J. Plasma Physics (2007), *vol.* 73, *part* 4, *pp.* 473–484. © 2006 Cambridge University Press
doi:10.1017/S0022377806004739 First published online 25 July 2006 Printed in the United Kingdom

A two-dimensional analysis of mode conversion at a resonance layer with high-magnetic field side reflection

NORA NASSIRI-MOFAKHAM and AKBAR PARVAZIAN

Department of Physics, Isfahan University of Technology, Isfahan, Iran
(n.mofakham@gmail.com)

(Received 17 April 2006)

Abstract. Mode converted Bernstein waves potentially allow the implementation of local heating and current drive in spherical torus devices, which are not directly accessible to low-harmonic cyclotron waves. The mode conversion method of Cairns and Lashmore-Davies previously used to study the usual mode conversion in non-planar geometry is extended to include the effect of the high-magnetic-field-side cutoff, and is solved analytically. The analytic solutions to the triplet, cutoff–resonance–cutoff, equations give the reflection and conversion coefficients in terms of parameters defining the local behavior of the dispersion relation. The variation of mode conversion efficiency depends not only on the coupling parameter but also on the phasing effect introduced by the high-field-side cutoff. The change in characteristic phases, which are concerned with the coupling parameter, brings an additional degree of freedom allowing optimization via the position of the high-field cutoff. A discrete spectrum of phases exists for which complete mode conversion of the incident wave for a transit of the resonance region can be achieved. The results we obtain here give the general conditions for efficient Bernstein wave heating in two-dimensional geometry.

1. Introduction

In a previous paper we have generalized the mode conversion method of Cairns and Lashmore-Davies to plasmas with two-dimensional inhomogeneities and have applied it to the absorption of the ordinary (O) mode and extraordinary (X) mode at the electron cyclotron resonance (Nassiri-Mofakham and Sabzevari 2006). Previously, our studies on the mode conversion process have analyzed the propagation of incident electromagnetic waves in a region that includes the low-magnetic-field side cutoff (LFC) and the upper hybrid resonance (UHR). The description is equivalent to the scattering of a wave by a potential having a resonance and a cutoff. Regarding this process as the mode conversion to the electron Bernstein waves, it was shown that the maximum of 25% of the wave power can be mode converted to electron Bernstein waves (EBWs). Electron Bernstein waves have no density limit, they propagate towards the dense plasma core and are suitable for core heating in high-β spherical tokamak plasmas.

Here we develop this mode conversion technique further. The wave dispersion characteristics show that the quasi-transverse extraordinary electromagnetic mode

https://doi.org/10.1017/S0022377806004739 Published online by Cambridge University Press

sees the UHR, right- and left-hand cutoffs, while the O mode sees only the plasma cutoff, beyond which the electromagnetic modes do not propagate. In other words, when the UHR is inside a typical tokamak plasma, there are two cutoffs located in the region of the plasma: on the high-magnetic-field side, and on the low-magnetic-field side of the UHR (Stix 1992). The incident wave from the LFC is coupled to an internal resonator system, composed of LFC, resonance and high-magnetic field side cutoff (HFC), is dissipated in this resonator at the resonance layer, and is converted to Bernstein waves which damp on electrons. The HFC can have a significant effect on the mode conversion efficiency (Majeski et al. 1994). The condition for efficient mode conversion corresponds to the situation in which the incident wave is coupled to an internal plasma resonator system. Then there is no reflection of the wave toward the low-field side, and the incident wave power is totally absorbed. The resonator is intrinsic to the plasma since it is formed by the cutoffs, which are intrinsic to the plasma. A physically phase-integral analysis of the problem has shown that, in contrast to the Budden problem, a maximum of 100% of the incident electromagnetic wave power can be mode converted to EBWs (Fuchs et al. 1995).

In the following sections we shall give a modified version of our previous study and analytically solve the two-dimensional coupled wave equation coming from the local dispersion relation around the mode conversion region. We use the technique to calculate the conversion coefficients of the electromagnetic wave to Bernstein wave mode conversion process for triplet, cutoff–resonance–cutoff, equation. Our object is to investigate mode conversion under more realistic assumptions, the high-magnetic field, side effect in plasmas with two-dimensional inhomogeneities using the unified method proposed by Cairns and Lashmore-Davies (1983). In this method it is not necessary to consider polarization vectors and eigenvalues of dispersion matrices around coupling points. Instead, the frequencies ω_1 and ω_2 of the uncoupled modes belonging to two different dispersion equations are considered as functions of the space variable $\mathbf{x}$ and the wave vector $\mathbf{k}$, and are coupled together via a small quantity η, which is only significant around the coupling point. The essential point in the method of Cairns and Lashmore-Davies is that this method is equivalent to S-matrix methods, where the singularity in the interaction region is circumvented by asymptotic methods. Hence, concerning singularities, no mathematical obscurity appears in this method.

In Sec. 2, the method will be shown to give two-dimensional equations that can be solved analytically for transmission, reflection and conversion coefficients that depend on the local behavior of the dispersion relation. In Sec. 3 we drive an analytic solution of the mode conversion efficiency when the wave is incident from the low-field side and encounters the LFC, the resonance and the HFC. The approximate solutions are obtained for the case when this cutoff is far from the resonance. For this case, we determine the mode conversion and show the possibility of achieving 100% mode conversion efficiency.

