Proc. Camb. Phil. Soc. (1970), **68**, 571
PCPS 68–60
Printed in Great Britain

Internal labelling: the classical groups†

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Abstract. The internal labelling problems for the group-subgroup combinations $\mathrm{Sp}(2n) \supset \mathrm{Sp}(2n-2) \times \mathrm{Sp}(2)$ and $O(n) \supset O(n-1)$ are solved by the method of elementary multiplets.

1. *Introduction.* The classical groups—special unitary, orthogonal and symplectic—have received much attention in recent years, in part because they describe hypothetical or actual symmetries of particles and nuclei. Behrends, Dreitlein, Fronsdal and Lee(1) and Konuma, Shima and Wada(2) give excellent reviews with detailed consideration of groups of rank 2 and 3 respectively.

The internal labelling (or state labelling) problem for a group G is solved by specifying or defining unambiguously the basis vectors for an irrep‡ of G. Usually one decomposes the irrep into multiplets‡ of a subgroup H. The representation labels and internal labels of H then serve as internal labels for G. In the case of the special unitary group $\mathrm{SU}(n)$ the subgroup $\mathrm{SU}(n-1)$ together with the n-hypercharge Z_n provides enough labels and solves the internal labelling problem(3). Similarly for the orthogonal group $O(n)$ the subgroup $O(n-1)$ provides all the necessary labels(4). The largest subgroup of the symplectic group $\mathrm{Sp}(2n)$ is $\mathrm{Sp}(2n-2) \times \mathrm{Sp}(2)$; for $n > 2$ it does not provide enough labels and the solution of this labelling problem is the main object of this paper. Solutions have been given for special cases; Pajas and Raczka(5) have found explicit bases for irreps of $\mathrm{Sp}(2n)$ corresponding to Young tableaux with one or two rows. Hegerfeldt(6) has given the $\mathrm{Sp}(2n-2)$ content of $\mathrm{Sp}(2n)$ without, however, labelling the basis vectors.

The method of elementary multiplets has been used occasionally in the past to solve specific internal labelling problems(3, 7); recently it was applied systematically to a number of group-subgroup combinations(8). The approach is based on the fact that the stretched§ product of multiplets from one or more irreps defines‖ in general a multiplet of a higher irrep. If all multiplets of all irreps can be represented in this way in terms of a finite number of elementary multiplets the internal labelling problem is

† Research supported by the National Research Council of Canada.

‡ For clarity different words irrep or multiplet are used for irreducible representation of the group or subgroup respectively. Also we use irrep or multiplet for the longer basis for irrep or multiplet; no confusion should arise since transformation matrices are not referred to in this article.

§ Stretched product of a number of multiplets means the multiplet containing the highest weight basis vector in their Kronecker product.

‖ We say defines rather than equals because only the part belonging to the stretched product of the irreps concerned must be retained; parts belonging to lower irreps should be projected out.

https://doi.org/10.1017/S030500410004634X Published online by Cambridge University Press

solved—the required labels are simply the indices of the powers of the elementary multiplets which occur in the general multiplet. For more details see ref.(8).

Of the classical groups the case $SU(n) \supset SU(n-1)$ is already treated in ref.(8). The orthogonal and symplectic groups are dealt with in the succeeding sections.

2. *Orthogonal groups.* Although the labelling problem for $O(n)$ is solved by the branching theorems given by Boerner(4) together with the observation that $O(n-1)$ provides enough labels, it is still useful to have a solution in terms of elementary multiplets. The stretched product of elementary multiplets which characterizes a particular multiplet is the leading term in a certain expansion of the states(8); it is a useful starting point in calculation of actual states and sometimes is all that is required if it is desired to expand some expression in basis states as in calculation of transformation or generator matric elements or Clebsch–Gordan coefficients.

The simple cases $O(4) \supset O(3)$, $O(5) \supset O(4)$ and $O(6) \supset O(5)$ are dealt with in ref.(8). In the general case it is necessary to treat separately $O(2n+1) \supset O(2n)$ and $O(2n) \supset O(2n-1)$.

