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Generalized eigenfunctions of interval exchange maps

MICHAEL BOSHERNITZAN† and ARNALDO NOGUEIRA‡

† *Department of Mathematics, Rice University, Houston, TX 77251, USA*
(e-mail: michael@math.rice.edu)
‡ *Institut de Mathétiques de Luminy, 163 avenue de Luminy, Case 907, 13288 Marseille Cedex 9, France*
(e-mail: nogueira@iml.univ-mrs.fr)

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Abstract. Let f be a generalized eigenfunction of an interval exchange transformation, T, on the unit interval, which satisfies the infinite distinct orbit condition (IDOC). We assume that the minimum spacing, $\epsilon_n(T)$, of the partition defined by T^n is of the order $1/n$ for infinitely many n. This assumption is generic. Given $\delta > 0$ we prove that for sufficiently large n and for every interval J satisfying $|J| = \epsilon_n(T)$, there exists $x_0 \in J$ such that

$$|\{x \in J : |f(x) - f(x_0)| \geq \delta\}| < \delta|J|.$$

This provides a specific sufficient generic diophantine condition for Veech's result [**V2**, Lemma 7.3]. Above $|\cdot|$ denotes the linear measure of the set. Moreover, if T is uniquely ergodic, then any bounded variation continuous f must be of the form $f(x) = \gamma \exp(2\pi i a x)$ almost everywhere, where $\gamma \in \mathbb{C}$ and $a \in \mathbb{R}$.

1. *Introduction*

Let $m \in \mathbb{Z}$, $m > 1$. We denote by Λ_m the positive cone $\mathbb{R}^m_+$ and we let

$$\Delta_{m-1} = \{\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda_m : \lambda_1 + \cdots + \lambda_m = 1\},$$

the so-called $(m-1)$-dimensional simplex. For $\lambda \in \Lambda_m$ we define

$$|\lambda| = \lambda_1 + \cdots + \lambda_m, \quad I^\lambda = I(\lambda) = [0, |\lambda|)$$
$$\beta_0(\lambda) = 0, \quad \beta_j(\lambda) = \beta_{j-1}(\lambda) + \lambda_j, \quad I^\lambda_j = I_j(\lambda) = [\beta_{j-1}, \beta_j), \quad 1 \leq j \leq m.$$

Let S_m be the group of permutations on $\{1, \ldots, m\}$ and, for $\pi \in S_m$, set $\lambda^\pi_i = \lambda_{\pi^{-1} i}$, for $1 \leq i \leq m$. The (λ, π)-*interval exchange transformation* (IET) is the one-to-one onto map on I^λ, denoted by $T = T_{(\lambda,\pi)}$, defined by

$$Tx = \beta_{\pi i - 1}(\lambda^\pi) + x - \beta_{i-1}(\lambda),$$

for all $x \in I^\lambda_i$, $1 \leq i \leq m$. The discontinuities of T belong to the set $\{\beta_1, \ldots, \beta_{m-1}\}$.

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Let $\lambda \in \Delta_{m-1}$, so $T = T_{(\lambda,\pi)}$ is an IET on the unit interval $I = [0, 1)$. A function E on the interval I is *T-piecewise constant* if it is constant on every interval $I_j(\lambda)$ exchanged by T, $j = 1, \ldots, m$. A function f on I is a *generalized eigenfunction* of T, if there exists a T-piecewise constant function E such that

$$f(Tx) = E(x)f(x), \tag{1.1}$$

for almost every $x \in I$. For example, $f(x) = \gamma \exp(2\pi i a x)$, where $\gamma \in \mathbb{C}$ and $a \in \mathbb{R}$, is a generalized eigenfunction of any IET.

We need a notion regarding the permutation π.

Definition 1.1. A permutation π is *irreducible*, if, for $T = T_{(\lambda,\pi)}$, $T[0, y) = [0, y)$, for some $y > 0$, implies $y = |\lambda|$. This means that if $\pi\{1, \ldots, k\} = \{1, \ldots, k\}$, then $k = m$. We denote by S_m^0 the set of all irreducible permutations on m letters.