2. Two-dimensional wave-amplitude equations

The frequencies ω_1 and ω_2 of the uncoupled modes belonging to two different dispersion equations are considered as functions of the space variable $\mathbf{x}$ and the wave vector $\mathbf{k}$, and are coupled together via a small quantity η, which is only

https://doi.org/10.1017/S0022377806004739 Published online by Cambridge University Press

significant around the coupling point

$$[\omega - \omega_1(\mathbf{x},\mathbf{k})][\omega - \omega_2(\mathbf{x},\mathbf{k})] = \eta(\mathbf{x},\mathbf{k}). \tag{2.1}$$

We emphasis here that ω_1 and ω_2 belong to two different dispersion equations that do not result from a dispersion matrix. By expanding these two different modes ω_1 and ω_2 around a point $(\mathbf{x}_0,\mathbf{k}_0)$ where they become equal, constructing a coupled system of ordinary differential equations and finding asymptotic solutions, the conversion coefficients are derived (Cairns and Lashmore-Davies 1983).

We consider an inhomogeneous plasma where both the density and magnetic field are functions of position. If a wave of given frequency ω_0 propagates into the location $(\mathbf{x},\mathbf{k})$, the incident wave and the fundamental or second cyclotron harmonic wave will be in resonance, that is, $\omega_0 = \omega_1(\mathbf{x},\mathbf{k}) = \omega_2(\mathbf{x},\mathbf{k})$. Expanding about the resonance point gives

$$\omega_1 = \omega_0 + \mathbf{a}\cdot\boldsymbol{\delta} + \mathbf{b}\cdot\boldsymbol{\xi}, \quad \omega_2 = \omega_0 + \mathbf{f}\cdot\boldsymbol{\delta} + \mathbf{g}\cdot\boldsymbol{\xi}, \tag{2.2}$$

where $\boldsymbol{\delta} = \mathbf{k}-\mathbf{k}_0$, $\boldsymbol{\xi} = \mathbf{x}-\mathbf{x}_0$ and the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{f}$ and $\mathbf{g}$ are the partial derivatives of ω_1 and ω_2. Then around $\mathbf{x}_0$, by substituting (2.2) into (2.1) and by Fourier transformation for $\boldsymbol{\delta} = -i\partial/\partial\boldsymbol{\xi}$, the behavior of the waves is described by two coupled partial differential equations for the two modes (Nassiri-Mofakham and Sabzevari 2006),

$$\begin{aligned} \mathbf{a}\cdot\frac{\partial u_1}{\partial\boldsymbol{\xi}} - i(\mathbf{a}\cdot\mathbf{k}_0 - \mathbf{b}\cdot\boldsymbol{\xi})u_1 &= i\sqrt{\eta_0}u_2, \\ \mathbf{f}\cdot\frac{\partial u_2}{\partial\boldsymbol{\xi}} - i(\mathbf{f}\cdot\mathbf{k}_0 - \mathbf{g}\cdot\boldsymbol{\xi})u_2 &= i\sqrt{\eta_0}u_1, \end{aligned} \tag{2.3}$$

where η_0 is the value of $\eta(\mathbf{x},\mathbf{k})$ at $(\mathbf{x}_0,\mathbf{k}_0)$. By eliminating u_2 from (2.3) we arrive at a second-order partial differential equation with two variables that is of the hyperbolic type (Smirnov 1964). By introducing the appropriate characteristic curves, the partial differential equation can be transformed into the canonical form

$$\mathscr{L}(u_1) = \frac{\partial^2 u_1}{\partial x\partial y} + D(x,y)\frac{\partial u_1}{\partial x} + E(x,y)\frac{\partial u_1}{\partial y} + F(x,y)u_1 = 0, \tag{2.4}$$

with

$$\begin{aligned} D(x,y) &= -i\frac{a_1}{\alpha}\left[\mathbf{f}\cdot\mathbf{k}_0 + \frac{f_1}{\alpha}(\mathbf{a}\cdot\mathbf{g})x - \frac{a_1}{\alpha}(\mathbf{f}\cdot\mathbf{g})y\right], \\ E(x,y) &= i\frac{f_1}{\alpha}\left[\mathbf{a}\cdot\mathbf{k}_0 + \frac{f_1}{\alpha}(\mathbf{a}\cdot\mathbf{b})x - \frac{a_1}{\alpha}(\mathbf{f}\cdot\mathbf{b})y\right], \\ F(x,y) &= -\frac{a_1 f_1}{\alpha^2}(i\mathbf{f}\cdot\mathbf{b} + \eta_0) + D(x,y)E(x,y), \\ \alpha &= a_1 f_2 - a_2 f_1. \end{aligned} \tag{2.5}$$

Equation (2.4) can be solved analytically by using Riemann's method (for more details see Nassiri-Mofakham and Sabzevari (2006)). The general form of this

https://doi.org/10.1017/S0022377806004739 Published online by Cambridge University Press

solution is

$$u_1(x,y;x_0,y_0) = \exp\left[\frac{if_1\mathbf{a}\cdot\mathbf{k}_0}{\alpha}(x-x_0) - \frac{ia_1f_1(\mathbf{f}\cdot\mathbf{b})}{\alpha^2}y(x-x_0) + \frac{if_1^2(\mathbf{a}\cdot\mathbf{b})}{2\alpha^2}(x^2-x_0^2)\right] \times \exp\left[-\frac{ia_1\mathbf{f}\cdot\mathbf{k}_0}{\alpha}(y-y_0) - \frac{ia_1f_1(\mathbf{a}\cdot\mathbf{g})}{\alpha^2}x(y-y_0) + \frac{ia_1^2(\mathbf{f}\cdot\mathbf{g})}{2\alpha^2}(y^2-y_0^2)\right] \times W(x,y;\,x_0,y_0). \tag{2.6}$$