According to Racah(9) the number of internal labels required is $\frac{1}{2}(l_G - r_G)$ where l_G and r_G are the order (number of generators) and rank (number of representation labels, or number of fundamental irreps) of the group G. If l_H and r_H are the corresponding quantities for the subgroup it is apparent that $\frac{1}{2}(r_G + l_G + r_H - l_H)$ is the number of independent elementary multiplets required.

For $O(2n)$, $r = n$ and $l = n(2n-1)$; for $O(2n+1)$, $r = n$ and $l = n(2n+1)$. It follows that $2n$ elementary multiplets are needed for $O(2n+1) \supset O(2n)$ and $2n-1$ for $O(2n) \supset O(2n-1)$.

We follow the notation of Boerner(4). According to his branching theorem 12·1*b* the multiplets $(m'_1, \ldots, m'_n)$ of $O(2n)$ contained (once each) in the irrep $(m_1, \ldots, m_n)$ of $O(2n+1)$ are those satisfying

$$m_1 \geqslant m'_1 \geqslant m_2 \geqslant \ldots \geqslant m'_{n-1} \geqslant m_n \geqslant |m'_n|. \tag{2·1}$$

Thus each fundamental irrep contains two multiplets:

$$\left.\begin{aligned} (1^\nu 0^{n-\nu}) &= (1^\nu 0^{n-\nu})_\nu + (1^{\nu-1} 0^{n-\nu+1})_\nu \quad (\nu = 1, \ldots, n-1), \\ (\tfrac{1}{2}^n) &= (\tfrac{1}{2}^n) + (\tfrac{1}{2}^{n-1} - \tfrac{1}{2}). \end{aligned}\right\} \tag{2·2}$$

The $2n$ multiplets on the right of (2·2) are just the desired elementary multiplets. $(1^\nu 0^{n-\nu})_\nu$ increases $m'_\nu - m_{\nu+1}$ by unity, $\nu = 1, \ldots, n-1$; $(1^{\nu-1} 0^{n-\nu+1})_\nu$ increases $m_\nu - m'_\nu$ by unity, $\nu = 1, \ldots, n-1$; $(\frac{1}{2}^n)$ increases all m by $\frac{1}{2}$ and $m_{n-1} + m'_n$ by unity; $(\frac{1}{2}^n - \frac{1}{2})$ increases $m_n - m'_n$ by unity. It follows that stretched products of the elementary multiplets correspond one-one to multiplets of all irreps. Moreover, since the conventional m labels are additive for stretched products it is easy to relate them to the indices of the elementary multiplets.

According to Boerner's branching Theorem 12·1*a*, the multiplets $(m'_1, \ldots, m'_{n-1})$ of $O(2n-1)$ contained in the irrep $(m_1, \ldots, m_n)$ of $O(2n)$ are those with

$$m_1 \geqslant m'_1 \geqslant m_2 \geqslant \ldots \geqslant m'_{p-1} \geqslant |m_p|. \tag{2·3}$$

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Thus each fundamental irrep contains two multiplets except the last two which contain one each:

$$\left.\begin{aligned}(1^\nu 0^{n-\nu}) &= (1^\nu 0^{n-\nu-1})_\nu + (1^{\nu-1}0^{n-\nu})_\nu,\\ (\tfrac{1}{2}^n) &= (\tfrac{1}{2}^{n-1})_1,\\ (\tfrac{1}{2}^{n-1}-\tfrac{1}{2}) &= (\tfrac{1}{2}^{n-1})_2.\end{aligned}\right\} \tag{2·4}$$

The right of (2·4) provide $2n-2$ elementary multiplets, one too few. Consider the non-fundamental irrep $(1^{n-1}0)$ which is the stretched product of the fundamental irreps $(\frac{1}{2}^n)$ and $(\frac{1}{2}^{n-1}-\frac{1}{2})$:

$$(1^{n-1}0) = (1^{n-1}) + (1^{n-2}0). \tag{2·5}$$

We adopt $(1^{n-2}0)$ as the missing elementary multiplet. $(1^\nu 0^{n-\nu-1})_\nu$ increases $m'_\nu - m_{\nu+1}$ by unity, $\nu = 1, \ldots, n-2$; $(1^{\nu-1}0^{n-\nu})_\nu$ increases $m_\nu - m'_\nu$ by unity, $\nu = 1, \ldots, n-2$; $(1^{n-2}0)$ increases $m_{n-1} - m'_{n-1}$ by unity; $(\frac{1}{2}^{n-1})_1$ increases all m by $\frac{1}{2}$ and $m'_n + m_n$ by unity; $(\frac{1}{2}^{n-1})_2$ increases $m'_{n-1} - m_n$ by unity. It follows that stretched products of the elementary multiplets correspond one-one to multiplets of all irreps. Again the conventional m-labels are easily related to the indices.