We say that $\pi \in S_m^0$ is of *rotation class*, if there exists $1 \le j < m$ and for every $1 \le i \le m$, $\pi(i) = i + j$ if $1 \le i \le m - j$ and $\pi(i) = i - (m - j)$ otherwise. It is conjectured that if π is not of rotation class, then, for almost every $\lambda \in \Lambda_m$, $T = T_{(\lambda,\pi)}$ is weak mixing. It means that 1 is the only eigenvalue of T. It has been proved by Katok and Stepin [**KS**] and Veech [**V2**] that the claim holds for certain permutations. For example, the claim is true for the permutation $\sigma_m(i) = m - i + 1$, for every $1 \le i \le m$, when m is odd. It is unknown if the claim holds for m even, $m \ge 4$. Recently, it has been shown by Chaves and Nogueira [**CN**] that the permutations considered in [**KS**] and [**V2**] force, in a naive way, the eigenvalue of the IET to be 1.

In [**NR**] a weaker claim is proved. If π is not of rotation class, then for almost every $\lambda \in \Lambda_m$, $T = T_{(\lambda,\pi)}$ is topological weak mixing. This means that continuous eigenfunctions must be constant.

Definition 1.2. We say that $T = T_{(\lambda,\pi)}$ satisfies the *infinite distinct orbit condition (IDOC)* if:

(1) $O(\beta_j) = \{T^n(\beta_j) : n \ge 0\}$ is an infinite set for every $1 \le j \le m - 1$;
(2) $O(\beta_i) \cap O(\beta_j) = \emptyset$ for every $1 \le i < j \le m - 1$.

Let $T = T_{(\lambda,\pi)}$ be an IET on the unit interval and $D_1 = \{\beta_1, \ldots, \beta_{m-1}\}$. For all $n \ge 1$, let

$$D_n = \bigcup_{0 \le k \le n-1} T^{-k}(D_1),$$

which defines a partition of $[0, 1)$. Let $\epsilon_n = \epsilon_n(T)$ be the length of the shortest interval of the partition D_n. We now state our main result, which holds under the assumption that $\limsup n\epsilon_n > 0$. Below, $|\cdot|$ denotes the measure of a linear set.

THEOREM 1.3. *Let T be an ergodic IET which satisfies the IDOC and f be a generalized eigenfunction of T. Assume that there exists $c > 0$ such that $n\epsilon_n(T) \ge c$ infinitely often. Then given $\delta > 0$ for infinitely many n for all subintervals $J \subset I^\lambda$ with $|J| = \epsilon_n$, there exists $x_0 \in J$ such that*

$$|\{x \in J : |f(x) - f(x_0)| > \delta\}| < \delta|J|.$$

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This property is related to the spectral properties of the IET, as we can see in [**V2**] and [**CN**]. Regarding the topological weak mixing property of IET, the result proved in [**NR**] strengthens the conjecture that almost every IET is indeed weak mixing. The study of this property is also the subject of Arnoux [**A**], Ferenczi *et al*. [**FHZ**] and Katok and Stepin [**KS**].

This paper is organized as follows. In §2 we state some known facts about IETs and define the Rauzy induction [**R**]. The proof of Theorem 1.3 is left to §3. In §4 we consider bounded variation generalized eigenfunctions f of IETs. We prove that, if the IET satisfies the IDOC, then f must be continuous. Moreover, we show that for uniquely ergodic IETs on the unit interval, such a generalized eigenfunction of bounded variation must have a very specific form: $f(x) = \gamma \exp(2\pi i \alpha x)$, for some $\gamma \in \mathbb{C}$ and $a \in \mathbb{R}$.

2. *Basic properties of IET*

In [**R**] a suitable inductive process for IETs was introduced which has been used to prove the dynamical and ergodic properties. The so-called Rauzy induction is applied to IETs which satisfy the IDOC. In order to carry on the induction, we need a notion regarding the vector λ.