Recalling the time dependence, that $u_1 \sim \exp(-i\omega t)$, suppressed here. Equation (2.4) can be transformed to a differential equation with one variable by an appropriate change of variables (see, e.g., Sommerfeld (1964)). By introducing the transformations

$$z = (x-x_0)(y-y_0), \qquad W = z^{-1/2}\exp\left[\frac{ia_1f_1}{2\alpha^2}(\mathbf{a}\cdot\mathbf{g}+\mathbf{f}\cdot\mathbf{b})z\right]G(z), \tag{2.7}$$

(2.4) reduces to the form for the confluent hypergeometric function

$$\frac{d^2G}{dz^2} + \left\{\frac{1}{4z^2} + \frac{a_1f_1}{2\alpha^2}[i(\mathbf{a}\cdot\mathbf{g}-\mathbf{f}\cdot\mathbf{b})-2\eta_0]\frac{1}{z} + \frac{1}{4}\left(\frac{a_1f_1}{\alpha^2}(\mathbf{a}\cdot\mathbf{g}-\mathbf{f}\cdot\mathbf{b})\right)^2\right\}G = 0. \tag{2.8}$$

This equation describes the propagation of the electromagnetic wave through the resonance and cutoffs. Equation (2.8) can be expressed in the standard Whittaker form (Abramovitz and Stegun 1970)

$$\frac{d^2G}{d\zeta^2} + \left(\frac{1}{4\zeta^2} + \frac{i\kappa}{\zeta} - \frac{1}{4}\right)G(\zeta) = 0, \tag{2.9}$$

where $\zeta = -i2\sqrt{\gamma}z$ and $\kappa = \tilde{\gamma}/2\sqrt{\gamma}$ with

$$\tilde{\gamma} = \frac{a_1f_1}{2\alpha^2}[i(\mathbf{a}\cdot\mathbf{g}-\mathbf{f}\cdot\mathbf{b})-2\eta_0], \qquad \gamma = \left(\frac{a_1f_1}{2\alpha^2}(\mathbf{a}\cdot\mathbf{g}-\mathbf{f}\cdot\mathbf{b})\right)^2 = \left(\frac{a_1f_1}{\alpha^2}\frac{\eta_0}{\beta}\right)^2, \qquad \beta = \frac{2\eta_0}{(\mathbf{a}\cdot\mathbf{g}-\mathbf{f}\cdot\mathbf{b})}. \tag{2.10}$$

In the usual Budden problem for low-field incidence, the relevant solution to (2.9) is $W_{-i\kappa,0}(-\zeta)$. Then, using the asymptotic properties of the Whittaker functions (see Appendix A), it can be shown that the power transmission, power reflection and power mode conversion coefficients are

$$T = \mid e^{\pi\kappa} \mid^2 = e^{-\pi\beta} = \exp\left\{-\frac{2\pi\eta_0}{\mathbf{a}\cdot\mathbf{g}-\mathbf{f}\cdot\mathbf{b}}\right\}, \qquad R = (1-T)^2, \qquad C = 1-T-R = T(1-T), \tag{2.11}$$

respectively. The description for T is just the same relation for the transmission coefficient, which has been derived previously (Nassiri-Mofakham and Sabzevari 2006, Equation (2.32)). The power mode conversion coefficient attains a maximum value of 0.25 for $T = 0.5$, that is, the maximum fraction of the incident wave power that can be converted to the Bernstein waves is 25%.

https://doi.org/10.1017/S0022377806004739 Published online by Cambridge University Press

3. Solution to the triplet (cutoff–resonance–cutoff) equation

In order to study the effect of the HFC, let us assume that for a wave incident from the low-field side ($z > 0$) there is a reflection region at the point $z = z_{\mathrm{HFC}}$. Then the wave transmitted through the resonance layer is reflected back toward the resonance from the high-field side. In this case, there is no transmission beyond the cutoff, since waves do not propagate for $z < z_{\mathrm{HFC}}$. There is only a power reflection coefficient and power mode conversion coefficient. The general solution to (2.9) is

$$G(z) = AW_{i\kappa,0}(\zeta) + BW_{-i\kappa,0}(-\zeta), \tag{3.1}$$

where A and B are arbitrary constants that depend on the boundary conditions and $W_{i\kappa,0}(\zeta)$ and $W_{-i\kappa,0}(-\zeta)$ are the Whittaker functions.