3. *Symplectic groups.* According to Dynkin(10) the largest subgroup of $\mathrm{Sp}(2n)$ is $\mathrm{Sp}(2n-2) \times \mathrm{Sp}(2)$; we use $\mathrm{Sp}(2n-2) \times \mathrm{Sp}(2)$ in what follows to label the basis vectors of $\mathrm{Sp}(2n)$. All our considerations are for $n \geqslant 2$ ($\mathrm{Sp}(2)$ is isomorphic to $\mathrm{SU}(2)$). For $\mathrm{Sp}(2n)$, $r = n$ and $l = n(2n+1)$. It follows that there are $n-2$ missing labels and that $3n-2$ independent elementary multiplets are needed.

We use two notations for $\mathrm{Sp}(2n)$ irreps, $(\lambda_1, \ldots, \lambda_n)$ and $(l_1, \ldots, l_n)$. The λ's are Cartan labels; λ_i is the degree of the ith fundamental irrep of $\mathrm{Sp}(2n)$ in the irrep $(\lambda_1 \ldots \lambda_n)$. The l's are related to the λ's and to the l's of Weyl(11) by

$$l_i = \left(\sum_{j=1}^{n} \lambda_j\right) - i + 1 = l_i^W - n + 1, \tag{3·1}$$

where l_i^W is the l_i appearing in Weyl's ((11) (8·6) p. 219). α and a play the same role for $\mathrm{Sp}(2n-2)$ as λ and l respectively for $\mathrm{Sp}(2n)$. Thus the general $\mathrm{Sp}(2n-2) \times \mathrm{Sp}(2)$ multiplet is denoted by $(\alpha_1 \ldots \alpha_{n-1}, \alpha_n)$ or $(a_1 \ldots a_{n-1}, a_n)$; $\alpha_n = a_n$ is the $\mathrm{Sp}(2)$ representation label. Numerical values of representation labels are always λ's (or α's).

We begin by listing the elementary multiplets for $\mathrm{Sp}(2n) \supset \mathrm{Sp}(2n-2) \times \mathrm{Sp}(2)$; they are discovered by examining low-lying irreps of $\mathrm{Sp}(2n)$. The proof that they solve the labelling problem in general will then take up the remainder of the section.

The ith fundamental irrep of $\mathrm{Sp}(2n)$ is that with $\lambda_i = 1$ and all other λ's equal to zero. For $1 < i < n$ it contains three multiplets of $\mathrm{Sp}(2n-2) \times \mathrm{Sp}(2)$. The first, which we denote by A_1^i has $\alpha_i = 1$ and all other α's equal to zero. The second which we call A_2^i has $\alpha_{i-1} = \alpha_n = 1$ and all other α's equal to zero. The third, A_3^i, has $\alpha_{i-2} = 1$ and all other α's equal to zero (A_3^2 has all α's equal to zero, i.e., is an $\mathrm{Sp}(2n-2) \times \mathrm{Sp}(2)$ scalar). The first and nth fundamental irreps of $\mathrm{Sp}(2n)$ contain only two multiplets each. For the first, A_1^1 is as above with $i = 1$, A_2^1 has $\alpha_n = 1$ and all other α's equal to zero and there is no A_3^1. For the nth there is no A_1^n and A_2^n, A_3^n are as above with $i = n$. Thus we get $3n-2$ elementary multiplets

$$\left.\begin{aligned}&A_1^i\ (i = 1, \ldots, n-1),\\ &A_2^i\ (i = 1, \ldots, n),\\ &A_3^i\ (i = 2, \ldots, n).\end{aligned}\right\} \tag{3·2}$$

https://doi.org/10.1017/S030500410004634X Published online by Cambridge University Press

Equation (3·2) contains the correct number $3n-2$ of elementary multiplets but alas the picture is more complicated; certain additional composite elementary multiplets A^{ij} $(i \leqslant j-2)$ are required. A^{ij} belongs to the Sp$(2n)$ irrep with $\lambda_i = \lambda_j = 1$ and all other λ's equal to zero; as an Sp$(2n-2) \times$ Sp(2) multiplet it has $\alpha_{i-1} = \alpha_{j-1} = 1$ and all other α's equal to zero (for $i = 1$, A^{ij} has $\alpha_{j-1} = 1$ and all other α's equal to zero).