Definition 2.1. We say that $\lambda \in \Lambda_m$ is *irrational* if $\lambda_1, \ldots, \lambda_m$ are linearly independent over $\mathbb{Q}$.

The notion of an irrational vector together with the notion of an irreducible permutation allow one to state a theorem proved in [**Ke**].

THEOREM 2.2. [**Ke**] *Let $\pi \in S_m^0$ and $\lambda \in \Lambda_m$ be irrational. Then two discontinuities of $T = T_{(\lambda,\pi)}$ never lie on the same orbit. Moreover, every orbit of T is dense on I^λ.*

This result leads to the notion of the IDOC (Definition 1.1). By Theorem 2.2, if $\pi \in S_m^0$ and $\lambda \in \Lambda_m$ are irrational, then $T_{(\lambda,\pi)}$ satisfies the IDOC.

Let $\pi \in S_m$ and $\mathcal{A}$ be an assertion. We say $T_{(\lambda,\pi)}$ satisfies $\mathcal{A}$ for almost every λ if $\{\lambda : T_{(\lambda,\pi)} \text{ satisfies } \mathcal{A}\}$ is a full Lebesgue measure set. According to [**Bo1**], once $\pi \in S_m^0$ is fixed almost every IET satisfies the hypothesis of Theorem 1.3.

THEOREM 2.3. [**Bo1**] *Let $\pi \in S_m^0$, then for almost every $\lambda \in \Lambda_m$ $\limsup_n n\epsilon_n(T_{(\lambda,\pi)}) > 0$.*

More precisely, in [**Bo1**] a sufficient condition for the unique ergodicity of T is formulated in terms of the asymptotic behavior of $\epsilon_n(T)$. A convenient improved version of this condition is given by the following.

THEOREM 2.4. [**V3**] *Let T be an IDOC IET which is not uniquely ergodic. Then $\lim_{n\to\infty} n\epsilon_n(T) = 0$.*

Let $\pi \in S_m^0$, $\lambda \in \Lambda_m$ be irrational, $\tau(\lambda, \pi) = \max\{\beta_{m-1}(\lambda), \beta_{m-1}(\lambda^\pi)\}$ and

$$J(\lambda, \pi) = [0, \tau(\lambda, \pi)). \tag{2.1}$$

We denote the Poincaré first return map induced by T on the subinterval $I' = J(\lambda, \pi)$ by T_1. There exists $(\lambda', \pi_1) \in \Lambda_m \times S_m^0$ such that $T_1 = T_{(\lambda',\pi_1)}$. Moreover, there exists a unique $m \times m$ matrix E_1 whose entries are 0 and 1 and $\lambda = E_1\lambda'$.

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Since λ is irrational, λ' is also irrational. Therefore, the inductive process can be continued with T_1. Recursively we define for every $k \geq 1$ the Poincaré first return map induced by T_{k-1} on the subinterval $I^{(k)} = I(\lambda^{(k)}) = J(\lambda^{(k-1)}, \pi_{k-1})$ (2.1), denoted by $T_k = T(\lambda^{(k)}, \pi_k)$. Moreover, there exists a unique $m \times m$ matrix whose entries are 0 and 1, denoted by E_k, such that $\lambda^{(k-1)} = E_k \lambda^{(k)}$. We define $A^{(k)} = E_1 \dots E_k$, then $\lambda = A^{(k)} \lambda^{(k)}$. We write $A^{(k)} = (A_{ij}^{(k)})$. We have that the entry $A_{ij}^{(k)}$ is the number of visits to $I_i(\lambda)$ of any point of $I_j(\lambda^{(k)})$ before returning to $I(\lambda^{(k)})$. The return time of any point of $I_j(\lambda^{(k)})$ to $I(\lambda^{(k)})$ is $\sum_{i=1}^{m} A_{ij}^{(k)}$, i.e. the sum of the entries of the jth column of $A^{(k)}$.

The above inductive process associates to (λ, π) the sequence $(\lambda^{(k)}, \pi_k)$.