In order to determine the scattering coefficients for (2.8), we need to know only the asymptotic forms of the Whittaker functions. According to the asymptotic forms of the Whittaker functions (see Appendix A) for $z \to -\infty$, $W_{i\kappa,0}(\zeta)$ represents an incoming wave toward $z = 0$, while $W_{-i\kappa,0}(-\zeta)$ represents an outgoing wave. For $z \to +\infty$, $W_{i\kappa,0}(\zeta)$ represents an outgoing wave while $W_{-i\kappa,0}(-\zeta)$ is a combination of incoming and outgoing waves. Then, using the asymptotic properties of the Whittaker functions for $z \to \infty$ (see Appendix A), the power reflection coefficient is

$$R = \left| e^{2\pi\kappa} \left[\frac{A}{B} - \frac{2\pi i e^{-\pi\kappa}}{\left[\Gamma\left(\frac{1}{2} + i\kappa\right)\right]^2} \right] \right|^2. \tag{3.2}$$

Using the properties of the Gamma function (Abramovitz and Stegun 1970), this can be written as

$$\begin{aligned} R &= |e^{2\pi\kappa}|^2 \left| \frac{A}{B} - ie^{-2i\theta}(1 + e^{-2\pi\kappa}) \right|^2 \\ &= e^{-2\pi\beta} \left| \frac{A}{B} - i(1 - e^{\pi\beta})e^{-2i\theta} \right|^2, \end{aligned} \tag{3.3}$$

where θ is the phase of $\Gamma(-i\beta/2)$ and depends on the parameter β as

$$\theta = \frac{\beta}{2} \left[-C + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \tan^{-1} \frac{\beta/2}{n-1} \right) \right],$$

with $C = 0.577215\ldots$, is the Euler constant (Flügge 1999).

The ratio A/B is determined by matching the solution for the region $z > 0$ to the solution in the region $z < 0$ at $z = 0$. In the region $z > 0$, the solutions are given by the Whittaker functions. In the region $z < 0$, there does not exist a closed form solution in terms of known functions. We will use the method of matched asymptotic expansions (Bender and Orszag 1978) to construct the approximate solution to the differential equation in the region $z < 0$. With this solution we can determine the ratio A/B needed to calculate the reflection coefficient in (3.3). Region $z < 0$ is divided into three subregions: region 1, corresponding to $z \approx 0$, between the LFC and the resonance point; region 2, corresponding to $z_{\mathrm{HFC}} < z < 0$, between the resonance and the HFC; region 3, corresponding to $z < z_{\mathrm{HFC}}$, beyond the HFC. By matching the approximate solutions of the corresponding subregions, we determine the solution in the region $z < 0$.

In region 1, the solution to $G(z)$ is in terms of the Whittaker functions

$$G_1(z) = A_1 W_{i\kappa,0}(\zeta) + B_1 W_{-i\kappa,0}(-\zeta), \tag{3.4}$$

where A_1 and B_1 are constants.

https://doi.org/10.1017/S0022377806004739 Published online by Cambridge University Press

In region 2, which is assumed to be sufficiently far from the HFC and from the resonance point, we assume a solution of the Wentzel-Kramer-Brillouin (WKB) form (Bender and Orszag 1978)

$$G_2(z) = A_2[Q(z)]^{-1/4}\exp\left(i\int_{z_{\rm HFC}}^{z}\sqrt{Q(z')}\,dz'\right) + B_2\,[Q(z)]^{-1/4}\exp\left(-i\int_{z_{\rm HFC}}^{z}\sqrt{Q(z')}\,dz'\right), \tag{3.5}$$

where A_2 and B_2 are constants and the scattering potential is

$$Q(z) = \frac{1}{4z^2} + \frac{\tilde{\gamma}}{z} + \gamma, \tag{3.6}$$

where the notation is as defined in (2.10). The parameter $\sqrt{\gamma}$ is the normalized wave number and $\tilde{\gamma}/\gamma$ defines the normalized distance between the resonance and the HFC. The cutoff is given by $Q(z) = 0$ for $z < 0$. We assume that near the HFC $1/4z^2$ is small compared to $\tilde{\gamma}/z$ and γ, so that this cutoff is approximately given by $z_{\rm HFC} \approx -\tilde{\gamma}/\gamma$. Since for $z < z_{\rm HFC}$ we assume that the solutions to (2.8) are evanescent, it follows that $\tilde{\gamma} > 0$ and $\gamma > 0$.

In region 3, the solution to $G(z)$ is of the form

$$G_3(z) = A_3 W_{i\kappa,0}(\zeta) + B_3 W_{-i\kappa,0}(-\zeta), \tag{3.7}$$

where A_3 and B_3 are constants. For $z \to -\infty$, there is no transmitted wave, so that from the asymptotic of the Whittaker functions (see Appendix A) $B_3 = 0$.

The constants A_1, B_1, A_2 and B_2 can be expressed in terms of A_3, by the method of uniform asymptotic matching (Bender and Orszag 1978), which requires that

$$\lim_{z\to+\infty} G_3(z) = \lim_{z\to z_{\rm HFC}} G_2(z), \tag{3.8}$$

$$\lim_{z\to 0} G_2(z) = \lim_{z\to-\infty} G_1(z). \tag{3.9}$$

Equation (3.8) matches the WKB solution to the Whittaker function solution near the HFC while (3.9) matches the WKB solution to the Whittaker function solution near the resonance. The right-hand side of (3.8) can be evaluated from (3.6) by approximating

$$\int_{z_{\rm HFC}}^{z}\sqrt{Q(z')}\,dz' = \sqrt{\gamma}z + \frac{\tilde{\gamma}}{2\sqrt{\gamma}} + \frac{\tilde{\gamma}}{2\sqrt{\gamma}}\left[\ln z + \ln\left(\frac{2\gamma}{-\tilde{\gamma}}\right)\right]. \tag{3.10}$$