The A^{ij} are of course not independent of the $A^i_{1,2,3}$. They satisfy relations of the form $(i < k < j)$

$$a_1 A^{ij}A^k_2 + a_2 A^{ik}A^j_2 + a_3 A^{kj}A^i_2 = 0 \tag{3·3}$$

where $a_1\, a_2\, a_3$ are constants. In (3·3) if $i = k-1$, A^{ik} should be replaced by $A^i_1 A^k_3$ and similarly for A^{kj} if $k = j-1$. In addition, the A^{ij} satisfy relations among themselves of the form $(i < k < j < h)$

$$b_1 A^{ij}A^{kh} + b_2 A^{ik}A^{jh} + b_3 A^{ih}A^{kj} = 0. \tag{3·4}$$

If $i = k-1$, $k = j-1$ or $j = h-1$ then A^{ik} should be replaced by $A^i_1 A^k_3$, etc. Equations (3·3) and (3·4) can be used to express the first term on the left side in terms of the other two; the eliminated terms must be regarded as redundant combinations of elementary multiplets. The situation is described by considering that A^{ij} is incompatible with A^k_2 for $i < k < j$ and with A^{kh} if $i < k < j < h$ or $k < i < h < j$; incompatible pairs must not appear together for the purpose of forming higher multiplets.†

According to Weyl ((11) (8·6), p. 219), the characters of Sp$(2n)$ irreps are given by

$$\chi(l_1 \ldots l_n) = |P_l, P_{l+1} + P_{l-1}, \ldots, P_{l+n-1} + P_{l-n+1}|. \tag{3·5}$$

We have followed Weyl in representing the determinant

$$\begin{vmatrix} P_{l_1} & P_{l_1+1} + P_{l_1-1} & \cdots & P_{l_1+n-1} + P_{l_1-n+1} \\ \vdots & \vdots & & \vdots \\ P_{l_n} & P_{l_n+1} + P_{l_n-1} & \cdots & P_{l_n+n-1} + P_{l_n-n+1} \end{vmatrix}$$

by the abbreviated symbol on the right of (3·5); similar abbreviations for determinants are used in what follows. $P_l \equiv p(\epsilon_1, \ldots, \epsilon_n)$ is defined as the coefficient of z^l in the expansion

$$\frac{1}{\phi(z)} = \sum_{l=0}^{\infty} p_l(\epsilon_1, \ldots, \epsilon_n) z^l \tag{3·6}$$

with

$$\phi(z) = \prod_{i=1}^{n} (1 - \epsilon_i z)(1 - z/\epsilon_i). \tag{3·7}$$

For l negative $p_l(\epsilon_1, \ldots, \epsilon_n)$ is defined to vanish; it follows from the definitions that $P_0 = 1$ and

$$P_l = \sum_{a=0}^{l} p'_a P'_{l-a} \tag{3·8}$$

† In the unlikely event that a_1 or b_1 vanish in equation (3·3) or (3·4) one of the other two terms in the equation in question must be regarded as redundant; this would not change in any essential way the solution of the labelling problem or the proof which follows. The explicit values of the coefficients a, b are being investigated.

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where $P'_a \equiv p_a(\epsilon_1, \ldots, \epsilon_{n-1})$ and $p'_a \equiv p_a(\epsilon_n)$; P'_a and p'_a are the p_l appropriate to Sp $(2n-2)$ and Sp (2) respectively.

The proof that the conjectured elementary multiplets give the correct subgroup content proceeds by induction. We assume they are correct for irreps with $\lambda_i = 0$, i.e. $l_i = -i+1$ for $m+1 \leqslant i \leqslant n$ and show they continue to give the correct multiplets for λ_m increased by unity (to start the induction it can be easily checked that the elementary multiplets are good for $m = 2$).

