider a smooth Mallows process $\mathcal{M} = (\mathcal{M}_t)_{t\in[0,\infty)}$ taking values in $\mathcal{S}_n$ such that $\langle \mathcal{G}_{\mathcal{M}} \rangle = \mathcal{T}^a$. Then define a Mallows process $\tilde{\mathcal{M}} = (\tilde{\mathcal{M}}_t)_{t\in[0,\infty)}$ taking values in $\mathcal{S}_{n-1}$ by

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$$\tilde{\mathcal{M}}_t(i) = \begin{cases} \mathcal{M}_t(i) & \text{if } i < \mathcal{M}_t^{-1}(n) \\ \mathcal{M}_t(i+1) & \text{if } \mathcal{M}_t^{-1}(n) \le i < n-1 \end{cases}.$$

In other words, if $\mathcal{M}_t = (s_1, \dots, s_{j-1}, n, s_{j+1}, \dots, s_n)$, then $\tilde{\mathcal{M}}_t = (s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_n)$, which simply corresponds to removing the value n in the permutation $\mathcal{M}_t$. It is then easy to verify that $(\tilde{\mathcal{M}}_t)_{t\in[0,\infty)}$ is a smooth Mallows process with $\langle \mathcal{G}_{\tilde{\mathcal{M}}} \rangle = \mathcal{T}^a$.

5.2 Another Gaussian process

In Section 4.1, we studied properties of the uniform Mallows process and showed in Theorem 4.4 that there was an embedded, asymptotically Gaussian process related to the amount of information of this process (more precisely a Brownian bridge).

In parallel, the birth Mallows process also appears to have a natural associated asymptotically Gaussian process, obtained by an appropriate normalization of the number of jumps. More precisely, let the *number of jumps* of $(\mathcal{M}_t)_{t\in[0,\infty)}$ be the process $(J_t)_{t\in[0,\infty)}$ defined by $J_t = |\{k \le \binom{n}{2} : T_k \le t\}|$. Then, by applying the Lindeberg–Feller theorem [17, Theorem 3.4.10], it is straightforward to prove the following fact.

Fact 5.3. *As n goes to infinity, for any $t \in [0, 1)$, we have that*

$$\frac{1}{\sqrt{n}}\Big[(1-t)J_t - nt\Big] \xrightarrow{d} \text{NORMAL}(0, t)\,.$$

This suggests that $([(1-t)J_t - nt]/\sqrt{n})_{t\in[0,1)}$ converges to some Gaussian process. Proving that this process indeed converges, and characterizing its distribution remains open.

Conjecture 5.4. *As n goes to infinity we have that*

$$\left(\frac{1}{\sqrt{n}}\Big[(1-t)J_t - nt\Big]\right)_{t\in[0,1)} \xrightarrow{d} (G_t)_{t\in[0,1)}\,,$$

where $(G_t)_{t\in[0,1)}$ is a centred Gaussian and Markovian process.

5.3 Relation to random sorting networks

From the results of Theorem 1.2, we know that any smooth Mallows process $\mathcal{M} = (\mathcal{M}_t)_{t\in[0,\infty)}$ transitions using transpositions. This means that for any $(\mathcal{M}_t)_{t\in[0,\infty)}$, there exists $\tau_1, \dots, \tau_N \in \mathcal{T}$ where $N = \binom{n}{2}$ such that $\mathcal{M}_{T_k} = \tau_1 \cdot \ldots \cdot \tau_k$, which implies that $\tau_1 \cdot \ldots \cdot \tau_N = (n, n-1, \dots, 1)$. This naturally suggests an investigation of the relation between Mallows processes and sorting networks.

Sorting networks are sequences $\tau_1, \dots, \tau_N \in \mathcal{T}^a$ such that $\tau_1 \cdot \ldots \cdot \tau_N = (n, n-1, \dots, 1)$. They were first considered by Stanley [36], who managed to give an exact formula for the number of such sorting networks of size n. Since then, further work on proving this formula through diverse methods was developed [18, 29], and these works recently led several authors [5, 7, 14, 15] to study properties of uniformly sampled *random sorting networks*. In these works, various results of random sorting networks are proven, such as the asymptotic distribution of τ_1, a convergence for the scaled swap process, and properties of the trajectories $(\tau_1 \cdot \ldots \cdot \tau_k(i))_{0\le k\le N}$ as k varies from 0 to N and for a fixed $i \in [n]$. Mallows processes do not directly correspond to sorting networks, since they might use transpositions in $\mathcal{T} \setminus \mathcal{T}^a$, but many of the properties of the sorting networks that have previously been studied have analogues for Mallows processes and it would be natural to investigate these in more detail.

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It is worth mentioning that a recent work by Adamczak and Kotowski [2] studies the local and global limit of Mallows processes. As a local limit, Mallows processes converge to a process on permutations of $\mathbb{Z}$ with close ties to the two-sided infinite Mallows permutation introduced by Gnedin and Olshanski [23]. However, as a global limit, the authors characterise the trajectories of the entries in a large Mallows process, thus extending the corresponding results from the case of random sorting networks [14].

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