Then using the asymptotic expansion for the Whittaker function (see Appendix A) we obtain

$$A_2 = \frac{\gamma^{1/4}e^{\pi\kappa/2}e^{-i\kappa}}{(-\tilde{\gamma})^{i\kappa}}A_3, \quad B_2 = \frac{\gamma^{1/4}e^{-3\pi\kappa/2}e^{i\kappa}}{(-\tilde{\gamma})^{-i\kappa}}A_3. \tag{3.11}$$

The left-hand side of (3.9) requires that the phase integral in (3.5) be evaluated from the HFC to a location slightly to the high-field side of the resonance. The phase-integral in (3.5) is evaluated by using the approximation in which we assume that in the WKB regime $1/4z^2 \ll \gamma + \tilde{\gamma}/z$. So that we can keep the leading order contribution, we find that

$$\int_{z_{\rm HFC}}^{z}\sqrt{Q(z')}\,dz' = \sqrt{\gamma}z + \frac{\tilde{\gamma}}{2\sqrt{\gamma}}[\ln z + \ln(4\gamma)], \tag{3.12}$$

https://doi.org/10.1017/S0022377806004739 Published online by Cambridge University Press

in which we approximate $[Q(z)]^{1/4}$ by $\gamma^{1/4}$. Then using the asymptotic forms of the Whittaker functions (Heading 1962) we find

$$A_1 = \gamma^{-1/4}(2\sqrt{\gamma})^{i\kappa}e^{-\pi\kappa/2}A_2,$$
$$B_1 = \gamma^{-1/4}(2\sqrt{\gamma})^{-i\kappa}e^{3\pi\kappa/2}B_2. \tag{3.13}$$

Moreover, the solution in region 1 is matched to the solution $G(z)$ of (3.1) at $z = 0$, and it is clear that

$$\lim_{z\to 0^+} G(z) = \lim_{z\to 0^-} G_1(z). \tag{3.14}$$

Thus

$$A = A_1 \quad \text{and} \quad B = B_1. \tag{3.15}$$

Then, from (3.11) and (3.13),

$$\frac{A}{B} = \left(-\frac{\tilde{\gamma}}{2\sqrt{\gamma}}\right)^{-i\tilde{\gamma}/\sqrt{\gamma}} \exp\left(\frac{-i\tilde{\gamma}}{\sqrt{\gamma}}\right). \tag{3.16}$$

Substituting this into (3.3) completely determines the reflection coefficient. Since from (3.16) $\mid A/B \mid = 1$, we can then express $A/B = e^{i(\pi+\phi)}$. Therefore, we find that the phase ϕ is given by

$$\phi = \sqrt{\gamma}z_{\mathrm{HFC}} + \sqrt{\gamma}z_{\mathrm{HFC}} \ln\left(\frac{\sqrt{\gamma}}{2}z_{\mathrm{HFC}}\right) - \pi, \tag{3.17}$$

where $z_{\mathrm{HFC}} = -\tilde{\gamma}/\gamma < 0$ is the location of the HFC, and ϕ is the phase difference between the mode propagating toward the HFC and the reflected component propagating toward the resonance. Thus the phase introduced by this cutoff is related to the distance between the resonance and the HFC.

If we assume that there is no wave damping between the resonance and the HFC, then A and B differ by an arbitrary phase. Let us assume that $B = Ae^{-i(\pi/2+2\phi)}$. Then the power reflection coefficient, from (3.3), is

$$\begin{aligned} R(\beta,\phi) &= e^{-2\pi\beta}|1 + (1 - e^{\pi\beta})e^{-i(2\phi+2\theta)}|^2 \\ &= e^{-2\pi\beta}[1 + (1 - e^{\pi\beta})^2 + 2(1 - e^{\pi\beta})\cos(2\phi + 2\theta)] \\ &= 1 - 4T(1 - T)\cos^2(\phi + \theta), \end{aligned} \tag{3.18}$$

where $T = e^{-\pi\beta}$. Then the power mode conversion coefficient $C = 1 - R$ is

$$C(\beta,\phi) = 4T(1 - T)\cos^2(\phi + \theta). \tag{3.19}$$

As a function of the phase, the reflection is minimized and the conversion is maximized when $\phi+\theta = n\pi$, for which the mode conversion coefficient is $C_{\max} = 4T(1-T)$ and the good coupling condition $T = 0.5$ is unchanged from the usual mode conversion problem. Hence with a HFC the maximum power mode conversion coefficient can be 100%, in which $T = 0.5$ and $(\phi + \theta)$ is an integer multiple of π. This shows the effect of constructive or destructive interference between the incoming and outgoing waves. A build-up of the field in the resonant region results in more efficient mode conversion of the incoming wave. Thus adding the effect of a HFC reflection in our previous study can significantly alter the power mode converted. Figure 1 illustrates the dependence of conversion coefficient on β and ϕ

https://doi.org/10.1017/S0022377806004739 Published online by Cambridge University Press