Definition 2.5. Let $\pi \in S_m^0$; the *Rauzy class* of π is the set $R(\pi)$ consisting of all permutations reached by the Rauzy induction beginning at π:

$$R(\pi) = \{\pi_k(\lambda) : k \geq 0, \pi_0 = \pi, \lambda \in \Lambda_m\}.$$

Let $\pi_0 \in S_m^0$ be fixed. The Rauzy induction defines a two-to-one map

$$(\lambda, \pi) \in \Lambda_m \times R(\pi_0) \mapsto (\lambda', \pi_1) \in \Lambda_m \times R(\pi_0).$$

For more information on the subject, we suggest to the reader [**V1**] and [**NR**].

3. *Generalized eigenfunctions*

For simplicity, from now on $\pi \in S_m^0$ and $\lambda \in \Delta_{m-1}$, so $T = T_{(\lambda,\pi)}$ is an IET on the unit interval $I = [0, 1)$.

In [**V2**] the following property of IET is established using the so-called Rauzy induction. For almost every λ for any generalized eigenfunction f of $T_{(\lambda,\pi)}$ (1.1), given $\delta > 0$, for infinitely many k there exists $x_k \in I^{(k)}(\lambda)$ such that

$$|\{x \in I_j^{(k)} : |f(x) - f(x_k)| > \delta\}| < \delta |I_j^{(k)}|, \tag{3.1}$$

for every $1 \leq j \leq m$. In [**V2**] and [**CN**], (3.1) is used to study the spectral properties of IETs. The claim in Theorem 1.3 is stronger than (3.1) and does not use the Rauzy induction.

Next we state an assertion which follows directly from the definition of the minimum spacing $\epsilon_n = \epsilon_n(T)$, which was first pointed out in [**Bo2**].

PROPOSITION 3.1. *Let $T = T_{(\lambda,\pi)}$ satisfy the IDOC and let $n \geq 1$ be fixed. Then for every subinterval $J \subset I$ such that $|J| = \epsilon_n$, there exist integers $p, q \geq 0$ such that $p + q = n - 1$ and T^k acts as a translation on J for all $-p \leq k \leq q$. Moreover $T^{-p}(J), T^{-p+1}(J), \dots, J, \dots, T^{q-1}(J), T^q(J)$ are pairwise disjoint intervals.*

Now we prove our main result.

Proof of Theorem 1.3. There exist $0 < c < 1$ and infinitely many n such that $n\epsilon_n \geq c$. For n satisfying this condition, let $J \subset I$ be a subinterval such that $|J| = \epsilon_n$. Then by Proposition 3.1, there exist $p, q \geq 0$ such that $p + q = n - 1$ and the intervals $T^{-p}(J), T^{-p+1}(J), \dots, J, \dots, T^{q-1}(J), T^q(J)$ are pairwise disjoint. Let f be a generalized eigenfunction of T (1.1). As T is ergodic, $|f|$ is almost surely constant on I.

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This implies that $|E(x)| = 1$ for every $x \in I$, where E is the T-piecewise constant function which defines f. By Vitali–Lusin's theorem (see [**Be**, Theorem 2.10]), given $\gamma > 0$ there exists a continuous function g on $[0, 1]$ such that

$$|\{x \in I : f(x) \neq g(x)\}| < \gamma,$$

which implies

$$\left|\left\{x \in \bigcup_{-p \leq k \leq q} T^k(J) : f(x) \neq g(x)\right\}\right| < \gamma.$$

Therefore, there exists $-p \leq k \leq q$ such that

$$|\{x \in T^k(J) : f(x) \neq g(x)\}| < \frac{\gamma}{n}.$$

The uniform continuity of g implies that for n sufficiently large

$$|g(z) - g(y)| < \gamma,$$

for all $y, z \in I$ such that $|z - y| \leq \epsilon_n$. Therefore, there exists $x_0 \in J$ such that

$$g(T^k(x_0)) = f(T^k(x_0)) \quad \text{and} \quad |g(y) - g(T^k(x_0))| < \gamma,$$

for every $y \in T^k(J)$, and moreover (1.1) holds for $x = T^l(x_0)$, for every $l \in \mathbb{Z}$. We conclude that

$$|\{y \in T^k(J) : |f(y) - f(T^k(x_0))| \geq 2\gamma\}| < \frac{\gamma}{n}. \tag{3.2}$$