For $\lambda_i = 0$, $m+1 \leqslant i \leqslant n$, the character (3·5) reduces to an $m \times n$ determinant

$$\chi(l_1 \ldots l_m -) = |P_l, P_{l+1}+P_{l-1}, \ldots, P_{l+m-1}+P_{l-m+1}|. \tag{3·9}$$

The dash indicates that the missing l have the values $l_i = -i+1$ which implies $\lambda_i = 0$. Split off the term p'_{l+m-1} (see 3·8) from the last column of the determinant. Then $\chi = \chi_1 + \chi_R$ where

$$\chi_R(l_1 \ldots l_m —) = |P_l, P_{l+1}+P_{l-1}, \ldots, P_{l+m-2}+P_{l-m+2}, p'_{l+m-1}| \tag{3·10}$$

and, with the help of (3·8),

$$\chi_1(l_1 \ldots l_m —) = \sum_{a_1=0}^{l_1+m-2} \cdots \sum_{a_m=0}^{l_m+m-2} p'_{a_1} \ldots p'_{a_m} \times |P'_{l-a}, P'_{l-a+1}+P'_{l-a-1}, \ldots, P'_{l-a+m-1}+P'_{l-a-m+1}|. \tag{3·11}$$

We want to show that $\chi_1(l_1 \ldots l_m —)$ is the part of $\chi(l_1 \ldots l_m —)$ corresponding to multiplets which contain the elementary factor A_1^m or, in the case $m = n$, that $\chi_1(l_1 \ldots l_n)$ vanishes (there is no elementary factor A_1^n).

First suppose $m = n$. Then χ_1 is a linear combination of determinants of the form

$$|P'_l, P'_{l+1}+P'_{l-1}, \ldots, P'_{l+n-1}+P'_{l-n+1}|, \tag{3·12}$$

where $l+n-1 \geqslant 1$. Our proof that such a determinant vanishes parallels the similar proof by Weyl ((11), p. 204) for the group $GL(n)$. Expand (3·7)

$$\phi(z) = c_n z^n + c_{n-1}(z^{n+1}+z^{n-1}) + \ldots + c_0(1+z^{2n})$$

and multiply by the right side of (3·6). Equate coefficients of z^{l+n}. Then for $l+n \geqslant 1$,

$$c_n P_l + c_{n-1}(P_{l+1}+P_{l-1}) + \ldots + c_0(P_{l+n}+P_{l-n}) = 0;$$

writing down $n+1$ such equations and eliminating the c's, which do not vanish, we conclude that the determinant of their coefficients vanishes. But the determinant, with n reduced by unity, just (3·12).

Now suppose $m < n$. To show χ_1 contains all multiplets involving A_1^m it is sufficient to prove that χ, with all l reduced by 1 and the subscripts of all P' increased by 1, is equal to χ_1. χ can be written, by (3·8)

$$\chi(l_1 \ldots l_m —) = \sum_{a_1=0}^{l_1+m-1} \cdots \sum_{a_m=0}^{l_m+m-1} p'_{a_1} \ldots p'_{a_m} \times |P'_{l-a}, P'_{l-a+1}+P'_{l-a-1}, \ldots, P'_{l-a+m-1}+P'_{l-a-m+1}|.$$

It is easy to see that the result of the above operation on χ results in χ_1 as given by (3·11).

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It remains to be shown that χ_R contains all multiplets not involving A_1^m. Now χ_R can be written

$$\chi_R(l_1 \dots l_m—) = \sum_{a_1=a_2+1}^{l_1} \dots \sum_{a_{m-2}=a_{m-1}+1}^{l_{m-2}} \sum_{a_{m-1}=-m+2}^{l_{m-1}} \chi'(a_1 \dots a_m—) \times |p'_{l-a_1}, \dots, p'_{l-a_{m-1}}, p'_{l+m-1}|. \quad (3{\cdot}13)$$

To see this use (3·8) to expand the P's in χ_R as given by (3·10) and in each Sp $(2n-2)$ character relabel the a's so the largest is a_1 etc. The coefficient of each Sp $(2n-2)$ character $\chi'(a_1 \dots a_{m-1}—)$ is then a determinant of p' as in (3·13).