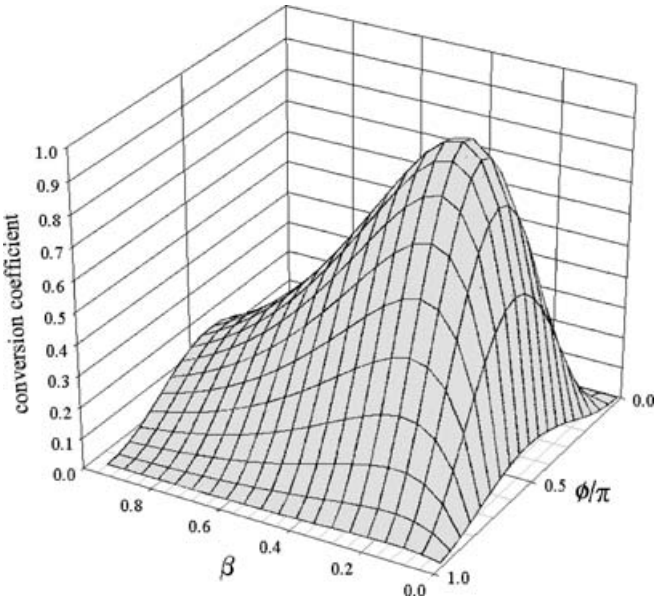


Figure 1. Mesh plot of the constant mode conversion coefficient as a function of β and ϕ.

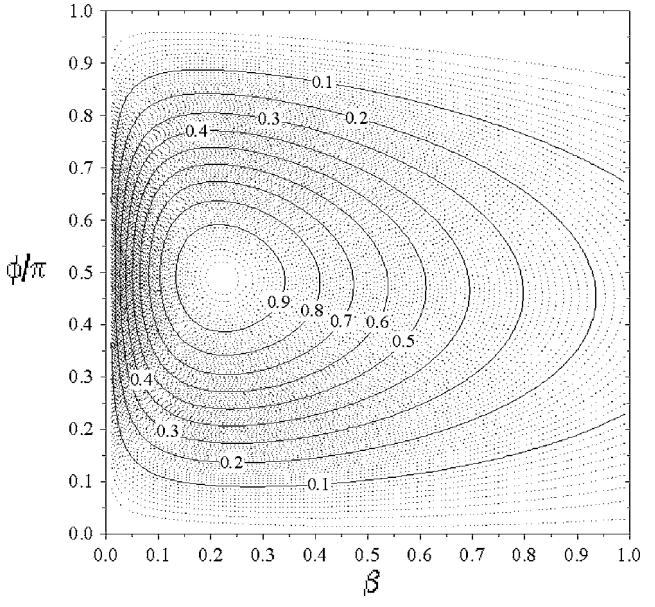


Figure 2. Contours of the constant mode conversion coefficient as a function of β and ϕ.

in a three-dimensional mesh plot. In Fig. 2 the contours of constant $C(\beta, \phi)$ have been plotted as a function of β and ϕ. The contours are periodic in integer multiples of π. These figures show that the maximum value of C is strongly dependent on β, which depends on the size of the coupling region through the parameter η_0 itself. Figure 2 shows that for $\beta < 0.2$ and $\beta > 0.25$ it is not possible to attain ∼100%

https://doi.org/10.1017/S0022377806004739 Published online by Cambridge University Press

mode conversion, regardless of the value of the phase ϕ. The analysis reveals that efficient mode conversion of more than 90% occurs where $0.15 \leq \beta \leq 0.35$ and $0.4\pi \leq \phi \leq 0.6\pi$. Then, from (3.16), the phase ϕ is related to the position of the HFC, z_{HFC}, by

$$\phi = \frac{\sqrt{\gamma}}{2} z_{\mathrm{HFC}} + \frac{\sqrt{\gamma}}{2} z_{\mathrm{HFC}} \ln\left(\frac{\sqrt{\gamma}}{2} z_{\mathrm{HFC}}\right) - \frac{\pi}{4}. \tag{3.20}$$

Therefore, we find that

$$\frac{1.3\pi}{\sqrt{\gamma}} \leq z_{\mathrm{HFC}} + z_{\mathrm{HFC}} \ln\left(\frac{\sqrt{\gamma}}{2} z_{\mathrm{HFC}}\right) \leq \frac{1.7\pi}{\sqrt{\gamma}}. \tag{3.21}$$

This relation implies the limits for z_{HFC} to achieve high mode conversion efficiency. Moreover, for $\beta = \ln(2)/\pi \approx 0.22$, we have $\phi \approx 0.49\pi$ and the phase of $\Gamma(-i\beta/2)$ is $\theta \approx 0.51\pi$. Here there is no transmission, the fields are localized between the resonance and the HFC and the conversion coefficient attains the maximum ratio 100%. For $\phi \approx 0.49\pi$, the distance between the resonance and the HFC, from (3.20), is given by

$$z_{\mathrm{HFC}} + z_{\mathrm{HFC}} \ln\left(\frac{\sqrt{\gamma}}{2} z_{\mathrm{HFC}}\right) \approx \frac{1.48\pi}{\sqrt{\gamma}}. \tag{3.22}$$

Then, if the HFC is not too close to the resonance, the appropriate plasma conditions for complete mode conversion are determined for a given plasma configuration.