Next we prove that for almost every $y \in J$,

$$|f(T^k(x_0)) - f(T^k(y))| = |f(x_0) - f(y)|.$$

We have that $T^l(J)$ is a translation of the subinterval J, for every $-p + 1 \leq l \leq q - 1$; in addition, the points $T^l(x_0)$ and $T^l(y)$ belong to the same interval I_j, for some $1 \leq j \leq m$, therefore $E(T^l(x_0)) = E(T^l(y))$ and (1.1) holds for $x = T^l(y)$, for every $-p + 1 \leq l \leq q - 1$. We deduce

$$f(T^l(x)) = E(T^{l-1}(x)) \dots E(x) f(x), \tag{3.3}$$

for $x \in \{x_0, y\}$. To conclude we recall that $|E(\cdot)| = 1$.

We rewrite (3.2),

$$|\{x \in J : |f(T^k(x)) - f(T^k(x_0))| \geq 2\gamma\}| < \frac{\gamma}{n}.$$

Using (3.3), we deduce

$$|\{x \in J : |f(x) - f(x_0)| \geq 2\gamma\}| < \frac{\gamma}{n} < \frac{\gamma}{c}|J|.$$

Taking $c < \frac{1}{2}$ and $\delta = \gamma/c$, the claim is proved. □

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4. *Bounded variation generalized eigenfunctions*

The *variation* of $\psi : \mathbb{R} \longrightarrow \mathbb{C}$ on $[a, b]$ is $\Delta_\psi([a, b]) = \psi(b) - \psi(a)$. The function ψ has *bounded variation* on $[a, b]$, if there exists $C > 0$ such that for every finite partition $a_0 = a < a_1 < \cdots < a_k = b$ of the interval $[a, b]$

$$\sum_{i=1}^{k} |\Delta_\psi([a_{i-1}, a_i])| \leq C.$$

The next results deal with bounded variation generalized eigenfunctions (1.1).

LEMMA 4.1. *Let $T = T_{(\lambda,\pi)}$ be an ergodic IET which satisfies the IDOC and let f be a bounded variation generalized eigenfunction of T, then there exists a continuous eigenfunction g such that the set $\{x \in I : f(x) \neq g(x)\}$ is countable, therefore the function $E(\cdot)$ (1.1) is constant and g is an eigenfunction of T. Moreover, if c is a discontinuity of f,*

$$f(T^{l+1}(c)) = E(T^l(c)) f(T^l(c))$$

for at most finitely many l.

Proof. Let f be a generalized eigenfunction of T (1.1). As T is ergodic, $|f|$ is almost surely constant on I. This implies that $|E(x)| = 1$ for every $x \in I$, where E is the T-piecewise constant function which defines f. We assume that f is discontinuous at $0 < c < 1$, and there exist $\epsilon > 0$ and $\delta > 0$ such that for all x_1 and x_2, $0 < c - \delta < x_1 < c < x_2 < c + \delta < 1$,

$$|f(x_2) - f(x_1)| > \epsilon. \tag{4.1}$$

Let $n > 0$ be such that $\epsilon_n(T) < \delta$ and $x_1 < c < x_2$, $x_2 - x_1 < \epsilon_n(T)$, be such that (1.1) holds for all $x \in \{T^l(x_j) : j = 1, 2, l \in \mathbb{Z}\}$. As T satisfies the IDOC, Proposition 3.1 holds: there exist $p, q \geq 0$ such that $p + q = n - 1$ and $T^k(J)$ is a translation of the interval $J = [x_1, x_2]$, for all $-p \leq k \leq q$, moreover the intervals $T^{-p}(J), \ldots, T^q(J)$ are pairwise disjoint. The latter implies that $|\Delta_f T^k(J)| = |f(x_2) - f(x_1)|$, since $|E(x)| = 1$ for every x. By (4.1) we conclude that