The multiplets in χ_R which contain the elementary factor A_2^m are obtained from χ_R by reducing all l by 1 and increasing by 1 all a in $\chi'(a_1 \dots a_m—)$ *and* increasing by 1 the subscript of the resultant p'. The resultant p' may be increased by 1 by replacing p'_{l+m-2} in the last column of the determinant in (3·13) by $p'_{l+m-1} - p_{\bar{l}+m-1}$ where $p_{\bar{l}+m-1}$ is an instruction: reduce subscript of operand p' by $l+m-1$, and is to be regarded as operating on its cofactor when the determinant in (3·13) is expanded by its last column. Carrying out the above operation on χ_R we get for χ_2, the part of χ_R containing A_2^m:

$$\chi_2(l_1 \dots l_m—) = \sum_{a_1=a_2+1}^{l_1} \dots \sum_{a_{m-2}=a_{m-1}+1}^{l_{m-2}} \sum_{a_{m-1}=-m+3}^{l_{m-1}} \chi'(a_1 \dots a_{m-1}—) \times |p'_{l-a_1}, \dots, p'_{l-a_{m-1}}, p'_{l+m-1} - p_{\bar{l}+m-1}|. \quad (3{\cdot}14)$$

Subtracting χ_2 from χ_R leaves two terms χ_3 and χ_4:

$$\chi_3(l_1 \dots l_m—) = \sum_{a_1=a_2+1}^{l_1} \dots \sum_{a_{m-2}=-m+3}^{l_{m-2}} \chi'(a_1 \dots a_{m-2}—) \times |p'_{l-a_1}, \dots, p'_{l-a_{m-2}}, p'_{l+m-2}, p'_{l+m-1}|, \quad (3{\cdot}15)$$

$$\chi_4(l_1 \dots l_m—) = \sum_{a_1=a_2+1}^{l_1} \dots \sum_{a_{m-2}=a_{m-1}+1}^{l_{m-2}} \sum_{a_{m-1}=-m+3}^{l_{m-1}} \chi'(a_1 \dots a_{m-1}—) \times |p'_{l-a_1}, \dots, p'_{l-a_{m-1}}, p_{\bar{l}+m-1}|. \quad (3{\cdot}16)$$

χ_3 contains all multiplets with no A_1^m, A_2^m, A^{im}, A_1^{m-1} and hence contains just $l_m+m-1=\lambda_m$ factors A_3^m. To see this we start by observing that because of the vanishing of (3·12) for the case Sp (2) it is permissible to keep reducing by 1 the subscripts of the last two columns of the determinant in (3·15) until the second last column has bottom subscript -1. The determinant vanishes unless $a_{m-2} > l_{m+1}$; otherwise two columns would be equal during the reduction process. So the bottom row becomes zero except for the last element which is 1 and χ_3 can be written

$$\chi_3(l_1 \dots l_m—) = \sum_{a_1=a_2+1}^{l_1} \dots \sum_{a_{m-2}=a_{m-2}+1}^{l_{m-2}} \sum_{a_{m-2}=l_m+2}^{l_{m-2}} \chi'(a_1 \dots a_{m-2}—) \times |p'_{l-a_1}, \dots, p'_{l-a_{m-2}}, p_{l-l_m-1}|. \quad (3{\cdot}17)$$

To complete the identification of χ_3 it is sufficient to show that

$$\chi_R(l_1-l_m-m+1, \dots, l_{m-1}-l_m-m+1—) \quad \text{(i.e. } \chi_R(\lambda_1 \dots \lambda_{m-1} 0 \dots 0)\text{)}$$

agrees with χ_3 when expanded in the form (3·15) with the subscripts of the a's in χ' increased by l_m+m-1. This is easily verified when χ_3 is written in the form (3·17).

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Finally, we must show χ_4 contains those multiplets in which the only elementary multiplets contributing to λ_m are A^{im} and A_3^m; if there are no A^{im} there must be at least one A_1^{m-1}.