The relations (3.21) and (3.22) imply that the variation of z_{HFC} is strongly dependent on the parameter $\sqrt{\gamma}$, i.e. the normalized wave number. This parameter, which is limited by the values of β itself, depends on the coupling parameter η_0. In previous formulations for a high-density finite-temperature plasma, the parameter η_0 has been derived as a function of the spatial variable $\mathbf{x}$, wave vector $\mathbf{k}$ and the local properties of the plasma: density, temperature and magnetic field profiles. The parameter η_0 is varied by manipulating the parallel (to the magnetic field) wave vector $k_\parallel$, the wave frequency or the peak density. Thus, for a given incident wave frequency, $f = \omega/2\pi$, and the local parameters in question, the coupling parameter η_0 can be obtained at the resonance point (Nassiri-Mofakham and Sabzevari 2006). Hence, the equations of η_0 and z_{HFC} are simultaneously solved to give more conditions to the problem in the appropriate range of β and ϕ. These additional criteria control the properties of the incident wave (wave frequency, $k_\parallel$ and launch position), the spatial location of the resonance and the plasma conditions to achieve the efficient and/or complete mode conversion in a high-density plasma with two-dimensional geometry.

4. Conclusions

Through a detailed analysis we have generalized the theory of mode conversion to Bernstein modes to non-planar geometry. We have developed the mode conversion method of Cairns and Lashmore-Davies considering the effect of the high-magnetic-field-side cutoff, which provides a simple way of analyzing the triplet (cutoff–resonance–cutoff) problem in which the HFC adjusts the resonator while the spacing between the LFC and the resonance adjusts the coupling. Once the basic analysis has been carried out, transmission, reflection and conversion coefficients are obtained by means of elementary algebraic manipulations of the local dispersion relation. We have shown that the triplet problem in two-dimensional

https://doi.org/10.1017/S0022377806004739 Published online by Cambridge University Press

geometry can be solved by this method, and the mode conversion coefficient obtained by this analysis varies from a minimum of 0% to a maximum of 100%. The variation depends on the phasing effect introduced by the HFC.

The coupling between the incident wave and internal resonator (composed of the cutoff–resonance–cutoff plasma system) is mainly determined by β (or, from (2.10), by η_0), which gives $T = \exp(-\pi\beta)$. Furthermore, the resonator is specified by the phases ϕ and θ. In other words, a triplet configuration is characterized not only by the resonance and the two cutoffs but also by its characteristic phases. Owing to the intrinsically inhomogeneous plasma, these phases are not simply related to some integer multiple of half wavelength between the LFC and the HFC. We have shown that the phases are related to the distance between the resonance and the HFC, which is concerned with β itself. The phase ϕ then also depends on the form of the potential Q, given by (3.6). Roughly speaking, ϕ/π is the number of local wavelengths that fit between the resonance and the HFC. The phase θ is directly dependent on the parameter β. Thus, the important parameter in achieving high mode conversion efficiency is β.

The analysis reveals that more than 90% of the incident wave energy can be mode converted to Bernstein modes, when we have $0.15 \leq \beta \leq 0.35$ and $0.4\pi \leq \phi \leq 0.6\pi$. These are the general conditions for efficient mode conversion. Moreover, the limited β and ϕ result in a bounding coupling region and a resonant region via the coupling parameter η_0 and the position of z_{HFC} of the HFC, respectively. An analytic expression for η_0 relevant to high-density plasma in non-planar geometry has previously been proposed (Nassiri-Mofakham and Sabzevari 2006), which can determine the range of η_0 whose presence contributes to the expressions (3.21) and (3.22). Also, the boundaries of z_{HFC} may be calculated from (3.21). Thus the limits on η_0 and z_{HFC} give additional conditions to the problem, which must be simultaneously satisfied to achieve high mode conversion efficiency for the given plasma configuration in question. In the special case, where we have chosen parameters such that $\beta = \ln(2)/\pi \approx 0.22$, the phase of $\Gamma(-i\beta/2)$ is $\theta \approx 0.51\pi$ and, from (3.19), $\phi \approx n\pi - 0.51\pi$, where n is an integer. Then, if the HFC, given by (3.22), is not too close to the resonance, the incident wave is critically coupled to the resonator and, for a discrete spectrum of phases, the mode conversion efficiency is 100%. Here there is no reflection toward the LFC and the fields are localized between the resonance and the HFC.

In regard to the outlined optimization, however, the phase and T are not completely independent variables, both depending on the parameter β (or the coupling parameter η_0). The additional degree of freedom allowing optimization in practice is the change in phase via the position of z_{HFC} of the HFC. As the position z_{HFC} is varied by manipulating $k_\parallel$ or the peak density within experimentally realistic limits, a critical coupling may actually never occur. Nevertheless, the effect of the HFC results in enhancing over mode conversion efficiency, and that each configuration should be considered in its particular details.