$$\sum_{k=-p}^{q} |\Delta_f T^k(J)| > n\epsilon.$$

The right-hand side tends to infinity as $n \to \infty$. This contradicts the fact that f has bounded variation. We conclude that, for every $0 < c < 1$, the limits below are finite and

$$\lim_{x \to c+} f(x) = \lim_{x \to c-} f(x).$$

Therefore there exists g continuous such that $\{x \in I : f(x) \neq g(x)\}$ is a countable set. Moreover, if $f(c) \neq g(c)$, $f(T^{l+1}(c)) = E(T^l(c)) f(T^l(c))$, for only finitely many l. This concludes the proof of the lemma. □

Masur [**M**] and Veech [**V1**] independently proved that, for a fixed irreducible permutation π, for almost every λ, $T_{(\lambda,\pi)}$ is uniquely ergodic. The next lemma characterizes bounded variation generalized eigenfunctions of uniquely ergodic IETs.

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LEMMA 4.2. *Let T be a uniquely ergodic IET and f a continuous bounded variation generalized eigenfunction of T. Then there exist $\gamma \in \mathbb{C}$ and $a \in \mathbb{R}$ such that, for every x,*

$$f(x) = \gamma e^{2\pi i a x}.$$

Proof. We assume that $|f(x)| = 1$, for every $x \in I = [0, 1)$, as $T = T_{(\lambda,\pi)}$ is ergodic and f is continuous. The continuity of f implies that there exists a continuous real-valued function ϕ on I such that for every $x \in I$,

$$f(x) = \exp(2\pi i \phi(x)).$$

Moreover for every $j = 1, \ldots, m$, there exists $\alpha_j \in \mathbb{R}$ such that

$$E(x) = e^{2\pi i \alpha_j},$$

for every $x \in I_j(\lambda)$. We have for every $x \in I_j(\lambda)$

$$f(T(x)) = \exp(2\pi i \phi(T(x))) = \exp(2\pi i(\alpha_j + \phi(x))).$$

Moreover, ϕ is also bounded variation, as f is bounded variation.

Let $y \in I$ and $\epsilon > 0$. The fact that T is uniquely ergodic implies that there exists $N = N(y, \epsilon)$ such that for all $k \geq N$ and for all $z \in I$,

$$\left| \frac{|\{0 \leq l < k : T^l(z) \leq y\}|}{k} - y \right| < \epsilon, \tag{4.2}$$

where $|\{\cdot\}|$ means the cardinality of the set $\{\cdot\}$. We consider the kth Rauzy induction of T which is defined on the interval

$$I^{(k)} = \bigcup_{1 \leq j \leq m} I_j^{(k)}.$$

The right-hand side is a pairwise disjoint union and $I_j^{(k)} = [\beta_{j-1}^{(k)}, \beta_j^{(k)})$, for $1 \leq j \leq m$. Let $t_j^{(k)}$ be the return time to the interval $I^{(k)}$ of any point of the interval $I_j^{(k)}$, for $1 \leq j \leq m$. We set

$$\Gamma_j^{(k)}(y) = \{0 \leq l < t_j^{(k)} : T^l(\beta_{j-1}^{(k)}) \leq y\} \quad \text{and} \quad s_j^{(k)} = |\Gamma_j^{(k)}(y)|.$$