Equation (3·16) shows there is a one-one correspondence between multiplets of $\chi_4(\lambda_1 \dots \lambda_m 0 \dots 0)$ and those of $\chi_4(\lambda_1 \dots \lambda_m + 1\, 0 \dots 0)$; the latter are obtained from the former by increasing α_{m-1} by unity and decreasing the Sp (2) label α_n by unity. The last sentence requires two qualifications. First there is no multiplet of $\chi_4(\lambda_1 \dots \lambda_m + 1\, 0 \dots 0)$ corresponding to a multiplet of $\chi_4(\lambda_1 \dots \lambda_m 0 \dots 0)$ which has $\alpha_n = 0$; it is wiped out by the reduction of α_n; second the remark applies only to $\alpha_{m-1} \geqslant 1$ in $\chi_4(\lambda_1 \dots \lambda_m 0 \dots 0)$, i.e. to $\alpha_{m-1} \geqslant 2$ in $\chi_4(\lambda_1 \dots \lambda_m + 1\, 0 \dots 0)$. To extend the considerations to $\alpha_{m-1} = 1$ in $\chi_4(\lambda_1 \dots \lambda_m + 1\, 0 \dots 0)$ it is necessary to add $\chi_3(\lambda_1 \dots \lambda_m 0 \dots 0)$ to $\chi_4(\lambda_1 \dots \lambda_m 0 \dots 0)$, but leave $\chi_4(\lambda_1 \dots \lambda_m + 1\, 0 \dots 0)$ as it is. It is possible to show that χ_3 is just the term obtained from χ_4 by putting $a_{m-1} = -m+2$ (i.e. $\alpha_{m-1} = 0$) instead of summing over a_{m-1} (the proof consists of noticing that $|p'_{l+m-2}, p'_{l+m-1}|$ and $|p'_{l+m-2}, p_{\bar{l}+m-1}|$ are equal when viewed as operators on an arbitrary p').

To show that the required correspondence between the multiplets of

$$\chi_3(\lambda_1 \dots \lambda_m 0 \dots 0) + \chi_4(\lambda_1 \dots \lambda_m 0 \dots 0)$$

and those of $\chi_4(\lambda_1 \dots \lambda_m + 1\, 0 \dots 0)$ is implied by the conjectured elementary multiplets we set up an explicit mapping. In an arbitrary multiplet of

$$\chi_3(\lambda_1 \dots \lambda_m 0 \dots 0) + \chi_4(\lambda_1 \dots \lambda_m 0 \dots 0)$$

let i be the largest superscript for which an A_2^i is present; replace A_2^i by A^{im} to get the corresponding multiplet of $\chi_4(\lambda_1 \dots \lambda_m + 1\, 0 \dots 0)$. In case $i = m-1$ replace A_2^{m-1} by $A_1^{m-1} A_3^m$. If no A_2^i is present the Sp (2) label of the multiplet vanishes and there is no corresponding multiplet of $\chi_4(\lambda_1 \dots \lambda_{m-1} \lambda_m + 1\, 0 \dots 0)$ as required. Going backwards, let i be the lowest index for which A^{im} is present in a multiplet of

$$\chi_4(\lambda_1 \dots \lambda_{m-1} \lambda_m + 1\, 0 \dots 0);$$

replace A^{im} by A_2^i to get the corresponding multiplet of

$$\chi_3(\lambda_1 \dots \lambda_m 0 \dots 0) + \chi_4(\lambda_1 \dots \lambda_m 0 \dots 0).$$

If there are no A^{im} replace $A_1^{m-1} A_3^m$ by A_2^{m-1}. That the correspondence is the same in both directions is guaranteed by the compatibility rules postulated for the elementary multiplets.

This completes the induction and the proof that the conjectured elementary multiplets and compatibility rules define correctly the Sp $(2n-2) \times$ Sp (2) multiplets of all Sp $(2n)$ irreps. Since they are additive for stretched products the conventional l, a (or λ, α) labels of multiplets are easily expressed in terms of the indices of the elementary multiplets.

Incidentally it is simple to infer from the elementary factors the number of labels needed for special irreps of Sp $(2n)$ in which some of the λ's vanish.

The author's thanks are due to C. S. Lam for helpful discussions and to M. Papadopoulos for pointing out relevant papers.

https://doi.org/10.1017/S030500410004634X Published online by Cambridge University Press

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https://doi.org/10.1017/S030500410004634X Published online by Cambridge University Press