Appendix A. Asymptotic forms of the Whittaker functions

The second-order differential equation

$$\frac{d^2G}{dz^2} + \left\{ \frac{1}{4z^2} + \frac{a_1 f_1}{2\alpha^2}[i(\mathbf{a}\cdot\mathbf{g} - \mathbf{f}\cdot\mathbf{b}) - 2\eta_0]\frac{1}{z} + \left(\frac{a_1 f_1}{2\alpha^2}(\mathbf{a}\cdot\mathbf{g} - \mathbf{f}\cdot\mathbf{b}) \right)^2 \right\} G(z) = 0, \quad \text{(A 1)}$$

https://doi.org/10.1017/S0022377806004739 Published online by Cambridge University Press

can be expressed in the standard Whittaker form (Abramovitz and Stegun 1970)

$$\frac{d^2G}{d\zeta^2} + \left(\frac{1}{4\zeta^2} + \frac{i\kappa}{\zeta} - \frac{1}{4}\right)G(\zeta) = 0, \tag{A 2}$$

where $\zeta = -i2\sqrt{\gamma z}$ and $\kappa = \tilde{\gamma}/2\sqrt{\gamma}$ with

$$\tilde{\gamma} = \frac{a_1 f_1}{2\alpha^2}[i(\mathbf{a}\cdot\mathbf{g} - \mathbf{f}\cdot\mathbf{b}) - 2\eta_0],$$

$$\gamma = \left(\frac{a_1 f_1}{2\alpha^2}(\mathbf{a}\cdot\mathbf{g} - \mathbf{f}\cdot\mathbf{b})\right)^2 = \left(\frac{a_1 f_1}{\alpha^2}\frac{\eta_0}{\beta}\right)^2, \tag{A 3}$$

$$\beta = \frac{2\eta_0}{(\mathbf{a}\cdot\mathbf{g} - \mathbf{f}\cdot\mathbf{b})}.$$

The two independent solutions are given by the Whittaker functions $W_{\nu,\mu}(\zeta)$ and $W_{-\nu,\mu}(-\zeta)$, where $\nu = i\kappa$, $\mu = 0$. There is a pole at $z = 0$. Since we assume a time dependence of the form $e^{-i\omega t}$, there is a branch cut in the upper half complex-z plane starting at $z = 0$ (Budden 1985). Thus, for $z < 0$ we define $z = |z|e^{-i\pi}$. With this convention, the asymptotic forms of the Whittaker functions are given by (Heading 1962):

for $z \to +\infty$,

$$W_{i\kappa,0}(\zeta) \sim \zeta^{i\kappa}e^{-\zeta/2} = |2\sqrt{\gamma}z|^{i\kappa}e^{\pi\kappa/2}e^{i\sqrt{\gamma}z}, \tag{A 4}$$

$$W_{-i\kappa,0}(-\zeta) \sim \left[\zeta^{-i\kappa}\,e^{\zeta/2} - \frac{2i\pi}{[\Gamma(\frac{1}{2} + i\kappa)]^2}\zeta^{i\kappa}e^{-\zeta/2}\right]e^{-\pi\kappa}$$

$$= |2\sqrt{\gamma}z|^{-i\kappa}e^{-3\pi\kappa/2}e^{-i\sqrt{\gamma}z} - \frac{2i\pi}{\left[\Gamma\left(\frac{1}{2} + i\kappa\right)\right]^2}|2\sqrt{\gamma}z|^{i\kappa}e^{\pi\kappa/2}e^{i\sqrt{\gamma}z}, \tag{A 5}$$

and for $z \to -\infty$,

$$W_{i\kappa,0}(\zeta) \sim \zeta^{i\kappa}e^{-\zeta/2} = |2\sqrt{\gamma}z|^{i\kappa}e^{\pi\kappa/2}e^{i\sqrt{\gamma}z}, \tag{A 6}$$

$$W_{-i\kappa,0}(-\zeta) \sim \zeta^{-i\kappa}e^{\zeta/2}e^{-\pi\kappa} = |2\sqrt{\gamma}z|^{-i\kappa}e^{-3\pi\kappa/2}e^{-i\sqrt{\gamma}z}, \tag{A 7}$$

where Γ is the Gamma function (Abramovitz and Stegun 1970).

References

Abramovitz, M. and Stegun, I. A. 1970 *Handbook of Mathematical Functions*. New York: Dover.

Bender, C. M. and Orszag, S. A. 1978 *Advanced Mathematical Methods for Scientists and Engineers*. New York: McGraw-Hill, ch. 10.

Budden, K. G. 1985 *The Propagation of Radio Waves*. Cambridge: Cambridge University Press, pp. 596–602.

Cairns, R. A. and Lashmore-Davies, C. N. 1983 *Phys. Fluids* **26**, 1268–1274.

Flügge S. 1999 *Practical Quantum Mechanics*. Berlin: Springer, pp. 259.

Fuchs, V., Ram, A. K., Schultz, S. D., Bers, A. and Lashmore-Davies, C. N. 1995 *Phys. Plasmas* **2**, 1637–1647.

Heading, J. 1962 *J. Math. Soc. London* **37**, 195–208.

Majeski, R., Phillips, C. K. and Wilson, J. R. 1994 *Phys. Rev. Lett.* **73**, 2204–2207.

https://doi.org/10.1017/S0022377806004739 Published online by Cambridge University Press

Nassiri-Mofakham, N. and Sabzevari, B. Sh. 2006 *J. Plasma Phys.* **72**, 71–83.

Smirnov, V. I. 1964 *A Course of Higher Mathematics*, Vol. IV. Oxford: Pergamon Press, pp. 387–426.

Sommerfeld, A. 1964 *Partial Differential Equations in Physics: Lectures on Theoretical Physics*. New York: Academic Press, §11.

Stix, T. H. 1992 *Waves in Plasmas*. New York: Springer, ch. 13.

Whittaker, E. T. and Watson, G. N. 1927 *A Course of Modern Analysis*, 4th edn. New York: Cambridge University Press.

https://doi.org/10.1017/S0022377806004739 Published online by Cambridge University Press