For k sufficiently large, $t_j^{(k)} \geq N(y, \epsilon)$, for all $1 \leq j \leq m$, then, by (4.2),

$$\left| \frac{s_j^{(k)}}{t_j^{(k)}} - y \right| < \epsilon.$$

Let J be a closed subinterval of I_j for some $j = 1, \ldots, m$, we note that

$$\Delta_\phi(T(J)) = \Delta_\phi(J).$$

We recall that, for all $j = 1, \ldots, m$ and $0 \leq l \leq t_j^{(k)}$, there exists $r = r(j, l) \in \{1, \ldots, m\}$ such that $T^l(I_j^{(k)}) \subset I_r$. We deduce

$$\phi(1) - \phi(0) = \sum_{1 \leq j \leq m} \sum_{0 \leq l < t_j^{(k)}} \Delta_\phi(T^l(I_j^{(k)})) = \sum_{1 \leq j \leq m} t_j^{(k)} \Delta_\phi(I_j^{(k)}).$$

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Regarding the variation of ϕ on the interval $[0, y]$, we have

$$\sum_{1\le j\le m}\sum_{l\in\Gamma_j^{(k)}}\Delta_\phi(T^l(I_j^{(k)})) = \sum_{1\le j\le m} s_j^{(k)}\Delta_\phi(I_j^{(k)}) = \sum_{1\le j\le m}\frac{s_j^{(k)}}{t_j^{(k)}}t_j^{(k)}\Delta_\phi(I_j^{(k)}).$$

Using that ϕ has bounded variation,

$$\left|\sum_{1\le j\le m}\frac{s_j^{(k)}}{t_j^{(k)}}t_j^{(k)}\Delta_\phi(I_j^{(k)})\right| \le (y+\epsilon)\sum_{1\le j\le m} t_j^{(k)}|\Delta\phi(I_j^{(k)})| < (y+\epsilon)C.$$

Then

$$\lim_{k\to\infty}\sum_{1\le j\le m} s_j^{(k)}\Delta\phi(I_j^{(k)})$$

exists and is finite. The measure of the remaining set

$$[0, y] - \bigcup_{1\le j\le m}\bigcup_{l\in\Gamma_j^{(k)}} T^l(I_j^{(k)})$$

tends to zero, as $k\to\infty$. We conclude that for every y,

$$\phi(y) = \phi(0) + y(\phi(1) - \phi(0)).$$ □

5. *Weak mixing for special permutations*

Here an alternative approach is given to a question considered in [**V2**] and [**CN**]. Let $\pi \in S_m^0$. By recurrence, we define the finite sequence (see [**CN**, Lemma 3.1])

$$a_0 = 1, \quad a_p = \pi^{-1}(\pi a_{p-1} - 1) + 1, \quad \text{if } a_{p-1} \notin \{\pi^{-1}1, m+1\}.$$

Let a_r be the last term of the sequence, then $a_r = \pi^{-1}1$ or $a_r = m+1$. These two possibilities determine a partition of S_m^0:

$$S_m^0 = \{\pi \in S_m^0 : a_r = \pi^{-1}1\} \cup \{\pi \in S_m^0 : a_r = m+1\}.$$

Definition 5.1. With notation as above, we say $\pi \in S_m^0$ is $\mathcal{W}$-*type*, if the last term of the sequence $a_r = \pi^{-1}1$. An IET is $\mathcal{W}$-*type* if it is defined by a $\mathcal{W}$-type permutation.

Regarding $\mathcal{W}$-type permutations, the following result is known.

THEOREM 5.2. [**V2**] *Let $m > 2$ and $\pi \in S_m$ be a $\mathcal{W}$-type permutation. Then for almost every $\lambda \in \Lambda_m$, $T_{(\lambda,\pi)}$ is weakly mixing.*

An alternative proof of Theorem 5.2 is given in [**CN**]. In the case where $n = 3$ and the permutation is $\sigma_3 : \sigma_3(1) = 3, \sigma_3(2) = 2, \sigma_3(3) = 1$, the result was proved earlier in [**KS**]. We prove the following claim.

THEOREM 5.3. *Let $m > 2$, $\lambda \in \Lambda_m$ and $\pi \in S_m^0$ be a $\mathcal{W}$-type permutation. Let $T = T_{(\lambda,\pi)}$ be IDOC and ergodic and suppose there exists $c > 0$ such that infinitely often $n\epsilon_n(T) \ge c$. Then T is weakly mixing.*

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Proof. We assume $|\lambda| = 1$. Let f be an eigenfunction of T, that is $f(Tx) = \alpha f(x)$, for almost every $x \in I$. Since T is ergodic we have $|\alpha| = 1$ and we can consider $|f(x)| = 1$, for every $x \in I$.

Given $\delta > 0$, by Theorem 1.3, there exist infinitely many n and $\gamma \in \mathbb{C}$, $|\gamma| = 1$, such that

$$|\{x \in J_0 : |f(x) - \gamma| > \delta\}| < \delta |J_0|,$$

where $J_0 = [0, \epsilon_n)$, then

$$|\{x \in T(J_0) : |f(x) - \alpha\gamma| > \delta\}| < \delta |T(J_0)|.$$

Let $K_0 = [T0 - \epsilon_n/2, T0 + \epsilon_n/2)$, then

$$|\{x \in K_0 : |f(x) - \alpha\gamma| > 2\delta\}| < \delta |K_0|.$$

By recurrence we obtain $J_i = [\beta_{a_i - 1}, \beta_{a_i - 1} + \epsilon_n)$, for $1 \leq i \leq r$, and

$$|\{x \in J_i : |f(x) - \gamma| > 4^i \delta\}| < \delta |J_i|.$$

Since $r = \pi^{-1}1$, $T(J_r) = J_0$ and we conclude

$$|\{x \in J_0 : |f(x) - \alpha\gamma| > 4^r \delta\}| < \delta |J_0|.$$

We deduce that $|\gamma - \alpha\gamma| = |1 - \alpha| \leq 4^r \delta$. As $\delta > 0$ is arbitrary, we conclude $\alpha = 1$. Therefore, T is weakly mixing. □

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References

[A] P. Arnoux. Un exemple de semi-conjugaision entre une échange d'intervalles et une translation sur le tore. *Bull. Soc. Math. France* **116** (1988), 489–500.

[Be] J. J. Benedetto. *Real Variables and Integration*. B. G. Teubner, Stuttgart, 1976.

[Bo1] M. Boshernitzan. A condition for minimal interval exchange map to be uniquely ergodic. *Duke Math. J.* **52**(3) (1985), 723–752.

[Bo2] M. Boshernitzan. Rank two interval exchange maps. *Ergod. Th. & Dynam. Sys.* **8**(3) (1988), 379–394.

[CN] J. Chaves and A. Nogueira. Spectral properties of interval exchange maps. *Monatsh. Math.* **134**(2) (2001), 89–102.

[FHZ] S. Ferenczi, C. Holton and L. Zamboni. Structure of 3-interval exchange transformations III: ergodic and spectral properties. *Preprint*.

[KS] A. Katok and A. Stepin. Approximations in ergodic theory. *Usp. Mat. Nauk.* **22**(5)(137) (1967), 81–106 (Engl. transl. *Russian Math. Surveys* **22**(5) (1967), 77–102).

[Ke] M. Keane. Interval exchange transformations. *Math. Z.* **141** (1975), 25–31.

[M] H. Masur. Interval exchange transformations and measured foliations. *Ann. Math.* **115** (1982), 169–200.

[NR] A. Nogueira and D. Rudolph. Topological weak-mixing of interval exchange maps. *Ergod. Th. & Dynam. Sys.* **17** (1997), 1183–1209.

[R] G. Rauzy. Échanges d'intervalles et transformations induites. *Acta Arith.* **34** (1979), 315–328.

[V1] W. Veech. Gauss measures for transformations on spaces of interval exchange maps. *Annal. Math.* **115**(1) (1982), 201–242.

[V2] W. Veech. The metric theory of interval exchange transformations I. Generic spectral properties. *Amer. J. Math.* **106**(6) (1984), 1331–1359.

[V3] W. Veech. Boshernitzan's criterion for unique ergodicity of an interval exchange transformation. *Ergod. Th. & Dynam. Sys.* **7**(1) (1987), 149–153.

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